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Backward stochastic Navier-Stokes equations in two dimensions

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BACKWARD STOCHASTIC NAVIER-STOKES EQUATIONS IN TWO DIMENSIONS

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A List of Notations

G	a bounded domain in $\mathbb{R}^2$ with smooth boundary
$\mathcal{V}$	$\{\varphi \in C_0^\infty(G)^2 \vert \mathrm{Div}\varphi = 0\}$
H	closure of $\mathcal{V}$ in $L^2(G)^2$
V	closure of $\mathcal{V}$ in $H_0^1(G)^2$
V'	the dual space of V
P	orthogonal Leray projection from $L^2(G)^2$ to H
$tr(A)$	the trace of the square matrix A.
x^T	the transpose of the vector (or matrix) x.
$\langle \cdot, \cdot \rangle_K$	inner product in some Hilbert space K.
$\langle \cdot, \cdot \rangle_{V',V}$	the duality pairing for space V and the dual V'
I_A	the indicator function of the set A.
$C([0,T];\mathbb{K})$	the set of all continuous functions φ: $[0,T] \to \mathbb{K}$.
$C_b(U)$	the set of all uniformly bounded, continuous functions on U, where U is a metric space.
$L^p(0,T;\mathbb{K})$	the set of all Lebesgue measurable functions φ: $[0,T] \to \mathbb{K}$ such that $\int_0^T \Vert\varphi(t)\Vert_{\mathbb{K}}^p dt < \infty$ where $p \in [1,\infty)$.

$L^\infty(0,T;\mathbb{K})$	the set of all essentially bounded measurable functions φ: $[0,T] \to \mathbb{K}$.
$L^p_{\mathcal{G}}(\Omega;\mathbb{K})$	the set of all $\mathbb{K}$-valued $\mathcal{G}$-measurable random variables X such that $E\|X\|^p_{\mathbb{K}} < \infty$ where $p \in [1,\infty)$. Here $\mathcal{G}$ is a σ-algebra.
$L^\infty_{\mathcal{G}}(\Omega;\mathbb{K})$:	the set of all essentially bounded $\mathbb{K}$-valued $\mathcal{G}$-measurable random variables.
$L^p_{\mathcal{F}}(\Omega;L^p(0,T;\mathbb{K}))$	the set of all $\{\mathcal{F}_t\}_{t\geq 0}$-adapted $\mathbb{K}$-valued processes $X(\cdot)$ such that $E\int_0^T \|X(t)\|^p_{\mathbb{K}}dt < \infty$.
$L_{\mathcal{F}}(\Omega;L^\infty(0,T;\mathbb{K}))$	the set of all $\{\mathcal{F}_t\}_{t\geq 0}$-adapted $\mathbb{K}$-valued essentially bounded processes.
$L^p_{\mathcal{F}}(\Omega;C([0,T];\mathbb{K}))$	the set of all $\{\mathcal{F}_t\}_{t\geq 0}$-adapted $\mathbb{K}$-valued continuous processes $X(\cdot)$ such that $E\sup_{t\in[0,T]}\|X(t)\|^p_{\mathbb{K}} < \infty$, where $p \in [0,\infty)$.

Abstract

There are two parts in this dissertation. The backward stochastic Lorenz system is studied in the first part. Suitable a priori estimates for adapted solutions of the backward stochastic Lorenz system are obtained. The existence and uniqueness of solutions is shown by the use of suitable truncations and approximations. The continuity of the adapted solutions with respect to the terminal data is also established.

The backward two-dimensional stochastic Navier-Stokes equations (BSNSEs, for short) corresponding to incompressible fluid flow in a bounded domain G are studied in the second part. Suitable a priori estimates for adapted solutions of the BSNSEs are obtained which reveal a surprising pathwise $L^{\infty}(H)$ bound on the solutions. The existence of solutions is shown by using a monotonicity argument. Uniqueness is proved by using a novel method that uses finite-dimensional projections, linearization, and truncations. The continuity of the adapted solutions with respect to the terminal data and the external body force is also established.

Chapter 1
Introduction

1.1 Lorenz System

The first part of this dissertation focuses on the backward stochastic Lorenz system. In a celebrated work, Edward N. Lorenz introduced a nonlinear system of ordinary differential equations describing fluid convection of nonperiodic flows (Lorenz [15]). The derivation of these equations is from a model of fluid flow within a region of uniform depth and with higher temperature at the bottom (Rayleigh [23]).

Lorenz introduced three time-dependent variables. The variable X is proportional to the intensity of the convective motion, Y is proportional to the temperature difference between ascending and descending currents, and Z is proportional to distortion of the vertical temperature profile from linearity. The model consists of the following three equations:

$$\begin{cases} \dot{X} = -aX + aY \\ \dot{Y} = -XZ + bX - Y \\ \dot{Z} = XY - cZ \end{cases} \tag{1.1.1}$$

where a is the *Prandtl number*, b is the temperature difference of the heated layer and c is related to the size of the fluid cell. The numbers a, b, and c are all positive.

In the past 40 years, ranging from physics (Sparrow [27]) to physiology of the human brain (Weiss [30]), Lorenz systems have been widely studied in many areas for a variety of parameter values. Randomness has also been introduced into Lorenz system and some properties of the forward system have been studied (Schmalfuß [26] and Keller [11]).

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ be a complete probability space with the filtration $\{\mathcal{F}_t\}_{t\geq 0}$, and $\{W(t)\}$ be a 3-dimensional Wiener process defined on it. Define a matrix A as follows:

$$A = \begin{pmatrix} a & -a & 0 \\ -b & 1 & 0 \\ 0 & 0 & c \end{pmatrix}$$

For any $y = \left(\begin{smallmatrix} y_1 \\ y_2 \\ y_3 \end{smallmatrix}\right)$ and $\bar{y} = \left(\begin{smallmatrix} \bar{y}_1 \\ \bar{y}_2 \\ \bar{y}_3 \end{smallmatrix}\right)$ in $\mathbb{R}^3$, we define an operator B as follows:

$$B(y, \bar{y}) = \begin{pmatrix} 0 \\ y_1\bar{y}_3 \\ -y_1\bar{y}_2 \end{pmatrix}$$

Then the *backward stochastic Lorenz system* corresponding to equation (1.1.1) is given by the following terminal value problem:

$$\begin{cases} dY(t) = -(AY(t) + B(Y(t), Y(t)))dt + Z(t)dW(t) \\ Y(T) = \xi \end{cases}$$

for $t \in [0, T]$ and $\xi \in Ł^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^3)$. The integral form of the backward stochastic Lorenz system is as follows:

$$Y(t) = \xi + \int_t^T (AY(s) + B(Y(s), Y(s)))ds - \int_t^T Z(s)dW(s).$$

The problem consists in finding a pair of adapted solutions $\{(Y(t), Z(t))\}_{t\in[0,T]}$.

Linear backward stochastic differential equations were introduced by Bismut in 1973 ([1]), and the systematic study of general backward stochastic differential equations (BSDEs for short) were put forward first by Pardoux and Peng in 1990

([22]). Since the theory of BSDEs is well connected with nonlinear partial differential equations, nonlinear semigroups and stochastic controls, it has been intensively studied in the past two decades. There are also various applications of BSDEs in the theory of mathematical finance. For instance, the hedging and pricing of a contingent claim can be described as linear BSDEs.

In the present work, since the coefficient B is nonlinear and unbounded, the existing theory of BSDEs does not apply. To overcome this difficulty, a truncation of the coefficient and an approximation scheme have been used.

1.2 Navier-Stokes Equations

A major part of this dissertation involves the backward stochastic Navier-Stokes equations. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ be a stochastic basis satisfying the usual conditions over which a Hilbert space valued Wiener process $\{W(t)\}$ with a nuclear covariance operator Q is defined. The two-dimensional stochastic Navier-Stokes system in a bounded domain $G \subset \mathbb{R}^2$ with no-slip condition is given by

$$\begin{aligned} \partial \mathbf{u} + \{(\mathbf{u} \cdot \nabla)\mathbf{u} - \nu \Delta \mathbf{u}\} dt &= \{-\nabla p + \mathbf{f}(t)\} dt + \sigma(t, \mathbf{u})\, dW(t) \\ \nabla \cdot \mathbf{u} &= 0 \end{aligned}$$

with $\mathbf{u}(t, x) = 0 \quad x \in \partial G$, and $\mathbf{u}(0, x) = \mathbf{u}_0(x) \quad x \in G$. Here p denotes pressure, a real-valued random field, and $\mathbf{u}_0$ is the initial condition. The solution consists of $(\mathbf{u}, p)$, where $\mathbf{u}$ is a two-dimensional random field. It is well-known (as explained in the next section) that the above system can be written in the abstract evolution equation setup as

$$d\mathbf{u}(t) + \{\nu \mathbf{A}\mathbf{u}(t) + \mathbf{B}(\mathbf{u}(t))\} dt = \mathbf{f}(t) dt + \sigma(t, \mathbf{u}(t))\, dW(t) \tag{1.2.1}$$

with $\mathbf{u}(0) = \mathbf{u}_0$. In this equation the pressure p doesn't appear. However, it can be determined from the solution $\mathbf{u}$ of (1.2.1) and the no-slip boundary condition. The backward stochastic Navier-Stokes equation corresponding to the equation (1.2.1) is given by the following terminal value problem: for $0 \leq t \leq T$,

$$\begin{cases} d\mathbf{u}(t) = -\nu \mathbf{A}\mathbf{u}(t)dt - \mathbf{B}(\mathbf{u}(t))dt + \mathbf{f}(t)dt + Z(t)dW(t) \\ \mathbf{u}(T) = \xi \end{cases}$$

The above stochastic equation is understood in the integral form:

$$\mathbf{u}(t) = \xi + \int_t^T \{\nu \mathbf{A}\mathbf{u}(s) + \mathbf{B}(\mathbf{u}(s)) - \mathbf{f}(s)\}ds - \int_t^T Z(s)dW(s)$$

The problem consists in finding an adapted solution $\{\mathbf{u}(t), Z(t)\}$ for $0 \leq t \leq T$.

A backward stochastic Navier-Stokes system can be viewed as an inverse problem wherein the velocity profile at a time T is observed and given, and the noise coefficient has to be ascertained from the given terminal data. Such a motivation arises naturally when one understands the importance of inverse problems in partial differential equations (see J. L. Lions [13], [14]). A systematic study of backward stochastic differential equations was initiated by Pardoux and Peng, Ma, Protter, Yong, Zhou, and several other authors. Ma and Yong have studied linear degenerate backward stochastic differential equations motivated by stochastic control theory. Later, Hu, Ma and Yong [8] considered the semi-linear equations as well. Backward stochastic partial differential equations were shown to arise naturally in stochastic versions of the Black-Scholes formula by Ma and Yong [17]. A nice introduction to backward stochastic differential equations is presented in the book by Yong and Zhou [31], with various applications.

In the present work, it is worthwhile to note that the drift coefficient in the backward stochastic Navier-Stokes equation (BSNSE) is nonlinear and unbounded, so that the usual methods of proving existence and uniqueness of solutions do not apply. The drift coefficient is monotone on bounded $L^4(G)$ balls in V, which was first observed by Menaldi and Sritharan [19]. The method of monotonicity is used in this chapter to prove the existence and uniqueness of solutions to BSNSEs. Much of the work owes itself to a surprising a priori estimate on $\sup_{[0,T]} \|\mathbf{u}(t)\|_H^2$ that holds almost surely. Such an estimate seldom holds for stochastic Navier-Stokes equations that move forward in time. The proof of the uniqueness of solutions is wrought by establishing the closeness of (a) solutions of finite-dimensional projections of the BSNSE, (b) solutions of a linearized projected BSNSE, and (c) finite-dimensional projections of solutions of the BSNSE. Existence and uniqueness of solutions is shown to hold under the natural square integrability condition $E\|\xi\|_H^2 < \infty$ on the terminal value ξ. Continuity of the solution with respect to the data at the terminal time T and the external body force $\mathbf{f}(s)$ is also established in this article.

1.3 The Organization

The organization of this dissertation is as follows. In chapter 2, some background knowledge of backward stochastic differential equations is introduced and some frequently used results are stated.

In chapter 3, the backward stochastic Lorenz system is studied. In section 2, a priori estimates for the solutions of systems with uniformly bounded terminal values are obtained and a truncation of the system is introduced. In section 3, we prove the existence and uniqueness of the solution to the Lorenz system using an approximation scheme. Section 4 is devoted to the continuity of the solutions with respect to terminal data.

In chapter 4, backward stochastic Navier-Stokes equations are studied. In section 1, the setup of the problem, the function spaces, and the background results are presented. The a priori estimates for the solutions are proved in section 2. The existence of solutions to the backward stochastic Navier-Stokes equations is shown by the Minty-Browder monotonicity argument in section 3 for bounded terminal data. In section 4, uniqueness of solutions is proved along with the estimates that are needed in the proof of uniqueness. The boundedness condition on the terminal data is relaxed to integrability conditions in section 5. In section 6, the continuity properties of the solution are studied.

Chapter 2

Backward Stochastic Differential Equations

2.1 Introduction

Backward stochastic differential equations (BSDEs for short) have received considerable attention in the last two decades. The theory of BSDEs has been greatly developed because of its natural connections with stochastic partial differential equations, stochastic controls, mathematical finance, etc.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ be a complete probability space with the filtration $\{\mathcal{F}_t\}_{t\geq 0}$, and $\{W(t)\}$ be a standard one-dimensional Brownian motion. An example of a BSDE is the following:

$$\begin{cases} dY(t) = h(t, Y(t), Z(t))dt + Z(t)dW(t) \\ Y(T) = \xi, \end{cases}$$

where $t \in [0,T]$, $Y(\cdot)\in L^2_{\mathcal{F}}(\Omega; C([0,T];\mathbb{R}))$, $Z(\cdot)\in L^2_{\mathcal{F}}(\Omega; L^2(0,T;\mathbb{R}))$, $\xi \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R})$, and

$$h(t,x,y) : [0,T] \times \mathbb{R} \times \mathbb{R} \times \Omega \to \mathbb{R}.$$

The integral form of this BSDE is

$$Y(t) = \xi - \int_t^T h(s, Y(s), Z(s))ds - \int_t^T Z(s)dW(s)$$

A BSDE can not be considered as a time-reversing transformation of an initial valued problem. Since all the integrals are in the sense of Itô type stochastic calculus, simply reversing the time would destroy the adaptedness. So the solution of a BSDE turns out to be a pair of adapted processes $(Y(\cdot), Z(\cdot))$, i.e., $Z(\cdot)$ is also unknown and it is used to correct the nonadaptedness caused by the backward nature of the problem and the terminal value ξ.

Definition 2.1.1. A pair of processes $(Y(t), Z(t)) \in \mathcal{M}[0,T]$ is called an adapted solution of the above BSDE if the following holds:

$$Y(t) = \xi - \int_t^T h(s, Y(s), Z(s))ds - \int_t^T Z(s)dW(s) \quad \forall t \in [0,T], \text{ P-a.s.}$$

Here

$$\mathcal{M}[0,T] = L^2_{\mathcal{F}}(\Omega; C([0,T]; \mathbb{R})) \times L^2_{\mathcal{F}}(\Omega; L^2(0,T; \mathbb{R}))$$

and it is equipped with the norm

$$\|Y(\cdot), Z(\cdot)\|_{\mathcal{M}[0,T]} = \{E(\sup_{0\le t\le T} |Y(t)|^2) + E\int_0^T |Z(t)|^2 dt\}^{\frac{1}{2}}.$$

2.2 Backward Gronwall's Inequality

Let us first introduce the forward Gronwall inequality.

Proposition 2.2.1. *Suppose that $g(t)$, $\alpha(t)$, $\beta(t)$ and $\gamma(t)$ are integrable functions, and $\beta(t), \gamma(t) \ge 0$. For $0 \le s \le t$, if*

$$g(t) \le \alpha(t) + \beta(t)\int_s^t \gamma(\rho)g(\rho)d\rho \tag{2.2.1}$$

then

$$g(t) \le \alpha(t) + \beta(t)\int_s^t \alpha(\eta)\gamma(\eta)e^{\int_\eta^t \beta(\rho)\gamma(\rho)d\rho}d\eta.$$

In particular, if $\alpha(t) \equiv \alpha$, $\beta(t) \equiv \beta$ and $\gamma(t) \equiv 1$, then

$$g(t) \le \alpha e^{\beta(t-s)}$$

Proof:

$$\frac{d}{dt}\Big(e^{-\int_s^t \beta(\rho)\gamma(\rho)d\rho}\int_s^t \gamma(\rho)g(\rho)d\rho\Big)$$

$$
\begin{aligned}
&= -\beta(t)\gamma(t)e^{-\int_s^t \beta(\rho)\gamma(\rho)d\rho}\int_s^t \gamma(\rho)g(\rho)d\rho + e^{-\int_s^t \beta(\rho)\gamma(\rho)d\rho}\gamma(t)g(t) \\
&= \gamma(t)e^{-\int_s^t \beta(\rho)\gamma(\rho)d\rho}\Big[g(t) - \beta(t)\int_s^t \gamma(\rho)g(\rho)d\rho\Big] \\
&\leq \alpha(t)\gamma(t)e^{-\int_s^t \beta(\rho)\gamma(\rho)d\rho}, \text{ by (2.2.1).}
\end{aligned}
$$

Thus

$$
e^{-\int_s^t \beta(\rho)\gamma(\rho)d\rho}\int_s^t \gamma(\rho)g(\rho)d\rho \leq \int_s^t \alpha(\eta)\gamma(\eta)e^{-\int_s^\eta \beta(\rho)\gamma(\rho)d\rho}d\eta
$$

and

$$
\int_s^t \gamma(\rho)g(\rho)d\rho \leq e^{\int_s^t \beta(\rho)\gamma(\rho)d\rho}\int_s^t \alpha(\eta)\gamma(\eta)e^{-\int_s^\eta \beta(\rho)\gamma(\rho)d\rho}d\eta.
$$

Hence from (2.2.1) we get

$$
\begin{aligned}
g(t) &\leq \alpha(t) + \beta(t)e^{\int_s^t \beta(\rho)\gamma(\rho)d\rho}\int_s^t \alpha(\eta)\gamma(\eta)e^{-\int_s^\eta \beta(\rho)\gamma(\rho)d\rho}d\eta \\
&= \alpha(t) + \beta(t)\int_s^t \alpha(\eta)\gamma(\eta)e^{\int_\eta^t \beta(\rho)\gamma(\rho)d\rho}d\eta.
\end{aligned}
$$

□

Another more general version of Gronwall's inequality is given below.

Proposition 2.2.2. *Let μ be a Borel measure on $[0,\infty)$, let $\epsilon \geq 0$, and let f be a Borel measurable function that is bounded on bounded intervals and satisfies*

$$
0 \leq f(t) \leq \epsilon + \int_{[0,t)} f(s)\mu(ds),\ t \geq 0. \tag{2.2.2}
$$

Then

$$
f(t) \leq \epsilon e^{\mu[0,t)},\ t \geq 0.
$$

In particular, if $M > 0$ and

$$
0 \leq f(t) \leq \epsilon + M\int_0^t f(s)ds,\ t \geq 0,
$$

then

$$
f(t) \leq \epsilon e^{Mt},\ t \geq 0.
$$

Proof: Iteration of (2.2.2) with itself gives

$$\begin{aligned} f(t) &\leq \epsilon + \epsilon \sum_{k=1}^{\infty} \int_{[0,t)} \int_{[0,s_1)} \cdots \int_{[0,s_{k-1})} \mu(ds_k) \cdots \mu(ds_2)\mu(ds_1) \\ &= \epsilon + \epsilon \sum_{k=1}^{\infty} \frac{1}{k!} (\mu[0,t))^k \\ &= \epsilon e^{\mu[0,t)}. \end{aligned}$$

□

Now let us discuss Gronwall's inequality for backward differential equations.

Proposition 2.2.3. *Suppose that $g(t)$, $\alpha(t)$, $\beta(t)$ and $\gamma(t)$ are integrable functions, and $\beta(t), \gamma(t) \geq 0$. For $0 \leq t \leq T$, if*

$$g(t) \leq \alpha(t) + \beta(t) \int_t^T \gamma(\rho) g(\rho) d\rho \tag{2.2.3}$$

then

$$g(t) \leq \alpha(t) + \beta(t) \int_t^T \alpha(\eta)\gamma(\eta) e^{\int_\eta^t \beta(\rho)\gamma(\rho)d\rho} d\eta.$$

In particular, if $\alpha(t) \equiv \alpha$, $\beta(t) \equiv \beta$ and $\gamma(t) \equiv 1$, then

$$g(t) \leq \alpha(2 - e^{-\beta(T-t)})$$

Proof:

$$\begin{aligned} & \frac{d}{dt}\Big(e^{-\int_t^T \beta(\rho)\gamma(\rho)d\rho} \int_t^T \gamma(\rho) g(\rho) d\rho \Big) \\ &= \beta(t)\gamma(t) e^{-\int_t^T \beta(\rho)\gamma(\rho)d\rho} \int_t^T \gamma(\rho) g(\rho) d\rho - e^{-\int_t^T \beta(\rho)\gamma(\rho)d\rho} \gamma(t) g(t) \\ &= \gamma(t) e^{-\int_t^T \beta(\rho)\gamma(\rho)d\rho} \Big[-g(t) + \beta(t) \int_t^T \gamma(\rho) g(\rho) d\rho \Big] \\ &\geq -\alpha(t)\gamma(t) e^{-\int_t^T \beta(\rho)\gamma(\rho)d\rho}, \text{ by (2.2.3)}. \end{aligned}$$

Thus

$$-e^{-\int_t^T \beta(\rho)\gamma(\rho)d\rho} \int_t^T \gamma(\rho) g(\rho) d\rho \geq -\int_t^T \alpha(\eta)\gamma(\eta) e^{-\int_\eta^T \beta(\rho)\gamma(\rho)d\rho} d\eta.$$

and

$$\int_t^T \gamma(\rho)g(\rho)d\rho \leq e^{\int_t^T \beta(\rho)\gamma(\rho)d\rho} \int_t^T \alpha(\eta)\gamma(\eta)e^{-\int_\eta^T \beta(\rho)\gamma(\rho)d\rho}d\eta.$$

Hence from (2.2.3) we get

$$\begin{aligned} g(t) &\leq \alpha(t) + \beta(t)e^{\int_t^T \beta(\rho)\gamma(\rho)d\rho} \int_t^T \alpha(\eta)\gamma(\eta)e^{-\int_\eta^T \beta(\rho)\gamma(\rho)d\rho}d\eta \\ &= \alpha(t) + \beta(t)\int_t^T \alpha(\eta)\gamma(\eta)e^{\int_\eta^t \beta(\rho)\gamma(\rho)d\rho}d\eta. \end{aligned}$$

□

2.3 The Backward Itô Formula

A very useful tool is the backward version of the Itô formula. We will first state the one-dimensional Itô formula.

Proposition 2.3.1. *([20]) Let $X(t)$ be an Itô process given by*

$$dX(t) = a(t, X(t))dt + b(t, X(t))dW(t)$$

where a is $\mathcal{F}_t$-adapted and integrable, and b is $\mathcal{F}_t$-adapted and square integrable. Let $f(t,x) \in C^2([0,\infty) \times \mathbb{R})$. Then

$$\begin{aligned} df(t, X(t)) =& [f_1'(t, X(t)) + a(t, X(t))f_2'(t, X(t)) + \frac{1}{2}f_2''(t, X(t))b^2(t, X(t))]dt \\ &+ f_2'(t, X(t))b(t, X(t))dW(t), \end{aligned}$$

or, in integral form,

$$\begin{aligned} f(t, X(t)) &= f(T, X(T)) - \int_t^T \Big[f_1'(s, X(s)) + f_2'(s, X(s))a(s, X(s)) \\ &\quad + \frac{1}{2}f_2''(s, X(s))b^2(s, X(s))\Big]ds - \int_t^T f_2'(s, X(s))b(s, X(s))dW(s). \end{aligned}$$

Remark 2.3.2. The Itô formula in the usual sense is as follows:

$$f(t, X(t)) = f(0, X(0)) + \int_0^t [f_1'(s, X(s)) + f_2'(s, X(s))a(s, X(s))$$

$$+\frac{1}{2}f_2''(s,X(s))b^2(s,X(s))]ds+\int_0^t f_2'(s,X(s))b(s,X(s))dW(s).$$

Now let us state the n-dimensional Itô formula.

Proposition 2.3.3. *([20]) Let $X(t)$ be an n-dimensional Itô process given by*

$$dX(t)=a(t,X(t))dt+b(t,X(t))dW(t),$$

where $X(t)=\begin{pmatrix} X_1(t)\\ \vdots \\ X_n(t)\end{pmatrix}$, $a(t,X(t))=\begin{pmatrix} a_1(t,X(t))\\ \vdots \\ a_n(t,X(t))\end{pmatrix}$, $W(t)=\begin{pmatrix} W_1(t)\\ \vdots \\ W_m(t)\end{pmatrix}$, *and*

$$b(t,X(t))=\begin{pmatrix} b_{11}(t,X(t)) & \cdots & b_{1m}(t,X(t))\\ \vdots & & \vdots \\ b_{n1}(t,X(t)) & \cdots & b_{nm}(t,X(t))\end{pmatrix}$$

where m is any natural number. We also assume that a is $\mathcal{F}_t$-adapted and integrable, and b is $\mathcal{F}_t$-adapted and square integrable. Let $f(t,x)=(f_1(t,x),\cdots,f_p(t,x))$ be a C^2 map from $[0,\infty)\times\mathbb{R}^n$ into $\mathbb{R}^p$. Then the process $f(t,X(t))$ is a p-dimensional Itô process, whose k-th component is given by

$$\begin{aligned}f_k(t,X(t))=&f_k(T,X(T))-\int_t^T\frac{\partial f_k(s,X(s))}{\partial s}ds-\sum_i\int_t^T\frac{\partial f_k(s,X(s))}{\partial x_i}dX_i\\&-\frac{1}{2}\sum_{i,j}\frac{\partial^2 f_k(s,X(s))}{\partial x_ix_j}dX_idX_j.\end{aligned}$$

2.4 Some General Results on Backward Stochastic Differential Equations

2.4.1 Linear Backward Stochastic Differential Equations

Theorem 2.4.1. *(Yong and Zhou [31]) Let $A(\cdot),B_1(\cdot),\dots,B_m(\cdot)\in L_{\mathcal{F}}(\Omega;L^\infty(0,T;\mathbb{R}^{k\times k}))$. Then, for any $f\in L^2_{\mathcal{F}}(\Omega;L^2(0,T;\mathbb{R}^k))$ and $\xi\in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^k)$, the following* *backward stochastic differential equation,*

$$\begin{cases} dY(t) = & \Big\{A(t)Y(t) + \sum_{j=1}^{m} B_j(t)Z_j(t) + f(t)\Big\}dt \\ & +Z(t)dW(t),\ t \in [0,T], \\ Y(T) = & \xi, \end{cases} \tag{2.4.1}$$

admits a unique adapted solution $(Y(\cdot), Z(\cdot))$ *in the space* $L^2_{\mathcal{F}}(\Omega; C([0,T]; \mathbb{R}^k)) \times L^2_{\mathcal{F}}(\Omega; L^2(0,T; \mathbb{R}^{k\times m}))$*, and there exists a constant* $K > 0$ *such that*

$$\begin{aligned} &E\big(\sup_{t\in[0,T]} |Y(t)|^2\big) + \sum_{j=1}^{m} E\int_0^T |Z_j(t)|^2 dt \\ \le &K\Big\{E|\xi|^2 + E\int_0^T |f(t)|^2 dt\Big\}. \end{aligned} \tag{2.4.2}$$

Proof: We first consider two SDEs for $\mathbb{R}^{k\times k}$-valued processes:

$$\begin{cases} d\Phi(t) = \Big\{A(t)\Phi(t) + \sum_{j=1}^{m} B_j(t)B_j(t)\Phi(t)\Big\}dt \\ \qquad\qquad + \sum_{j=1}^{m} B_j(t)\Phi(t)dW^j(t), \\ \Phi(0) = I, \end{cases} \tag{2.4.3}$$

$$\begin{cases} d\Psi(t) = -\Psi(t)A(t)dt - \sum_{j=1}^{m} B_j(t)\Psi(t)dW^j(t), \\ \Psi(0) = I. \end{cases} \tag{2.4.4}$$

Since (2.4.3) and (2.4.4) are usual linear SDEs with bounded coefficients, they admit unique strong solutions $\Phi(t)$ and $\Psi(t)$, which are $\{\mathcal{F}_t\}_{t\ge 0}$-adapted. Now by the Itô formula,

$$d[\Psi(t)\Phi(t)] = -\,\Psi(t)A(t)\Phi(t)dt - \sum_{j=1}^{m} \Psi(t)B_j(t)\Phi(t)dW^j(t)$$

$$+\Psi(t)\big[A(t)\Phi(t)+\sum_{j=1}^{m}B_j(t)B_j(t)\Phi(t)\big]dt \tag{2.4.5}$$
$$+\sum_{j=1}^{m}\Psi(t)B_j(t)\Phi(t)dW^j(t)-\sum_{j=1}^{m}\Psi(t)B_j(t)B_j(t)\Phi(t)dt=0.$$

Since we have $\Phi(0)=\Psi(0)=I$, we must have

$$\Psi(t)^{-1}=\Phi(t), \qquad \forall t\in[0,T],\ \text{P-a.s.} \tag{2.4.6}$$

Next, we suppose $(Y(\cdot),Z(\cdot))$ is an adapted solution of (2.4.1). Applying the Itô formula to $\Psi(t)Y(t)$, we have

$$\begin{aligned} d[\Psi(t)Y(t)]=&-\Psi(t)A(t)Y(t)dt-\sum_{j=1}^{m}\Psi(t)B_j(t)Y(t)dW^j(t)\\ &+\Psi(t)\big[A(t)Y(t)+\sum_{j=1}^{m}B_j(t)Z_j(t)+f(t)\big]dt \\ &+\sum_{j=1}^{m}\Psi(t)Z_j(t)dW^j(t)-\sum_{j=1}^{m}\Psi(t)B_j(t)Z_j(t)dt\\ =&\ \Psi(t)f(t)dt+\sum_{j=1}^{m}\Psi(t)[Z_j(t)-B_j(t)Y(t)]dW^j(t)\end{aligned} \tag{2.4.7}$$

Thus,

$$\begin{aligned}\Psi(t)Y(t)=&\ \Psi(T)\xi-\int_t^T\Psi(s)f(s)ds\\ &-\sum_{j=1}^{m}\int_t^T\Psi(s)[Z_j(s)-B_j(s)Y(s)]dW^j(s)\\ =&\ \theta+\int_0^t\Psi(s)f(s)ds \\ &-\sum_{j=1}^{m}\int_t^T\Psi(s)[Z_j(s)-B_j(s)Y(s)]dW^j(s),\end{aligned} \tag{2.4.8}$$

where

$$\theta\triangleq\Psi(T)\xi-\int_0^T\Psi(s)f(s)ds. \tag{2.4.9}$$

Taking $E(\cdot|\mathcal{F}_t)$ on both sides of (2.4.8), we get

$$\Psi(t)Y(t) = \int_0^t \Psi(s)f(s)ds + E(\theta|\mathcal{F}_t), \qquad t \in [0,T]. \tag{2.4.10}$$

Note that the right-hand side of (2.4.10) depends only on ξ and $f(\cdot)$.

From (2.4.10), we see that one should define

$$Y(t) = \Phi(t)\Big\{\int_0^t \Psi(s)f(s)ds + E(\theta|\mathcal{F}_t)\Big\}, \qquad t \in [0,T], \tag{2.4.11}$$

with $\Phi(\cdot)$ and $\Psi(\cdot)$ being the solutions of (2.4.3) and (2.4.4), respectively, and θ being defined by (2.4.9). We now prove that the $Y(\cdot)$ defined by (2.4.11) together with some $Z(\cdot)$ will be an adapted solution of (2.4.1).

First of all, by (2.4.11) and (2.4.9), we have

$$Y(T) = \Phi(T)\Big\{\int_0^T \Psi(s)f(s)ds + \theta\Big\} = \Phi(T)\Psi(T)\xi = \xi. \tag{2.4.12}$$

Next, since $E(\theta|\mathcal{F}_t)$ is a square-integrable martingale, by the martingale representation theorem we can find a unique $\eta \equiv (\eta_1, \ldots, \eta_m) \in L^2_{\mathcal{F}}(\Omega; L^2(0,T;\mathbb{R}^{k\times m}))$ such that

$$E(\theta|\mathcal{F}_t) = E\theta + \sum_{j=1}^m \int_0^t \eta_j(s)dW^j(s), \qquad \forall t \in [0,T], \text{ P-a.s.} \tag{2.4.13}$$

Hence it follows from (2.4.11) and (2.4.13) that

$$\begin{aligned} Y(t) &= \Phi(t)\Big\{\int_0^t \Psi(s)f(s)ds + \sum_{j=1}^m \int_0^t \eta_j(s)dW^j(s) + E\theta\Big\}, \\ &\triangleq \Phi(t)r(t), \qquad t \in [0,T], \end{aligned} \tag{2.4.14}$$

Applying Itô formula, we obtain

$$dY(t) = \Big\{A(t)\Phi(t)r(t) + \sum_{j=1}^m B_j(t)B_j(t)\Phi(t)r(t)\Big\}dt$$

$$
\begin{aligned}
&+\sum_{j=1}^{m} B_j(t)\Phi(t)r(t)dW^j(t) + \Phi(t)\Psi(t)f(t)dt \\
&+\sum_{j=1}^{m} \Phi(t)\eta_j(t)dW^j(t) + \sum_{j=1}^{m} B_j(t)\Phi(t)\eta_j(t)dt \qquad (2.4.15)\\
=&\Big\{A(t)Y(t) + \sum_{j=1}^{m} B_j(t)[B_j(t)Y(t) + \Phi(t)\eta_j(t)] + f(t)\Big\}dt \\
&+\sum_{j=1}^{m}[B_j(t)Y(t) + \Phi(t)\eta_j(t)]dW^j(t).
\end{aligned}
$$

Therefore, by setting $Z(\cdot) = (Z_1(\cdot), \dots, Z_m(\cdot))$ with

$$
Z_j(t) \triangleq B_j(t)Y(t) + \Phi(t)\eta_j(t), \qquad t \in [0,T], \text{ P-a.s., } 1 \le j \le m, \qquad (2.4.16)
$$

and using (2.4.12) and (2.4.15), we conclude that $(Y(\cdot), Z(\cdot)) \in L^{2,loc}_{\mathcal{F}}(\Omega; L^2(0,T;\mathbb{R}^k)) \times L^{2,loc}_{\mathcal{F}}(\Omega; L^2(0,T;\mathbb{R}^{k\times m}))$ satisfy (2.4.1).

Next we show that in fact $(Y(\cdot), Z(\cdot)) \in L^2_{\mathcal{F}}(\Omega; C([0,T];\mathbb{R}^k) \times L^2_{\mathcal{F}}(\Omega; L^2(0,T;\mathbb{R}^{k\times m}))$. Now for each n we define an $\{\mathcal{F}\}_{t\ge 0}$-stopping time

$$
\tau_n \triangleq \inf\{t \ge 0 : \int_0^t |Z(s)|^2 ds \ge n\} \wedge T.
$$

It is clear that τ_n increases to T P-a.s. as $n \to \infty$. Applying Itô's formula to $|Y(t\wedge\tau_n)|^2$, we obtain

$$
\begin{aligned}
&E|Y(0)|^2 + E\int_0^{T\wedge\tau_n} \sum_{j=1}^{m} |Z_j(s)|^2 ds \\
=&\ E|Y(T\wedge\tau_n)|^2 \\
&- 2E\int_0^{T\wedge\tau_n} \langle Y(s), A(s)Y(s) + \sum_{j=1}^{m} B_j(s)Z_j(s) + f(s)\rangle ds \qquad (2.4.17)\\
\le&\ E|Y(T\wedge\tau_n)|^2 + KE\int_0^{T\wedge\tau_n} \{|Y(s)|^2 + |f(s)|^2\} ds \\
&+ \frac{1}{2}E\int_0^{T\wedge\tau_n} \sum_{j=1}^{m} |Z_j(s)|^2 ds.
\end{aligned}
$$

for some constant K. Hence,

$$\sum_{j=1}^{m} E\int_0^{T\wedge\tau_n} |Z_j(s)|^2 ds \leq K\Big\{E|Y(T\wedge\tau_n)|^2 + E\int_0^{T\wedge\tau_n} |f(s)|^2 ds\Big\}. \qquad (2.4.18)$$

Letting $n\to\infty$ and using Fatou's lemma, we conclude that

$$\sum_{j=1}^{m} E\int_0^{T} |Z_j(s)|^2 ds \leq K\Big\{E|\xi|^2 + E\int_0^{T} |f(s)|^2 ds\Big\}. \qquad (2.4.19)$$

This shows that $Z(\cdot)\in L^2_{\mathcal{F}}(\Omega; L^2(0,T;\mathbb{R}^{k\times m}))$.

We now prove $Y(\cdot)\in L^2_{\mathcal{F}}(\Omega; C([0,T];\mathbb{R}^k))$. From (2.4.15) and (2.4.12), we obtain

$$Y(t) = \xi - \int_t^T h(s)ds - \int_t^T Z(s)dW(s), \qquad t\in[0,T], \qquad (2.4.20)$$

for some $h(\cdot)\in L^2_{\mathcal{F}}(\Omega; L^2(0,T;\mathbb{R}^k))$. Thus, by Bürkholder-Davis-Gundy's inequality and Doob's inequality, we have

$$\begin{aligned}
&\Big(E\Big\{\sup_{t\in[0,T]} |Y(t)|^2\Big\}\Big)^{\frac{1}{2}} \\
\leq&(E|\xi|^2)^{\frac{1}{2}} + \Big(E\Big\{\int_0^T |h(s)|ds\Big\}^2\Big)^{\frac{1}{2}} \\
&+ \Big(E\Big|\int_0^T Z(s)dW(s)\Big|^2\Big)^{\frac{1}{2}} + \Big(E\sup_{t\in[0,T]}\Big|\int_0^t Z(s)dW(s)\Big|^2\Big)^{\frac{1}{2}} \\
\leq&(E|\xi|^2)^{\frac{1}{2}} + \sqrt{T}\Big(E\int_0^T |h(s)|^2 ds\Big)^{\frac{1}{2}} + 3\Big(E\int_0^T |Z(s)|^2 ds\Big)^{\frac{1}{2}} < \infty.
\end{aligned}$$

This implies $Y(\cdot)\in L^2_{\mathcal{F}}(\Omega; C([0,T];\mathbb{R}^k))$, and therefore $(Y(\cdot), Z(\cdot))$ is an adapted solution of (2.4.1). Combined with (2.4.19), we obtain the estimate (2.4.2). The uniqueness follows, as the equation is linear.

□

2.4.2 Nonlinear Backward Stochastic Differential Equations

For the rest of this section, we assume the following.

For any $(y,z) \in \mathbb{R}^k \times \mathbb{R}^{k\times m}$, $h(t,y,z) : [0,T] \times \mathbb{R}^k \times \mathbb{R}^{k\times m} \times \Omega \to \mathbb{R}^k$ is $\{\mathcal{F}_t\}_{t\geq 0}$-adapted with $h(\cdot,0,0) \in L^2_{\mathcal{F}}(\Omega; L^2(0,T;\mathbb{R}^k))$. Moreover, there exists an $L > 0$, such that

$$|h(t,y,z) - h(t,\bar{y},\bar{z})| \leq L\{|y-\bar{y}| + |z-\bar{z}|\}$$

$\forall t \in [0,T]$, $y, \bar{y} \in \mathbb{R}^k$ and $z, \bar{z} \in \mathbb{R}^{k\times m}$ P-a.s.

Theorem 2.4.2. *(Yong and Zhou [31]) Under the previous assumption, for any given* $\xi \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^k)$*, the BSDE*

$$\begin{cases} dY(t) = h(t,Y(t),Z(t))dt + Z(t)dW(t), \quad t \in [0,T), \ \textit{a.s.} \\ Y(T) = \xi, \end{cases} \tag{2.4.21}$$

admits a unique adapted solution $(Y(\cdot), Z(\cdot)) \in \mathcal{M}[0,T]$.

Proof: For any fixed $(y(\cdot), z(\cdot)) \in \mathcal{M}[0,T]$, it follows from the previous assumption that

$$h(\cdot) \equiv h(\cdot, y(\cdot), z(\cdot)) \in L^2_{\mathcal{F}}(\Omega; L^2(0,T;\mathbb{R}^k)). \tag{2.4.22}$$

Consider the following BSDE:

$$\begin{cases} dY(t) = h(t,y(t),z(t))dt + Z(t)dW(t), \quad t \in [0,T), \ \text{a.s.} \\ Y(T) = \xi. \end{cases} \tag{2.4.23}$$

This is a linear BSDE. By Theorem 2.4.1, it admits a unique adapted solution $(Y(\cdot), Z(\cdot)) \in \mathcal{M}_\beta[0,T]$. Hence, we can define an operator $\tau : \mathcal{M}_\beta[0,T] \to \mathcal{M}_\beta[0,T]$ by $(y,z) \mapsto (Y,Z)$ via the BSDE (2.4.23). We are going to prove that for some $\beta > 0$ and $\forall (y,z), (\bar{y},\bar{z}) \in \mathcal{M}_\beta[0,T]$,

$$\|\tau(y,z) - \tau(\bar{y},\bar{z})\|_{\mathcal{M}_\beta[0,T]} \leq \frac{1}{2}\|(y,z) - (\bar{y},\bar{z})\|_{\mathcal{M}_\beta[0,T]}. \tag{2.4.24}$$

This means that τ is a contraction mapping on the Banach space $\mathcal{M}_\beta[0,T]$. Then we can use the contraction mapping theorem to claim the existence and uniqueness of the fixed point of τ, which is the unique adapted solution of (2.4.21).

To prove (2.4.24), take any $(y(\cdot), z(\cdot)), (\bar{y}(\cdot), \bar{z}(\cdot)) \in \mathcal{M}[0,T]$, and let

$$(Y(\cdot), Z(\cdot)) = \tau(y,z), (\bar{Y}(\cdot), \bar{Z}(\cdot)) = \tau(\bar{y}, \bar{z}).$$

Define

$$\begin{cases} \hat{Y}(t) \triangleq Y(t) - \bar{Y}(t), \hat{Z}(t) \triangleq Z(t) - \bar{Z}(t), \\ \hat{y}(t) \triangleq y(t) - \bar{y}(t), \hat{z}(t) \triangleq z(t) - \bar{z}(t), \\ \hat{h}(t) \triangleq h(t, y(t), z(t)) - h(t, \bar{y}(t), \bar{z}(t)). \end{cases} \tag{2.4.25}$$

Let $\beta > 0$ be undetermined. Applying the Itô formula to $|\hat{Y}(t)|^2 e^{2\beta t}$, we have

$$\begin{aligned} &|\hat{Y}(t)|^2 e^{2\beta t} + \int_t^T |\hat{Z}(s)|^2 e^{2\beta s} ds \\ =& - \int_t^T \{2\beta |\hat{Y}(s)|^2 + 2\langle \hat{Y}(s), \hat{h}(s)\rangle\} e^{2\beta s} ds \\ &- \int_t^T 2e^{2\beta s} \langle \hat{Y}(s), \hat{Z}(s) dW(s)\rangle \\ \leq& \int_t^T \{-2\beta |\hat{Y}(s)|^2 + 2L|\hat{Y}(s)|(|\bar{y}(s)| + |\bar{z}(s)|)\} e^{2\beta s} ds \\ &- \int_t^T 2e^{2\beta s} \langle \hat{Y}(s), \hat{Z}(s) dW(s)\rangle \\ \leq& \int_t^T \{(-2\beta + \frac{2L^2}{\lambda})|\hat{Y}(s)|^2 + \lambda(|\bar{y}(s)|^2 + |\bar{z}(s)|^2)\} e^{2\beta s} ds \\ &- \int_t^T 2e^{2\beta s} \langle \hat{Y}(s), \hat{Z}(s) dW(s)\rangle \end{aligned} \tag{2.4.26}$$

where we take $\lambda \triangleq \frac{2L^2}{\beta} > 0$. Then the above implies

$$\begin{aligned} &|\hat{Y}(t)|^2 e^{2\beta t} + \int_t^T |\hat{Z}(s)|^2 e^{2\beta s} ds \\ \leq& \lambda(T+1)\{ \sup_{t\in[0,T]} (|\hat{y}(t)|^2 e^{2\beta t}) + \int_0^T |\hat{z}(s)|^2 e^{2\beta s} ds\} \end{aligned} \tag{2.4.27}$$

$$-\int_t^T 2e^{2\beta s}\langle \hat{Y}(s), \hat{Z}(s)dW(s)\rangle.$$

By taking the expectation, one obtains

$$E\{|\hat{Y}(t)|^2 e^{2\beta t} + \int_t^T |\hat{Z}(s)|^2 e^{2\beta s} ds\} \leq \lambda(T+1)\|(\hat{y},\hat{z})\|^2_{\mathcal{M}_\beta[0,T]}. \tag{2.4.28}$$

On the other hand, by the Bürkholder-Davis-Gundy inequality, we have

$$\begin{aligned}
&E\{\sup_{0\leq t\leq T} |\int_t^T e^{2\beta s}\langle \hat{Y}(s), \hat{Z}(s)dW(s)\rangle|\} \\
\leq &E|\int_0^T e^{2\beta s}\langle \hat{Y}(s), \hat{Z}(s)dW(s)\rangle| \\
&+ E\{\sup_{0\leq t\leq T} |\int_0^t e^{2\beta s}\langle \hat{Y}(s), \hat{Z}(s)dW(s)\rangle|\} \\
\leq &2E\{\sup_{0\leq t\leq T} |\int_0^t e^{2\beta s}\langle \hat{Y}(s), \hat{Z}(s)dW(s)\rangle|\} \\
\leq &KE\{\int_0^T |\hat{Y}(s)|^2|\hat{Z}(s)|^2 e^{4\beta s} ds\}^{\frac{1}{2}} \qquad (2.4.29) \\
\leq &KE\{(\sup_{0\leq t\leq T}(|\hat{Y}(t)|^2 e^{2\beta t}))^{\frac{1}{2}}(\int_0^T \|\hat{Z}(s)\|^2 e^{2\beta s} ds)^{\frac{1}{2}}\} \\
\leq &\frac{1}{4}E(\sup_{0\leq t\leq T}(|\hat{Y}(t)|^2 e^{2\beta t})) + K^2 E\int_0^T \|\hat{Z}(s)\|^2 e^{2\beta s} ds \\
\leq &\frac{1}{4}E(\sup_{0\leq t\leq T}(|\hat{Y}(t)|^2 e^{2\beta t})) + K^2\lambda(T+1)\|(\hat{y},\hat{z})\|^2_{\mathcal{M}_\beta[0,T]}.
\end{aligned}$$

Consequently, from (2.4.27), we obtain

$$\begin{aligned}
&E(\sup_{0\leq t\leq T}(|\hat{Y}(t)|^2 e^{2\beta t})) \\
\leq &\lambda(T+1)\|(\hat{y},\hat{z})\|^2_{\mathcal{M}_\beta[0,T]} \\
&+ 2E\{\sup_{0\leq t\leq T} |\int_t^T e^{2\beta s}\langle \hat{Z}^T(s)\hat{Y}(s), dW(s)\rangle|\} \qquad (2.4.30) \\
\leq &\frac{1}{2}E(\sup_{0\leq t\leq T}(|\hat{Y}(t)|^2 e^{2\beta t})) + (1+2K^2)\lambda(T+1)\|(\hat{y},\hat{z})\|^2_{\mathcal{M}_\beta[0,T]}.
\end{aligned}$$

Combining (2.4.28) and (2.4.30), we get

$$\|(\hat{Y},\hat{Z})\|^2_{\mathcal{M}_\beta[0,T]} \leq \frac{2(3+4K^2)(T+1)L^2}{\beta}\|(\hat{y},\hat{z})\|^2_{\mathcal{M}_\beta[0,T]}. \tag{2.4.31}$$

Then we can choose $\beta > 0$ large enough to get the contractivity of the operator τ on $\mathcal{M}_\beta[0,T]$, which in turn implies the existence and the uniqueness of the adapted solution to (2.4.21). $\square$

Chapter 3
An Example: Lorenz System

3.1 Introduction

Lorenz original derivation of the equations

$$\begin{cases} \dot{X} = -aX + aY \\ \dot{Y} = -XZ + bX - Y \\ \dot{Z} = XY - cZ \end{cases}$$

are from a model for fluid flow of the atmosphere: a two-dimensional fluid cell is warmed from below and cooled from above and the resulting convective motion is modeled by a partial differential equation. The variables are expanded into an infinite number of modes and all except three of them are put to zero. In this chapter, the randomness are introduced into Lorenz system and some properties of the inverse problem of the forward system are studied.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ be a complete probability space with the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ and $\{W(t)\}$ be a 3-dimensional Wiener process defined on it. Define a matrix A as follows:

$$A = \begin{pmatrix} a & -a & 0 \\ -b & 1 & 0 \\ 0 & 0 & c \end{pmatrix}$$

For any $y = \left(\begin{smallmatrix} y_1 \\ y_2 \\ y_3 \end{smallmatrix}\right)$ and $\bar{y}= \left(\begin{smallmatrix} \bar{y}_1 \\ \bar{y}_2 \\ \bar{y}_3 \end{smallmatrix}\right)$ in $\mathbb{R}^3$, we define an operator B as follows:

$$B(y, \bar{y}) = \begin{pmatrix} 0 \\ y_1\bar{y}_3 \\ -y_1\bar{y}_2 \end{pmatrix}$$

Then the *backward stochastic Lorenz system* corresponding to equation (1.1.1) is given by the following terminal value problem:

$$\begin{cases} dY(t) = -(AY(t) + B(Y(t), Y(t)))dt + Z(t)dW(t) \\ Y(T) = \xi \end{cases} \tag{3.1.1}$$

for $t \in [0, T]$ and $\xi \in Ł^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^3)$. The integral form of the backward stochastic Lorenz system is as follows:

$$Y(t) = \xi + \int_t^T (AY(s) + B(Y(s), Y(s)))ds - \int_t^T Z(s)dW(s).$$

The problem consists in finding a pair of adapted solutions $\{(Y(t), Z(t))\}_{t\in[0,T]}$.

Definition 3.1.1. A pair of processes $(Y(t), Z(t))\in \mathcal{M}[0, T]$ is called an adapted solution of (3.1.1) if the following holds:

$$Y(t) = \xi + \int_t^T (AY(s) + B(Y(s), Y(s)))ds - \int_t^T Z(s)dW(s) \quad \forall t \in [0, T], \text{P-a.s.}$$

Here

$$\mathcal{M}[0, T] = L^2_{\mathcal{F}}(\Omega; C([0, T]; \mathbb{R}^3)) \times L^2_{\mathcal{F}}(\Omega; L^2(0, T; \mathbb{R}^{3\times3}))$$

and it is equipped with the norm

$$\|Y(\cdot), Z(\cdot)\|_{\mathcal{M}[0,T]} = \{E(\sup_{0\le t\le T} |Y(t)|^2) + E\int_0^T |Z(t)|^2dt\}^{\frac{1}{2}}.$$

3.2 A Priori Estimates

A frequently used property of B is stated below:

Proposition 3.2.1. *For any y and $\bar{y} \in \mathbb{R}^3$, we have*

$$\langle B(y, \bar{y}), \bar{y} \rangle = 0$$

and

$$|B(y, y) - B(\bar{y}, \bar{y})|^2 \leq (|y|^2 + |\bar{y}|^2)|y - \bar{y}|^2.$$

Let $E^{\mathcal{F}_t} X$ to be the conditional expectation $E(X|\mathcal{F}_t)$, and let us list few assumptions:

A1 : $|\xi|^2 \leq K$ for some constant K, P-a.s.

A2 : $\xi \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n)$

Proposition 3.2.2. *Under Assumption* **A1**, *if $(Y(t), Z(t))$ is an adapted solution for Lorenz system* (3.1.1), *then there exists a constant N_0, such that $|Y(t)| \leq N_0$ for all $t \in [0, T]$, P-a.s.*

Proof: Applying the Itô formula to $|Y(t)|^2$,

$$d|Y(t)|^2 = \{-2\langle Y(t), AY(t) + B(Y(t), Y(t))\rangle + \|Z(t)\|^2\}dt + 2\langle Y(t), Z(t)\rangle dW(t).$$

Therefore

$$\begin{aligned} |Y(t)|^2 = &|\xi|^2 + \int_t^T 2\langle Y(s), AY(s) + B(Y(s), Y(s))\rangle ds \\ &- \int_t^T \|Z(s)\|^2 ds - 2\int_t^T \langle Y(s), Z(s)\rangle dW(s). \end{aligned}$$

Since $\langle Y(t), B(Y(t), Y(t))\rangle = 0$ by Proposition 3.2.1,

$$|Y(t)|^2 + \int_t^T \|Z(s)\|^2 ds = |\xi|^2 + \int_t^T 2\langle Y(s), AY(s)\rangle ds$$

$$-2\int_t^T \langle Y(s), Z(s)\rangle dW(s).$$

For all $0 \le r \le t \le T$, we have:

$$\begin{aligned}
& E^{\mathcal{F}_r}|Y(t)|^2 + E^{\mathcal{F}_r}\int_t^T \|Z(s)\|^2 ds \\
= \;& E^{\mathcal{F}_r}|\xi|^2 + 2E^{\mathcal{F}_r}\int_t^T \langle Y(s), AY(s)\rangle ds \\
\le \;& E^{\mathcal{F}_r}|\xi|^2 + 2\|A\|\int_t^T E^{\mathcal{F}_r}|Y(s)|^2 ds.
\end{aligned}$$

By Gronwall's inequality (2.2.3),

$$\begin{aligned}
& E^{\mathcal{F}_r}|Y(t)|^2 + E^{\mathcal{F}_r}\int_t^T \|Z(s)\|^2 ds \\
\le & E^{\mathcal{F}_r}|\xi|^2 + 2\|A\|\int_t^T E^{\mathcal{F}_r}|\xi|^2 e^{\int_s^t 2\|A\| dv} ds \\
= & E^{\mathcal{F}_r}|\xi|^2 + 2\|A\| E^{\mathcal{F}_r}|\xi|^2 \int_t^T e^{2\|A\|(t-s)} ds \\
= & E^{\mathcal{F}_r}|\xi|^2 + 2\|A\| E^{\mathcal{F}_r}|\xi|^2 \frac{1}{2\|A\|}(1 - e^{-2\|A\|(T-t)}) \\
= & E^{\mathcal{F}_r}|\xi|^2 (2 - e^{-2\|A\|(T-t)}).
\end{aligned}$$

Letting r to be t, and by Assumption **A1**, it follows that $|Y(t)|^2 \le N_0$ for some constant $N_0 > 0$ which is only related to K. □

Definition 3.2.3. Let $b(y) = Ay + B(y, y)$, and for all $N \in \mathbb{N}$ and $y \in \mathbb{R}^3$, we define

$$b^N(y) = \begin{cases} b(y) & \text{if } |y| \le N \\ b(\frac{y}{|y|}N) & \text{if } |y| > N. \end{cases}$$

The *truncated Lorenz system* is the following BSDE:

$$\begin{cases} dY^N(t) = -b^N(Y^N(t))dt + Z^N(t)dW(t) \\ Y^N(T) = \xi \end{cases} \tag{3.2.1}$$

where $W(t)$ is a 3-dimensional Wiener process defined on a complete probability space $(\Omega, \mathcal{F}, P)$ and $\xi \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^3)$.

Corollary 3.2.4. *Under Assumption* **A1***, if* $(Y^N(t), Z^N(t))$ *is an adapted solution for truncated Lorenz system* (3.2.1)*, then there exists a constant* N_0*, such that* $|Y^N(t)| \leq N_0$ *for all* $N \in \mathbb{N}$ *and* $t \in [0, T]$*, P-a.s.*

Proof: Apply Itô formula to $|Y^N(t)|^2$, we get:

$$\begin{aligned} d|Y^N(t)|^2 =& \{-2\langle Y^N(t), b^N(Y^N(t))\rangle + \|Z^N(t)\|^2\}dt \\ &+ 2\langle Y^N(t), Z^N(t)\rangle dW(t). \end{aligned}$$

So we have

$$\begin{aligned} |Y^N(t)|^2 =& |\xi|^2 + \int_t^T 2\langle Y^N(s), b^N(Y^N(s))\rangle ds - \int_t^T \|Z^N(s)\|^2 ds \\ &- 2\int_t^T \langle Y^N(s), Z^N(s)\rangle dW(s). \end{aligned}$$

If $|Y^N(t)| \leq N$, then $\langle Y^N(t), B(Y^N(t), Y^N(t))\rangle = 0$. If $|Y^N(t)| > N$, then we also have $\langle Y^N(t), B(\frac{Y^N(t)}{|Y^N(t)|}N, \frac{Y^N(t)}{|Y^N(t)|}N)\rangle = 0$. Let

$$a^N(y) = \begin{cases} Ay & \text{if } |y| \leq N \\ A\frac{y}{|y|}N & \text{if } |y| > N. \end{cases}$$

Then $\langle Y^N(t), b^N(Y^N(t))\rangle = \langle Y^N(t), a^N(Y^N(t))\rangle$ and

$$\begin{aligned} |Y^N(t)|^2 + \int_t^T \|Z^N(s)\|^2 ds =& |\xi|^2 + \int_t^T 2\langle Y^N(s), a^N(Y^N(s))\rangle ds \\ &- 2\int_t^T \langle Y^N(s), Z^N(s)\rangle dW(s). \end{aligned}$$

For all $0 \leq r \leq t \leq T$, we have:

$$E^{\mathcal{F}_r}|Y^N(t)|^2 + E^{\mathcal{F}_r}\int_t^T \|Z^N(s)\|^2 ds$$

$$
\begin{aligned}
&= E^{\mathcal{F}_r}|\xi|^2 + 2E^{\mathcal{F}_r}\int_t^T \langle Y^N(s), a^N(Y^N(s))\rangle ds \\
&\leq E^{\mathcal{F}_r}|\xi|^2 + 2\|A\|\int_t^T E^{\mathcal{F}_r}|Y^N(s)|^2 ds.
\end{aligned}
$$

By Gronwall's inequality (2.2.3),

$$
\begin{aligned}
& E^{\mathcal{F}_r}|Y^N(t)|^2 + E^{\mathcal{F}_r}\int_t^T \|Z^N(s)\|^2 ds \\
\leq\ & E^{\mathcal{F}_r}|\xi|^2 + 2\|A\|\int_t^T E^{\mathcal{F}_r}|\xi|^2 e^{\int_s^t 2\|A\| dv} ds \\
=\ & E^{\mathcal{F}_r}|\xi|^2 + 2\|A\| E^{\mathcal{F}_r}|\xi|^2 \int_t^T e^{2\|A\|(t-s)} ds \\
=\ & E^{\mathcal{F}_r}|\xi|^2 + 2\|A\| E^{\mathcal{F}_r}|\xi|^2 \frac{1}{2\|A\|}(1 - e^{-2\|A\|(T-t)}) \\
=\ & E^{\mathcal{F}_r}|\xi|^2 (2 - e^{-2\|A\|(T-t)}).
\end{aligned}
$$

Now, let r to be t, and by Assumption **A1**, we get $|Y^N(t)|^2 \leq N_0$ for some constant $N_0 > 0$ which is only related to K.

□

3.3 Existence and Uniqueness of Solutions

Proposition 3.3.1. *The function b^N is Lipschitz continuous on $\in \mathbb{R}^3$.*

Proof: For any y and $\bar{y} \in \mathbb{R}^3$, let us assume that $y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ and $\bar{y} = \begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \\ \bar{y}_3 \end{pmatrix}$.

Case I: $|y|$, $|\bar{y}| \leq N$. Then we have

$$
\begin{aligned}
& |b^N(y) - b^N(\bar{y})| \\
=\ & |b(y) - b(\bar{y})| \\
=\ & |Ay + B(y,y) - A\bar{y} - B(\bar{y},\bar{y})| \\
=\ & \left| \begin{pmatrix} a & -a & 0 \\ -b & 1 & 0 \\ 0 & 0 & c \end{pmatrix} (y - \bar{y}) + \begin{pmatrix} 0 \\ y_1 y_3 \\ -y_1 y_2 \end{pmatrix} - \begin{pmatrix} 0 \\ \bar{y}_1 \bar{y}_3 \\ -\bar{y}_1 \bar{y}_2 \end{pmatrix} \right|
\end{aligned}
$$

$$\leq \|A\||y-\bar{y}| + \sqrt{(y_1y_3 - \bar{y}_1\bar{y}_3)^2 + (y_1y_2 - \bar{y}_1\bar{y}_2)^2}$$

$$\leq |ac - abc||y-\bar{y}| + N|y-\bar{y}|.$$

Let $L_N = |ac - abc| + N$. Thus $|b^N(y) - b^N(\bar{y})| \leq L_N|y-\bar{y}|$.

Case II: $|y| \leq N$, but $|\bar{y}| > N$. Then by Case I, we have

$$|b^N(y) - b^N(\bar{y})| = |b(y) - b(\frac{\bar{y}}{|\bar{y}|}N)| \leq L_N|y - \frac{\bar{y}}{|\bar{y}|}N|.$$

Let us prove that $|y - \frac{\bar{y}}{|\bar{y}|}N| \leq |y-\bar{y}|$.

By carefully choosing a coordinate system, it is possible to make $\bar{y} = (\bar{y}_1, 0, 0)$. Under such coordinate system, we have

$$|y - \frac{\bar{y}}{|\bar{y}|}N| = [(y_1 - sign(\bar{y}_1)N)^2 + y_2^2 + y_3^2]^{\frac{1}{2}} \quad \text{and}$$

$$|y-\bar{y}| = [(y_1 - \bar{y}_1)^2 + y_2^2 + y_3^2]^{\frac{1}{2}}.$$

Since $|y_1| \leq N < |\bar{y}_1|$, it is clear that $|y_1 - sign(\bar{y}_1)N| \leq |y_1 - \bar{y}_1|$. So $|y - \frac{\bar{y}}{|\bar{y}|}N| \leq |y-\bar{y}|$ and thus

$$|b^N(y) - b^N(\bar{y})| \leq L_N|y-\bar{y}|.$$

Case III: $|y| > N$ and $|\bar{y}| > N$. Then by Case I, we have

$$|b^N(y) - b^N(\bar{y})| = |b(\frac{y}{|y|}N) - b(\frac{\bar{y}}{|\bar{y}|}N)| \leq L_N|\frac{y}{|y|}N - \frac{\bar{y}}{|\bar{y}|}N|.$$

Without lose of generality, let us assume that $|y| \leq |\bar{y}|$, Consider $|y|$ as N in Case II. It is clear that

$$|\frac{y}{|y|}N - \frac{\bar{y}}{|\bar{y}|}N| = \frac{N}{|y|}|y - \frac{|y|}{|\bar{y}|}\bar{y}| \leq \frac{N}{y}|y-\bar{y}|$$

Thus we have shown that

$$|b^N(y) - b^N(\bar{y})| \leq L_N|y-\bar{y}| \quad \text{for Case III}$$

and the proof is complete. □

Theorem 3.3.2. *Under Assumption* **A1***, the Lorenz system* (3.1.1) *has a unique solution.*

Proof: First let us prove the existence of the solution of Lorenz system (3.1.1). By Proposition 3.3.1, b^N is Lipschitz. Thus b^N satisfies the assumption of Theorem 2.4.2. By Theorem 2.4.2, there exists a unique solution $(Y^N(t), Z^N(t))$ of truncated Lorenz system (3.2.1) with such b^N for each $N \in \mathbb{N}$.

Because of Assumption **A1**, by Corollary 3.2.4, there exists a natural number N_0, such that

$$|Y^N(t)| \leq N_0 \quad \text{for all } N \in \mathbb{N}.$$

Since $|Y^{N_0}(t)| \leq N_0$, $b^{N_0}(Y^{N_0}(t)) = b(Y^{N_0}(t))$ by the definition of $b^N(y)$. Thus for N_0, truncated Lorenz system (3.2.1) is the same as Lorenz system (3.1.1). Hence $(Y^{N_0}(t), Z^{N_0}(t))$ is also solution of Lorenz system (3.1.1).

Let $(Y(t), Z(t))$ and $(\bar{Y}(t), \bar{Z}(t))$ be two pairs of solutions of Lorenz system (3.1.1). By Proposition 3.2.2, there exists a natural number N_0, such that $|Y(t)| \leq N_0$ and $|\bar{Y}(t)| \leq N_0$. Since Lorenz system (3.1.1) and truncated Lorenz system (3.2.1) for $N = N_0$ are the same, $(Y(t), Z(t))$ and $(\bar{Y}(t), \bar{Z}(t))$ are also solutions of truncated Lorenz system (3.2.1) for $N = N_0$. By Theorem 2.4.2, we know that truncated Lorenz system (3.2.1) for $N = N_0$ has unique solution. Thus $(Y(t), Z(t))$ $= (\bar{Y}(t), \bar{Z}(t))$ P-a.s. Thus the uniqueness of the solution has been shown.

□

Definition 3.3.3. For any ξ satisfies Assumption **A2** and $n \in \mathbb{N}$, we define

$$\xi^n = \xi \vee (-n) \wedge n.$$

The *n-Lorenz system* is the following BSDE:

$$\begin{cases} dY^n(t) = -b(Y^n(t))dt + Z^n(t)dW(t) \\ Y^n(T) = \xi^n \end{cases} \tag{3.3.1}$$

where $W(t)$ is a 3-dimensional Wiener process defined on a complete probability space $(\Omega, \mathcal{F}, P)$.

Proposition 3.3.4. *Under Assumption* **A2***,the solutions of n-Lorenz systems are Cauchy in* $\mathcal{M}[0,T]$.

Proof: Since ξ^n is bounded by n, the existence and uniqueness of the solution of n-Lorenz system is guaranteed by Proposition 3.2.2.

For all n and $m \in \mathbb{N}$, let $(Y^n(t), Z^n(t))$ and $(Y^m(t), Z^m(t))$ be the unique solutions of n-Lorenz system and m-Lorenz system, respectively. Define

$$\begin{aligned} \widetilde{Y}(t) &= Y^n(t) - Y^m(t), \\ \widetilde{Z}(t) &= Z^n(t) - Z^m(t) \text{ and} \\ \widetilde{\xi} &= \xi^n - \xi^m. \end{aligned}$$

Then we have

$$\begin{aligned} \widetilde{Y}(t) =& \widetilde{\xi} + \int_t^T (A\widetilde{Y}(s) + B(Y^n(s), Y^n(s)) - B(Y^m(s), Y^m(s)))ds \\ &- \int_t^T \widetilde{Z}(s)dW(s). \end{aligned}$$

Apply Itô formula to $|\widetilde{Y}(t)|^2$ to get

$$\begin{aligned} |\widetilde{Y}(t)|^2 &= |\widetilde{\xi}|^2 + \int_t^T 2\langle \widetilde{Y}(s), A\widetilde{Y}(s) + B(Y^n(s), Y^n(s)) - B(Y^m(s), Y^m(s))\rangle ds \\ &\quad - \int_t^T \|\widetilde{Z}(s)\|^2 ds - 2\int_t^T \langle \widetilde{Y}(s), \widetilde{Z}(s)\rangle dW(s). \end{aligned}$$

Hence

$$
\begin{aligned}
&|\widetilde{Y}(t)|^2 + \int_t^T \|\widetilde{Z}(s)\|^2 ds \\
=&|\widetilde{\xi}|^2 + \int_t^T 2\langle \widetilde{Y}(s), A\widetilde{Y}(s) + B(Y^n(s), Y^n(s)) - B(Y^m(s), Y^m(s))\rangle ds \\
&- 2\int_t^T \langle \widetilde{Y}(s), \widetilde{Z}(s)\rangle dW(s) \\
\leq&|\widetilde{\xi}|^2 + \int_t^T 2\|A\||\widetilde{Y}(s)|^2 ds + 2\int_t^T \langle \widetilde{Y}(s), B(Y^n(s), Y^n(s)) - B(Y^m(s), Y^m(s))\rangle ds \\
&- 2\int_t^T \langle \widetilde{Y}(s), \widetilde{Z}(s)\rangle dW(s) \qquad (3.3.2)
\end{aligned}
$$

Since

$$
\begin{aligned}
&\langle \widetilde{Y}(s), B(Y^n(s), Y^n(s)) - B(Y^m(s), Y^m(s))\rangle \\
=&\langle Y^n(s) - Y^m(s), B(Y^n(s), Y^n(s)) - B(Y^m(s), Y^m(s))\rangle \\
=&-\langle Y^n(s), B(Y^m(s), Y^m(s))\rangle - \langle Y^m(s), B(Y^n(s), Y^n(s))\rangle \\
=&\langle Y^m(s), B(Y^m(s), Y^n(s))\rangle - \langle Y^m(s), B(Y^n(s), Y^n(s))\rangle \\
=&\langle Y^m(s), B(Y^m(s) - Y^n(s), Y^n(s))\rangle \\
=&\langle Y^m(s) - Y^n(s), B(Y^m(s) - Y^n(s), Y^n(s))\rangle \\
=&\langle \widetilde{Y}(s), B(\widetilde{Y}(s), Y^n(s))\rangle \\
\leq&|\widetilde{Y}(s)|^2 |Y^n(s)|
\end{aligned}
$$

From Proposition 3.2.2, we know that

$$|Y^n(t)|^2 \leq (2 - e^{-2\|A\|(T-t)}) E^{\mathcal{F}_t}|\xi^n|^2 \leq 2n^2.$$

So $|Y^n(t)| \leq \sqrt{2}n \ \forall t$.

Hence

$$|\langle \widetilde{Y}(s), B(Y^n(s), Y^n(s)) - B(Y^m(s), Y^m(s))\rangle|$$

$$\leq |\widetilde{Y}(s)|^2 |Y^n(s)| \tag{3.3.3}$$

$$\leq \sqrt{2}n |\widetilde{Y}(s)|^2. \tag{3.3.4}$$

From (3.3.2) and (3.3.4), it follows that

$$|\widetilde{Y}(t)|^2 + \int_t^T \|\widetilde{Z}(s)\|^2 ds \leq |\widetilde{\xi}|^2 + (2\|A\| + 2\sqrt{2}n) \int_t^T |\widetilde{Y}(s)|^2 ds$$
$$- 2 \int_t^T \langle \widetilde{Y}(s), \widetilde{Z}(s) \rangle dW(s). \tag{3.3.5}$$

After taking expectation on both sides of (3.3.5), one gets

$$E|\widetilde{Y}(t)|^2 + E \int_t^T \|\widetilde{Z}(s)\|^2 ds \leq E|\widetilde{\xi}|^2 + (2\|A\| + 2\sqrt{2}n) \int_t^T E|\widetilde{Y}(s)|^2 ds. \tag{3.3.6}$$

Hence by Gronwall's inequality, it is easy to see that

$$\begin{aligned}
& E|\widetilde{Y}(t)|^2 + E \int_t^T \|\widetilde{Z}(s)\|^2 ds \\
\leq & E|\widetilde{\xi}|^2 + (2\|A\| + 2\sqrt{2}n) \int_t^T E|\widetilde{\xi}|^2 e^{(2\|A\| + 2\sqrt{2}n)(t-s)} ds \\
= & E|\widetilde{\xi}|^2 + E|\widetilde{\xi}|^2 (1 - e^{-(2\|A\| + 2\sqrt{2}n)(T-t)}) \\
\leq & 2E|\widetilde{\xi}|^2.
\end{aligned} \tag{3.3.7}$$

Since $E|\xi|^2 < \infty$ and $|\widetilde{\xi}|^2 \leq 2|\xi|^2$, an application of the Lebesgue dominated convergence theorem yeilds

$$\lim_{m,n \to \infty} E|\widetilde{\xi}|^2 = E \lim_{m,n \to \infty} |\widetilde{\xi}|^2 = 0 \tag{3.3.8}$$

So it follows from (3.3.7) that

$$\lim_{m,n \to \infty} E \int_t^T \|\widetilde{Z}(s)\|^2 ds = 0 \text{ and } \lim_{m,n \to \infty} E|\widetilde{Y}(t)|^2 = 0, \quad \forall t$$

i.e.

$$\lim_{m,n \to \infty} E \int_0^T \|\widetilde{Z}(s)\|^2 ds = 0 \tag{3.3.9}$$

On the other hand, by means of the Bürkholder-Davis-Gundy inequality (p. 160 [24]), one obtains

$$
\begin{aligned}
&E\{\sup_{t\le\rho\le T}|\int_\rho^T \langle \widetilde{Y}(s),\widetilde{Z}(s)\rangle dW(s)|\}\\
\le& E|\int_t^T \langle \widetilde{Y}(s),\widetilde{Z}(s)\rangle dW(s)| + E\sup_{t\le\rho\le T}|\int_t^\rho \langle \widetilde{Y}(s),\widetilde{Z}(s)\rangle dW(s)|\\
\le& 2E\sup_{t\le\rho\le T}|\int_t^\rho \langle \widetilde{Y}(s),\widetilde{Z}(s)\rangle dW(s)|\\
\le& 4\sqrt{2}E\{\int_t^T |\widetilde{Y}(s)|^2\|\widetilde{Z}(s)\|^2 ds\}^{\frac{1}{2}}\\
\le& 4\sqrt{2}E\{(\sup_{t\le\rho\le T}|\widetilde{Y}(\rho)|^2)^{\frac{1}{2}}(\int_t^T \|\widetilde{Z}(s)\|^2 ds)^{\frac{1}{2}}\}\\
\le& \frac{1}{4}E(\sup_{t\le\rho\le T}|\widetilde{Y}(\rho)|^2) + 32E\int_t^T \|\widetilde{Z}(s)\|^2 ds
\end{aligned}
\tag{3.3.10}
$$

Thus from (3.3.2),(3.3.3) and (3.3.10), one gets

$$
\begin{aligned}
&E(\sup_{t\le\rho\le T}|\widetilde{Y}(\rho)|^2) + E\int_t^T \|\widetilde{Z}(s)\|^2 ds\\
\le& E|\widetilde{\xi}|^2 + E\int_t^T (2\|A\| + 2|Y^n(s)|)|\widetilde{Y}(s)|^2 ds\\
&+ 2E(\sup_{t\le\rho\le T}|\int_\rho^T \langle \widetilde{Y}(s),\widetilde{Z}(s)\rangle dW(s)|)\\
\le& E|\widetilde{\xi}|^2 + E\int_t^T (2\|A\| + 2|Y^n(s)|)|\widetilde{Y}(s)|^2 ds\\
&+ \frac{1}{2}E(\sup_{t\le\rho\le T}|\widetilde{Y}(\rho)|^2) + 64E\int_t^T \|\widetilde{Z}(s)\|^2 ds
\end{aligned}
$$

Hence it follows from (3.3.7) and the above inequality that

$$
\frac{1}{2}E(\sup_{t\le\rho\le T}|\widetilde{Y}(\rho)|^2) \le 127E|\widetilde{\xi}|^2 + E\int_t^T (2\|A\| + 2|Y^n(s)|)|\widetilde{Y}(s)|^2 ds \tag{3.3.11}
$$

From Proposition 3.2.2, one has

$$
|Y^n(s)|^2 \le E^{\mathcal{F}_s}|\xi^n|^2(2 - e^{-2\|A\|(T-t)}) \le 2E^{\mathcal{F}_s}|\xi|^2 \tag{3.3.12}
$$

Clearly $\{E^{\mathcal{F}_s}|\xi|^2\}_{s\in[0,T]}$ is a $\mathcal{F}$-adapted martingale. By Doob's submartingale inequality, for any $\lambda > 0$,

$$P\{\sup_{0\leq s\leq T} E^{\mathcal{F}_s}|\xi|^2 \geq \lambda\} \leq \frac{1}{\lambda}EE^{\mathcal{F}_T}|\xi|^2 = \frac{1}{\lambda}E|\xi|^2 \to 0 \ \text{ as } \ \lambda \to \infty$$

Let $\tau_R = \inf\{t : E^{\mathcal{F}_t}|\xi|^2 > R\} \wedge T$ for $R > 0$. It is easy to show that $\tau_R \to T$ a.s. as $R \to \infty$. From (3.3.11) and Gronwall's inequality, one gets

$$\begin{aligned}
&E \sup_{t\leq\rho\leq T} |\widetilde{Y}(\rho\wedge\tau_R)|^2 \\
\leq& 254E|\widetilde{\xi}|^2 + 2E\int_t^T (2\|A\| + 2|Y^n(s\wedge\tau_R)|)|\widetilde{Y}(s\wedge\tau_R)|^2 ds \\
\leq& 254E|\widetilde{\xi}|^2 + 4(\|A\| + R)\int_t^T E\sup_{s\leq\rho\leq T}|\widetilde{Y}(\rho\wedge\tau_R)|^2 ds \\
\leq& 508E|\widetilde{\xi}|^2 \quad \text{for all } t \in [0,T]
\end{aligned}$$

An application of monotone convergence theorem yields

$$\begin{aligned}
&\lim_{m,n\to\infty} E(\sup_{0\leq\rho\leq T}|\widetilde{Y}(\rho)|^2) \\
=&\lim_{m,n\to\infty} E(\lim_{R\to\infty}\sup_{0\leq\rho\leq T}|\widetilde{Y}(\rho\wedge\tau_R)|^2) \\
=&\lim_{m,n\to\infty}\lim_{R\to\infty} E(\sup_{0\leq\rho\leq T}|\widetilde{Y}(\rho\wedge\tau_R)|^2) \\
\leq&\lim_{m,n\to\infty} 508E|\widetilde{\xi}|^2 = 0
\end{aligned} \tag{3.3.13}$$

From (3.3.9), (3.3.13), and the definition of the norm of $\mathcal{M}[0,T]$, we know that the solutions of n-Lorenz systems are Cauchy. □

Since $\mathcal{M}[0,T]$ is a Banach space, we know that there exists $(Y,Z) \in \mathcal{M}[0,T]$, such that

$$\lim_{n\to\infty}\{E(\sup_{0\leq t\leq T}|Y^n(t) - Y(t)|^2) + E\int_0^T |Z^n(t) - Z(t)|^2 dt\} = 0$$

We want to show that (Y,Z) is actually the solution of Lorenz system (3.1.1).

Theorem 3.3.5. *Under Assumption* 2°*, Lorenz system* (3.1.1) *has a unique solution.*

Proof: By Proposition 3.3.4, we know that $Y^n(t)$ converge to $Y(t)$ uniformly. Since $Y^n(t)$ and $Y(t)$ are all bounded, it is easy to see that

$$\lim_{n\to\infty} E\int_t^T |b(Y^n(s)) - b(Y(s))|^2 ds = 0$$

By Itô isometry, we have

$$\lim_{n\to\infty} E(\int_t^T (Z^n(s) - Z(s))dW(s))^2 = \lim_{n\to\infty} E\int_0^T |Z^n(s) - Z(s)|^2 ds = 0$$

Thus we have shown that $(Y(t), Z(t))$ satisfies Lorenz system (3.1.1), that is, $(Y(t), Z(t))$ is a solution of Lorenz system (3.1.1).

Now assume that $(Y(t), Z(t))$ and $(Y'(t), Z'(t))$ are two solutions of Lorenz system (3.1.1). Similar to the proof of Proposition 3.3.4, we can get

$$E(\sup_{0\le t\le T} |Y(t) - Y'(t)|^2) = 0 \text{ and } E\int_0^T |Z(t) - Z'(t)|^2 dt = 0,$$

That is, $\|(Y(t), Z(t)) - (Y'(t), Z'(t))\|_{\mathcal{M}[0,T]} = 0$. Thus we have proved the uniqueness of the solution. □

3.4 Continuity with Respect to Terminal Data

Theorem 3.4.1. *Assume that* $\xi \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n)$*. Then the solution of* (3.1.1) *is continuous with respect to the terminal data.*

Proof: For any $\xi, \zeta \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n)$, let $(Y(t), Z(t))$ and $(X(t), V(t))$ be solutions of (3.1.1) under terminal values ξ and ζ, respectively. Let $(Y^n(t), Z^n(t))$ and $(X^n(t), V^n(t))$ be solutions of corresponding n-Lorenz system. By Proposition 3.3.4, we know that

$$\lim_{n\to\infty} E\sup_{0\le t\le T} |Y(t) - Y^n(t)|^2 = \lim_{n\to\infty} E\int_0^T \|Z(t) - Z^n(t)\|^2 dt = 0 \quad \text{and}$$

$$\lim_{n\to\infty} E \sup_{0\le t\le T} |X(t) - X^n(t)|^2 = \lim_{n\to\infty} E \int_0^T \|V(t) - V^n(t)\|^2 dt = 0$$

Similar to the proof of Proposition 3.3.4, one can show that

$$E \int_0^T \|Z^n(t) - V^n(t)\|^2 dt \le 2E|\xi^n - \zeta^n|^2$$

and

$$E \sup_{0\le t\le T} |Y^n(t) - X^n(t)|^2 \le 508E|\xi^n - \zeta^n|^2$$

Hence

$$E \sup_{0\le t\le T} |Y^n(t) - X^n(t)|^2 + E \int_0^T \|Z^n(t) - V^n(t)\|^2 dt \le 510E|\xi^n - \zeta^n|^2$$

Thus

$$\begin{aligned}
& E \sup_{0\le t\le T} |Y(t) - X(t)|^2 + E \int_0^T \|Z(t) - V(t)\|^2 dt \\
\le & \lim_{n\to\infty} 3E \sup_{0\le t\le T} (|Y(t) - Y^n(t)|^2 + |Y^n(t) - X^n(t)|^2 + |X(t) - X^n(t)|^2) \\
& + \lim_{n\to\infty} 3E \int_0^T (\|Z(t) - Z^n(t)\|^2 + \|Z^n(t) - V^n(t)\|^2 + \|V(t) - V^n(t)\|^2) dt \\
\le & \lim_{n\to\infty} 1530E|\xi^n - \zeta^n|^2 \\
= & 1530E|\xi - \zeta|^2
\end{aligned}$$

□

Chapter 4

The Backward Stochastic Navier-Stokes Equation

4.1 Preliminaries

Let G be a bounded domain in $\mathbb{R}^2$ with a smooth boundary, and for $t\in [0,T]$, let $\mathbf{u}(t)=\binom{u_1(t)}{u_2(t)} \in H$ where $H = \{\mathbf{u} \in (L^2(G))^2 : \mathrm{div}(\mathbf{u}) = \nabla \cdot \mathbf{u} = 0 \text{ and } \gamma(\mathbf{u}) = \mathbf{u} \cdot \mathbf{n}_G = 0\}$, where $\mathbf{n}_G$ stands for the outer normal to ∂G.

Definition 4.1.1. Let A be an operator on a separable Hilbert space K with CONS $\{e_j\}_{j=1}^{\infty}$. If $\langle Ax, y\rangle=\langle x, A^* y\rangle$ for any $x, y \in K$, then A^* is called the *adjoint* of A. If $A = A^*$, then A is called *self-adjoint*.

Definition 4.1.2. Let A be a linear operator from a separable Hilbert space K with CONS $\{e_j\}_{j=1}^{\infty}$ to a separable Hilbert space H.

(a) We denote by $L(K, H)$ the class of all bounded linear operators with the uniform operator norm $\|\cdot\|_L$.

(b) If $\|A\|_{L_1}=\sum_{k=1}^{\infty}\langle (A^*A)^{\frac{1}{2}} e_k, e_k\rangle_K < \infty$, then A is called a *trace class(nuclear) operator*. We denote by $L_1(K, H)$ the class of trace class operators with norm $\|\cdot\|_{L_1}$.

(c) We also denote by $L_2(K, H)$ the class of *Hilbert-Schmidt operators* with norm $\|\cdot\|_{L_2}$ given by $\|A\|_{L_2}=(\sum_{k=1}^{\infty}\langle Ae_k, Ae_k\rangle_H)^{\frac{1}{2}}$. Sometimes $\|\cdot\|_{L_2}$ is also denoted by $\|\cdot\|_{H.S.}$

(d) Let $Q \in L_1(K, K)$ be self-adjoint and positive definite. Let K_0 be the Hilbert subspace of K with inner product

$$\langle f, g\rangle_{K_0} = \langle Q^{-\frac{1}{2}} f, Q^{-\frac{1}{2}} g\rangle_K,$$

and we denote $L_Q = L_2(K_0, H)$ with the inner product

$$\langle F, G\rangle_{L_Q} = tr(FQG^*) = tr(GQF^*), \qquad F, G \in L_Q.$$

Definition 4.1.3. $W(t)$ is an H-valued Q-Wiener process, where Q is a trace class operator on H, if $W(t)$ satisfies the following:

(a) $W(t)$ has continuous sample paths in H-norm with $W(0) = 0$.

(b) $(W(t), h)$ has stationary independent increments for all $h \in H$.

(c) $W(t)$ is a Gaussian process with mean zero and covariance operator Q, i.e. $E(W(t), g)(W(s), h)$=$(t \wedge s)(Qg, h)$ for all $g, h \in H$.

Consider the stochastic Navier-Stokes equation for a viscous incompressible flow with no-slip condition at the boundary. Displaying the external forces on the right side of the equation, we have, for $\nu > 0$,

$$\begin{cases} \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f}(t) + \sigma(t, \mathbf{u}) \frac{dW(t)}{dt} \\ \nabla \cdot \mathbf{u} = 0 \end{cases} \tag{4.1.1}$$

with $\mathbf{u}(t, x) = 0$ for $x \in \partial G$, and $\mathbf{u}(0, x) = \mathbf{u}_0(x)$ for $x \in G$. In the above, p denotes pressure and is a scalar-valued function. The process $\{W_t\}$ is a H-valued Q-Wiener process, and ν is the coefficient of viscosity. The solution of the above system is $(\mathbf{u}, p)$ where $\mathbf{u}$ is a two-dimensional vector.

The stochastic Navier-Stokes equation can be written in the abstract evolution equation setup (see Temam [29]) for bounded domains. Let P be the orthogonal

Leray projector:

$$\mathsf{P} : (L^2(G))^2 \to H$$

By applying the Leray projection to each term of the Navier-Stokes system, and invoking the result of Helmholtz that $L^2(G)$ admits an orthogonal decomposition into divergence free and irrotational components, namely $L^2(G) = H + H^{\perp}$ where $H^{\perp}$ can be characterized by

$$H^{\perp} = \{\mathbf{g} \in (L^2(G))^2 : \mathbf{g} = \nabla h \text{ where } h \in W^{1,2}(G)\}, \tag{4.1.2}$$

we can write the system (4.1.1) as

$$\begin{cases} d\mathbf{u}(t) = -\nu \mathbf{A}\mathbf{u}(t)dt - \mathbf{B}(\mathbf{u}(t))dt + \mathbf{f}(t)dt + Z(t)dW(t) \\ \mathbf{u}(0) = \mathbf{u}_0(x) \end{cases}$$

where $\mathbf{B}(\mathbf{u}, \mathbf{v}) \doteq \mathsf{P}((\mathbf{u} \cdot \nabla)\mathbf{v})$ with the notation $\mathbf{B}(\mathbf{u}) = \mathbf{B}(\mathbf{u}, \mathbf{u})$, and $\mathbf{A}\mathbf{u} \doteq -\mathsf{P}(\Delta \mathbf{u})$.

Now we fix the terminal value by letting $\mathbf{u}(T) = \xi$. Then

$$\begin{cases} d\mathbf{u}(t) = -\nu \mathbf{A}\mathbf{u}(t)dt - \mathbf{B}(\mathbf{u}(t))dt + \mathbf{f}(t)dt + Z(t)dW(t) \\ \mathbf{u}(T) = \xi \end{cases} \tag{4.1.3}$$

is called the backward stochastic Navier-Stokes equation.

Let $V = \{\mathbf{u} \in (H^1(G))^2 : \nabla \cdot \mathbf{u} = 0 \text{ and } \gamma_0(\mathbf{u}) = \mathbf{u}|_{\partial G} = 0\}$ and V' be the dual of V. From the definition of V and H, we see that they are both separable Hilbert spaces, V is a dense subset of H, and the embedding $V \hookrightarrow H$ is dense, continuous and compact.

We identify H' with H. For any $\mathbf{h} \in H$, there exist an $\mathbf{h}' \in V'$, such that $\langle \mathbf{h}', \mathbf{v} \rangle_{V',V} = \langle \mathbf{h}, \mathbf{v} \rangle_H$. Then the mapping $\mathbf{h} \mapsto \mathbf{h}'$ is linear, injective, compact and continuous. We may identify $\mathbf{h}'$ with $\mathbf{h}$. In this sense, H is a dense subset of V'. Thus we have evolution triple

$$V \subset H \subset V' \tag{4.1.4}$$

and $\|\mathbf{v}\|_H \le C\|\mathbf{v}\|_V$, $\|\mathbf{h}\|_{V'} \le C\|\mathbf{h}\|_H$, and $\langle \mathbf{h}, \mathbf{v}\rangle_{V',V} = \langle \mathbf{h}, \mathbf{v}\rangle_H$ for all $\mathbf{v} \in V$, $\mathbf{h} \in H$ and some constant C. From now on, we may consider $\mathbf{A}$ and $\mathbf{B}$ as mappings that map V into V'.

Remark 4.1.4. Let $\{e_j\}_{j=1}^{\infty}$ be a complete orthonormal system in H such that there exists an increasing sequence of positive numbers $\{\lambda_j\}_{j=1}^{\infty}$, $\lim_{j\to\infty} \lambda_j = \infty$ and $\mathbf{A}e_j = \lambda_j e_j$ for all j. Let $Qe_k = q_k e_k$, and $\{b^k(t)\}$ be a sequence of iid Brownian motions in $\mathbb{R}$. The Wiener process $\{W(t)\}$ is taken as $W(t)=\sum_{k=1}^{\infty} \sqrt{q_k} b^k(t) e_k$ with $\sum_{k=1}^{\infty} q_k < \infty$.

For any Banach space $\mathbb{K}$, let $L^p_{\mathcal{F}}(\Omega; L^p(0,T;\mathbb{K}))$ be the set of all $\{\mathcal{F}\}_{t\ge 0}$-adapted $\mathbb{K}$-valued processes $X(\cdot)$ such that $E\int_0^T \|X(t)\|^p_{\mathbb{K}} dt < \infty$.

Definition 4.1.5. A pair of $\mathcal{F}_t$-adapted processes $(\mathbf{u}(t), Z(t))$ is called a solution of backward stochastic differential equation (4.1.3) if the following holds:

(1) $\mathbf{u}(t) = \xi + \int_t^T (\nu \mathbf{A}\mathbf{u}(s) + \mathbf{B}(\mathbf{u}(s)) - \mathbf{f}(s))ds - \int_t^T Z(t)dW(t)$ holds P-a.s. in V',

(2) $\mathbf{u}(\cdot) \in L^2_{\mathcal{F}}(\Omega; L^2(0,T;V)) \cap L^\infty(0,T; L^2_{\mathcal{F}}(\Omega; H))$.

(3) $Z(\cdot) \in L^2_{\mathcal{F}}(\Omega; L^2(0,T;L_Q))$.

Proposition 4.1.6. *For* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, *we have*

(1) $\langle \mathbf{A}\mathbf{u}, \mathbf{w}\rangle_{V',V} = \sum_{i,j} \int_G \partial_i u_j \partial_i w_j dx = \langle \mathbf{A}\mathbf{w}, \mathbf{u}\rangle_{V',V} = \langle \mathbf{u}, \mathbf{w}\rangle_V$

(2) $\langle \mathbf{B}(\mathbf{u},\mathbf{v}),\mathbf{w}\rangle_{V',V}=\sum_{i,j}\int_G u_i(\partial_i v_j)w_j dx$

(3) $\langle \mathbf{B}(\mathbf{u},\mathbf{v}),\mathbf{w}\rangle_{V',V}=-\langle \mathbf{B}(\mathbf{u},\mathbf{w}),\mathbf{v}\rangle_{V',V}$

Proof: Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w} \in \mathcal{V}=\{\varphi \in C_0^\infty(G)^2 | \text{Div } \varphi = 0\}$.

(1)By integration by parts and the fact that $w_i = 0$ on ∂G, we have

$$\begin{aligned}
&\langle \mathbf{A}\mathbf{u},\mathbf{w}\rangle_{V',V} = \langle -\mathsf{P}\Delta\mathbf{u},\mathbf{w}\rangle_H = \langle -\Delta\mathbf{u},\mathsf{P}\mathbf{w}\rangle_H = \langle -\Delta\mathbf{u},\mathbf{w}\rangle_H \\
=&-\int_G \Delta\mathbf{u}\cdot\mathbf{w}dx = -\sum_i\int_G \Delta u_i w_i dx = -\sum_{i,j}\int_G \frac{\partial^2 u_i}{\partial x_j^2} w_i dx \\
=&-\sum_{i,j}\int_{\partial G}\frac{\partial u_i}{\partial x_j} w_i \mathbf{n}_j d\sigma + \sum_{i,j}\int_G \frac{\partial u_i}{\partial x_j}\frac{\partial w_i}{\partial x_j}dx \\
=&\sum_{i,j}\int_G \frac{\partial u_i}{\partial x_j}\frac{\partial w_i}{\partial x_j}dx \\
=&\langle \mathbf{A}\mathbf{w},\mathbf{u}\rangle_{V',V}
\end{aligned}$$

In the above derivation, $\mathbf{n}_j$ is the jth coordinate of the outer normal vector $\mathbf{n}_G$ to G. Since V=closure of $\mathcal{V}$ in $H_0^1(G)^2$, by a density argument, the result holds for $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w} \in V$.

(2)Since

$$\langle \mathbf{B}(\mathbf{u},\mathbf{v}),\mathbf{w}\rangle_{V',V} = \langle \mathsf{P}(\mathbf{u}\cdot\nabla)\mathbf{v},\mathbf{w}\rangle_H = \langle (\mathbf{u}\cdot\nabla)\mathbf{v},\mathsf{P}\mathbf{w}\rangle_H = \langle (\mathbf{u}\cdot\nabla)\mathbf{v},\mathbf{w}\rangle_H$$

and

$$(\mathbf{u}\cdot\nabla)\mathbf{v} = \sum_i u_i\frac{\partial \mathbf{v}}{\partial x_i} = (\sum_i u_i\frac{\partial v_1}{\partial x_i}, \sum_i u_i\frac{\partial v_2}{\partial x_i}, \cdots, \sum_i u_i\frac{\partial v_n}{\partial x_i}),$$

we have

$$\langle (\mathbf{u}\cdot\nabla)\mathbf{v},\mathbf{w}\rangle_H = \sum_{i,j}\int_G u_i\frac{\partial v_j}{\partial x_i}w_j dx.$$

By a density argument, the result holds for $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w} \in V$ and the derivatives are in the weak sense. Thus we have shown (2).

(3)By integration by parts, we have

$$\begin{aligned}\langle \mathbf{B}(\mathbf{u},\mathbf{v}),\mathbf{w}\rangle_{V',V} &= \sum_{i,j}\int_G u_i \frac{\partial v_j}{\partial x_i} w_j dx \\ =&\sum_{i,j}\int_{\partial G} u_i v_j w_j \mathbf{n}_i d\sigma - \sum_{i,j}\int_G v_j \frac{\partial (u_i w_j)}{\partial x_i} dx \\ =&-\sum_{i,j}\int_G v_j \frac{\partial u_i}{\partial x_i} w_j dx - \sum_{i,j}\int_G u_i v_j \frac{\partial w_j}{\partial x_i} dx \\ =&-\sum_{i,j}\int_G u_i v_j \frac{\partial w_j}{\partial x_i} dx = -\langle \mathbf{B}(\mathbf{u},\mathbf{w}),\mathbf{v}\rangle_{V',V}.\end{aligned}$$

By a density argument, the result holds for $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w} \in V$. □

Corollary 4.1.7. *For any* $\mathbf{u},\mathbf{v} \in V$, $\langle \mathbf{B}(\mathbf{u},\mathbf{v}),\mathbf{v}\rangle_{V',V} = 0$

Proposition 4.1.8. *(Constantin and Foias [4]) Let* $G \subset \mathbb{R}^n$ *be bounded, open and of class* C^l *where* $l \geq 1$. *Let* s_1, s_2, s_3 *be real numbers,* $0 \leq s_1 \leq l$, $0 \leq s_2 \leq l-1$, $0 \leq s_3 \leq l$. *Let us assume that*

(1) $s_1 + s_2 + s_3 \geq n/2$ *if* $s_i \neq n/2$ *for all* $i = 1, 2, 3$

or

(2) $s_1 + s_2 + s_3 > n/2$ *if* $s_i = n/2$ *for at least one* i

Then there exists a constant $C_G > 0$ *depending on* s_1, s_2, s_3, *and* G, *a scale invariant constant, such that*

$$\begin{aligned}|\langle \mathbf{B}(\mathbf{u},\mathbf{v}),\mathbf{w}\rangle_{V',V}| \leq & C_G \|\mathbf{u}\|_{[s_1],G}^{1+[s_1]-s_1} \|\mathbf{u}\|_{[s_1]+1,G}^{s_1-[s_1]} \|\mathbf{v}\|_{[s_2]+1,G}^{1+[s_2]-s_2} \\ & \|\mathbf{v}\|_{[s_2]+2,G}^{s_2-[s_2]} \|\mathbf{w}\|_{[s_3],G}^{1+[s_3]-s_3} \|\mathbf{w}\|_{[s_3]+1,G}^{s_3-[s_3]}\end{aligned}$$

for all $\mathbf{u},\mathbf{v},\mathbf{w} \in C^\infty(\bar{G})^n$.

Lemma 4.1.9. *[29] Assume that* $G \subset \mathbb{R}^2$ *is bounded and of class* $\mathcal{C}^2$. *If* $\mathbf{u} \in V \cap H^2(G)$, *then* $\mathbf{B}(\mathbf{u}) \in H \subset L^2(G)$ *and*

$$\|\mathbf{B}(\mathbf{u})\|_H \leq C_1 \|\mathbf{u}\|_H^{\frac{1}{2}} \|\mathbf{u}\|_V \|\mathbf{A}\mathbf{u}\|_H^{\frac{1}{2}}$$

4.2 A Priori Estimates

Let $P_N : H \to H_N$ be the projection where H_N =span$\{e_1, e_2, \cdots, e_N\}$. Notice the fact that $V_N = H_N = V_N'$ for all N. First we restrict the domain of $\mathbf{A}$ and $\mathbf{B}$ to H_N and still denote it by $\mathbf{A}$ and $\mathbf{B}$. Now We introduce the following projection:

$$\mathbf{A}^N = P_N\mathbf{A} \text{ and } \mathbf{B}^N = P_N\mathbf{B}$$

Then the projected backward Navier-Stokes equation is defined by

$$\begin{cases} d\mathbf{u}^N(t) = -\nu\mathbf{A}^N\mathbf{u}^N(t)dt - \mathbf{B}^N(\mathbf{u}^N(t))dt + \mathbf{f}^N(t)dt + Z^N(t)dW^N(t) \\ \mathbf{u}^N(T) = \xi^N \end{cases} \tag{4.2.1}$$

for $0 \le t \le T$, where $\mathbf{f}^N = P_N f$, $W^N(t) = \sum_{k=1}^{N} \sqrt{q_k} b^k(t) e_k$, $\xi^N = P_N\xi$, and $Z^N(t) : [0,T] \times \Omega \to L(H_N, H_N)$.

Proposition 4.2.1. *Assume that* $\mathbf{f} \in L^2(0,T;V')$*, and* $\|\xi\|_H^2 \le K$ *for almost all* $\omega \in \Omega$ *and some constant* K*. If* $(\mathbf{u}^N(t), Z^N(t))$ *is an adapted solution for the projected system* (4.2.1)*, then there exists a constant* K_0*, independent of* N*, such that*

$$\sup_{0\le t\le T} \|\mathbf{u}^N(t)\|_H^2 + E\int_0^T \|\mathbf{u}^N(t)\|_V^2 dt + E\int_0^T \|Z^N(t)\|_{L_Q}^2 dt \le K_0 \tag{4.2.2}$$

Proof: An application of the multidimensional Itô's formula to $\|\mathbf{u}^N(t)\|_H^2$ yields

$$\begin{aligned} \|\mathbf{u}^N(t)\|_H^2 =& \|\xi^N\|_H^2 - \int_t^T 2\langle -\nu\mathbf{A}^N\mathbf{u}^N(s) - \mathbf{B}^N(\mathbf{u}^N(s)), \mathbf{u}^N(s)\rangle_{V',V} ds \\ & - \int_t^T 2\langle \mathbf{f}^N(s), \mathbf{u}^N(s)\rangle_{V',V} ds - 2\int_t^T \langle (Z^N(s))^*(\mathbf{u}^N(s)), dW^N(s)\rangle_H \\ & - \int_t^T tr[Z^N(s)Q(Z^N(s))^*]ds \\ =& \|\xi^N\|_H^2 - 2\int_t^T \langle -\nu\mathbf{A}^N\mathbf{u}^N(s), \mathbf{u}^N(s)\rangle_{V',V} ds \end{aligned}$$

$$
\begin{aligned}
&- 2\int_t^T \langle \mathbf{f}^N(s), \mathbf{u}^N(s)\rangle_{V',V} ds - 2\int_t^T \langle (Z^N(s))^*(\mathbf{u}^N(s)), dW^N(s)\rangle_H \\
&- \int_t^T \|Z^N(s)\|^2_{L_Q} ds
\end{aligned}
$$

where $(Z^N)^*$ is the adjoint of Z^N, and the duality pairing $\langle \cdot, \cdot \rangle_{V',V}$ is just the H-norm. For $0 < r \leq t \leq T$, we take the conditional expectation as in Proposition 3.3.4 in the previous chapter to obtain

$$
\begin{aligned}
&E^{\mathcal{F}_r}\|\mathbf{u}^N(t)\|^2_H + E^{\mathcal{F}_r}\int_t^T \|Z^N(s)\|^2_{L_Q} ds \\
=&E^{\mathcal{F}_r}\|\xi^N\|^2_H + 2E^{\mathcal{F}_r}\int_t^T \langle \nu \mathbf{A}^N \mathbf{u}^N(s), \mathbf{u}^N(s)\rangle_{V',V} ds \\
&- 2E^{\mathcal{F}_r}\int_t^T \langle \mathbf{f}^N(s), \mathbf{u}^N(s)\rangle_{V',V} ds
\end{aligned}
$$

Thus

$$
\begin{aligned}
&E^{\mathcal{F}_r}\|\mathbf{u}^N(t)\|^2_H + E^{\mathcal{F}_r}\int_t^T \|Z^N(s)\|^2_{L_Q} ds + E^{\mathcal{F}_r}\int_t^T \langle \mathbf{A}^N \mathbf{u}^N(s), \mathbf{u}^N(s)\rangle_{V',V} ds \\
=&E^{\mathcal{F}_r}\|\xi^N\|^2_H + 2E^{\mathcal{F}_r}\int_t^T \langle (\nu + \frac{1}{2})\mathbf{A}^N \mathbf{u}^N(s), \mathbf{u}^N(s)\rangle_{V',V} ds \\
&- 2E^{\mathcal{F}_r}\int_t^T \langle \mathbf{f}^N(s), \mathbf{u}^N(s)\rangle_{V',V} ds \\
\leq&E^{\mathcal{F}_r}\|\xi^N\|^2_H + (2\nu + 1)\lambda_N \int_t^T E^{\mathcal{F}_r}\|\mathbf{u}^N(s)\|^2_H ds + 2\int_0^T \|\mathbf{f}^N(s)\|^2_{V'} ds \\
&+ \frac{1}{2}\int_t^T E^{\mathcal{F}_r}\|\mathbf{u}^N(s)\|^2_V ds
\end{aligned}
$$

where λ_N is the N^{th} eigenvalue of $\mathbf{A}$. By Gronwall's inequality and Proposition 4.1.6, we have

$$
\begin{aligned}
&E^{\mathcal{F}_r}\|\mathbf{u}^N(t)\|^2_H + E^{\mathcal{F}_r}\int_t^T \|Z^N(s)\|^2_{L_Q} ds + \frac{1}{2}E^{\mathcal{F}_r}\int_t^T \|\mathbf{u}^N(s)\|^2_V ds \\
\leq&E^{\mathcal{F}_r}\|\xi^N\|^2_H + 2\int_0^T \|\mathbf{f}^N(s)\|^2_{V'} ds \\
&+ (2\nu\lambda_N + \lambda_N)\int_t^T \{E^{\mathcal{F}_r}\|\xi^N\|^2_H + 2\int_0^T \|\mathbf{f}^N(s)\|^2_{V'} ds\} e^{(2\nu\lambda_N + \lambda_N)(t-s)} ds
\end{aligned}
$$

$$
\begin{aligned}
&=\{E^{\mathcal{F}_r}\|\xi^N\|_H^2 + 2\int_0^T \|\mathbf{f}^N(s)\|_{V'}^2 ds\}(2 - e^{-(2\nu\lambda_N+\lambda_N)(T-t)}) \\
&\leq 2E^{\mathcal{F}_r}\|\xi^N\|_H^2 + 4\int_0^T \|\mathbf{f}(s)\|_{V'}^2 ds
\end{aligned} \tag{4.2.3}
$$

Omitting the first term on the left hand side of the above inequality and taking expectation on both sides, one gets

$$
E\int_t^T \|Z^N(s)\|_{L_Q}^2 ds + \frac{1}{2}E\int_t^T \|\mathbf{u}^N(s)\|_V^2 ds \leq 2E\|\xi^N\|_H^2 + 4\int_0^T \|\mathbf{f}(s)\|_{V'}^2 ds
$$

Since $\mathbf{f} \in L^2(0,T;V')$ and $\|\xi\|_H^2 \leq K$, we know that for some constant K', such that

$$
E\int_t^T \|Z^N(s)\|_{L_Q}^2 ds + \frac{1}{2}E\int_t^T \|\mathbf{u}^N(s)\|_V^2 ds \leq K'
$$

Taking r to be t and omitting the last two terms on the left hand side of inequality (4.2.3), we know that $\|\mathbf{u}^N(t)\|_H^2 \leq K'$

So for some constant K_0,

$$
\sup_{0\leq t\leq T} \|\mathbf{u}^N(t)\|_H^2 + E\int_0^T \|\mathbf{u}^N(t)\|_V^2 dt + E\int_0^T \|Z^N(t)\|_{L_Q}^2 dt \leq K_0.
$$

□

Corollary 4.2.2. *Let the conditions in Proposition 4.2.1 hold. Additionally, we assume that* $\mathbf{f} \in L^4(0,T;V')$. *Then there exists a constant* K_1, *independent of* N, *such that*

$$
\sup_{0\leq t\leq T} \|\mathbf{u}^N(t)\|_H^4 + E\{\int_0^T \|\mathbf{u}^N(t)\|_V^2 dt\}^2 \leq K_1
$$

i.e. $\{\mathbf{u}^N(t)\}$ *is bounded in* $L^\infty(\Omega\times[0,T];H)\cap L^4_{\mathcal{F}}(\Omega;L^2(0,T;V))$.

Proof: Applying Itô formula to $\|\mathbf{u}^N(t)\|_H^2$, we get

$$
\|\mathbf{u}^N(t)\|_H^2 + \int_t^T \|\mathbf{u}^N(s)\|_V^2 ds
$$

$$\leq \|\xi^N\|_H^2 + (2\nu+1)\int_t^T \langle \mathbf{A}^N\mathbf{u}^N(s), \mathbf{u}^N(s)\rangle_{V',V} ds$$
$$-2\int_t^T \langle \mathbf{f}^N(s), \mathbf{u}^N(s)\rangle_{V',V} ds - 2\int_t^T \langle (Z^N(s))^*(\mathbf{u}^N(s)), dW^N(s)\rangle_H$$
$$\leq \|\xi^N\|_H^2 + (2\nu\lambda_N + \lambda_N)\int_t^T \|\mathbf{u}^N(s)\|_H^2 ds$$
$$+2\int_t^T \|\mathbf{f}^N(s)\|_{V'}^2 ds + \frac{1}{2}\int_t^T \|\mathbf{u}^N(s)\|_V^2 ds - 2\int_t^T \langle (Z^N(s))^*(\mathbf{u}^N(s)), dW^N(s)\rangle_H$$

Squaring both sides of the last inequality, we get

$$\|\mathbf{u}^N(t)\|_H^4 + (\frac{1}{2}\int_t^T \|\mathbf{u}^N(s)\|_V^2 ds)^2$$
$$\leq 4\|\xi^N\|_H^4 + 4(2\nu\lambda_N + \lambda_N)^2\int_t^T \|\mathbf{u}^N(s)\|_H^4 ds$$
$$+16\int_t^T \|\mathbf{f}^N(s)\|_{V'}^4 ds + 16(\int_t^T \langle (Z^N(s))^*(\mathbf{u}^N(s)), dW^N(s)\rangle_H)^2$$

Taking expectation for $0 \leq r \leq t$, we get

$$E^{\mathcal{F}_r}\|\mathbf{u}^N(t)\|_H^4 + \frac{1}{4}E^{\mathcal{F}_r}(\int_t^T \|\mathbf{u}^N(s)\|_V^2 ds)^2$$
$$\leq 4E^{\mathcal{F}_r}\|\xi^N\|_H^4 + 4(2\nu\lambda_N + \lambda_N)^2\int_t^T E^{\mathcal{F}_r}\|\mathbf{u}^N(s)\|_H^4 ds \tag{4.2.4}$$
$$+16\int_0^T \|\mathbf{f}^N(s)\|_{V'}^4 ds + 16E^{\mathcal{F}_r}(\int_t^T \langle (Z^N(s))^*(\mathbf{u}^N(s)), dW^N(s)\rangle_H)^2$$

For the last term in (4.2.4), we have

$$E^{\mathcal{F}_r}(\int_t^T \langle (Z^N(s))^*(\mathbf{u}^N(s)), dW^N(s)\rangle_H)^2$$
$$=E^{\mathcal{F}_r}(\sum_{i=1}^N \int_t^T \langle (Z^N(s))^*(\mathbf{u}^N(s)), \sqrt{q_i}e_i\rangle_H db^i(s))^2$$
$$=\sum_{i=1}^N E^{\mathcal{F}_r}(\int_t^T \langle \sqrt{q_i}\mathbf{u}^N(s), Z^N(s)(e_i)\rangle_H db^i(s))^2$$
$$=\sum_{i=1}^N E^{\mathcal{F}_r}\int_t^T \langle \sqrt{q_i}\mathbf{u}^N(s), Z^N(s)(e_i)\rangle_H^2 ds$$

$$\leq K_0 \sum_{i,j=1}^{N} E^{\mathcal{F}_r} \int_t^T \langle \sqrt{q_i} e_j, Z^N(s)(e_i) \rangle_H^2 ds$$
$$\leq K_0 \sum_{i,j=1}^{\infty} E^{\mathcal{F}_r} \int_t^T \langle e_j, Q^{\frac{1}{2}} Z^N(s)(e_i) \rangle_H^2 ds$$
$$= K_0 E^{\mathcal{F}_r} \int_t^T \|Q^{\frac{1}{2}} Z^N(s)\|_{H.S.}^2 ds$$
$$= K_0 E^{\mathcal{F}_r} \int_t^T tr((Z^N(s))^* Q Z^N(s)) ds$$
$$= K_0 E^{\mathcal{F}_r} \int_t^T \|Z^N(s)\|_{L_Q}^2 ds$$

Thus by (4.2.3) in Proposition 4.2.1, there exists a constant K_1, independent of N, such that

$$E^{\mathcal{F}_r} \|\mathbf{u}^N(t)\|_H^4 + \frac{1}{4} E^{\mathcal{F}_r} (\int_t^T \|\mathbf{u}^N(s)\|_V^2 ds)^2$$
$$\leq \frac{1}{8} K_1 + 4(2\nu\lambda_N + \lambda_N)^2 \int_t^T E^{\mathcal{F}_r} \|\mathbf{u}^N(s)\|_H^4 ds$$

Using Gronwall's inequality, it follows that

$$\sup_{0 \leq t \leq T} \|\mathbf{u}^N(t)\|_H^4 + E\{\int_0^T \|\mathbf{u}^N(t)\|_V^2 dt\}^2 \leq K_1$$

□

Corollary 4.2.3. *Let the conditions in Proposition 4.2.1 hold. Additionally, let $\|\xi\|_V^2 \leq K$ for almost all $\omega \in \Omega$ and some constant K. Then there exists a constant K_2, independent of N, such that*

$$\sup_{0 \leq t \leq T} \|\mathbf{u}^N(t)\|_V^2 \leq K_2$$

Proof: The proof is similar to that of Proposition 4.2.1. However, it is given in full since slight variations are needed.

First, by Proposition 4.1.6 and equation (4.2.1), we get

$$
\begin{aligned}
&\frac{1}{2}d\|\mathbf{u}^N(t)\|_V^2 = \langle d\mathbf{u}^N(t), \mathbf{u}^N(t)\rangle_V = \langle d\mathbf{u}^N(t), \mathbf{A}^N\mathbf{u}^N(t)\rangle_H \\
=& -\langle \nu\mathbf{A}^N\mathbf{u}^N(t), \mathbf{A}^N\mathbf{u}^N(t)\rangle_H dt - \langle \mathbf{B}^N(\mathbf{u}^N(t)), \mathbf{A}^N\mathbf{u}^N(t)\rangle_H dt \\
&+ \langle \mathbf{f}^N(t), \mathbf{A}^N\mathbf{u}^N(t)\rangle_H + \langle Z^N(t)dW^N(t), \mathbf{A}^N\mathbf{u}^N(t)\rangle_H
\end{aligned}
$$

Integrating from t to T, and taking conditional expectation for $0 \leq r \leq t \leq T$, we get

$$
\begin{aligned}
E^{\mathcal{F}_r}\|\mathbf{u}^N(t)\|_V^2 \leq & E^{\mathcal{F}_r}\|\xi^N\|_V^2 + 2E^{\mathcal{F}_r}\int_t^T \nu\|\mathbf{A}^N\mathbf{u}^N(s)\|_H^2 ds \\
&+ 2E^{\mathcal{F}_r}\int_t^T \langle \mathbf{B}^N(\mathbf{u}^N(s)), \mathbf{A}^N\mathbf{u}^N(s)\rangle_H ds \\
&+ \int_t^T \|\mathbf{f}^N(s)\|_{V'}^2 ds + E^{\mathcal{F}_r}\int_t^T \|\mathbf{A}^N\mathbf{u}^N(s)\|_V^2 ds
\end{aligned}
\tag{4.2.5}
$$

By the definition of $\mathbf{A}^N$, we know that

$$
\|\mathbf{A}^N\mathbf{u}^N(t)\|_H \leq \lambda_N\|\mathbf{u}^N(t)\|_H \leq \lambda_N\|\mathbf{u}^N(t)\|_V
$$

By Lemma 4.1.9 and Proposition 4.2.1, we have

$$
\begin{aligned}
&|\langle \mathbf{B}^N(\mathbf{u}^N(t)), \mathbf{A}^N\mathbf{u}^N(t)\rangle_H| \leq C_1\|\mathbf{u}^N(t)\|_H^{\frac{1}{2}}\|\mathbf{u}^N(t)\|_V\|\mathbf{A}^N\mathbf{u}^N(t)\|_H^{\frac{3}{2}} \\
\leq & C_1\lambda_N^{\frac{3}{2}}\|\mathbf{u}^N(t)\|_H^2\|\mathbf{u}^N(t)\|_V \leq \frac{C_1^2K_0^2}{4} + \lambda_N^3\|\mathbf{u}^N(t)\|_V^2
\end{aligned}
$$

Thus (4.2.5) becomes

$$
\begin{aligned}
E^{\mathcal{F}_r}\|\mathbf{u}^N(t)\|_V^2 \leq & E^{\mathcal{F}_r}\|\xi^N\|_V^2 + (2\lambda_N^3 + 2\nu\lambda_N^2 + \lambda_N^2)\int_t^T E^{\mathcal{F}_r}\|\mathbf{u}^N(s)\|_V^2 ds \\
&+ \frac{C_1^2K_0^2T}{2} + \int_0^T \|\mathbf{f}^N(s)\|_{V'}^2 ds
\end{aligned}
$$

Denote $2\lambda_N^3 + 2\nu\lambda_N^2 + \lambda_N^2$ by C_N. An application of Gronwall's inequality yields

$$
E^{\mathcal{F}_r}\|\mathbf{u}^N(t)\|_V^2
$$

$$\leq \{E^{\mathcal{F}_r}\|\xi^N\|_V^2 + \frac{C_1^2K_0^2T}{2} + \int_0^T \|\mathbf{f}^N(s)\|_{V'}^2 ds\}\{1 + C_N\int_t^T e^{C_N(t-s)}ds\}$$
$$= \{E^{\mathcal{F}_r}\|\xi^N\|_V^2 + \frac{C_1^2K_0^2T}{2} + \int_0^T \|\mathbf{f}^N(s)\|_{V'}^2 ds\}\{2 - e^{-C_N(T-t)}\}$$
$$\leq 2E^{\mathcal{F}_r}\|\xi^N\|_V^2 + C_1^2K_0^2T + 2\int_0^T \|\mathbf{f}^N(s)\|_{V'}^2 ds$$

Taking r to be t, we get

$$\sup_{0\leq t\leq T}\|\mathbf{u}^N(t)\|_V^2 \leq K_2$$

for some constant K_2 independent of N. □

Now for every $M \in \mathbb{N}$, we define L_M to be a Lipschitz C^∞ function as follows:

$$L_M(\|\mathbf{u}\|_V) = \begin{cases} 1 & \text{if } \|\mathbf{u}\|_V < M \\ 0 & \text{if } \|\mathbf{u}\|_V > M+1 \\ 0 \leq L_M(\|\mathbf{u}\|_V) \leq 1 & \text{otherwise} \end{cases}$$

Proposition 4.2.4. *$\|L_M(\|x\|_V)\mathbf{B}^N(x) - L_M(\|y\|_V)\mathbf{B}^N(y)\|_H \leq C_{N,M}\|x-y\|_V$ for any $x, y \in H_N$ and $M \in \mathbb{N}$.*

Proof: For any $x, y \in H_N$,

$$\|L_M(\|x\|_V)\mathbf{B}^N(x) - L_M(\|y\|_V)\mathbf{B}^N(y)\|_H^2$$
$$= \sum_{i=1}^N |\langle L_M(\|x\|_V)\mathbf{B}(x) - L_M(\|y\|_V)\mathbf{B}(y), e_i\rangle_{V',V}|^2$$
$$= \sum_{i=1}^N |\langle L_M(\|y\|_V)\mathbf{B}(x) - L_M(\|y\|_V)\mathbf{B}(y)$$
$$+ L_M(\|x\|_V)\mathbf{B}(x) - L_M(\|y\|_V)\mathbf{B}(x), e_i\rangle_{V',V}|^2$$
$$\leq 2\sum_{i=1}^N |L_M(\|y\|_V)\langle\mathbf{B}(x) - \mathbf{B}(y), e_i\rangle_{V',V}|^2$$

$$+ 2\sum_{i=1}^{N}|\langle \mathbf{B}(x), e_i\rangle_{V',V}|^2|L_M(\|x\|_V) - L_M(\|y\|_V)|^2$$

$$\leq 2\sum_{i=1}^{N}L_M^2(\|y\|_V)|\langle \mathbf{B}(x) - \mathbf{B}(y), e_i\rangle_{V',V}|^2 + C\sum_{i=1}^{N}|\langle \mathbf{B}(x), e_i\rangle_{V',V}|^2\|x-y\|_V^2$$

since L_M is Lipschitz.

Let us denote $\langle \mathbf{B}(x,y), z\rangle_{V',V}$ by $b(x,y,z)$. For any i,

$$\begin{aligned}
&|\langle \mathbf{B}(x) - \mathbf{B}(y), e_i\rangle_{V',V}| \\
=&|b(x,x,e_i) - b(y,y,e_i)| \\
=&|b(x-y,x,e_i) + b(y,x,e_i) - b(y,y,e_i)| \\
\leq&|b(x-y,x,e_i)| + |b(y,x-y,e_i)|
\end{aligned}$$

By Proposition 4.1.8 and the Poincaré inequality, we have

$$\begin{aligned}
|b(x-y,x,e_i)| &\leq C_G\|x-y\|_H^{\frac{1}{2}}\|x-y\|_V^{\frac{1}{2}}\|x\|_V\|e_i\|_H^{\frac{1}{2}}\|e_i\|_V^{\frac{1}{2}} \\
&\leq C_G\|x\|_V\|e_i\|_V\|x-y\|_V
\end{aligned}$$

for some constant C_G depending G.

Similarly,

$$\begin{aligned}
|b(y,x-y,e_i)| &\leq C_G\|y\|_H^{\frac{1}{2}}\|y\|_V^{\frac{1}{2}}\|x-y\|_V\|e_i\|_H^{\frac{1}{2}}\|e_i\|_V^{\frac{1}{2}} \\
&\leq C_G\|y\|_V\|e_i\|_V\|x-y\|_V
\end{aligned}$$

and

$$|\langle \mathbf{B}(x), e_i\rangle_{V',V}| \leq C_G\|e_i\|_V\|x\|_V^2.$$

Thus

$$\|L_M(\|x\|_V)\mathbf{B}^N(x) - L_M(\|y\|_V)\mathbf{B}^N(y)\|_H^2$$

$$\leq 4 \max_{1\leq i\leq N} \|e_i\|_V^2 NC_G^2[L_M^2(\|y\|_V)(\|x\|_V^2 + \|y\|_V^2) + \|x\|_V^4]\|x-y\|_V^2.$$

Without lose of generality, we assume that $\|x\|_V \leq \|y\|_V$, and let us discuss it in the following 3 cases:

Case I. $\|y\|_V \leq M+1$.

In this case, $\|x\|_V$ and $\|y\|_V$ are all bounded by $M+1$. Thus $\|L_M(\|x\|_V)\mathbf{B}^N(x) - L_M(\|y\|_V)\mathbf{B}^N(y)\|_H^2 \leq C_{N,M}\|x-y\|_V^2$, where $C_{N,M}$ is only related to N, M, and G.

Case II. $\|y\|_V > M+1$ and $\|x\|_V \leq M+1$.

Then by the definition of L_M, $L_M(\|y\|_V) = 0$. Thus

$$\|L_M(\|x\|_V)\mathbf{B}^N(x) - L_M(\|y\|_V)\mathbf{B}^N(y)\|_H^2 \leq 4 \max_{1\leq i\leq N} \|e_i\|_V^2 NC_G^2(M+1)^4\|x-y\|_V^2.$$

Case III. $\|y\|_V > M+1$ and $\|X\|_V > M+1$.

By the definition of L_M, $\|L_M(\|x\|_V)\mathbf{B}^N(x) - L_M(\|y\|_V)\mathbf{B}^N(y)\|_H^2 = 0 \leq \|x-y\|_V^2$.

Thus we have shown that

$$\|L_M(\|x\|_V)\mathbf{B}^N(x) - L_M(\|y\|_V)\mathbf{B}^N(y)\|_H \leq C_{N,M}\|x-y\|_V$$

where $C_{N,M}$ is a constant which is only related to N, M and G. □

Proposition 4.2.5. *Assume that* $\mathbf{f} \in L^2(0,T;V')$*, and* $\|\xi\|_H^2 \leq K$ *for almost all* $\omega \in \Omega$ *and some constant* K*. Then the projected system* (4.2.1) *admits a unique adapted solution* $(\mathbf{u}^N(t), Z^N(t))$ *for each* N *and*

$$E(\sup_{0\leq t\leq T} \|\mathbf{u}^N(t)\|_H^2) + E\int_0^T \|Z^N(s)\|_{L_Q}^2 ds < \infty \tag{4.2.6}$$

Proof: First, we introduce some notations.

For $1 \leq i \leq N$, suppose that $\langle \mathbf{u}^N(t), e_i\rangle_H = \hat{u}_i^N(t)$ and we denote

$$\hat{\mathbf{u}}^N(t) = \begin{pmatrix} \hat{u}_1^N(t) \\ \vdots \\ \hat{u}_N^N(t) \end{pmatrix}.$$

For $\mathbf{A}^N$, we denote

$$\langle \mathbf{A}^N \mathbf{u}^N(t), e_i \rangle_{V',V} = \langle \sum_{i=1}^{N} \lambda_i \hat{u}_i^N(t) e_i, e_i \rangle_{V',V} = \lambda_i \hat{u}_i^N(t).$$

For $\mathbf{B}^N$, we have

$$\begin{aligned}
&\langle \mathbf{B}^N(\mathbf{u}^N(t)), e_i \rangle_{V',V} = \langle P_N \mathbf{B}(\mathbf{u}^N(t)), e_i \rangle_{V',V} \\
=&\langle \sum_{j=1}^{N} \langle \mathbf{B}(\mathbf{u}^N(t)), e_j \rangle_{V',V} e_j, e_i \rangle_{V',V} \\
=&\langle \sum_{j=1}^{N} b(\mathbf{u}^N(t), \mathbf{u}^N(t), e_j) e_j, e_i \rangle_{V',V} \\
=&b(\mathbf{u}^N(t), \mathbf{u}^N(t), e_i) \\
=&b(\sum_{k=1}^{N} \hat{u}_k^N(t) e_k, \sum_{l=1}^{N} \hat{u}_l^N(t) e_l, e_i) \\
=&\sum_{k,l=1}^{N} b(e_k, e_l, e_i) \hat{u}_k^N(t) \hat{u}_l^N(t)
\end{aligned}$$

Since we have

$$\begin{aligned}
\|\mathbf{u}^N(t)\|_V =&\langle \mathbf{A}\mathbf{u}^N(t), \mathbf{u}^N(t) \rangle_{V',V}^{\frac{1}{2}} = \langle \sum_{j=1}^{N} \lambda_j \hat{u}_j^N(t) e_j, \sum_{i=1}^{N} \hat{u}_i^N(t) e_i \rangle_{V',V}^{\frac{1}{2}} \\
=&\sqrt{\sum_{j=1}^{N} \lambda_j (\hat{u}_j^N(t))^2},
\end{aligned}$$

Now we define $\hat{\mathbf{A}}^N(\hat{\mathbf{u}}^N(t)) = \begin{pmatrix} \lambda_1 \hat{u}_1^N(t) \\ \lambda_2 \hat{u}_2^N(t) \\ \vdots \\ \lambda_N \hat{u}_N^N(t) \end{pmatrix}$ and

$$\hat{\mathbf{B}}^N(\hat{\mathbf{u}}^N(t)) = \begin{pmatrix} \sum_{k,l=1}^{N} b(e_k, e_l, e_1) \hat{u}_k^N(t) \hat{u}_l^N(t) \\ \sum_{k,l=1}^{N} b(e_k, e_l, e_2) \hat{u}_k^N(t) \hat{u}_l^N(t) \\ \vdots \\ \sum_{k,l=1}^{N} b(e_k, e_l, e_N) \hat{u}_k^N(t) \hat{u}_l^N(t) \end{pmatrix}$$

We also denote $\langle \mathbf{f}^N(t), e_i\rangle_{V',V} = \hat{f}_i^N(t)$, $\hat{\mathbf{f}}^N(t) = \begin{pmatrix} \hat{f}_1^N(t) \\ \vdots \\ \hat{f}_N^N(t) \end{pmatrix}$, and $\hat{\xi}^N = \begin{pmatrix} \langle \xi^N, e_1\rangle_H \\ \vdots \\ \langle \xi^N, e_N\rangle_H \end{pmatrix}$.

Since

$$\begin{aligned}
&\langle \int_t^T Z^N(s) dW^N(s), e_i\rangle_H \\
=&\langle \sum_{k=1}^N \sqrt{q_k} \int_t^T Z^N(s)(e_k) db^k(s), e_i\rangle_H \\
=&\langle \sum_{k=1}^N \sqrt{q_k} \sum_{l=1}^N \int_t^T \langle Z^N(s)(e_k), e_l\rangle_H e_l db^k(s), e_i\rangle_H \\
=&\sum_{k=1}^N \sqrt{q_k} \int_t^T \langle Z^N(s)(e_k), e_i\rangle_H db^k(s) \\
=&\sum_{k=1}^N \int_t^T \langle \sqrt{q_k} e_k, (Z^N(s))^*(e_i)\rangle_H db^k(s) \\
=&\sum_{k=1}^N \int_t^T \langle Q^{\frac{1}{2}}(e_k), (Z^N(s))^*(e_i)\rangle_H db^k(s)
\end{aligned} \tag{4.2.7}$$

we define $\hat{Z}^N(t)$ as

$$\begin{pmatrix}
\langle Q^{\frac{1}{2}}(e_1), (Z^N(s))^*(e_1)\rangle_H, & \langle Q^{\frac{1}{2}}(e_2), (Z^N(s))^*(e_1)\rangle_H, & \cdots, & \langle Q^{\frac{1}{2}}(e_N), (Z^N(s))^*(e_1)\rangle_H \\
\langle Q^{\frac{1}{2}}(e_1), (Z^N(s))^*(e_2)\rangle_H, & \langle Q^{\frac{1}{2}}(e_2), (Z^N(s))^*(e_2)\rangle_H, & \cdots, & \langle Q^{\frac{1}{2}}(e_N), (Z^N(s))^*(e_2)\rangle_H \\
\vdots & \vdots & \vdots & \vdots \\
\langle Q^{\frac{1}{2}}(e_1), (Z^N(s))^*(e_N)\rangle_H, & \langle Q^{\frac{1}{2}}(e_2), (Z^N(s))^*(e_N)\rangle_H, & \cdots, & \langle Q^{\frac{1}{2}}(e_N), (Z^N(s))^*(e_N)\rangle_H
\end{pmatrix}$$

and $\hat{\mathcal{W}}^N(t) = \begin{pmatrix} b^1(t) \\ \vdots \\ b^N(t) \end{pmatrix}$ where $\{b^j(t)\}_1^N$ are N independent standard 1-dimensional Brownian motions.

Thus for any $N \in \mathbb{N}$, the projected system 4.2.1 is equivalent to

$$\begin{cases}
d\hat{\mathbf{u}}^N(t) = -\nu \hat{\mathbf{A}}^N \hat{\mathbf{u}}^N(t)dt - \hat{\mathbf{B}}^N(\hat{\mathbf{u}}^N(t))dt + \hat{\mathbf{f}}^N(t)dt + \hat{Z}^N(t)d\hat{\mathcal{W}}^N(t) \\
\hat{\mathbf{u}}^N(T) = \hat{\xi}^N
\end{cases} \tag{4.2.8}$$

Define the associated truncated system as follows:

$$\begin{cases} d\hat{\mathbf{u}}^{N,M}(t) & = -\nu \hat{\mathbf{A}}^N \hat{\mathbf{u}}^{N,M}(t)dt - L_M(\|\mathbf{u}^{N,M}(t)\|_V)\hat{\mathbf{B}}^N(\hat{\mathbf{u}}^{N,M}(t))dt + \hat{\mathbf{f}}^N(t)dt \\ & \quad +\hat{Z}^{N,M}(t)d\hat{\mathcal{W}}^N(t) \\ \hat{\mathbf{u}}^{N,M}(T) & = \hat{\xi}^N \end{cases} \tag{4.2.9}$$

Let $h^{N,M}(t,x) = -\nu\hat{\mathbf{A}}^N x - L_M(\|x\|_V)\hat{\mathbf{B}}^N(x) + \hat{\mathbf{f}}^N(t)$. Obviously, $h^{N,M}(t,x)$ is Lipschitz on $[0,T] \times \mathbb{R}^N$. By Theorem 2.4.2, we know that (4.2.9) admits a unique adapted solution $(\hat{\mathbf{u}}^{N,M}(t), \hat{Z}^{N,M}(t)) \in \mathcal{M}[0,T]$, where $\mathcal{M}[0,T]$ equipped with the norm $\|Y(\cdot), Z(\cdot)\|_{\mathcal{M}}$=$\{E(\sup_{0\le t\le T} |Y(t)|^2) + E\int_0^T |Z(t)|^2 dt\}^{\frac{1}{2}}$ and here $|Z|^2 = tr(ZZ^T)$.

Similar to the proof of Proposition 4.2.1, it can be shown that $\sup_{0\le t\le T} \|\mathbf{u}^{N,M}(t)\|_H^2$ $\le K_0$ for a constant K_0 independent of N, and M. For a fixed N, we make use of the fact that $V_N = H_N$ and $\|\cdot\|_V$ and $\|\cdot\|H$ are equivalent to each other for the finite dimensional case. So, $\sup_{0\le t\le T} \|\mathbf{u}^{N,M}(t)\|_V^2 \le K_{0,N}$ for a constant $K_{0,N}$ independent of M.

Thus for $M > K_{0,N}$, $L_M(\|\mathbf{u}^{N,M}(t)\|_V) = 1$, and the solution of (4.2.9) is also the solution of (4.2.8). The existence of a solution of (4.2.8) has thus been shown. Let $(\hat{\mathbf{u}}^N(t), \hat{Z}^N(t))$ and $(\hat{\mathbf{v}}^N(t), \hat{Y}^N(t))$ be two pairs of solutions of (4.2.8). We know that there exists an M_0, such that $\sup_{0\le t\le T} |\hat{\mathbf{u}}^N(t)|^2 \le M_0$ and $\sup_{0\le t\le T} |\hat{\mathbf{v}}^N(t)|^2 \le M_0$. Since (4.2.8) and (4.2.9) are the same for $M > M_0$, we know that $(\hat{\mathbf{u}}^N(t), \hat{Z}^N(t))$ and $(\hat{\mathbf{v}}^N(t), \hat{Y}^N(t))$ are also solutions of (4.2.9). The uniqueness of the solution of (4.2.9) implies the uniqueness of the solution of (4.2.8).

Since (4.2.1) and (4.2.8) are equivalent, we have shown that there is a unique adapted solution $(\mathbf{u}^N(t), Z^N(t))$ to the projected system (4.2.1).

Equation (4.2.6) is proved by the following:

$$
\begin{aligned}
&E(\int_0^T \hat{Z}^N(s)d\hat{\mathcal{W}}^N(s))^2 = E\int_0^T |\hat{Z}^N(s)|^2 ds = E\int_0^T tr(\hat{Z}^N(s)\hat{Z}^{N^T}(s))ds \\
=&E\sum_{k,l=1}^N \int_0^T \langle Q^{\frac{1}{2}}(e_k), (Z^N(s))^*(e_l)\rangle_H^2 ds = E(\int_0^T Z^N(s)dW^N(s))^2 \quad \text{by (4.2.7)} \\
=&E\sum_{k,l=1}^N \int_0^T \langle e_k, Q^{\frac{1}{2}}(Z^N(s))^*(e_l)\rangle_H^2 ds = E\sum_{k,l=1}^\infty \int_0^T \langle e_k, Q^{\frac{1}{2}}(Z^N(s))^*(e_l)\rangle_H^2 ds \\
=&E\sum_{l=1}^\infty \int_0^T \|Q^{\frac{1}{2}}(Z^N(s))^*(e_l)\|_H^2 ds = E\int_0^T \|Q^{\frac{1}{2}}(Z^N(s))^*\|_{H.S.}^2 ds \\
=&E\int_0^T tr(Z^N(s)Q(Z^N(s))^*)ds = E\int_0^T \|Z^N(s)\|_{L_Q}^2 ds.
\end{aligned}
$$

□

4.3 Existence of Solutions

First of all, let us prove some simple results.

Lemma 4.3.1.

$$
|\langle \mathbf{B}(\mathbf{u}) - \mathbf{B}(\mathbf{v}), \mathbf{u} - \mathbf{v}\rangle_{V',V}| \le \frac{\nu}{2}\|\mathbf{u}-\mathbf{v}\|_V^2 + \frac{C_G^2}{2\nu}\|\mathbf{u}-\mathbf{v}\|_H^2\|\mathbf{v}\|_V^2
$$

for all $\mathbf{u}, \mathbf{v} \in V$.

Proof: Denote $\mathbf{u}-\mathbf{v}$ by $\mathbf{w}$, then

$$
\begin{aligned}
&\langle \mathbf{B}(\mathbf{u}) - \mathbf{B}(\mathbf{v}), \mathbf{u} - \mathbf{v}\rangle_{V',V} = \langle \mathbf{B}(\mathbf{u}), \mathbf{w}\rangle_{V',V} - \langle \mathbf{B}(\mathbf{v}), \mathbf{w}\rangle_{V',V} \\
=& -\langle \mathbf{B}(\mathbf{u},\mathbf{w}), \mathbf{u}\rangle_{V',V} + \langle \mathbf{B}(\mathbf{v},\mathbf{w}), \mathbf{v}\rangle_{V',V} \\
=& -\langle \mathbf{B}(\mathbf{u},\mathbf{w}), \mathbf{w}\rangle_{V',V} - \langle \mathbf{B}(\mathbf{u},\mathbf{w}), \mathbf{v}\rangle_{V',V} + \langle \mathbf{B}(\mathbf{v},\mathbf{w}), \mathbf{v}\rangle_{V',V} \\
=& -\langle \mathbf{B}(\mathbf{u},\mathbf{w}), \mathbf{v}\rangle_{V',V} + \langle \mathbf{B}(\mathbf{v},\mathbf{w}), \mathbf{v}\rangle_{V',V} \\
=& -\langle \mathbf{B}(\mathbf{w}), \mathbf{v}\rangle_{V',V}
\end{aligned}
$$

By Proposition 4.1.8, we have

$$|\langle \mathbf{B}(\mathbf{u}) - \mathbf{B}(\mathbf{v}), \mathbf{u} - \mathbf{v}\rangle_{V',V}| = | - \langle \mathbf{B}(\mathbf{w}), \mathbf{v}\rangle_{V',V}|$$
$$=| - \langle \mathbf{B}(\mathbf{w}, \mathbf{v}), \mathbf{w}\rangle_{V',V}| \leq C_G \|\mathbf{w}\|_V \|\mathbf{w}\|_H \|\mathbf{v}\|_V$$
$$\leq \frac{\nu}{2}\|\mathbf{u} - \mathbf{v}\|_V^2 + \frac{C_G^2}{2\nu}\|\mathbf{u} - \mathbf{v}\|_H^2 \|\mathbf{v}\|_V^2.$$

□

Lemma 4.3.2.

$$\langle \nu \mathbf{A}\mathbf{w}, \mathbf{w}\rangle_{V',V} + \langle \mathbf{B}(\mathbf{u}) - \mathbf{B}(\mathbf{v}), \mathbf{w}\rangle_{V',V} + \frac{C_G^2}{2\nu}\|\mathbf{w}\|_H^2 \|\mathbf{v}\|_V^2 \geq \frac{\nu}{2}\|\mathbf{w}\|_V^2$$

for all $\mathbf{u}, \mathbf{v} \in V$*, where* $\mathbf{w} = \mathbf{u} - \mathbf{v}$*.*

Proof: From Lemma 4.3.1, we know that

$$\langle \mathbf{B}(\mathbf{u}) - \mathbf{B}(\mathbf{v}), \mathbf{w}\rangle_{V',V} \geq -\frac{\nu}{2}\|\mathbf{w}\|_V^2 - \frac{C_G^2}{2\nu}\|\mathbf{w}\|_H^2 \|\mathbf{v}\|_V^2$$

Thus by Proposition 4.1.6,

$$\langle \nu \mathbf{A}\mathbf{w}, \mathbf{w}\rangle_{V',V} + \langle \mathbf{B}(\mathbf{u}) - \mathbf{B}(\mathbf{v}), \mathbf{w}\rangle_{V',V} \geq \frac{\nu}{2}\|\mathbf{w}\|_V^2 - \frac{C_G^2}{2\nu}\|\mathbf{w}\|_H^2 \|\mathbf{v}\|_V^2$$

□

Corollary 4.3.3. *Let* $r_1(t) = \frac{C_G^2}{\nu}\int_0^t \|\mathbf{u}(s)\|_V^2 ds$ *and* $r_2(t) = \frac{C_G^2}{\nu}\int_0^t \|\mathbf{v}(s)\|_V^2 ds$ *for all* $\mathbf{u}, \mathbf{v} \in L^2_{\mathcal{F}}(\Omega; L^2(0,T;V))$*. Then*

$$\langle \nu \mathbf{A}\mathbf{w} + \mathbf{B}(\mathbf{u}) - \mathbf{B}(\mathbf{v}) + \frac{1}{2}\dot{r}_i(t)\mathbf{w}, \mathbf{w}\rangle_{V',V} \geq 0 \qquad i = 1, 2$$

for all $\mathbf{u}, \mathbf{v} \in L^2_{\mathcal{F}}(\Omega; L^2(0,T;V))$*, where* $\mathbf{w} = \mathbf{u} - \mathbf{v}$*.*

Proof: Notice that $\langle \frac{1}{2}\dot{r}_2(t)\mathbf{w}, \mathbf{w}\rangle_{V',V} = \frac{C_G^2}{2\nu}\|\mathbf{w}\|_H^2 \|\mathbf{v}\|_V^2$. Thus

$$\langle \nu \mathbf{A}\mathbf{w} + \mathbf{B}(\mathbf{u}) - \mathbf{B}(\mathbf{v}) + \frac{1}{2}\dot{r}_2(t)\mathbf{w}, \mathbf{w}\rangle_{V',V}$$

$$
\begin{aligned}
&=\langle \nu \mathbf{A}\mathbf{w} + \mathbf{B}(\mathbf{u}) - \mathbf{B}(\mathbf{v}), \mathbf{w}\rangle_{V',V} + \frac{C_G^2}{2\nu}\|\mathbf{w}\|_H^2\|\mathbf{v}\|_V^2 \\
&\geq \frac{\nu}{2}\|\mathbf{w}\|_V^2 \geq 0
\end{aligned}
$$

Similarly, we can prove it for the case that $i = 1$. □

Lemma 4.3.4. *For any* $\mathbf{u}$*,* $\mathbf{v}$*, and* $\mathbf{w} \in V$*, we have*

$$
|\langle \mathbf{B}(\mathbf{u}) - \mathbf{B}(\mathbf{v}), \mathbf{w}\rangle_{V',V}| \leq C(\|\mathbf{u}\|_H^{\frac{1}{2}}\|\mathbf{u}\|_V^{\frac{1}{2}} + \|\mathbf{v}\|_H^{\frac{1}{2}}\|\mathbf{v}\|_V^{\frac{1}{2}})\|\mathbf{u}-\mathbf{v}\|_H^{\frac{1}{2}}\|\mathbf{u}-\mathbf{v}\|_V^{\frac{1}{2}}\|\mathbf{w}\|_V
$$

Proof:

$$
\begin{aligned}
&\langle \mathbf{B}(\mathbf{u}) - \mathbf{B}(\mathbf{v}), \mathbf{w}\rangle_{V',V} \\
=& - \langle \mathbf{B}(\mathbf{u},\mathbf{w}), \mathbf{u}\rangle_{V',V} + \langle \mathbf{B}(\mathbf{v},\mathbf{w}), \mathbf{v}\rangle_{V',V} \\
=& - \langle \mathbf{B}(\mathbf{u},\mathbf{w}), \mathbf{u}-\mathbf{v}\rangle_{V',V} - \langle \mathbf{B}(\mathbf{u},\mathbf{w}), \mathbf{v}\rangle_{V',V} + \langle \mathbf{B}(\mathbf{v},\mathbf{w}), \mathbf{v}\rangle_{V',V} \\
=& - \langle \mathbf{B}(\mathbf{u},\mathbf{w}), \mathbf{u}-\mathbf{v}\rangle_{V',V} - \langle \mathbf{B}(\mathbf{u}-\mathbf{v},\mathbf{w}), \mathbf{v}\rangle_{V',V}
\end{aligned}
$$

Thus

$$
\begin{aligned}
&|\langle \mathbf{B}(\mathbf{u}) - \mathbf{B}(\mathbf{v}), \mathbf{w}\rangle_{V',V}| \\
\leq& C\|\mathbf{u}\|_H^{\frac{1}{2}}\|\mathbf{u}\|_V^{\frac{1}{2}}\|\mathbf{u}-\mathbf{v}\|_H^{\frac{1}{2}}\|\mathbf{u}-\mathbf{v}\|_V^{\frac{1}{2}}\|\mathbf{w}\|_V \\
+& C\|\mathbf{u}-\mathbf{v}\|_H^{\frac{1}{2}}\|\mathbf{u}-\mathbf{v}\|_V^{\frac{1}{2}}\|\mathbf{v}\|_H^{\frac{1}{2}}\|\mathbf{v}\|_V^{\frac{1}{2}}\|\mathbf{w}\|_V \\
=& C(\|\mathbf{u}\|_H^{\frac{1}{2}}\|\mathbf{u}\|_V^{\frac{1}{2}} + \|\mathbf{v}\|_H^{\frac{1}{2}}\|\mathbf{v}\|_V^{\frac{1}{2}})\|\mathbf{u}-\mathbf{v}\|_H^{\frac{1}{2}}\|\mathbf{u}-\mathbf{v}\|_V^{\frac{1}{2}}\|\mathbf{w}\|_V
\end{aligned}
$$

□

Lemma 4.3.5. *Let* $A : K \to K'$ *be linear and monotone, where* K *is a Banach space and* K' *is the dual. Then* A *is continuous.*

Proof: First, let us prove that A is locally bounded at 0, i.e., there exists a neighborhood U of 0, such that AU is bounded.

Suppose that A is not locally bounded at 0. Then there exists a sequence $\{x_i\}_{i=1}^{\infty}$ in K, such that

$$\|x_i\|_K \to 0 \text{ but } \|Ax_i\|_{K'} \to \infty$$

Let $a_i = \frac{1}{1+\|Ax_i\|_{K'}\|x_i\|_K}$. Since A is monotone, we know that for every $y \in K$, $a_i\langle A(x_i - y), x_i - y\rangle_{K',K} \geq 0$. Thus

$$a_i\langle Ax_i, x_i\rangle_{K',K} - a_i\langle Ay, x_i - y\rangle_{K',K} \geq a_i\langle Ax_i, y\rangle_{K',K},$$

which implies

$$\frac{\|Ax_i\|_{K'}\|x_i\|_K + \|Ay\|_{K'}\|x_i - y\|_K}{1 + \|Ax_i\|_{K'}\|x_i\|_K} \geq a_i\langle Ax_i, y\rangle_{K',K},$$

Similarly, since $a_i\langle A(x_i + y), x_i + y\rangle_{K',K} \geq 0$, we know that

$$a_i\langle Ax_i, y\rangle_{K',K} \geq -\frac{\|Ax_i\|_{K'}\|x_i\|_K + \|Ay\|_{K'}\|x_i + y\|_K}{1 + \|Ax_i\|_{K'}\|x_i\|_K}.$$

We know that $\|Ay\|_{K'}$ is bounded. $\|x_i + y\|_K$ and $\|x_i - y\|_K$ are bounded by $\|x_i\|_K + \|y\|_K$ and $\|x_i\|_K \to 0$. Since $\|Ax_i\|_{K'} \to \infty$, we know that

$$\sup_i |\langle a_i Ax_i, y\rangle_{K',K}| < \infty \qquad \text{for every } y \text{ in } K.$$

By Banach-Steinhaus theorem, we know that for some constant N,

$$\sup_i \|a_i Ax_i\|_{K'} \leq N,$$

i.e.,

$$\begin{aligned}
&\sup_i \|a_i Ax_i\|_{K'} \\
=&\sup_i \frac{\|Ax_i\|_{K'}}{1 + \|Ax_i\|_{K'}\|x_i\|_K} \\
=&\sup_i \frac{1}{\frac{1}{\|Ax_i\|_{K'}} + \|x_i\|_K} \leq N
\end{aligned}$$

This is an contradiction since $\frac{1}{\|Ax_i\|_{K'}}$ and $\|x_i\|_K$ both tends to 0 as i goes to ∞. Thus A is locally bounded at 0.

Since A is linear and bounded on a neighborhood of 0, A is also bounded on the closed unit ball, i.e., A is continuous. □

Theorem 4.3.6. *Assume that* $\|\xi\|_H^2 < K$ *for some constant* K*,* P*-a.s., and* $\mathbf{f} \in L^4(0,T;V')$*. Then the Navier-Stokes equation* (4.1.3) *admits an adapted solution* $(\mathbf{u}(t), Z(t)) \in L^\infty(\Omega \times [0,T]; H) \cap L^4_{\mathcal{F}}(\Omega; L^2(0,T;V)) \times L^2_{\mathcal{F}}(\Omega; L^2(0,T;L_Q))$.

Proof: We will prove the theorem in the following steps.

Step 1: First, let us find some bounds for the projected system. By Proposition 4.2.1, we know that $\{\mathbf{u}^N(t)\}_{N=1}^\infty$ is bounded in $L^2_{\mathcal{F}}(\Omega; L^2(0,T;V))$. Hence $\{\mathbf{u}^N(t)\}_{N=1}^\infty$ is bounded in $L^2_{\mathcal{F}}(\Omega; L^2(0,T;V'))$.

Since $\mathbf{A}$ is linear and monotone, i.e.,

$$\langle \mathbf{A}(\mathbf{u}-\mathbf{v}), \mathbf{u}-\mathbf{v}\rangle_{V',V} = \|\mathbf{u}-\mathbf{v}\|_V \geq 0,$$

by Lemma 4.3.5, we know that $\mathbf{A}$ is continuous. So there exists a constant C, such that $\|\mathbf{A}\mathbf{u}\|_{V'} \leq C\|\mathbf{u}\|_V$ for all $\mathbf{v} \in V$. Thus from (4.2.2), we know that $\{\mathbf{A}^N\mathbf{u}^N(t)\}_{N=1}^\infty$ is bounded in $L^2_{\mathcal{F}}(\Omega; L^2(0,T;V'))$.

By Proposition 4.1.8, for any $\mathbf{v} \in V$,

$$|\langle \mathbf{B}^N(\mathbf{u}^N(t)), \mathbf{v}\rangle_{V',V}| \leq C_G \|\mathbf{u}^N(t)\|_V \|\mathbf{u}^N(t)\|_H \|\mathbf{v}\|_V.$$

By Proposition 4.2.1,

$$\|\mathbf{B}^N(\mathbf{u}^N(t))\|_{V'} = \sup_{\|\mathbf{v}\|_V=1} |\langle \mathbf{B}^N(\mathbf{u}^N(t)), \mathbf{v}\rangle_{V',V}| \leq C_G\sqrt{K_0}\|\mathbf{u}^N(t)\|_V.$$

Since $\{\mathbf{u}^N(t)\}_{N=1}^\infty$ is bounded in $L^2_{\mathcal{F}}(\Omega; L^2(0,T;V'))$, so is $\{\mathbf{B}^N(\mathbf{u}^N(t))\}$.

It readily follows by Proposition 4.2.1 that $\{Z^N(t)\}$ is bounded in $L^2_{\mathcal{F}}(\Omega; L^2(0,T;L_Q))$.

Step 2: Clearly we have the following strong convergence:

$$\xi^N \to \xi \text{ and } \mathbf{f}^N(t) \to \mathbf{f}(t) \qquad \text{in } L^2_{\mathcal{F}}(\Omega; L^2(0,T;V'))$$

Since $L^2_{\mathcal{F}}(\Omega; L^2(0,T;V'))$ and $L^2_{\mathcal{F}}(\Omega; L^2(0,T;L_Q))$ are Hilbert spaces, and from the results in Step 1, there exist $\mathbf{u}(t)$, $Y(t)$, $G(t)$, $Z(t)$, and $\{N_k\}_{k=1}^{\infty}$, such that

$$\mathbf{u}^{N_k}(t) \xrightarrow{w} \mathbf{u}(t), \ \nu \mathbf{A}^{N_k}\mathbf{u}^{N_k}(t) \xrightarrow{w} Y(t), \text{ and } \mathbf{B}^{N_k}(\mathbf{u}^{N_k}(t)) \xrightarrow{w} G(t)$$

in $L^2_{\mathcal{F}}(\Omega; L^2(0,T;V'))$, and

$$Z^{N_k}(t) \xrightarrow{w} Z(t) \quad \text{in } L^2_{\mathcal{F}}(\Omega; L^2(0,T;L_Q)).$$

For every t, we define

$$\begin{aligned} \mathcal{L}_t : L^2_{\mathcal{F}}(\Omega; L^2(0,T;L_Q)) &\to L^2_{\mathcal{F}}(\Omega; L^2(0,T;V')) \\ M(t) &\mapsto \int_t^T M(s)dW(s) \end{aligned}$$

Then by Bürkholder-Davis-Gundy's inequality,

$$\begin{aligned} &E\int_0^T \| \int_t^T M(s)dW(s)\|^2_{V'}dt \\ \le &E\int_0^T \| \int_t^T M(s)dW(s)\|^2_H dt \\ \le &TE\sup_{0\le t\le T} \| \int_t^T M(s)dW(s)\|^2_H \\ \le &2TE\| \int_0^T M(s)dW(s)\|^2_H + 2TE\sup_{0\le t\le T} \| \int_0^t M(s)dW(s)\|^2_H \\ \le &4TE\sup_{0\le t\le T} \| \int_0^t M(s)dW(s)\|^2_H \\ \le &4TCE\int_0^T \|M(s)\|^2_{L_Q}ds \end{aligned}$$

for some constant C. This shows that $\mathcal{L}_t$ is a bounded linear operator. Hence $\mathcal{L}_t$ maps weakly convergent sequence $\{Z^{N_k}(t)\}_{k=1}^{\infty}$ to a weakly convergent sequence $\{\int_t^T Z^{N_k}(s)dW^{N_k}(s)\}_{k=1}^{\infty}$ with limit $\int_t^T Z(s)dW(s)$.

Here we have used the fact that $\int_t^T Z^N(s)dW(s) = \int_t^T Z^N(s)dW^N(s)$ by letting $Z^N(t)(e_i)=0$ for $i > N$.

Similarly, it can be shown that

$$\int_t^T (\nu \mathbf{A}^{N_k}\mathbf{u}^{N_k}(s) + \mathbf{B}^{N_k}(\mathbf{u}^{N_k}(s)))ds \xrightarrow{w} \int_t^T (Y(s) + G(s))ds$$

in $L^2_{\mathcal{F}}(\Omega; L^2(0,T;V'))$. Let

$$\begin{aligned} F^{N_k}(t) =& \xi^{N_k} + \int_t^T \{\nu \mathbf{A}^{N_k}\mathbf{u}^{N_k}(s) + \mathbf{B}^{N_k}(\mathbf{u}^{N_k}(s)) - \mathbf{f}^{N_k}(s)\}ds \\ & - \int_t^T Z^{N_k}(s)dW^{N_k}(s). \end{aligned}$$

Then $\mathbf{u}^{N_k}(t) = F^{N_k}(t)$ P-a.s. for every k and they both weakly convergent in $L^2_{\mathcal{F}}(\Omega; L^2(0,T;V'))$. Hence the weak limits agree P-a.s, i.e.,

$$\mathbf{u}(t) = \xi + \int_t^T (Y(s) + G(s) - \mathbf{f}(s))ds - \int_t^T Z(s)dW(s) \tag{4.3.1}$$

Step 3: Now let us prove the existence. From now on, we will denote the index of those convergent subsequences by N again, instead of N_k.

Let $r(t) = \frac{C_G^2}{\nu}\int_0^t \|\mathbf{v}(s)\|_V^2 ds$ for any $\mathbf{v}(t) \in L^\infty(\Omega \times [0,T]; V)$. Apply Itô's formula to $e^{-r(t)}\|\mathbf{u}^N(t)\|_H^2$, we get

$$\begin{aligned} & e^{-r(T)}\|\xi^N\|_H^2 - \|\mathbf{u}^N(0)\|_H^2 \\ =& - \int_0^T \dot{r}(t)e^{-r(t)}\|\mathbf{u}^N(t)\|_H^2 dt \\ & - 2\int_0^T e^{-r(t)}\langle \nu \mathbf{A}^N\mathbf{u}^N(t) + \mathbf{B}^N(\mathbf{u}^N(t)) - \mathbf{f}^N(t), \mathbf{u}^N(t)\rangle_{V',V} dt \\ & + 2\int_0^T e^{-r(t)}\langle Z^{N*}(t)(\mathbf{u}^N(t)), dW^N(t)\rangle_H + \int_0^T e^{-r(t)}\|Z^N(t)\|_{L_Q}^2 dt \\ =& - 2\int_0^T e^{-r(t)}\langle \nu \mathbf{A}^N\mathbf{u}^N(t) + \mathbf{B}^N(\mathbf{u}^N(t)) + \frac{1}{2}\dot{r}(t)\mathbf{u}^N(t), \mathbf{u}^N(t)\rangle_{V',V} dt \\ & + 2\int_0^T e^{-r(t)}\langle \mathbf{f}^N(t), \mathbf{u}^N(t)\rangle_{V',V} dt + 2\int_0^T e^{-r(t)}\langle Z^{N*}(t)(\mathbf{u}^N(t)), dW^N(t)\rangle_H \end{aligned}$$

$$+\int_0^T e^{-r(t)}\|Z^N(t)\|_{L_Q}^2 dt$$

Now by taking expectation, we get

$$\begin{aligned} &Ee^{-r(T)}\|\xi^N\|_H^2 - E\|\mathbf{u}^N(0)\|_H^2 - 2E\int_0^T e^{-r(t)}\langle \mathbf{f}^N(t), \mathbf{u}^N(t)\rangle_{V',V} dt \\ &- E\int_0^T e^{-r(t)}\|Z^N(t)\|_{L_Q}^2 dt \\ =& -2E\int_0^T e^{-r(t)}\langle \nu \mathbf{A}^N \mathbf{u}^N(t) + \mathbf{B}^N(\mathbf{u}^N(t)) + \frac{1}{2}\dot{r}(t)\mathbf{u}^N(t), \mathbf{u}^N(t)\rangle_{V',V} dt \end{aligned} \tag{4.3.2}$$

Likewise, equation (4.3.1) and the Itô formula applied to $e^{-r(t)}\|\mathbf{u}(t)\|_H^2$ yield

$$\begin{aligned} &E\|\mathbf{u}(0)\|_H^2 - Ee^{-r(T)}\|\xi\|_H^2 + 2E\int_0^T e^{-r(t)}\langle \mathbf{f}(t), \mathbf{u}(t)\rangle_{V',V} dt \\ &+ E\int_0^T e^{-r(t)}\|Z(t)\|_{L_Q}^2 dt \\ =&2E\int_0^T e^{-r(t)}\langle Y(t) + G(t) + \frac{1}{2}\dot{r}(t)\mathbf{u}(t), \mathbf{u}(t)\rangle_{V',V} dt \end{aligned} \tag{4.3.3}$$

Taking the limit, (4.3.2) becomes

$$\begin{aligned} &\varliminf_{N\to\infty} \{2E\int_0^T e^{-r(t)}\langle \nu \mathbf{A}^N \mathbf{u}^N(t) + \mathbf{B}^N(\mathbf{u}^N(t)) + \frac{1}{2}\dot{r}(t)\mathbf{u}^N(t), \mathbf{u}^N(t)\rangle_{V',V} dt\} \\ =& \varliminf_{N\to\infty} \{E\|\mathbf{u}^N(0)\|_H^2 - Ee^{-r(T)}\|\xi^N\|_H^2 + 2E\int_0^T e^{-r(t)}\langle \mathbf{f}^N(t), \mathbf{u}^N(t)\rangle_{V',V} dt \\ &+ E\int_0^T e^{-r(t)}\|Z^N(t)\|_{L_Q}^2 dt\} \end{aligned} \tag{4.3.4}$$

For the first term in (4.3.4), since

$$0 \le E\|\mathbf{u}^N(0) - \mathbf{u}(0)\|_H^2 = E\|\mathbf{u}^N(0)\|_H^2 + E\|\mathbf{u}(0)\|_H^2 - 2E\langle \mathbf{u}^N(0), \mathbf{u}(0)\rangle_H$$

and $\mathbf{u}^N(0) \xrightarrow{w} \mathbf{u}(0)$ in $L^2_{\mathcal{F}_0}(\Omega; H)$, we get

$$0 \le \varliminf_{N\to\infty} E\|\mathbf{u}^N(0)\|_H^2 + E\|\mathbf{u}(0)\|_H^2 - 2E\|\mathbf{u}(0)\|_H^2$$

i.e.

$$\varliminf_{N\to\infty} E\|\mathbf{u}^N(0)\|_H^2 \ge E\|\mathbf{u}(0)\|_H^2$$

For the second term in (4.3.4), since

$$|Ee^{-r(T)}\|\xi^N\|_H^2 - Ee^{-r(T)}\|\xi\|_H^2| \leq |Ee^{-r(T)}\|\xi^N - \xi\|_H^2| \to 0,$$

we know that $\varliminf_{N\to\infty} Ee^{-r(T)}\|\xi^N\|_H^2 = Ee^{-r(T)}\|\xi\|_H^2$

For the third term in (4.3.4),

$$\begin{aligned}
&|E\int_0^T e^{-r(t)}(\langle \mathbf{f}^N(t), \mathbf{u}^N(t)\rangle_{V',V} - \langle \mathbf{f}(t), \mathbf{u}(t)\rangle_{V',V})dt| \\
=&|E\int_0^T e^{-r(t)}(\langle \mathbf{f}^N(t) - \mathbf{f}(t), \mathbf{u}^N(t)\rangle_{V',V} + \langle \mathbf{f}(t), \mathbf{u}^N(t) - \mathbf{u}(t)\rangle_{V',V})dt| \\
\leq&E\int_0^T e^{-r(t)}\|\mathbf{f}^N(t) - \mathbf{f}(t)\|_{V'}\|\mathbf{u}^N(t)\|_V dt + |E\int_0^T \langle e^{-r(t)}\mathbf{f}(t), \mathbf{u}^N(t) - \mathbf{u}(t)\rangle_{V',V} dt| \\
\leq&\{\int_0^T \|\mathbf{f}^N(t) - \mathbf{f}(t)\|_{V'}^2 dt\}^{\frac{1}{2}} E\{\int_0^T e^{-2r(t)}\|\mathbf{u}^N(t)\|_V^2 dt\}^{\frac{1}{2}} \\
&+ |E\int_0^T \langle e^{-r(t)}\mathbf{f}(t), \mathbf{u}^N(t) - \mathbf{u}(t)\rangle_{V',V} dt|
\end{aligned}$$

Obviously, the first term above converges to 0 as N tends to ∞. Since $\mathbf{f}(t) \in L^2(0,T;V')$, and $0 < e^{-r(t)} \leq 1$, we know that $e^{-r(t)}\mathbf{f}(t) \in L^2_{\mathcal{F}}(\Omega; L^2(0,T;V'))$. Since $\mathbf{u}^N(t) \xrightarrow{w} \mathbf{u}(t)$ in $L^2_{\mathcal{F}}(\Omega; L^2(0,T;V))$, we know that the second term above also converges to 0 as N tends to ∞.

Now for the last term in (4.3.4), similarly to the first term, we have

$$\varliminf_{N\to\infty} E\int_0^T e^{-r(t)}\|Z^N(t)\|_{L_Q}^2 dt \geq E\int_0^T e^{-r(t)}\|Z(t)\|_{L_Q}^2 dt$$

Combine (4.3.2), (4.3.3) and (4.3.4), we get

$$\begin{aligned}
&\varliminf_{N\to\infty}\{2E\int_0^T e^{-r(t)}\langle \nu\mathbf{A}^N\mathbf{u}^N(t) + \mathbf{B}^N(\mathbf{u}^N(t)) + \frac{1}{2}\dot{r}(t)\mathbf{u}^N(t), \mathbf{u}^N(t)\rangle_{V',V} dt\} \\
\geq&E\|\mathbf{u}(0)\|_H^2 - Ee^{-r(T)}\|\xi\|_H^2 + 2E\int_0^T e^{-r(t)}\langle \mathbf{f}(t), \mathbf{u}(t)\rangle_{V',V} dt \\
&+ E\int_0^T e^{-r(t)}\|Z(t)\|_{L_Q}^2 dt \\
=&2E\int_0^T e^{-r(t)}\langle Y(t) + G(t) + \frac{1}{2}\dot{r}(t)\mathbf{u}(t), \mathbf{u}(t)\rangle_{V',V} dt
\end{aligned} \tag{4.3.5}$$

Now by Corollary 4.3.3, we have

$$E\int_0^T e^{-r(t)}\langle \nu \mathbf{A}^N(\mathbf{v}(t)-\mathbf{u}^N(t))+\mathbf{B}^N(\mathbf{v}(t))-\mathbf{B}^N(\mathbf{u}^N(t))$$
$$+\frac{1}{2}\dot{r}(t)(\mathbf{v}(t)-\mathbf{u}^N(t)),\mathbf{v}(t)-\mathbf{u}^N(t)\rangle_{V',V}dt\geq 0$$

Hence

$$E\int_0^T e^{-r(t)}\langle \nu \mathbf{A}^N\mathbf{u}^N(t)+\mathbf{B}^N(\mathbf{u}^N(t))+\frac{1}{2}\dot{r}(t)\mathbf{u}^N(t),\mathbf{v}(t)-\mathbf{u}^N(t)\rangle_{V',V}dt$$
$$\leq E\int_0^T e^{-r(t)}\langle \nu \mathbf{A}\mathbf{v}(t)+\mathbf{B}(\mathbf{v}(t))+\frac{1}{2}\dot{r}(t)\mathbf{v}(t),\mathbf{v}(t)-\mathbf{u}^N(t)\rangle_{V',V}dt$$

Taking the limit and by (4.3.5), we get

$$E\int_0^T e^{-r(t)}\langle Y(t)+G(t)+\frac{1}{2}\dot{r}(t)\mathbf{u}(t),\mathbf{v}(t)-\mathbf{u}(t)\rangle_{V',V}dt$$
$$\leq E\int_0^T e^{-r(t)}\langle \nu \mathbf{A}\mathbf{v}(t)+\mathbf{B}(\mathbf{v}(t))+\frac{1}{2}\dot{r}(t)\mathbf{v}(t),\mathbf{v}(t)-\mathbf{u}(t)\rangle_{V',V}dt \tag{4.3.6}$$

Since $L^\infty(\Omega\times[0,T];V)$ is dense in $L^4_{\mathcal{F}}(\Omega;L^2(0,T;V))$, (4.3.6) is true for all $\mathbf{v}(t)\in L^\infty(\Omega\times[0,T];H)\cap L^4_{\mathcal{F}}(\Omega;L^2(0,T;V))$.

Now we take $\mathbf{v}(t)=\mathbf{u}(t)+\lambda\mathbf{w}(t)$ for any $\mathbf{w}(t)\in L^\infty(\Omega\times[0,T];H)\cap L^4_{\mathcal{F}}(\Omega;L^2(0,T;V))$ and $\lambda>0$.

By Corollary 4.2.2, we know that $\mathbf{u}(t)$ is also in $L^\infty(\Omega\times[0,T];H)\cap L^4_{\mathcal{F}}(\Omega;L^2(0,T;V))$. Therefore

$$E\int_0^T \langle \dot{r}(t)\mathbf{w}(t),\mathbf{w}(t)\rangle_{V',V}dt$$
$$=E\int_0^T \dot{r}(t)\|\mathbf{w}(t)\|_H^2dt$$
$$\leq E(\sup_{0\leq t\leq T}\|\mathbf{w}(t)\|_H^2\int_0^T\frac{C_G^2}{\nu}\|\mathbf{v}(t)\|_V^2dt)$$
$$\leq E(\sup_{0\leq t\leq T}\|\mathbf{w}(t)\|_H^4)E(\int_0^T\frac{C_G^2}{\nu}\|\mathbf{v}(t)\|_V^2dt)^2<\infty$$

and (4.3.6) becomes

$$
\begin{aligned}
&E\int_0^T e^{-r(t)}\langle Y(t)+G(t)-\nu\mathbf{A}\mathbf{u}(t)-\mathbf{B}(\mathbf{u}(t)+\lambda\mathbf{w}(t)),\lambda\mathbf{w}(t)\rangle_{V',V}dt\\
\leq &E\int_0^T e^{-r(t)}\langle\lambda\nu\mathbf{A}\mathbf{w}(t)+\frac{\lambda}{2}\dot{r}(t)\mathbf{w}(t),\lambda\mathbf{w}(t)\rangle_{V',V}dt
\end{aligned}
$$

Cancelling λ, and using the fact that

$$
\begin{aligned}
&\langle\mathbf{B}(\mathbf{u}(t)+\lambda\mathbf{w}(t)),\mathbf{w}(t)\rangle_{V',V}\\
=&-\langle\mathbf{B}(\mathbf{u}(t)+\lambda\mathbf{w}(t),\mathbf{w}(t)),\mathbf{u}(t)+\lambda\mathbf{w}(t)\rangle_{V',V}\\
=&-\langle\mathbf{B}(\mathbf{u}(t)+\lambda\mathbf{w}(t),\mathbf{w}(t)),\mathbf{u}(t)\rangle_{V',V}\\
=&-\langle\mathbf{B}(\mathbf{u}(t),\mathbf{w}(t)),\mathbf{u}(t)\rangle_{V',V}-\lambda\langle\mathbf{B}(\mathbf{w}(t),\mathbf{w}(t)),\mathbf{u}(t)\rangle_{V',V}\\
=&\langle\mathbf{B}(\mathbf{u}(t)),\mathbf{w}(t)\rangle_{V',V}+\lambda\langle\mathbf{B}(\mathbf{w}(t),\mathbf{u}(t)),\mathbf{w}(t)\rangle_{V',V},
\end{aligned}
$$

we get

$$
\begin{aligned}
&E\int_0^T e^{-r(t)}\langle Y(t)+G(t)-\nu\mathbf{A}\mathbf{u}(t)-\mathbf{B}(\mathbf{u}(t)),\mathbf{w}(t)\rangle_{V',V}dt\\
\leq &\lambda E\int_0^T e^{-r(t)}\langle\nu\mathbf{A}\mathbf{w}(t)+\mathbf{B}(\mathbf{w}(t),\mathbf{u}(t))+\frac{1}{2}\dot{r}(t)\mathbf{w}(t),\mathbf{w}(t)\rangle_{V',V}dt
\end{aligned}
$$

Letting $\lambda\to 0$, since the right hand side of the last inequality is finite, we get

$$
E\int_0^T e^{-r(t)}\langle Y(t)+G(t)-\nu\mathbf{A}\mathbf{u}(t)-\mathbf{B}(\mathbf{u}(t)),\mathbf{w}(t)\rangle_{V',V}dt\leq 0
$$

for all $\mathbf{w}(t)\in L^\infty(\Omega\times[0,T];H)\cap L^4_{\mathcal{F}}(\Omega;L^2(0,T;V))$.

Hence $Y(t)+G(t)=\nu\mathbf{A}\mathbf{u}(t)+\mathbf{B}(\mathbf{u}(t))$ P-a.s. and this completes the proof of the existence of the solution. □

4.4 Uniqueness of Solutions

The following Lemma is used in the proof of the uniqueness.

Lemma 4.4.1. *Assume that* $\|\xi\|_H^2 \le K$ *for some constant* K*,* P*-a.s., and* $\mathbf{f} \in L^4(0,T;V')$*. Let* $(\mathbf{u}(t), Z(t)) \in L^\infty(\Omega\times[0,T];H)\cap L^4_{\mathcal{F}}(\Omega;L^2(0,T;V))\times L^2_{\mathcal{F}}(\Omega;L^2(0,T;L_Q))$ *be a pair of solution to* (4.1.3)*, then*

$$E^{\mathcal{F}_r}\int_0^T \|P_N\mathbf{u}(s)\|_V^2 ds \le K'$$

for all $r \in [0,T]$ *and some constant* K'*.*

Proof: Denote $\rho^N(t) = P_N\mathbf{u}(t)$ and $\sigma^N(t) = P_N Z(t)$. Then

$$\begin{cases} d\rho^N(t) = -\nu\mathbf{A}^N\rho^N(t)dt - \mathbf{B}^N(\mathbf{u}(t))dt + \mathbf{f}^N(t)dt + \sigma^N(t)dW^N(t) \\ \rho^N(T) = \xi^N \end{cases}$$

An application of Itô formula to $\|\rho^N(t)\|_H^2$ yields

$$\begin{aligned} \|\rho^N(t)\|_H^2 =& \|\xi^N\|_H^2 + 2\int_t^T \langle \nu\mathbf{A}^N\rho^N(s) + \mathbf{B}^N(\mathbf{u}(s)), \rho^N(s)\rangle_{V',V} ds \\ &- \int_t^T 2\langle \mathbf{f}^N(s), \rho^N(s)\rangle_{V',V} ds - 2\int_t^T \langle (\sigma^N(s))^*(\rho^N(s)), dW^N(s)\rangle_H \\ &- \int_t^T \|\sigma^N(s)\|_{L_Q}^2 ds \end{aligned} \tag{4.4.1}$$

Using Proposition 4.1.8, and the fact that $\mathbf{u} \in L^\infty(\Omega\times[0,T];H)$ and $\rho^N = P_N\mathbf{u}$, we get

$$\begin{aligned} &|\langle B^N(\mathbf{u}(s)), \rho^N(s)\rangle_{V',V}| \\ =&|\langle B^N(\mathbf{u}(s), \rho^N(s)), \mathbf{u}(s)\rangle_{V',V}| \\ =&|\langle B^N(\mathbf{u}(s), \rho^N(s)), \rho^N(s)\rangle_{V',V}| \\ \le& C\|\mathbf{u}(s)\|_H^{\frac{1}{2}}\|\mathbf{u}(s)\|_V^{\frac{1}{2}}\|\rho^N(s)\|_V^{\frac{3}{2}}\|\rho^N(s)\|_H^{\frac{1}{2}} \\ \le& C\lambda_N^{\frac{3}{2}}\|\mathbf{u}(s)\|_V^{\frac{1}{2}}\|\rho^N(s)\|_H \end{aligned}$$

So (4.4.1) becomes

$$\|\rho^N(t)\|_H^2 + 2\int_t^T \|\rho^N(s)\|_V^2 ds$$

$$
\begin{aligned}
\leq & \|\xi^N\|_H^2 + (2\nu+2)\lambda_N \int_t^T \|\rho^N(s)\|_H^2 ds + 2C\lambda_N^{\frac{3}{2}} \int_t^T \|\mathbf{u}(s)\|_V^{\frac{1}{2}} \|\rho^N(s)\|_H ds \\
& + 2\int_t^T \|\mathbf{f}^N(s)\|_{V'}^2 ds + \int_t^T \|\rho^N(s)\|_V^2 ds - 2\int_t^T \langle (\sigma^N(s))^*(\rho^N(s)), dW^N(s)\rangle_H
\end{aligned}
$$

Taking conditional expectation with respect to $r \in [0,t]$, one gets

$$
\begin{aligned}
& E^{\mathcal{F}_r} \|\rho^N(t)\|_H^2 + E^{\mathcal{F}_r} \int_t^T \|\rho^N(s)\|_V^2 ds \\
\leq & E^{\mathcal{F}_r} \|\xi^N\|_H^2 + (2\nu+2)\lambda_N(\lambda_N+1)^2 \int_t^T E^{\mathcal{F}_r} \|\rho^N(s)\|_H^2 ds \\
& + 2C\lambda_N^{\frac{3}{2}} \{E^{\mathcal{F}_r} \int_t^T \|\mathbf{u}(s)\|_V ds\}^{\frac{1}{2}} \{\int_t^T E^{\mathcal{F}_r} \|\rho^N(s)\|_H^2 ds\}^{\frac{1}{2}} + 2\int_t^T \|\mathbf{f}^N(s)\|_{V'}^2 ds
\end{aligned}
$$

It is clear that for some constant K_1,

$$
E^{\mathcal{F}_r} \|\xi^N\|_H^2 + 2\int_t^T \|\mathbf{f}^N(s)\|_{V'}^2 ds \leq K_1.
$$

Let

$$
\alpha(N) = (2\nu+2)\lambda_N(\lambda_N+1)^2,
$$

$$
\beta(N) = 2C\lambda_N^{\frac{3}{2}} \{E^{\mathcal{F}_r} \int_t^T \|\mathbf{u}(s)\|_V ds\}^{\frac{1}{2}}, \text{ and}
$$

$$
g(N,t,r) = \{\int_t^T E^{\mathcal{F}_r} \|\rho^N(s)\|_H^2 ds\}^{\frac{1}{2}}.
$$

Then

$$
\begin{aligned}
& \alpha(N)g^2(N,t) + \beta(N)g(N,t,r) \\
= & \alpha(N)(g(N,t,r) + \frac{\beta(N)}{2\alpha(N)})^2 - \frac{\beta^2(N)}{4\alpha(N)} \\
\leq & \alpha(N)(2g^2(N,t,r) + \frac{\beta^2(N)}{2\alpha^2(N)}) - \frac{\beta^2(N)}{4\alpha(N)} \\
= & 2\alpha(N)g^2(N,t,r) + \frac{\beta^2(N)}{4\alpha(N)}
\end{aligned}
$$

So

$$\begin{aligned}
&E^{\mathcal{F}_r}\|\rho^N(t)\|_H^2 + E^{\mathcal{F}_r}\int_t^T \|\rho^N(s)\|_V^2 ds \\
\leq& K_1 + \frac{\beta^2(N)}{4\alpha(N)} + 2\alpha(N)g^2(N,t,r)
\end{aligned}$$

Applying Gronwall's inequality, one gets

$$\begin{aligned}
&E^{\mathcal{F}_r}\|\rho^N(t)\|_H^2 + E^{\mathcal{F}_r}\int_t^T \|\rho^N(s)\|_V^2 ds \\
\leq& 2K_1 + \frac{\beta^2(N)}{2\alpha(N)}.
\end{aligned}$$

Since $\lambda_N \to \infty$ as $N \to \infty$, $\frac{\beta^2(N)}{2\alpha(N)}$ is uniformly bounded. This completes the proof. □

Theorem 4.4.2. *Assume the conditions in Theorem 4.3.6. The adapted solution of* (4.1.3) *is unique in* $L^\infty(\Omega\times[0,T];H)\cap L^4_{\mathcal{F}}(\Omega;L^2(0,T;V)) \times L^2_{\mathcal{F}}(\Omega;L^2(0,T;L_Q))$.

Proof: Let $(\mathbf{u}(t), Z(t))$, $(\mathbf{v}(t), \sigma(t)) \in L^\infty(\Omega\times[0,T];H) \cap L^4_{\mathcal{F}}(\Omega;L^2(0,T;V))$ $\times\ L^2_{\mathcal{F}}(\Omega;L^2(0,T;L_Q))$ be two solutions of (4.1.3).

Step 1: For every $N\in\mathbb{N}$, let $\rho^N(t) = P_N\mathbf{v}(t)$, and we define the following finite dimensional system

$$\begin{cases}
d\mathbf{x}^N(t) = -\nu\mathbf{A}^N\mathbf{x}^N(t)dt - \mathbf{B}^N(\rho^N(t))dt + \mathbf{f}^N(t)dt + Y^N(t)dW^N(t) \\
\mathbf{x}^N(T) = \xi^N
\end{cases} \tag{4.4.2}$$

Since $\mathbf{A}^N$ is Lipschitz, it is easy to see that (4.4.2) admits a unique adapted solution $(\mathbf{x}^N(t), Y^N(t))$.

Apply Itô's formula to $\|\mathbf{x}^N(t)\|_H^2$, we get

$$\|\mathbf{x}^N(t)\|_H^2 = \|\xi^N\|_H^2 + 2\int_t^T \langle \nu\mathbf{A}^N\mathbf{x}^N(s) + \mathbf{B}^N(\rho^N(s)), \mathbf{x}^N(s)\rangle_{V',V} ds$$

$$- 2\int_t^T \langle \mathbf{f}^N(s), \mathbf{x}^N(s)\rangle_{V',V} ds - 2\int_t^T \langle Y^N(s)dW^N(s), \mathbf{x}^N(s)\rangle_H$$
$$- \int_t^T \|Y^N(s)\|_{L_Q}^2 ds$$

Since

$$2|\langle \mathbf{B}^N(\rho^N(s)), \mathbf{x}^N(s)\rangle_{V',V}| \leq C\|\rho^N(s)\|_V^2 + \|\mathbf{x}^N(s)\|_V^2$$

and

$$2|\langle \mathbf{f}^N(s), \mathbf{x}^N(s)\rangle_{V',V}| \leq \|\mathbf{f}^N(s)\|_{V'}^2 + \|\mathbf{x}^N(s)\|_V^2$$

we get, for $0 \leq r \leq t \leq T$,

$$\begin{aligned}
&E^{\mathcal{F}_r}\|\mathbf{x}^N(t)\|_H^2 + E^{\mathcal{F}_r}\int_t^T \|Y^N(s)\|_{L_Q}^2 ds + 3E^{\mathcal{F}_r}\int_t^T \langle \mathbf{A}^N\mathbf{x}^N(s), \mathbf{x}^N(s)\rangle_{V',V} ds \\
\leq &E^{\mathcal{F}_r}\|\xi^N\|_H^2 + (2\nu+3)E^{\mathcal{F}_r}\int_t^T \langle \mathbf{A}^N\mathbf{x}^N(s), \mathbf{x}^N(s)\rangle_{V',V} ds + CE^{\mathcal{F}_r}\int_t^T \|\rho^N(s)\|_V^2 ds \\
&+ \int_t^T \|\mathbf{f}^N(s)\|_{V'}^2 ds + 2E^{\mathcal{F}_r}\int_t^T \|\mathbf{x}^N(s)\|_V^2 ds
\end{aligned}$$

Using the assumption and Lemma 4.4.1, we have, for some constant K which is independent of N,

$$\begin{aligned}
&E^{\mathcal{F}_r}\|\mathbf{x}^N(t)\|_H^2 + E^{\mathcal{F}_r}\int_t^T \|Y^N(s)\|_{L_Q}^2 ds + E^{\mathcal{F}_r}\int_t^T \|\mathbf{x}^N(s)\|_V^2 ds \\
\leq &E^{\mathcal{F}_r}\xi^N + E^{\mathcal{F}_r}\int_t^T \|\rho^N(s)\|_V^2 ds + \int_0^T \|\mathbf{f}(s)\|_{V'}^2 ds \\
&+ (2\nu+3)\lambda_N \int_t^T E^{\mathcal{F}_r}\|\mathbf{x}^N(s)\|_H^2 ds \\
\leq &\frac{1}{2}K + (2\nu+3)\lambda_N \int_t^T E^{\mathcal{F}_r}\|\mathbf{x}^N(s)\|_H^2 ds
\end{aligned}$$

Apply Gronwall's inequality, we get

$$\sup_{0\leq t\leq T}\|\mathbf{x}^N(t)\|_H^2 + E\int_0^T \|Y^N(s)\|_{L_Q}^2 ds + E\int_0^T \|\mathbf{x}^N(s)\|_V^2 ds \leq K \qquad (4.4.3)$$

Step 2: It is clear that ρ^N satisfies

$$\begin{cases} d\rho^N(t) = -\nu \mathbf{A}^N \rho^N(t)dt - \mathbf{B}^N(\mathbf{v}(t))dt + \mathbf{f}^N(t)dt + P_N\sigma(t)dW(t) \\ \rho^N(T) = \xi^N \end{cases} \tag{4.4.4}$$

Let $\mathbf{w}^N(t) = \mathbf{x}^N(t) - \rho^N(t)$, then

$$\begin{cases} d\mathbf{w}^N(t) = -\nu \mathbf{A}^N \mathbf{w}^N(t)dt - (\mathbf{B}^N(\rho^N(t)) - \mathbf{B}^N(\mathbf{v}(t)))dt \\ \qquad\qquad +(Y^N(t) - P_N\sigma(t))dW(t) \\ \mathbf{w}^N(T) = 0 \end{cases}$$

where, for convenience, we set $Y^N(t)(e_k) = 0$ for all $k > N$.

The Itô formula yields

$$\begin{aligned} \|\mathbf{w}^N(t)\|_H^2 =& 2\int_t^T \langle \nu \mathbf{A}^N \mathbf{w}^N(s) + (\mathbf{B}^N(\rho^N(s)) - \mathbf{B}^N(\mathbf{v}(s))), \mathbf{w}^N(s)\rangle_{V',V} ds \\ &- 2\int_t^T \langle (Y^N(s) - P_N\sigma(s))dW(s), \mathbf{w}^N(s)\rangle_H \qquad (4.4.5) \\ &- \int_t^T \|Y^N(s) - P_N\sigma(s)\|_{L_Q}^2 ds \qquad (4.4.6) \end{aligned}$$

First,

$$\begin{aligned} &\langle \mathbf{B}^N(\rho^N(s)) - \mathbf{B}^N(\mathbf{v}(s)), \mathbf{w}^N(s)\rangle_{V',V} \\ =&\langle \mathbf{B}^N(\rho^N(s)), \mathbf{w}^N(s)\rangle_{V',V} - \langle \mathbf{B}^N(\mathbf{v}(s)), \mathbf{w}^N(s)\rangle_{V',V} \\ =& - \langle \mathbf{B}^N(\rho^N(s), \mathbf{w}^N(s)), \rho^N(s)\rangle_{V',V} + \langle \mathbf{B}^N(\mathbf{v}(s), \mathbf{w}^N(s)), \mathbf{v}(s)\rangle_{V',V} \\ =& - \langle \mathbf{B}^N(\rho^N(s), \mathbf{w}^N(s)), \rho^N(s)\rangle_{V',V} + \langle \mathbf{B}^N(\mathbf{v}(s), \mathbf{w}^N(s)), \rho^N(s)\rangle_{V',V} \\ =&\langle \mathbf{B}^N(\mathbf{v}(s) - \rho^N(s), \mathbf{w}^N(s)), \rho^N(s)\rangle_{V',V} \end{aligned}$$

Note that $\mathbf{v}(t) \in L^\infty(\Omega \times [0,T]; H)$. By the above equality,

$$|\langle \mathbf{B}^N(\rho^N(s)) - \mathbf{B}^N(\mathbf{v}(s)), \mathbf{w}^N(s)\rangle_{V',V}|$$

$$
\begin{aligned}
&\leq C\|\mathbf{v}(s)-\rho^N(s)\|_H^{\frac{1}{2}}\|\mathbf{v}(s)-\rho^N(s)\|_V^{\frac{1}{2}}\|\mathbf{w}^N(s)\|_V\|\rho^N(s)\|_H^{\frac{1}{2}}\|\rho^N(s)\|_V^{\frac{1}{2}} \\
&\leq C_1\|\mathbf{v}(s)-\rho^N(s)\|_H^{\frac{1}{2}}\|\mathbf{v}(s)\|_V\|\mathbf{w}^N(s)\|_V \\
&\leq C_2\|\mathbf{v}(s)-\rho^N(s)\|_H^{\frac{1}{2}}(\|\mathbf{v}(s)\|_V^2+\|\mathbf{w}^N(s)\|_V^2) \\
&\leq C_2\|\mathbf{v}(s)-\rho^N(s)\|_H^{\frac{1}{2}}\|\mathbf{v}(s)\|_V^2+C_3\lambda_N\|\mathbf{w}^N(s)\|_H^2
\end{aligned}
$$

where C_3 is a bound for $C_2\|\mathbf{v}(s)-\rho^N(s)\|_H^{\frac{1}{2}}$, independent of s and ω. Such finite bound exists because of Theorem 4.3.6. Applying the above inequality to (4.4.5), and by Gronwall's inequality, we get

$$
\begin{aligned}
&E^{\mathcal{F}_r}\|\mathbf{w}^N(t)\|_H^2+E^{\mathcal{F}_r}\int_t^T\|Y^N(s)-P_N\sigma(s)\|_{L_Q}^2ds+E^{\mathcal{F}_r}\int_t^T\|\mathbf{w}^N(s)\|_V^2ds \\
\leq&2E^{\mathcal{F}_r}\int_r^T C_2\|\mathbf{v}(s)-\rho^N(s)\|_H^{\frac{1}{2}}\|\mathbf{v}(s)\|_V^2ds \\
&+(2\nu+2C_3+1)\lambda_N\int_t^T E^{\mathcal{F}_r}\|\mathbf{w}^N(s)\|_H^2ds \\
\leq&4E^{\mathcal{F}_r}\int_0^T C_2\|\mathbf{v}(s)-\rho^N(s)\|_H^{\frac{1}{2}}\|\mathbf{v}(s)\|_V^2ds
\end{aligned}
$$

for $0\leq r\leq t\leq T$.

Letting $r=t$, we get

$$
\begin{aligned}
&\|\mathbf{w}^N(t)\|_H^2+E^{\mathcal{F}_t}\int_t^T\|Y^N(s)-P_N\sigma(s)\|_{L_Q}^2ds+E^{\mathcal{F}_t}\int_t^T\|\mathbf{w}^N(s)\|_V^2ds \\
\leq&4E^{\mathcal{F}_t}\int_0^T C_2\|\mathbf{v}(s)-\rho^N(s)\|_H^{\frac{1}{2}}\|\mathbf{v}(s)\|_V^2ds \\
\leq&4C_2E^{\mathcal{F}_t}(\sup_{0\leq s\leq T}\|\mathbf{v}(s)-\rho^N(s)\|_H^{\frac{1}{2}}\int_0^T\|\mathbf{v}(s)\|_V^2ds) \\
\leq&4C_2[E^{\mathcal{F}_t}\sup_{0\leq s\leq T}\|\mathbf{v}(s)-\rho^N(s)\|_H]^{\frac{1}{2}}[E^{\mathcal{F}_t}(\int_0^T\|\mathbf{v}(s)\|_V^2ds)^2]^{\frac{1}{2}}
\end{aligned}
$$

Since $\mathbf{v}(s) \in L^\infty(\Omega \times [0,T]; H) \cap L^4_{\mathcal{F}}(\Omega; L^2(0,T;V))$, we apply the Lebesgue dominated convergence theorem to get

$$\lim_{N\to\infty} (\sup_{0\le t\le T} \|\mathbf{w}^N(t)\|_H^2 + E\int_0^T \|Y^N(s) - P_N\sigma(s)\|_{L_Q}^2 ds + E\int_0^T \|\mathbf{w}^N(s)\|_V^2 ds) = 0 \tag{4.4.7}$$

Step 3: Let $\tilde{\mathbf{w}}^N(t) = \mathbf{u}^N(t) - \mathbf{x}^N(t)$ and $\tilde{Z}^N(t) = Z^N(t) - Y^N(t)$. Take the difference of (4.2.1) and (4.4.2), we get

$$\begin{cases} d\tilde{\mathbf{w}}^N(t) = -\nu \mathbf{A}^N \tilde{\mathbf{w}}^N(t)dt - (\mathbf{B}^N(\mathbf{u}^N(t)) - \mathbf{B}^N(\rho^N(t)))dt + \tilde{Z}^N(t)dW(t) \\ \tilde{\mathbf{w}}^N(T) = 0. \end{cases} \tag{4.4.8}$$

Similar to Corollary 4.2.3, we have

$$\begin{aligned} &\frac{1}{2}d\|\tilde{\mathbf{w}}^N(t)\|_V^2 = \langle d\tilde{\mathbf{w}}^N(t), \tilde{\mathbf{w}}^N(t)\rangle_V \\ =&\langle \mathbf{A}^N\tilde{\mathbf{w}}^N(t), d\tilde{\mathbf{w}}^N(t)\rangle_{V',V} = \langle d\tilde{\mathbf{w}}^N(t), \mathbf{A}^N\tilde{\mathbf{w}}^N(t)\rangle_H \\ =&-\langle \nu\mathbf{A}^N\tilde{\mathbf{w}}^N(t), \mathbf{A}^N\tilde{\mathbf{w}}^N(t)\rangle_H dt - \langle \mathbf{B}^N(\mathbf{u}^N(t)) - \mathbf{B}^N(\rho^N(t)), \mathbf{A}^N\tilde{\mathbf{w}}^N(t)\rangle_H dt \\ &+ \langle \tilde{Z}^N(t)dW^N(t), \mathbf{A}^N\tilde{\mathbf{w}}^N(t)\rangle_H \end{aligned}$$

and

$$\|\tilde{\mathbf{w}}^N(t)\|_V^2 \tag{4.4.9}$$

$$\begin{aligned} =&2\int_t^T \nu\|\mathbf{A}^N\tilde{\mathbf{w}}^N(s)\|_H^2 ds + 2\int_t^T \langle \mathbf{B}^N(\mathbf{u}^N(s)) - \mathbf{B}^N(\rho^N(s)), \mathbf{A}^N\tilde{\mathbf{w}}^N(s)\rangle_H ds \\ &- 2\int_t^T \langle \tilde{Z}^N(s)dW^N(s), \mathbf{A}^N\tilde{\mathbf{w}}^N(s)\rangle_H \end{aligned} \tag{4.4.10}$$

Let us make the following notation:

$$K_1(N,t) = \|\mathbf{u}^N(t)\|_H^{\frac{1}{2}}\|\mathbf{u}^N(t)\|_V^{\frac{1}{2}} + \|\mathbf{x}^N(t)\|_H^{\frac{1}{2}}\|\mathbf{x}^N(t)\|_V^{\frac{1}{2}}$$

$$K_2(N,t) = \|\mathbf{x}^N(t)\|_H^{\frac{1}{2}}\|\mathbf{x}^N(t)\|_V^{\frac{1}{2}} + \|\rho^N(t)\|_H^{\frac{1}{2}}\|\rho^N(t)\|_V^{\frac{1}{2}}$$

$$K_3(N,r) = \frac{1}{2}E^{\mathcal{F}_r}\int_0^T C^2K_2^2(N,s)\|\mathbf{w}^N(s)\|_H\|\mathbf{w}^N(s)\|_V ds$$

and $\alpha(N) = (2\nu + 2C_1 + 2)\lambda_N^2$, $\beta(N,r) = 2\{E^{\mathcal{F}_r}\int_r^T K_1^4(N,s)ds\}^{\frac{1}{2}}$ and $g(N,t,r) = \{\int_t^T E^{\mathcal{F}_r}\|\tilde{\mathbf{w}}^N(s)\|_V^2 ds\}^{\frac{1}{2}}$.

Recall that $\sup_{0\le t\le T}\|\tilde{w}^N(t)\|_H^2 < C$ where C is independent of N. Using this fact along with Lemma 4.3.4, one obtains

$$\begin{aligned}
&|\langle \mathbf{B}^N(\mathbf{u}^N(t)) - \mathbf{B}^N(\rho^N(t)), \mathbf{A}^N\tilde{\mathbf{w}}^N(t)\rangle_H| \\
\le&|\langle \mathbf{B}^N(\mathbf{u}^N(t)) - \mathbf{B}^N(\mathbf{x}^N(t)), \mathbf{A}^N\tilde{\mathbf{w}}^N(t)\rangle_H| \\
&+ |\langle \mathbf{B}^N(\mathbf{x}^N(t)) - \mathbf{B}^N(\rho^N(t)), \mathbf{A}^N\tilde{\mathbf{w}}^N(t)\rangle_H| \\
\le&CK_1(N,t)\|\mathbf{A}^N\tilde{\mathbf{w}}^N(t)\|_V\|\tilde{\mathbf{w}}^N(t)\|_H^{\frac{1}{2}}\|\tilde{\mathbf{w}}^N(t)\|_V^{\frac{1}{2}} \\
&+ CK_2(N,t)\|\mathbf{A}^N\tilde{\mathbf{w}}^N(t)\|_V\|\mathbf{w}^N(t)\|_H^{\frac{1}{2}}\|\mathbf{w}^N(t)\|_V^{\frac{1}{2}} \\
\le&C_1\lambda_N^2\|\tilde{\mathbf{w}}^N(t)\|_V^2 + K_1^2(N,t)\|\tilde{\mathbf{w}}^N(t)\|_V \\
&+ \lambda_N^2\|\tilde{\mathbf{w}}^N(t)\|_V^2 + \frac{1}{4}C^2K_2^2(N,t)\|\mathbf{w}^N(t)\|_H\|\mathbf{w}^N(t)\|_V
\end{aligned}$$

Thus for $0 \le r \le t \le T$, (4.4.9) becomes

$$\begin{aligned}
E^{\mathcal{F}_r}\|\tilde{\mathbf{w}}^N(t)\|_V^2 \le&\alpha(N)\int_t^T E^{\mathcal{F}_r}\|\tilde{\mathbf{w}}^N(s)\|_V^2 ds + K_3(N,r) \\
&+ 2E^{\mathcal{F}_r}\int_t^T K_1^2(N,s)\|\tilde{\mathbf{w}}^N(s)\|_V ds \\
\le&\alpha(N)g^2(N,t,r) + K_3(N,r) + \beta(N,r)g(N,t,r) \qquad (4.4.11)
\end{aligned}$$

Step 4: Since

$$\begin{aligned}
&\alpha(N)g^2(N,t,r) + \beta(N,r)g(N,t,r) \\
=&\alpha(N)(g(N,t,r) + \frac{\beta(N,r)}{2\alpha(N)})^2 - \frac{\beta(N,r)^2}{4\alpha(N)}
\end{aligned}$$

$$\leq \alpha(N)(2g^2(N,t,r) + \frac{\beta(N,r)^2}{2\alpha^2(N)}) - \frac{\beta(N,r)^2}{4\alpha(N)}$$
$$= 2\alpha(N)g^2(N,t,r) + \frac{\beta(N,r)^2}{4\alpha(N)},$$

(4.4.11) becomes

$$E^{\mathcal{F}_r}\|\tilde{\mathbf{w}}^N(t)\|_V^2 \leq K_3(N,r) + \frac{\beta(N,r)^2}{4\alpha(N)} + 2\alpha(N)\int_t^T E^{\mathcal{F}_r}\|\tilde{\mathbf{w}}^N(s)\|_V^2 ds$$

It is clear that we have the integrability to apply Gronwall's inequality. Thus one gets

$$E^{\mathcal{F}_r}\|\tilde{\mathbf{w}}^N(t)\|_V^2 \leq 2K_3(N,r) + \frac{\beta(N,r)^2}{2\alpha(N)}$$

Note that α_N converges to ∞ as N goes to ∞. It is easy to see that $\beta(N,r)$ is bounded uniformly over N. $\lim_{N\to\infty} K_3(N,r) = 0$ by (4.4.7) and Lemma 4.4.1. Letting $r = t$ and applying Lebesgue dominated convergence theorem, we get

$$\lim_{N\to\infty} \sup_{0\leq t\leq T} \|\tilde{\mathbf{w}}^N(t)\|_V^2 = 0$$

Since $\mathbf{u}^N(t) - \rho^N(t) \xrightarrow{w} \mathbf{u}(t) - \mathbf{v}(t)$, we have

$$\sup_{0\leq t\leq T} \|\mathbf{u}(t) - \mathbf{v}(t)\|_H^2$$
$$\leq \varliminf_{N\to\infty} \sup_{0\leq t\leq T} \|\mathbf{u}^N(t) - \rho^N(t)\|_H^2$$
$$\leq \lim_{N\to\infty} \sup_{0\leq t\leq T} \|\tilde{\mathbf{w}}^N(t)\|_V^2 + \lim_{N\to\infty} \sup_{0\leq t\leq T} \|\mathbf{w}^N(t)\|_H^2$$
$$= 0$$

Thus we have shown that $(\mathbf{u}(t), Z(t))=(\mathbf{v}(t), \sigma(t))$ a.s. for all $t \in [0,T]$. □

4.5 An Improvement on the Terminal Value

For all $\xi \in L^2_{\mathcal{F}_T}(\Omega; H)$, we know that $\|\xi\|^2_H < \infty$ a.s.. Then for all $n \in \mathbb{N}$, we define

$$\xi^n(\omega) = \begin{cases} \xi(\omega) & \text{if} \quad \|\xi\|_H \le n \\ \frac{n}{\|\xi\|_H}\xi(\omega) & \text{if} \quad n < \|\xi\|_H < \infty \\ 0 & \text{if} \quad \|\xi\|_H = \infty \end{cases}$$

Then $\|\xi\|_H \le n$. From Theorem 4.3.6, there exists a unique adapted solution $(\mathbf{u}^n(t), Z^n(t))$ for

$$\begin{cases} d\mathbf{u}^n(t) = -\nu \mathbf{A}\mathbf{u}^n(t)dt - \mathbf{B}(\mathbf{u}^n(t))dt + \mathbf{f}(t)dt + Z^n(t)dW(t) \\ \mathbf{u}^n(T) = \xi^n \end{cases} \tag{4.5.1}$$

Proposition 4.5.1. *Assume that $\xi \in L^2_{\mathcal{F}_T}(\Omega; H)$ and $\mathbf{f} \in L^4(0, T; V')$. For any $n \in \mathbb{N}$, let $(\mathbf{u}^n(t), Z^n(t))$ be the solution of (4.5.1). Then $\{(\mathbf{u}^n(t), Z^n(t))\}_{n=1}^\infty$ is Cauchy in $L^2_{\mathcal{F}}(\Omega; L^2(0, T; V)) \times L^2_{\mathcal{F}}(\Omega; L^2(0, T; L_Q))$.*

Proof: For any $n, N \in \mathbb{N}$, let $\mathbf{u}^{n,N}(t) = P_N\mathbf{u}^n(t)$, $Z^{n,N}(t) = P_N Z^n(t)$ and $\xi^{n,N} = P_N\xi^n$, then

$$\begin{cases} d\mathbf{u}^{n,N}(t) = -\nu \mathbf{A}^N\mathbf{u}^{n,N}(t)dt - \mathbf{B}^N(\mathbf{u}^n(t))dt + \mathbf{f}^N(t)dt + Z^{n,N}(t)dW(t) \\ \mathbf{u}^{n,N}(T) = \xi^{n,N} \end{cases}$$

Since $\mathbf{A}^N$ is Lipschitz, there exists a unique adapted solution $((\mathbf{x}^{n,N}(t), Y^{n,N}(t))$ of

$$\begin{cases} d\mathbf{x}^{n,N}(t) = -\nu \mathbf{A}^N\mathbf{x}^{n,N}(t)dt - \mathbf{B}^N(\mathbf{u}^{n,N}(t))dt + \mathbf{f}^N(t)dt + Y^{n,N}(t)dW^N(t) \\ \mathbf{x}^{n,N}(T) = \xi^{n,N} \end{cases}$$

.

From the proof of Theorem 4.4.2, we know that

$$\lim_{N\to\infty}\Big(\sup_{0\le t\le T}\|\mathbf{u}^{n,N}(t)-\mathbf{x}^{n,N}(t)\|_H^2+E\int_0^T\|Z^{n,N}(s)-Y^{n,N}(s)\|_{L_Q}^2ds$$
$$+E\int_0^T\|\mathbf{u}^{n,N}(s)-\mathbf{x}^{n,N}(s)\|_V^2ds\Big)=0 \tag{4.5.2}$$

Let

$$\mathbf{w}^{m,n,N}(t)=\mathbf{x}^{m,N}(t)-\mathbf{x}^{n,N}(t),$$

$$Y^{m,n,N}(t)=Y^{m,N}(t)-Y^{n,N}(t),\ \text{and}$$

$$\xi^{m,n,N}(t)=\xi^{m,N}(t)-\xi^{n,N}(t).$$

Then for any $m,n\in\mathbb{N}$,

$$\begin{cases} d\mathbf{w}^{m,n,N}(t)= & -\nu\mathbf{A}^N\mathbf{w}^{m,n,N}(t)dt-(\mathbf{B}^N(\mathbf{u}^{m,N}(t))-\mathbf{B}^N(\mathbf{u}^{n,N}(t)))dt\\ & +Y^{m,n,N}(t)dW^N(t)\\ \mathbf{w}^{m,n,N}(T)= & \xi^{m,n,N}\end{cases}$$

Applying the Itô formula to $\|\mathbf{w}^{m,n,N}(t)\|_H^2$,

$$\|\mathbf{w}^{m,n,N}(t)\|_H^2=\|\xi^{m,n,N}\|_H^2$$
$$+2\int_t^T\langle\nu\mathbf{A}^N\mathbf{w}^{m,n,N}(s)+(\mathbf{B}^N(\mathbf{u}^{m,N}(s))-\mathbf{B}^N(\mathbf{u}^{n,N}(s))),\mathbf{w}^{m,n,N}(s)\rangle_{V',V}ds$$
$$-2\int_t^T\langle Y^{m,n,N}(s)dW^N(s),\mathbf{w}^{m,n,N}(s)\rangle_H-\int_t^T\|Y^{m,n,N}(s)\|_{L_Q}^2ds \tag{4.5.3}$$

From Lemma 4.3.4, it follows that

$$|\langle\mathbf{B}^N(\mathbf{u}^{m,N}(s))-\mathbf{B}^N(\mathbf{u}^{n,N}(s)),\mathbf{w}^{m,n,N}(s)\rangle_H|$$
$$\le C(\|\mathbf{u}^{m,N}(s)\|_H^{\frac{1}{2}}\|\mathbf{u}^{m,N}(s)\|_V^{\frac{1}{2}}+\|\mathbf{u}^{n,N}(s)\|_H^{\frac{1}{2}}\|\mathbf{u}^{n,N}(s)\|_V^{\frac{1}{2}})$$

$$\cdot \|\mathbf{u}^{m,N}(s)-\mathbf{u}^{n,N}(s)\|_H^{\frac{1}{2}}\|\mathbf{u}^{m,N}(s)-\mathbf{u}^{n,N}(s)\|_V^{\frac{1}{2}}\|\mathbf{w}^{m,n,N}(s)\|_V$$

$$\leq C\lambda_N(\|\mathbf{u}^{m,N}(s)\|_H+\|\mathbf{u}^{n,N}(s)\|_H)\|\mathbf{u}^{m,N}(s)-\mathbf{u}^{n,N}(s)\|_H\|\mathbf{w}^{m,n,N}(s)\|_H$$

$$\leq \frac{1}{40T}\|\mathbf{u}^{m,N}(s)-\mathbf{u}^{n,N}(s)\|_H^2+K(m,n,N)\|\mathbf{w}^{m,n,N}(s)\|_H^2$$

where $K(m,n,N)$ is a constant which is related to m, n, N and T.

Thus for $0\leq r\leq t\leq T$, (4.5.3) becomes

$$E^{\mathcal{F}_r}\|\mathbf{w}^{m,n,N}(t)\|_H^2+E^{\mathcal{F}_r}\int_t^T\|Y^{m,n,N}(s)\|_{L_Q}^2ds+E^{\mathcal{F}_r}\int_0^T\|\mathbf{w}^{m,n,N}(s)\|_V^2ds$$

$$\leq E^{\mathcal{F}_r}\|\xi^{m,n,N}\|_H^2+(2\nu\lambda_N+\lambda_N+2K(m,n,N))\int_t^T E^{\mathcal{F}_r}\|\mathbf{w}^{m,n,N}(s)\|_H^2ds$$

$$+\frac{1}{20T}E^{\mathcal{F}_r}\int_0^T\|\mathbf{u}^{m,N}(s)-\mathbf{u}^{n,N}(s)\|_H^2ds$$

Applying Gronwall's inequality and letting $r=t$ yield

$$\|\mathbf{w}^{m,n,N}(t)\|_H^2+E^{\mathcal{F}_t}\int_0^T\|Y^{m,n,N}(s)\|_{L_Q}^2ds+E^{\mathcal{F}_t}\int_0^T\|\mathbf{w}^{m,n,N}(s)\|_V^2ds$$

$$\leq 2E^{\mathcal{F}_t}\|\xi^{m,n,N}\|_H^2+\frac{1}{10T}E^{\mathcal{F}_t}\int_0^T\|\mathbf{u}^{m,N}(s)-\mathbf{u}^{n,N}(s)\|_H^2ds \tag{4.5.4}$$

From (4.5.4), we know that

$$\frac{1}{5}E\int_0^T\|\mathbf{u}^m(s)-\mathbf{u}^n(s)\|_H^2ds\leq E\int_0^T\|\mathbf{u}^m(s)-\mathbf{u}^{m,N}(s)\|_H^2ds$$

$$+T\sup_{0\leq t\leq T}\|\mathbf{u}^{m,N}(t)-\mathbf{x}^{m,N}(t)\|_H^2+E\int_0^T\|\mathbf{x}^{m,N}(s)-\mathbf{x}^{n,N}(s)\|_H^2ds$$

$$+T\sup_{0\leq t\leq T}\|\mathbf{u}^{n,N}(t)-\mathbf{x}^{n,N}(t)\|_H^2+E\int_0^T\|\mathbf{u}^n(s)-\mathbf{u}^{n,N}(s)\|_H^2ds$$

$$\leq E\int_0^T\|\mathbf{u}^m(s)-\mathbf{u}^{m,N}(s)\|_H^2ds+T\sup_{0\leq t\leq T}\|\mathbf{u}^{m,N}(t)-\mathbf{x}^{m,N}(t)\|_H^2$$

$$+(2E\int_0^T E^{\mathcal{F}_s}\|\xi^{m,n,N}\|_H^2ds+\frac{1}{10}E\int_0^T\|\mathbf{u}^{m,N}(s)-\mathbf{u}^{n,N}(s)\|_H^2ds)$$

$$+T\sup_{0\leq t\leq T}\|\mathbf{u}^{n,N}(t)-\mathbf{x}^{n,N}(t)\|_H^2+E\int_0^T\|\mathbf{u}^n(s)-\mathbf{u}^{n,N}(s)\|_H^2ds$$

Obviously, for any $m \in \mathbb{N}$,

$$\lim_{N\to\infty} E\int_0^T \|\mathbf{u}^m(s) - \mathbf{u}^{m,N}(s)\|_H^2 ds = 0$$

By Lebesgue dominated convergence theorem, we know that

$$\begin{aligned}
&\lim_{N\to\infty}(2E\int_0^T E^{\mathcal{F}_s}\|\xi^{m,n,N}\|_H^2 ds + \frac{1}{10}E\int_0^T \|\mathbf{u}^{m,N}(s) - \mathbf{u}^{n,N}(s)\|_H^2 ds)\\
=&2E\int_0^T E^{\mathcal{F}_s}\|\xi^m - \xi^n\|_H^2 ds + \frac{1}{10}E\int_0^T \|\mathbf{u}^m(s) - \mathbf{u}^n(s)\|_H^2 ds
\end{aligned}$$

Thus combined with (4.5.2), one gets

$$\begin{aligned}
&\frac{1}{5}E\int_0^T \|\mathbf{u}^m(s) - \mathbf{u}^n(s)\|_H^2 ds\\
\leq&2E\int_0^T E^{\mathcal{F}_s}\|\xi^m - \xi^n\|_H^2 ds + \frac{1}{10}E\int_0^T \|\mathbf{u}^m(s) - \mathbf{u}^n(s)\|_H^2 ds
\end{aligned}$$

The Lebesgue dominated convergence theorem implies

$$\lim_{m,n\to\infty} E\int_0^T E^{\mathcal{F}_s}\|\xi^m - \xi^n\|_H^2 ds = 0$$

Thus

$$\lim_{m,n\to\infty} E\int_0^T \|\mathbf{u}^m(s) - \mathbf{u}^n(s)\|_H^2 ds = 0$$

Similarly, one can show

$$\lim_{m,n\to\infty} E\int_0^T \|Z^m(s) - Z^n(s)\|_{L_Q}^2 ds$$

and

$$\lim_{m,n\to\infty} E\int_0^T \|\mathbf{u}^m(s) - \mathbf{u}^n(s)\|_V^2 ds = 0$$

□

Theorem 4.5.2. *Assume that* $\xi \in L^2_{\mathcal{F}_T}(\Omega; H)$ *and* $\mathbf{f} \in L^4(0,T;V')$*. Then the Navier-Stokes equation* (4.1.3) *admits a unique adapted solution* $(\mathbf{u}(t), Z(t)) \in L^2_{\mathcal{F}}(\Omega; L^2(0,T;V)) \cap L^\infty(0,T; L^2_{\mathcal{F}}(\Omega; H)) \times L^2_{\mathcal{F}}(\Omega; L^2(0,T;L_Q))$.

Proof: **Step 1:** First let us show the existence of a solution. For any $n \in \mathbb{N}$, let $(\mathbf{u}^n(t), Z^n(t))$ be the solution of (4.5.1). Then by Proposition 4.5.1, we know that there is $(\mathbf{u}(t), Z(t)) \in L^2_{\mathcal{F}}(\Omega; L^2(0,T;V)) \times L^2_{\mathcal{F}}(\Omega; L^2(0,T;L_Q))$, such that

$$\mathbf{u}^n(t) \to \mathbf{u}(t) \qquad \text{strongly in } \ L^2_{\mathcal{F}}(\Omega; L^2(0,T;V))$$

and

$$Z^n(t) \to Z(t) \qquad \text{strongly in } L^2_{\mathcal{F}}(\Omega; L^2(0,T;L_Q))$$

Since $\mathbf{A}$ is continuous, we also know that

$$\mathbf{A}\mathbf{u}^n(t) \to \mathbf{A}\mathbf{u}(t) \qquad \text{strongly in } \ L^2_{\mathcal{F}}(\Omega; L^2(0,T;V'))$$

Similar to Step 2 of Theorem 4.3.6, we can show that

$$\int_t^T Z^n(s)dW(s) \to \int_t^T Z(s)dW(s) \qquad \text{strongly in } L^2_{\mathcal{F}}(\Omega; L^2(0,T;V'))$$

and

$$\int_t^T \mathbf{A}\mathbf{u}^n(s)ds \to \int_t^T \mathbf{A}\mathbf{u}(s)ds \qquad \text{strongly in } \ L^2_{\mathcal{F}}(\Omega; L^2(0,T;V'))$$

Clearly, $\xi^n \to \xi \qquad$ strongly in $L^2_{\mathcal{F}}(\Omega; L^2(0,T;V'))$

Let

$$Y(t)dt = -d\mathbf{u}(t) - \nu\mathbf{A}\mathbf{u}(t)dt + \mathbf{f}(t)dt + Z(t)dW(t) \tag{4.5.5}$$

and

$$G(t) = \mathbf{u}(t) - \xi(t) - \int_t^T \nu\mathbf{A}\mathbf{u}(s)ds + \int_t^T \mathbf{f}(s)ds + \int_t^T Z(s)dW(s)$$

Then

$$\mathbf{u}^n(t)-\xi^n(t)-\int_t^T \nu\mathbf{A}\mathbf{u}^n(s)ds+\int_t^T \mathbf{f}(s)ds+\int_t^T Z^n(s)dW(s)\to G(t)$$

in $L^2_{\mathcal{F}}(\Omega;L^2(0,T;V'))$ and $G(t)=\int_t^T Y(s)ds$

Let $r(t)=\frac{C_G^2}{\nu}\int_0^t\|\mathbf{v}(s)\|_V^2ds$ for any $\mathbf{v}\in L^\infty(\Omega\times[0,T];V)$. Apply Itô's formula to $e^{-r(t)}\|\mathbf{u}^n(t)\|_H^2$, we get

$$\begin{aligned}
&e^{-r(T)}\|\xi^n\|_H^2-\|\mathbf{u}^n(0)\|_H^2=-\int_0^T \dot{r}(t)e^{-r(t)}\|\mathbf{u}^n(t)\|_H^2dt\\
&-2\int_0^T e^{-r(t)}\langle\nu\mathbf{A}\mathbf{u}^n(t)+\mathbf{B}(\mathbf{u}^n(t))-\mathbf{f}(t),\mathbf{u}^n(t)\rangle_{V',V}dt\\
&+2\int_0^T e^{-r(t)}\langle(Z^n(t))^*(\mathbf{u}^n(t)),dW(t)\rangle_H+\int_0^T e^{-r(t)}\|Z^n(t)\|_{L_Q}^2dt\\
=&-2\int_0^T e^{-r(t)}\langle\nu\mathbf{A}\mathbf{u}^n(t)+\mathbf{B}(\mathbf{u}^n(t))+\frac{1}{2}\dot{r}(t)\mathbf{u}^n(t),\mathbf{u}^n(t)\rangle_{V',V}dt\\
&+2\int_0^T e^{-r(t)}\langle\mathbf{f}(t),\mathbf{u}^n(t)\rangle_{V',V}dt+2\int_0^T e^{-r(t)}\langle(Z^n(t))^*(\mathbf{u}^n(t)),dW(t)\rangle_H\\
&+\int_0^T e^{-r(t)}\|Z^n(t)\|_{L_Q}^2dt
\end{aligned}$$

Now by taking expectation, we get

$$\begin{aligned}
&Ee^{-r(T)}\|\xi^n\|_H^2-E\|\mathbf{u}^n(0)\|_H^2-2E\int_0^T e^{-r(t)}\langle\mathbf{f}(t),\mathbf{u}^n(t)\rangle_{V',V}dt\\
&-E\int_0^T e^{-r(t)}\|Z^n(t)\|_{L_Q}^2dt\\
=&-2E\int_0^T e^{-r(t)}\langle\nu\mathbf{A}\mathbf{u}^n(t)+\mathbf{B}(\mathbf{u}^n(t))+\frac{1}{2}\dot{r}(t)\mathbf{u}^n(t),\mathbf{u}^n(t)\rangle_{V',V}dt
\end{aligned}\tag{4.5.6}$$

Similarly, by (4.5.5) and apply Itô's formula to $e^{-r(t)}\|\mathbf{u}(t)\|_H^2$, we get

$$\begin{aligned}
&E\|\mathbf{u}(0)\|_H^2-Ee^{-r(T)}\|\xi\|_H^2+2E\int_0^T e^{-r(t)}\langle\mathbf{f}(t),\mathbf{u}(t)\rangle_{V',V}dt\\
&+E\int_0^T e^{-r(t)}\|Z(t)\|_{L_Q}^2dt\\
=&2E\int_0^T e^{-r(t)}\langle\nu\mathbf{A}\mathbf{u}(t)+Y(t)+\frac{1}{2}\dot{r}(t)\mathbf{u}(t),\mathbf{u}(t)\rangle_{V',V}dt
\end{aligned}\tag{4.5.7}$$

Taking the limit, (4.5.6) becomes

$$\begin{aligned}
&\lim_{n\to\infty}\{2E\int_0^T e^{-r(t)}\langle \nu\mathbf{A}\mathbf{u}^n(t)+\mathbf{B}(\mathbf{u}^n(t))+\frac{1}{2}\dot{r}(t)\mathbf{u}^n(t),\mathbf{u}^n(t)\rangle_{V',V}dt\}\\
=&E\|\mathbf{u}(0)\|_H^2 - Ee^{-r(T)}\|\xi\|_H^2 + 2E\int_0^T e^{-r(t)}\langle \mathbf{f}(t),\mathbf{u}(t)\rangle_{V',V}dt\\
&+E\int_0^T e^{-r(t)}\|Z(t)\|_{L_Q}^2 dt \qquad (4.5.8)\\
=&2E\int_0^T e^{-r(t)}\langle \nu\mathbf{A}\mathbf{u}(t)+Y(t)+\frac{1}{2}\dot{r}(t)\mathbf{u}(t),\mathbf{u}(t)\rangle_{V',V}dt
\end{aligned}$$

Now by Corollary 4.3.3, we have

$$\begin{aligned}
E\int_0^T e^{-r(t)}\langle \nu\mathbf{A}(\mathbf{v}(t)-\mathbf{u}^n(t))+\mathbf{B}(\mathbf{v}(t))-\mathbf{B}(\mathbf{u}^n(t))\\
+\frac{1}{2}\dot{r}(t)(\mathbf{v}(t)-\mathbf{u}^n(t)),\mathbf{v}(t)-\mathbf{u}^n(t)\rangle_{V',V}dt \geq 0
\end{aligned}$$

Hence

$$\begin{aligned}
&E\int_0^T e^{-r(t)}\langle \nu\mathbf{A}\mathbf{u}^n(t)+\mathbf{B}(\mathbf{u}^n(t))+\frac{1}{2}\dot{r}(t)\mathbf{u}^n(t),\mathbf{v}(t)-\mathbf{u}^n(t)\rangle_{V',V}dt\\
\leq&E\int_0^T e^{-r(t)}\langle \nu\mathbf{A}\mathbf{v}(t)+\mathbf{B}(\mathbf{v}(t))+\frac{1}{2}\dot{r}(t)\mathbf{v}(t),\mathbf{v}(t)-\mathbf{u}^n(t)\rangle_{V',V}dt
\end{aligned}$$

Now we take the limit and by (4.5.8), we get

$$\begin{aligned}
&E\int_0^T e^{-r(t)}\langle \nu\mathbf{A}\mathbf{u}(t)+Y(t)+\frac{1}{2}\dot{r}(t)\mathbf{u}(t),\mathbf{v}(t)-\mathbf{u}(t)\rangle_{V',V}dt\\
\leq&E\int_0^T e^{-r(t)}\langle \nu\mathbf{A}\mathbf{v}(t)+\mathbf{B}(\mathbf{v}(t))+\frac{1}{2}\dot{r}(t)\mathbf{v}(t),\mathbf{v}(t)-\mathbf{u}(t)\rangle_{V',V}dt \qquad (4.5.9)
\end{aligned}$$

Since $L^\infty(\Omega\times[0,T];V)$ is dense in $L^2_{\mathcal{F}}(\Omega;L^2(0,T;V))$, (4.5.9) is true for all $\mathbf{v}(t)\in L^2_{\mathcal{F}}(\Omega;L^2(0,T;V))$. Now we take $\mathbf{v}(t)=\mathbf{u}(t)+\lambda\mathbf{w}(t)$ for any $\mathbf{w}(t)\in L^\infty(\Omega\times[0,T];V)$ and $\lambda>0$. Then

$$E\int_0^T \langle \dot{r}(t)\mathbf{w}(t),\mathbf{w}(t)\rangle_{V',V}dt$$

$$
\begin{aligned}
&= E\int_0^T \dot{r}(t)\|\mathbf{w}(t)\|_H^2 dt \\
&\le E(\sup_{0\le t\le T}\|\mathbf{w}(t)\|_H^2 \int_0^T \frac{C_G^2}{\nu}\|\mathbf{v}(t)\|_V^2 dt) \\
&\le \sup_{t,\omega}\|\mathbf{w}(t)\|_H^2 E\int_0^T \frac{C_G^2}{\nu}\|\mathbf{v}(t)\|_V^2 dt < \infty
\end{aligned}
$$

Thus (4.5.9) becomes

$$
\begin{aligned}
&E\int_0^T e^{-r(t)}\langle Y(t) - \mathbf{B}(\mathbf{u}(t)+\lambda\mathbf{w}(t)), \lambda\mathbf{w}(t)\rangle_{V',V} dt \\
\le& E\int_0^T e^{-r(t)}\langle \lambda\nu\mathbf{A}\mathbf{w}(t) + \frac{\lambda}{2}\dot{r}(t)\mathbf{w}(t), \lambda\mathbf{w}(t)\rangle_{V',V} dt
\end{aligned}
$$

Cancelling λ, and using the fact that

$$
\begin{aligned}
&\langle \mathbf{B}(\mathbf{u}(t)+\lambda\mathbf{w}(t)), \mathbf{w}(t)\rangle_{V',V} \\
=& -\langle \mathbf{B}(\mathbf{u}(t)+\lambda\mathbf{w}(t), \mathbf{w}(t)), \mathbf{u}(t)+\lambda\mathbf{w}(t)\rangle_{V',V} \\
=& -\langle \mathbf{B}(\mathbf{u}(t)+\lambda\mathbf{w}(t), \mathbf{w}(t)), \mathbf{u}(t)\rangle_{V',V} \\
=& -\langle \mathbf{B}(\mathbf{u}(t), \mathbf{w}(t)), \mathbf{u}(t)\rangle_{V',V} - \lambda\langle \mathbf{B}(\mathbf{w}(t), \mathbf{w}(t)), \mathbf{u}(t)\rangle_{V',V} \\
=& \langle \mathbf{B}(\mathbf{u}(t)), \mathbf{w}(t)\rangle_{V',V} + \lambda\langle \mathbf{B}(\mathbf{w}(t), \mathbf{u}(t)), \mathbf{w}(t)\rangle_{V',V},
\end{aligned}
$$

we get

$$
\begin{aligned}
&E\int_0^T e^{-r(t)}\langle Y(t) - \mathbf{B}(\mathbf{u}(t)), \mathbf{w}(t)\rangle_{V',V} dt \\
\le& \lambda E\int_0^T e^{-r(t)}\langle \nu\mathbf{A}\mathbf{w}(t) + \mathbf{B}(\mathbf{w}(t), \mathbf{u}(t)) + \frac{1}{2}\dot{r}(t)\mathbf{w}(t), \mathbf{w}(t)\rangle_{V',V} dt
\end{aligned}
$$

Now we let $\lambda \to 0$. Since the right hand side of the last inequality is finite, we get

$$
E\int_0^T e^{-r(t)}\langle Y(t) - \mathbf{B}(\mathbf{u}(t)), \mathbf{w}(t)\rangle_{V',V} dt \le 0
$$

for all $\mathbf{w}(t) \in L^\infty(\Omega \times [0,T]; V)$

Hence $Y(t) = \mathbf{B}(\mathbf{u}(t))$ P-a.s. and $(\mathbf{u}(t), Z(t))$ is a pair of solution of (4.1.3).

Similar to the proof of Proposition 4.2.1, one can show that

$$\|\mathbf{u}^n(t)\|_H^2 \leq 2E^{\mathcal{F}_t}\|\xi\|_H^2 + 4\int_0^T \|\mathbf{f}(s)\|_{V'}^2 ds$$

for all $n \in \mathbb{N}$. So

$$\sup_{0\leq t\leq T} E\|\mathbf{u}^n(t)\|_H^2 < \infty$$

for all $n \in \mathbb{N}$. Thus we have shown that $\mathbf{u} \in L^\infty(0, T; L^2_{\mathcal{F}}(\Omega; H))$ and the proof of the existence of a solution is complete.

Step 2: Now let us show the uniqueness of the solution. Suppose that there is another pair of solution $(\mathbf{v}(t), Y(t)) \in L^2_{\mathcal{F}}(\Omega; L^2(0, T; V)) \cap L^\infty(0, T; L^2_{\mathcal{F}}(\Omega; H)) \times L^2_{\mathcal{F}}(\Omega; L^2(0, T; L_Q))$.

Let $(\mathbf{u}^{n,N}(t), Z^{n,N}(t))$ be the solution of

$$\begin{cases} d\mathbf{u}^{n,N}(t) = & -\nu\mathbf{A}^N\mathbf{u}^{n,N}(t)dt - \mathbf{B}^N(\mathbf{u}^n(t))dt + \mathbf{f}^N(t)dt \\ & +Z^{n,N}(t)dW^N(t) \\ \mathbf{u}^{n,N}(T) = & \xi^{n,N} \end{cases}$$

where $\xi^{n,N}{=}P_N\xi^n$.

Clearly,

$$\lim_{N\to\infty} E\int_0^T \|\mathbf{u}^n(s) - \mathbf{u}^{n,N}(s)\|_V^2 ds = 0 \tag{4.5.10}$$

and by Proposition 4.5.1,

$$\lim_{n\to\infty} E\int_0^T \|\mathbf{u}(s) - \mathbf{u}^n(s)\|_V^2 ds = 0 \tag{4.5.11}$$

Let $\rho^N(t) = P_N\mathbf{v}(t)$ and $Y^N(t) = P_N Y(t)$, then

$$\begin{cases} d\rho^N(t) = & -\nu\mathbf{A}^N\rho^N(t)dt - \mathbf{B}^N(\mathbf{v}(t))dt + \mathbf{f}^N(t)dt \\ & +Y^N(t)dW(t) \\ \rho^N(T) = & \xi^N \end{cases}$$

Let $(\mathbf{x}^{n,N}(t), Y^{n,N}(t))$ be the solution of

$$\begin{cases} d\mathbf{x}^{n,N}(t) = & -\nu\mathbf{A}^N\mathbf{x}^{n,N}(t)dt - \mathbf{B}^N(\rho^N(t))dt + \mathbf{f}^N(t)dt \\ & +Y^{n,N}(t)dW^N(t) \\ \mathbf{x}^{n,N}(T) = & \xi^{n,N} \end{cases}$$

and let $\mathbf{w}^{n,N}(t)=\rho^N(t) - \mathbf{x}^{n,N}(t)$ and $\sigma^{n,N}(t)=Y^N(t) - Y^{n,N}(t)$, then

$$\begin{cases} d\mathbf{w}^{n,N}(t) = & -\nu\mathbf{A}^N\mathbf{w}^{n,N}(t)dt - (\mathbf{B}^N(\mathbf{v}(t)) - \mathbf{B}^N(\rho^N(t)))dt \\ & +\sigma^{n,N}(t)dW(t) \\ \mathbf{w}^{n,N}(T) = & \xi^N - \xi^{n,N} \end{cases}$$

The Itô formula yields

$$\begin{aligned} &\|\mathbf{w}^{n,N}(t)\|_H^2 + \int_t^T \|\sigma^{n,N}(s)\|_{L_Q}^2 ds = \|\xi^N - \xi^{n,N}\|_H^2 \\ &+2\int_t^T \langle \nu\mathbf{A}^N\mathbf{w}^{n,N}(s) + (\mathbf{B}^N(\mathbf{v}(s)) - \mathbf{B}^N(\rho^N(s))), \mathbf{w}^{n,N}(s)\rangle_{V',V} ds \\ &- 2\int_t^T \langle \sigma^{n,N}(s)dW(s), \mathbf{w}^N(s)\rangle_H \end{aligned} \tag{4.5.12}$$

First, we have

$$\begin{aligned} &|\langle \mathbf{B}^N(\mathbf{v}(s)) - \mathbf{B}^N(\rho^N(s)), \mathbf{w}^{n,N}(s)\rangle_{V',V}| \\ =&|\langle \mathbf{B}^N(\mathbf{v}(s)), \mathbf{w}^{n,N}(s)\rangle_{V',V} - \langle \mathbf{B}^N(\rho^N(s)), \mathbf{w}^{n,N}(s)\rangle_{V',V}| \\ =&| - \langle \mathbf{B}^N(\mathbf{v}(s), \mathbf{w}^{n,N}(s)), \mathbf{v}(s)\rangle_{V',V} + \langle \mathbf{B}^N(\rho^N(s), \mathbf{w}^{n,N}(s)), \rho^N(s)\rangle_{V',V}| \end{aligned}$$

$$
\begin{aligned}
&= | - \langle \mathbf{B}^N(\mathbf{v}(s), \mathbf{w}^{n,N}(s)), \rho^N(s) \rangle_{V',V} + \langle \mathbf{B}^N(\rho^N(s), \mathbf{w}^{n,N}(s)), \rho^N(s) \rangle_{V',V} | \\
&= | \langle \mathbf{B}^N(\rho^N(s) - \mathbf{v}(s), \mathbf{w}^{n,N}(s)), \rho^N(s) \rangle_{V',V} | \\
&\leq C \|\rho^N(s) - \mathbf{v}(s)\|_H^{\frac{1}{2}} \|\rho^N(s) - \mathbf{v}(s)\|_V^{\frac{1}{2}} \|\mathbf{w}^{n,N}(s)\|_V \|\rho^N(s)\|_H^{\frac{1}{2}} \|\rho^N(s)\|_V^{\frac{1}{2}} \\
&\leq 2C\sqrt{\lambda_N} \|\mathbf{v}(s)\|_H^{\frac{3}{2}} \|\rho^N(s) - \mathbf{v}(s)\|_V^{\frac{1}{2}} \|\mathbf{w}^{n,N}(s)\|_V
\end{aligned}
$$

For $0 \leq r \leq t \leq T$, we take the expectation in (4.5.12) and apply the above inequality to get

$$
\begin{aligned}
&E^{\mathcal{F}_r} \|\mathbf{w}^{n,N}(t)\|_H^2 + E^{\mathcal{F}_r} \int_t^T \|\sigma^{n,N}(s)\|_{L_Q}^2 ds \\
\leq & E^{\mathcal{F}_r} \|\xi^N - \xi^{n,N}\|_H^2 + 2\nu\lambda_N \int_t^T E^{\mathcal{F}_r} \|\mathbf{w}^{n,N}(s)\|_H^2 ds \\
&+ 4C\sqrt{\lambda_N} E^{\mathcal{F}_r} \int_t^T \|\mathbf{v}(s)\|_H^{\frac{3}{2}} \|\rho^N(s) - \mathbf{v}(s)\|_V^{\frac{1}{2}} \|\mathbf{w}^{n,N}(s)\|_V ds \\
\leq & E^{\mathcal{F}_r} \|\xi^N - \xi^{n,N}\|_H^2 + 2\nu\lambda_N(\lambda_N + 1) \int_t^T E^{\mathcal{F}_r} \|\mathbf{w}^{n,N}(s)\|_H^2 ds \\
&+ 4C\sqrt{\lambda_N} \{E^{\mathcal{F}_r} \int_r^T \|\mathbf{v}(s)\|_H^3 \|\rho^N(s) - \mathbf{v}(s)\|_V ds\}^{\frac{1}{2}} \{\int_t^T E^{\mathcal{F}_r} \|\mathbf{w}^{n,N}(s)\|_V^2 ds\}^{\frac{1}{2}}
\end{aligned}
$$

Let

$$
\alpha(N) = 2\nu\lambda_N(\lambda_N + 1)
$$

and

$$
\beta(N) = 4C\sqrt{\lambda_N} \{E^{\mathcal{F}_r} \int_r^T \|\mathbf{v}(s)\|_H^3 \|\rho^N(s) - \mathbf{v}(s)\|_V ds\}^{\frac{1}{2}}
$$

Similar the proof of Theorem 4.4.2, one obtains

$$
\begin{aligned}
&E^{\mathcal{F}_r} \|\mathbf{w}^{n,N}(t)\|_H^2 + E^{\mathcal{F}_r} \int_t^T \|\sigma^{n,N}(s)\|_{L_Q}^2 ds \\
\leq & E^{\mathcal{F}_r} \|\xi^N - \xi^{n,N}\|_H^2 + 2\alpha(N) \int_t^T E^{\mathcal{F}_r} \|\mathbf{w}^{n,N}(s)\|_H^2 ds + \frac{\beta^2(N)}{4\alpha(N)}
\end{aligned}
$$

Since $\mathbf{v}(s) \in L^2_{\mathcal{F}}(\Omega; L^2(0,T;V)) \cap L^\infty(0,T; L^2_{\mathcal{F}}(\Omega; H))$, there is the integrability to apply Gronwall's inequality. Then,

$$\begin{aligned}&E^{\mathcal{F}_r}\|\mathbf{w}^{n,N}(t)\|_H^2 + E^{\mathcal{F}_r}\int_t^T \|\sigma^{n,N}(s)\|_{L_Q}^2 ds\\ \leq& 2E^{\mathcal{F}_r}\|\xi^N - \xi^{n,N}\|_H^2 + \frac{\beta^2(N)}{2\alpha(N)}\end{aligned}$$

Letting $r = t$, and taking the limit, by Lebesgue dominated convergence theorem,

$$\lim_{N\to\infty}\{\sup_{0\leq t\leq T}\|\mathbf{w}^{n,N}(t)\|_H^2 + E\int_0^T \|\sigma^{n,N}(s)\|_{L_Q}^2 ds\} = 0 \tag{4.5.13}$$

Let $\tilde{\mathbf{w}}^{n,N}(t)$=$\mathbf{u}^{n,N}(t) - \mathbf{x}^{n,N}(t)$ and $\tilde{\sigma}^{n,N}(t)$=$Z^{n,N}(t) - Y^{n,N}(t)$, so that

$$\begin{cases} d\tilde{\mathbf{w}}^{n,N}(t) = -\nu\mathbf{A}^N\tilde{\mathbf{w}}^{n,N}(t)dt - (\mathbf{B}^N(\mathbf{u}^n(t)) - \mathbf{B}^N(\rho^N(t)))dt + \tilde{\sigma}^{n,N}(t)dW^N(t)\\ \tilde{\mathbf{w}}^{n,N}(T) = 0\end{cases}$$

By (4.4.3), we know that $\sup\limits_{0\leq t\leq T}\|\mathbf{x}^{n,N}(t)\|_H < C(n)$ for a constant $C(n)$ which is dependent only on n. Thus it is clear that $\sup\limits_{0\leq t\leq T}\|\tilde{\mathbf{w}}^{n,N}(t)\|_H < C(n)$

Since

$$\begin{aligned}&\frac{1}{2}d\|\tilde{\mathbf{w}}^{n,N}(t)\|_V^2 = \langle d\tilde{\mathbf{w}}^{n,N}(t), \tilde{\mathbf{w}}^{n,N}(t)\rangle_V\\ =&\langle \mathbf{A}^N\tilde{\mathbf{w}}^{n,N}(t), d\tilde{\mathbf{w}}^{n,N}(t)\rangle_{V',V} = \langle d\tilde{\mathbf{w}}^{n,N}(t), \mathbf{A}^N\tilde{\mathbf{w}}^{n,N}(t)\rangle_H\\ =&-\langle \nu\mathbf{A}^N\tilde{\mathbf{w}}^{n,N}(t), \mathbf{A}^N\tilde{\mathbf{w}}^{n,N}(t)\rangle_H dt - \langle \mathbf{B}^N(\mathbf{u}^n(t)) - \mathbf{B}^N(\rho^N(t)), \mathbf{A}^N\tilde{\mathbf{w}}^{n,N}(t)\rangle_H dt\\ &+\langle \tilde{\sigma}^{n,N}(t)dW^N(t), \mathbf{A}^N\tilde{\mathbf{w}}^{n,N}(t)\rangle_H\end{aligned}$$

we have

$$\begin{aligned}\|\tilde{\mathbf{w}}^{n,N}(t)\|_V^2 =& 2\int_t^T \nu\|\mathbf{A}^N\tilde{\mathbf{w}}^{n,N}(s)\|_H^2 ds\\ &+2\int_t^T \langle \mathbf{B}^N(\mathbf{u}^n(s)) - \mathbf{B}^N(\rho^N(s)), \mathbf{A}^N\tilde{\mathbf{w}}^{n,N}(s)\rangle_H ds\\ &-2\int_t^T \langle \tilde{\sigma}^{n,N}(s)dW^N(s), \mathbf{A}^N\tilde{\mathbf{w}}^{n,N}(s)\rangle_H\end{aligned} \tag{4.5.14}$$

The following notation is introduced:

$$\begin{aligned}
K_1(n,N,t) =& \|\mathbf{u}^n(t)\|_H^{\frac{1}{2}}\|\mathbf{u}^n(t)\|_V^{\frac{1}{2}} + \|\mathbf{u}^{n,N}(t)\|_H^{\frac{1}{2}}\|\mathbf{u}^{n,N}(t)\|_V^{\frac{1}{2}} \\
K_2(n,N,t) =& \|\mathbf{u}^{n,N}(t)\|_H^{\frac{1}{2}}\|\mathbf{u}^{n,N}(t)\|_V^{\frac{1}{2}} + \|\mathbf{x}^{n,N}(t)\|_H^{\frac{1}{2}}\|\mathbf{x}^{n,N}(t)\|_V^{\frac{1}{2}} \\
K_3(n,N,t) =& \|\mathbf{x}^{n,N}(t)\|_H^{\frac{1}{2}}\|\mathbf{x}^{n,N}(t)\|_V^{\frac{1}{2}} + \|\rho^N(t)\|_H^{\frac{1}{2}}\|\rho^N(t)\|_V^{\frac{1}{2}} \\
K_4(n,N,t) =& K_1^2(n,N,t)\|\mathbf{u}^n(t)-\mathbf{u}^{n,N}(t)\|_H\|\mathbf{u}^n(t)-\mathbf{u}^{n,N}(t)\|_V \\
& + K_3^2(n,N,t)\|\mathbf{w}^{n,N}\|_H\|\mathbf{w}^{n,N}\|_V
\end{aligned}$$

By Lemma 4.3.4,

$$\begin{aligned}
& |\langle \mathbf{B}^N(\mathbf{u}^n(t)) - \mathbf{B}^N(\rho^N(t)), \mathbf{A}^N\tilde{\mathbf{w}}^{n,N}(t)\rangle_H| \\
\leq & |\langle \mathbf{B}^N(\mathbf{u}^n(t)) - \mathbf{B}^N(\mathbf{u}^{n,N}(t)), \mathbf{A}^N\tilde{\mathbf{w}}^{n,N}(t)\rangle_H| \\
& + |\langle \mathbf{B}^N(\mathbf{u}^{n,N}(t)) - \mathbf{B}^N(\mathbf{x}^{n,N}(t)), \mathbf{A}^N\tilde{\mathbf{w}}^{n,N}(t)\rangle_H| \\
& + |\langle \mathbf{B}^N(\mathbf{x}^{n,N}(t)) - \mathbf{B}^N(\rho^N(t)), \mathbf{A}^N\tilde{\mathbf{w}}^{n,N}(t)\rangle_H| \\
\leq & CK_1(n,N,t)\|\mathbf{A}^N\tilde{\mathbf{w}}^{n,N}(t)\|_V\|\mathbf{u}^n(t)-\mathbf{u}^{n,N}(t)\|_H^{\frac{1}{2}}\|\mathbf{u}^n(t)-\mathbf{u}^{n,N}(t)\|_V^{\frac{1}{2}} \\
& + CK_2(n,N,t)\|\mathbf{A}^N\tilde{\mathbf{w}}^{n,N}(t)\|_V\|\tilde{\mathbf{w}}^{n,N}(t)\|_H^{\frac{1}{2}}\|\tilde{\mathbf{w}}^{n,N}(t)\|_V^{\frac{1}{2}} \\
& + CK_3(n,N,t)\|\mathbf{A}^N\tilde{\mathbf{w}}^{n,N}(t)\|_V\|\mathbf{w}^{n,N}(t)\|_H^{\frac{1}{2}}\|\mathbf{w}^{n,N}(t)\|_V^{\frac{1}{2}} \\
\leq & C\|\mathbf{A}^N\tilde{\mathbf{w}}^{n,N}(t)\|_V^2 + K_1^2(n,N,t)\|\mathbf{u}^n(t)-\mathbf{u}^{n,N}(t)\|_H\|\mathbf{u}^n(t)-\mathbf{u}^{n,N}(t)\|_V \\
& + C\|\mathbf{A}^N\tilde{\mathbf{w}}^{n,N}(t)\|_V^2\|\tilde{\mathbf{w}}^{n,N}(t)\|_H + K_2^2(n,N,t)\|\tilde{\mathbf{w}}^{n,N}(t)\|_V \\
& + C\|\mathbf{A}^N\tilde{\mathbf{w}}^{n,N}(t)\|_V^2 + K_3^2(n,N,t)\|\mathbf{w}^{n,N}(t)\|_H\|\mathbf{w}^{n,N}(t)\|_V \\
\leq & (2C + C(n))\lambda_N^2\|\tilde{\mathbf{w}}^{n,N}(t)\|_V^2 + K_2^2(n,N,t)\|\tilde{\mathbf{w}}^{n,N}(t)\|_V + K_4(n,N,t)
\end{aligned}$$

Denote

$$\alpha(n,N) = (4C + 2C(n) + 2\nu)\lambda_N^2$$

and

$$\beta(n,N) = 2\{E^{\mathcal{F}_r}\int_0^T K_2^4(n,N,s)ds\}^{\frac{1}{2}}.$$

For $0 \le r \le t \le T$, taking the conditional expectation in (4.5.14), and based on the above inequality, it is easy to obtain

$$\begin{aligned}
E^{\mathcal{F}_r}\|\tilde{\mathbf{w}}^{n,N}(t)\|_V^2 &\le \alpha(n,N)\int_t^T E^{\mathcal{F}_r}\|\tilde{\mathbf{w}}^{n,N}(s)\|_V^2 ds \\
&+ 2E^{\mathcal{F}_r}\int_t^T K_2^2(n,N,s)\|\tilde{\mathbf{w}}^{n,N}(s)\|_V ds + 2E^{\mathcal{F}_r}\int_0^T K_4(n,N,s)ds \\
\le& 2E^{\mathcal{F}_r}\int_0^T K_4(n,N,s)ds + \alpha(n,N)\int_t^T E^{\mathcal{F}_r}\|\tilde{\mathbf{w}}^{n,N}(s)\|_V^2 ds \\
&+ 2\{E^{\mathcal{F}_r}\int_0^T K_2^4(n,N,s)ds\}^{\frac{1}{2}}\{\int_0^T E^{\mathcal{F}_r}\|\tilde{\mathbf{w}}^{n,N}(s)\|_V^2 ds\}^{\frac{1}{2}} \\
=& 2E^{\mathcal{F}_r}\int_0^T K_4(n,N,s)ds + \alpha(n,N)\int_t^T E^{\mathcal{F}_r}\|\tilde{\mathbf{w}}^{n,N}(s)\|_V^2 ds \\
&+ \beta(n,N)\{\int_0^T E^{\mathcal{F}_r}\|\tilde{\mathbf{w}}^{n,N}(s)\|_V^2 ds\}^{\frac{1}{2}} \\
\le& 2E^{\mathcal{F}_r}\int_0^T K_4(n,N,s)ds + 2\alpha(n,N)\int_t^T E^{\mathcal{F}_r}\|\tilde{\mathbf{w}}^{n,N}(s)\|_V^2 ds \\
&+ \frac{\beta^2(n,N)}{4\alpha(n,N)}
\end{aligned}$$

Here the technique used to get the last inequality first appeared in the proof of Theorem 4.4.2.

Clearly $K_4(n,N,t)$ is integrable for all $n,N \in \mathbb{N}$, and $\beta(n,N) < K(n)$ for some $K(n)$ independent of N. Thus we have the integrability to apply Gronwall's inequality, and

$$E^{\mathcal{F}_r}\|\tilde{\mathbf{w}}^{n,N}(t)\|_V^2 \le \frac{\beta^2(n,N)}{2\alpha(n,N)} + 4E^{\mathcal{F}_r}\int_0^T K_4(n,N,s)ds$$

Letting $r = t$, and taking the limit, by Lebesgue dominated convergence theorem,

$$\lim_{N\to\infty}\sup_{0\le t\le T}\|\tilde{\mathbf{w}}^{n,N}(t)\|_V^2 = 0 \tag{4.5.15}$$

for all $n \in \mathbb{N}$.

Finally, by the definition of $\rho^N(t)$, it is easy to see that

$$\lim_{N\to\infty} E\int_0^T \|\rho^N(s) - \mathbf{v}(s)\|_H^2 ds = 0 \qquad (4.5.16)$$

Since

$$E\int_0^T \|\mathbf{u}(s) - \mathbf{v}(s)\|_H^2 ds = E\int_0^T \|\mathbf{u}(s) - \mathbf{u}^n(s)\|_H^2 ds$$
$$+ E\int_0^T \|\mathbf{u}^n(s) - \mathbf{u}^{n,N}(s)\|_H^2 ds + E\int_0^T \|\mathbf{u}^{n,N}(s) - \mathbf{x}^{n,N}(s)\|_H^2 ds$$
$$+ E\int_0^T \|\mathbf{x}^{n,N}(s) - \rho^N(s)\|_H^2 ds + E\int_0^T \|\rho^N(s) - \mathbf{v}(s)\|_H^2 ds$$

By (4.5.11), (4.5.10), (4.5.15), (4.5.13), and (4.5.16), we get

$$E\int_0^T \|\mathbf{u}(s) - \mathbf{v}(s)\|_H^2 ds = 0$$

So $\mathbf{u}(t) = \mathbf{v}(t)$ P-a.s.. Thus we also have $Z(t) = Y(t)$ P-a.s. and the proof of uniqueness is complete. □

4.6 Continuity of the Solution

Theorem 4.6.1. *Let the conditions in Theorem 4.5.2 hold. Then the solution* $(\mathbf{u}(t), Z(t))$ *is continuous with respect to the terminal value and the external body force.*

Proof: Let $\xi_1, \xi_2 \in L^2_{\mathcal{F}_T}(\Omega; H)$ and $\mathbf{f}_1, \mathbf{f}_2 \in L^4(0,T;V')$. Let $(\mathbf{u}(t), Z(t))$ be the solution of (4.1.3) with respect to terminal value ξ_1 and external force $\mathbf{f}_1$, and we define $(\mathbf{u}^n(t), Z^n(t))$ and $(\mathbf{u}^{n,N}(t), Z^{n,N}(t))$ as in Theorem 4.5.2.

Let $(\mathbf{v}(t), Y(t))$ be the solution of (4.1.3) with respect to terminal value ξ_2 and external force $\mathbf{f}_2$, and we define $(\mathbf{v}^n(t), Y^n(t))$ and $(\mathbf{v}^{n,N}(t), Y^{n,N}(t))$ similarly.

Let $\mathbf{w}^{n,N}(t)=\mathbf{u}^{n,N}(t)-\mathbf{v}^{n,N}(t)$ and $\sigma^{n,N}(t)=Z^{n,N}(t)-Y^{n,N}(t)$, then

$$\begin{cases} d\mathbf{w}^{n,N}(t)= & -\nu\mathbf{A}^N\mathbf{w}^{n,N}(t)dt-(\mathbf{B}^N(\mathbf{u}^n(t))-\mathbf{B}^N(\mathbf{v}^n(t)))dt \\ & +(\mathbf{f}_1^N(t)-\mathbf{f}_2^N(t))dt+\sigma^{n,N}(t)dW(t) \\ \mathbf{w}^{n,N}(T)= & \xi_1^{n,N}-\xi_2^{n,N} \end{cases}$$

Applying Itô's formula to $\|w^{n,N}(t)\|_H^2$ and taking conditional expectation with respect to $r\in[0,t]$,

$$\begin{aligned} &E^{\mathcal{F}_r}\|\mathbf{w}^{n,N}(t)\|_H^2+E^{\mathcal{F}_r}\int_t^T\|\sigma^{n,N}(s)\|_{L_Q}^2ds=E^{\mathcal{F}_r}\|\xi_1^{n,N}-\xi_2^{n,N}\|_H^2 \\ &+2E^{\mathcal{F}_r}\int_t^T\langle\nu\mathbf{A}^N\mathbf{w}^{n,N}(s)+(\mathbf{B}^N(\mathbf{u}^n(s))-\mathbf{B}^N(\mathbf{v}^n(s))),\mathbf{w}^{n,N}(s)\rangle_{V',V}ds \qquad (4.6.1) \\ &-2E^{\mathcal{F}_r}\int_t^T\langle\mathbf{f}_1^N(s)-\mathbf{f}_2^N(s),\mathbf{w}^{n,N}(s)\rangle_{V,'V}ds \end{aligned}$$

By Lemma 4.3.4 and the fact that $\mathbf{u}^n(t),\mathbf{v}^n(t)\in L^\infty(\Omega\times[0,T];H)$, one obtains

$$\begin{aligned} &|\langle\mathbf{B}^N(\mathbf{u}^n(s))-\mathbf{B}^N(\mathbf{v}^n(s)),\mathbf{w}^{n,N}(s)\rangle_{V',V}| \\ \le& C(\|\mathbf{u}^n(s)\|_H^{\frac{1}{2}}\|\mathbf{u}^n(s)\|_V^{\frac{1}{2}}+\|\mathbf{v}^n(s)\|_H^{\frac{1}{2}}\|\mathbf{v}^n(s)\|_V^{\frac{1}{2}})\|\mathbf{w}^{n,N}(s)\|_H^{\frac{1}{2}}\|\mathbf{w}^{n,N}(s)\|_V^{\frac{3}{2}} \\ \le& C\lambda^{\frac{3}{2}}(\|\mathbf{u}^n(s)\|_V^{\frac{1}{2}}+\|\mathbf{v}^n(s)\|_V^{\frac{1}{2}})\|\mathbf{w}^{n,N}(s)\|_H \end{aligned}$$

Thus (4.6.1) becomes

$$\begin{aligned} &E^{\mathcal{F}_r}\|\mathbf{w}^{n,N}(t)\|_H^2+E^{\mathcal{F}_r}\int_t^T\|\sigma^{n,N}(s)\|_{L_Q}^2ds\le E^{\mathcal{F}_r}\|\xi_1^{n,N}-\xi_2^{n,N}\|_H^2 \\ &+\int_t^T\|\mathbf{f}_1^N(s)-\mathbf{f}_2^N(s)\|_{V'}^2ds+(2\nu\lambda_N+\lambda_N^2)\int_t^T E^{\mathcal{F}_r}\|\mathbf{w}^{n,N}(s)\|_H^2ds \\ &+2C\lambda_N^{\frac{3}{2}}E^{\mathcal{F}_r}\int_t^T(\|\mathbf{u}^n(s)\|_V^{\frac{1}{2}}+\|\mathbf{v}^n(s)\|_V^{\frac{1}{2}})\|\mathbf{w}^{n,N}(s)\|_H ds \end{aligned}$$

Similar to Step 2 of the proof of Theorem 4.5.2, it follows that

$$\lim_{N\to\infty}\{\|\mathbf{w}^{n,N}(t)\|_H^2+E\int_0^T\|\sigma^{n,N}(s)\|_{L_Q}^2ds\}$$

$$\leq 2\lim_{N\to\infty}\{E^{\mathcal{F}_t}\|\xi_1^{n,N}-\xi_2^{n,N}\|_H^2+\int_0^T\|\mathbf{f}_1^N(s)-\mathbf{f}_2^N(s)\|_{V'}^2 ds\}$$

Using the Lebesgue dominated convergence theorem and the monotone convergence theorem, we get

$$\begin{aligned}
&E\int_0^T\|\mathbf{u}(t)-\mathbf{v}(t)\|_H^2 dt+TE\int_0^T\|Z(s)-Y(s)\|_{L_Q}^2 ds\\
\leq& 2\int_0^T\{\int_0^T\|\mathbf{f}_1(s)-\mathbf{f}_2(s)\|_{V'}^2 ds+E(E^{\mathcal{F}_t}\|\xi_1-\xi_2\|_H^2)\}dt\\
\leq& 2T\int_0^T\|\mathbf{f}_1(s)-\mathbf{f}_2(s)\|_{V'}^2 ds+2TE\|\xi_1-\xi_2\|_H^2
\end{aligned}$$

and this completes the proof. □

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Vita

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