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Yuji Hibino
Saga University, Saga, 8408502, JAPAN, hibinoy@cc.saga-u.ac.jp

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CONSTRUCTION OF THE CANONICAL REPRESENTATION FROM A NONCANONICAL REPRESENTATION

YUJI HIBINO*

Abstract. A centered Gaussian process is determined by its covariance. However, the method to construct the canonical representation from the covariance has not been obtained. In this paper, we propose a new method to construct the canonical representation for a Gaussian process by using a noncanonical representation.

1. Introduction

P. Lévy [4] pointed out that the following two process

$$X_1(t) = \int_0^t (2t-u)dB_1(u) \tag{1.1}$$

and

$$X_2(t) = \int_0^t (3t-4u)dB_2(u) \tag{1.2}$$

have the same law, that is to say, they express the same process. After that, the difference of (1.1) and (1.2) was clarified as canonical representation theory. Namely, (1.1) is canonical because $\mathcal{F}_t(X_1) = \mathcal{F}_t(B_1)$ for any $t \geq 0$, while (1.2) is not canonical because $\mathcal{F}_t(X_2) \subset \mathcal{F}_t(B_2)$ for any $t \geq 0$, more precisely $H_t(X_2) = H_t(B_2) \ominus LS\{\int_0^t u^2 dB_2(u)\}$ for any $t \geq 0$, where $\mathcal{F}_t(X)$ means the minimal σ-field in which $\{X(s); s \leq t\}$ is measurable and $H_t(X)$ means the closed linear span of $\{X(s); s \leq t\}$.

As a definition, for some function F, a stochastic integral

$$X(t) = \int_0^t F(t,u)dB(u)$$

is said to be a *canonical representation with respect to* B if X generates the same filtration as the Brownian motion B, namely, it satisfies

$$\mathcal{F}_t(X) = \mathcal{F}_t(B) \text{ for any } t \geq 0.$$

Here, B is called an *innovation process*, and F is called a *canonical kernel function.* In the case of Gaussian processes, the condition is equivalent to

$$H_t(X) = H_t(B) \text{ for any } t \geq 0.$$

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* Corresponding author.

The canonical representation is essentially unique if it exists. (For more detail, refer to [3])

In general, a centered Gaussian process is determined by its covariance function. However, it has been open to find the canonical kernel function from the covariance except the cases of stationary processes and some classes.

Historically Lévy started the investigation of canonical representation theory since he found the noncanonical representation (1.2) in the study of the multi-parameter Brownian motion. Thus noncanonical representations have been considered as junk rather than byproduct. However the author focus on gap of the filtration since the gap characterizes the noncanonical property. In the paper [1], we have found how to construct the noncanonical representation having N-dimensional orthogonal complement:

Theorem 1.1 ([1]). *Let L^2_{loc}-functions $g_1, g_2, \ldots, g_N$ be linearly independent. Put $\left(\Gamma^{ij}(s)\right)_{1\le i,j\le N} = \left(\left(\int_0^s g_i(v)g_j(v)dv\right)_{1\le i,j\le N}\right)^{-1}$. Then*

$$B_{\mathbf{g}}(t) = \int_0^t \left(1 - \int_u^t \sum_{i,j=1}^N g_i(s)\Gamma^{ij}(s)g_j(u)ds\right) dB(u) \tag{1.3}$$

is a noncanonical representation of a Brownian motion satisfying

$$H_t(B) = H_t(B_{\mathbf{g}}) \oplus LS\left\{\int_0^t g_i(u)dB(u); i = 1, 2, \ldots, N\right\}.$$

Here, we prefer to write (1.3) as $\dot{B}_{\mathbf{g}} = (I - K_{\mathbf{g}})\dot{B}$, where the Volterra operator $K_{\mathbf{g}}$ is defined by

$$K_{\mathbf{g}}\alpha(s) = \int_0^s \sum_{i,j=1}^N g_i(s)\Gamma^{ij}(s)g_j(u)\alpha(u)du, \alpha \in L^2.$$

By using this notation, the representation kernel function in (1.3) is expressed as $(I - K^*_{\mathbf{g}})1_{(0,t)}(u)$, where the adjoint operator $K^*_{\mathbf{g}}$ is defined by

$$K^*_{\mathbf{g}}\beta(u) = \int_u^\infty \sum_{i,j=1}^N g_i(s)\Gamma^{ij}(s)g_j(u)\beta(s)ds, \beta \in L^2.$$

Since (1.3) is a Brownian motion, the operator $K_{\mathbf{g}}$ satisfies

$$(I - K_{\mathbf{g}})(I - K^*_{\mathbf{g}}) = I. \tag{1.4}$$

We note that $I - K_{\mathbf{g}}$ is surjective and that $I - K^*_{\mathbf{g}}$ is injective.

Though the theorem above describes a Brownian motion, the similar result is obtained for any Gaussian process having the canonical representation, as follows:

Theorem 1.2 ([1]). *Let $X(t) = \int_0^t F(t,u)dB(u)$ be the canonical representation. Then by using the same notation as the theorem above,*

$$X_{\mathbf{g}}(t) = \int_0^t \left(F(t,u) - \int_u^t \sum_{i,j=1}^N g_i(s)\Gamma^{ij}(s)g_j(u)F(t,s)ds\right) dB(u)$$

is a noncanonical representation of X satisfying

$$H_t(X) = H_t(B) = H_t(X_{\mathbf{g}}) \oplus LS\left\{\int_0^t g_i(u)dB(u); i = 1, 2, \ldots, N\right\}.$$

Thus we acquire a new tool for analyzing the canonical representation. In the next section we propose a new recipe for constructing the canonical representation from a noncanonical one.

2. Canonical Representation

For a noncanonical representation if the gap of the filtration is known, we can construct the canonical representation from the noncanonical one.

Theorem 2.1. *If $X(t) = \int_0^t F(t,u)dB(u)$ is a noncanonical representation satisfying*

$$H_t(B) = H_t(X) \oplus LS\left\{\int_0^t g_i(u)dB(u); i = 1, 2, \ldots, N\right\},$$

then $B_{\mathbf{g}}$, defined by (1.3)*, is an innovation process of X and the canonical representation of X is*

$$X(t) = \int_0^t \left(F(t,s) - \int_0^s \sum_{i,j=1}^N g_i(s)\Gamma^{ij}(s)g_j(u)F(t,u)du \right) dB_{\mathbf{g}}(s).$$

Proof. From the hypothesis we have $H_t(X) = H_t(B_{\mathbf{g}})$. Namely, $B_{\mathbf{g}}$ is an innovation process and there exists a canonical kernel function $\tilde{F}$ such that $X(t) = \int_0^t \tilde{F}(t,u)dB_{\mathbf{g}}(u)$ is a canonical representation. By using the Fubini theorem, we have

$$\begin{aligned}
&\int_0^t \tilde{F}(t,s)dB_{\mathbf{g}}(s) \\
&= \int_0^t \tilde{F}(t,s)dB(s) - \int_0^t \tilde{F}(t,v)\int_0^v \sum_{i,j=1}^N g_i(v)\Gamma^{ij}(v)g_j(s)dB(s)dv \\
&= \int_0^t \tilde{F}(t,s)dB(s) - \int_0^t \int_s^t \sum_{i,j=1}^N g_i(v)\Gamma^{ij}(v)g_j(s)\tilde{F}(t,v)dvdB(s).
\end{aligned}$$

Therefore, $F(t,\bullet) = (I - K_{\mathbf{g}}^*)\tilde{F}(t,\bullet)$. By the use of (1.4), we have $\tilde{F}(t,\bullet) = (I - K_{\mathbf{g}})F(t,\bullet)$. □

In order to show the usefulness of the theorem above, we consider a kind of perturbation of the operator in Theorem 1.1 for $N = 1$. Though the result itself was already known in [2, 5], the method here gives an alternative proof.

Theorem 2.2. *For any given $g \in L^2_{\text{loc}}$, let*

$$X_\lambda(t) = \int_0^t \left(1 - \lambda \int_u^t g(s)\Gamma(s)g(u)ds\right) dB(u), \tag{2.1}$$

where $\Gamma(s) = 1/\int_0^s g(v)^2 dv$. *If* $\lambda \le 1/2$ *then* (2.1) *is canonical with respect to* B, *and if* $\lambda > 1/2$ *then* (2.1) *is noncanonical with*

$$H_t(X_\lambda) = H_t(B) \ominus LS\left\{\int_0^t g(u)\left(\int_0^u g(v)^2 dv\right)^{\lambda-1} dB(u)\right\}.$$

Proof. Suppose $\int_0^t \varphi(u)dB(u)$ is independent of $H_t(X_\lambda)$. Then

$$E[X(r)\int_0^t \varphi(u)dB(u)] = 0, \text{ for any } r \le t.$$

Thus

$$\int_0^r \left(1 - \lambda\int_u^r g(s)\Gamma(s)g(u)ds\right)\varphi(u)du = 0, \text{ for any } r \le t.$$

We have the ordinary differential equation:

$$\frac{\varphi'(r)}{\varphi(r)} = \frac{g'(r)}{g(r)} + (\lambda - 1)\frac{g(r)^2}{\int_0^r g(v)^2 dv}$$

The solution is

$$\varphi(r) = Cg(r)\left(\int_0^r g(v)^2 dv\right)^{\lambda-1} \quad (C : \text{constant}). \tag{2.2}$$

The condition for $\varphi \in L^2_{\mathrm{loc}}$ is verified by

$$\int_0^t \varphi(r)^2 dr = \int_0^t g(r)^2\left(\int_0^r g(v)^2 dv\right)^{2\lambda-2} dr = \int_0^{\|g\|_t^2} x^{2\lambda-2}dx,$$

where $\|g\|_t^2 := \int_0^t g(v)^2 dv$ is finite because $g \in L^2_{\mathrm{loc}}$. Thus, $\varphi \in L^2_{\mathrm{loc}}$ if and only if $\lambda > 1/2$. This means that (2.1) is noncanonical with respect to B, satisfying

$$H_t(X_\lambda) = H_t(B) \ominus LS\left\{\int_0^t \varphi(u)dB(u)\right\}.$$

□

The theorem above says that (2.1) is noncanonical if $\lambda > 1/2$. However, we can find the canonical representation of X_λ by using Theorem 2.1.

Theorem 2.3. *For* $\lambda > 1/2$, *the canonical representation of* X_λ *is*

$$X_\lambda(t) = \int_0^t \left(1 - (1-\lambda)\int_u^t g(s)\Gamma(s)g(u)ds\right)dB_\varphi(u),$$

where φ *is defined by* (2.2) *(here we put* $C = 1$ *for convenience) and* B_φ *is a Brownian motion defined by*

$$B_\varphi(t) = \int_0^t \left(1 - \int_u^t \varphi(s)\Phi(s)\varphi(u)ds\right)dB(u),$$

where $\Phi(s) = 1/\int_0^s \varphi(v)^2 dv$.

Proof. Due to Theorems 2.1 and 2.2, the innovation of X_λ is B_φ. Thus X_λ can be canonically represented by B_φ, and the canonical kernel function is

$$(I - K_\varphi)(I - \lambda K_g^*)1_{(0,t)}.$$

By the use of

$$\int_0^t \varphi(s)^2 ds = \frac{1}{2\lambda - 1}\left(\int_0^t g(v)^2 dv\right)^{2\lambda-1},$$

after tedious calculation, we have

$$K_\varphi K_g^* = \frac{1}{\lambda} K_\varphi + \frac{2\lambda - 1}{\lambda} K_g^*.$$

Therefore, the result follows since

$$(I - K_\varphi)(I - \lambda K_g^*)1_{(0,t)} = (I - (1-\lambda)K_g^*)1_{(0,t)}.$$

□

This means that X_λ and $X_{1-\lambda}$ have the same law and that the former is non-canonical while the latter is canonical when $\lambda > 1/2$, which is similar to the relation between (1.2) and (1.1). This fact was already known for $N \geq 1$ in [2]. However, since the orthogonal complement has not been obtained for $N > 1$, our method cannot be applied.

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Yuji Hibino: Faculty of Science and Engineering, Saga University, Saga, 8408502, JAPAN

Email address: `hibinoy@cc.saga-u.ac.jp`