Journal of Stochastic Analysis

Volume 3 | Number 1 Article 2

March 2022

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Recommended Citation

John, Majnu and Wu, Yihren (2022) "Dynamic Correlation Estimators for Bivariate Brownian and Geometric Brownian Motions," *Journal of Stochastic Analysis*: Vol. 3 : No. 1 , Article 2.
DOI: 10.31390/josa.3.1.02
Available at: https://digitalcommons.lsu.edu/josa/vol3/iss1/2

Journal of Stochastic Analysis
Vol. 3, No. 1 (2022) Article 2 (27 pages)
DOI: 10.31390/josa.3.1.02

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DYNAMIC CORRELATION ESTIMATORS FOR BIVARIATE BROWNIAN AND GEOMETRIC BROWNIAN MOTIONS

MAJNU JOHN* AND YIHREN WU

Abstract. Estimating dynamic correlation between a pair of time series is of importance in many applications. We present new estimators for the dynamic correlation between a pair of correlated Brownian motions and separately for dynamic correlation between a pair of correlated Geometric Brownian motions. We show that, as the sample size increases, all estimators presented in this paper converge in probability to the underlying true dynamic correlation.

1. Introduction

Dynamic correlation between a pair of time series is of long-standing interest in many fields such as economics [1, 7] and neuroscience [5]. In neuroimaging fMRI studies, for example, the connectivity (-not anatomical connectivity but functional connectivity-) of different brain regions are often hypothesized to change over time. Such dynamic functional connectivity reveals underlying biological mechanisms and are often studied. In neurophysiological studies of rodent brains, correlations between local field potential time series from a pair of regions may change over time. Dynamic changes of correlation in such settings reveal time segments during which the corresponding brain regions work in tandem (e.g. during certain specific tasks or in response to a stimulus) and other time segments when they do not.

Recently, a nonparametric estimator of dynamic correlation, robust in the presence of extreme values, was introduced in [4]. The new estimator may be conceptualized based on a particular algorithm for conversion of time series into a weighted graph and considering the edge-weights of the graph. In this paper we consider extensions of the estimators presented in [4]. Specifically, we present estimators for a pair of correlated bivariate Brownian motions and estimators for a pair of correlated Geometric Brownian motions. The main focus of the paper is in showing weak consistency of the estimators.

The paper is structured as follows. The new estimators are presented in section 2. Next, in section 3, the weak consistency results are presented. Section 4 summarizes the conclusions and includes a brief discussion.

Received 2021-3-21; Accepted 2021-8-1; Communicated by the editors.

1991 *Mathematics Subject Classification.* Primary 62H20; Secondary 62M10, 60J65.

Key words and phrases. Dynamic correlation, correlated bivariate Brownian motion, correlated bivariate geometric Brownian motion, weak consistency.

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2. New Estimators

First we focus on a correlated (bivariate) Brownian motion pair denoted by $(X_t, Y_t), t \in [0, T]$, and present a weakly-consistent estimator for the dynamic correlation, $\rho_t = \gamma_t/\sigma_t^x \sigma_t^y = \gamma_t/t$ between X_t and Y_t. Assume n points are sampled from each time series at time points $\{t_1, \ldots, t_n = T\} \equiv I$. We assume without loss of generality, $\mathbb{E}(X_t) = 0 = \mathbb{E}(Y_t), \forall t$ and for convenience, we take $t_i = i, \forall i$; however, our results will hold in general, even without these simplifications. The estimators that we consider are $\hat{\rho}_u^{q,p} = \hat{\gamma}_u^{q,p}/(\hat{\sigma}_u^{x,q,p}\hat{\sigma}_u^{y,q,p})$, $u \in I, q, p \geq 0$, where

$$\hat{\gamma}_u^{q,p} = \frac{1}{T-1} \sum_{\substack{v \in I \\ v \neq u}} \frac{(v^q X_u - v^{-p} X_v)(v^q Y_u - v^{-p} Y_v)}{(u-v)^2}, \tag{2.1}$$

$$(\hat{\sigma}_u^{x,\,q,p})^2 = \frac{1}{T-1} \sum_{\substack{v \in I \\ v \neq u}} \frac{(v^q X_u - v^{-p} X_v)^2}{(u-v)^2},$$

$$(\hat{\sigma}_u^{y,\,q,p})^2 = \frac{1}{T-1} \sum_{\substack{v \in I \\ v \neq u}} \frac{(v^q Y_u - v^{-p} Y_v)^2}{(u-v)^2}.$$

The estimators that we originally considered in [4] were special cases of the estimator presented in (2.1), specifically with $p = q = 0$. The main reason that we consider the more general version is that, while studying the weak consistency of the original estimators theoretically, we realized that the asymptotic bias is non-zero for those estimators. In other words, weak consistency does not hold in the special case $p = q = 0$. In the general case, weak consistency do hold for certain specific range of values for p and q, as shown in section 3 below.

Next we consider a correlated (bivariate) Geometric Brownian motion pair (R_t, S_t), $t \in [0, T]$. The most general form that we should consider is

$$R_t = R_0 \exp\left\{\left(\mu_R - \frac{\sigma_R^2}{2}\right)t + \sigma_R W_t\right\} \text{ and } S_t = S_0 \exp\left\{\left(\mu_S - \frac{\sigma_S^2}{2}\right)t + \sigma_S U_t\right\},$$

where (W_t, U_t) is a correlated Brownian motion pair. For ease of exposition, we restrict our attention to the case with $R_0 = S_0 = 1$ and $\mu_R = \mu_S = \sigma_R^2/2 = \sigma_S^2/2$ and $\sigma_R = \sigma_S = \sigma$ so that

$$R_t = e^{\sigma W_t} \text{ and } S_t = e^{\sigma U_t}$$

is the Geometric Brownian motion pair that we consider. Note that in this case we have $\mathbb{E}(R_t) = \mathbb{E}(S_t) = e^{\sigma^2 t/2}$ and $\text{Var}(R_t) = \text{Var}(S_t) = e^{\sigma^2 t}(e^{\sigma^2 t} - 1)$. We also mention here that we use the same notation (ρ_t and γ_t, respectively) for the correlation and covariance between R_t and S_t as well. That is, whether ρ_t (similarly γ_t) stands for the correlation (covariance) between the Brownian motion pair (X_t, Y_t) or for the Geometric Brownian motion pair (R_t, S_t) will hopefully be understood from the context. We consider two estimators for the dynamic correlation $\rho_t = \gamma_t/\sigma_t^R\sigma_t^S = \gamma_t/(\sigma_{t,W}\sigma_{t,U})$ between R_t and S_t. In the first case

$$\hat{\gamma}_t = \frac{1}{e^{c\sigma^2 T}}\sum_{k=1}^{T}\left\{\left[e^{\frac{-b\sigma^2 k}{2}}\left(e^{\sigma W_k} - e^{\frac{\sigma^2 k}{2}}\right) - e^{\frac{a\sigma^2 k}{2}}\left(e^{\sigma W_t} - e^{\frac{\sigma^2 t}{2}}\right)\right]\right.$$
$$\left.\times\left[e^{\frac{-b\sigma^2 k}{2}}\left(e^{\sigma U_k} - e^{\frac{\sigma^2 k}{2}}\right) - e^{\frac{a\sigma^2 k}{2}}\left(e^{\sigma U_t} - e^{\frac{\sigma^2 t}{2}}\right)\right]\right\}. \quad (2.2)$$

In the second case, we have the estimator for γ_t as

$$\hat{\gamma}_t = e^{-c\sigma^2 T}\sum_{k=1}^{T}\left\{e^{a\sigma^2 k}\left[(e^{\sigma W_t} - e^{\sigma^2 t/2})(e^{\sigma U_t} - e^{\sigma^2 t/2})\right]\right.$$
$$\left. - e^{-b\sigma^2 k}\left[(e^{\sigma W_k} - e^{\sigma^2 k/2})(e^{\sigma U_k} - e^{\sigma^2 k/2})\right]\right\}. \quad (2.3)$$

By considering $W_k = U_k$ for all k (and $W_t = U_t$) in equations (2.2) and (2.3) we get the corresponding estimates for $\sigma^2_{t,W}$ and $\sigma^2_{t,U}$. Here a, b and c are real-valued constants with appropriate ranges given in section 3.

If the correlation between W_t and U_t is denoted by r_t, then it is easy to verify that

$$r_t = \frac{1}{\sigma^2 t}\log\left[1 + \rho_t(e^{\sigma^2 t} - 1)\right] \text{ or } \rho_t = \frac{e^{r_t\sigma^2 t} - 1}{e^{\sigma^2 t} - 1}.$$

Based on this, we may also consider an estimate for ρ_t by plugging in an estimate for r_t (for example, the one given in (2.1)). However we do not pursue such estimators in this paper as our focus is on estimators that are generalizations of the original estimators that we considered in [4].

3. Results

3.1. Weak consistency of dynamic correlation estimators in the Brownian Motion case.

Our main result in this subsection is the following theorem.

Theorem 3.1. *$\hat{\rho}_u^{q,p} \to \rho_u$ in probability for each u and $p > q = 1/2$, as $T \to \infty$.*

We first make a few remarks which will be used later.

Remark 3.2. A random variable V follows a variance-gamma distribution with parameters $r > 0$, $\theta \in \mathbb{R}$, $\sigma > 0$, $\mu \in \mathbb{R}$ (denoted $\mathrm{VG}(r, \theta, \sigma, \mu)$) if it has probability density function given by

$$p_{VG}(x; r, \theta, \sigma, \mu) = \frac{1}{\sigma\sqrt{\pi}\Gamma(\frac{r}{2})}\exp\left(\frac{\theta}{\sigma^2}(x-\mu)\right)\left(\frac{|x-\mu|}{2\sqrt{\theta^2+\sigma^2}}\right)^{\frac{r-1}{2}}$$
$$\times K_{\frac{r-1}{2}}\left(\frac{\sqrt{\theta^2+\sigma^2}}{\sigma^2}|x-\mu|\right),$$

where $x \in \mathbb{R}$ and $K_\nu(\cdot)$ is a modified Bessel function of the second kind [3, 6]. We have $\mathbb{E}(V) = \mu + r\theta$, $\mathrm{Var}(V) = r(\sigma^2 + 2\theta^2)$ [2].

Remark 3.3. If (X, Y) denote a bivariate normal random vector with zero means, variances (σ_x^2, σ_x^2) and correlation coefficient ρ, then $Z = XY$ follows $\mathrm{VG}(1, \rho\sigma_x\sigma_y, \sigma_x\sigma_y\sqrt{1-\rho^2}, 0)$ [3]. In particular, based on Remark 3.2, $\mathbb{E}(Z) =$

$\rho\sigma_x\sigma_y$, $\mathrm{Var}(Z) = (1+\rho^2)\sigma_x^2\sigma_y^2$ and $\mathbb{E}(Z^2) = (1+2\rho^2)\sigma_x^2\sigma_y^2$, which could also be obtained using direct calculation with bivariate density.

Remark 3.4. Based on Remark 3.3, for the bivariate Brownian motion (X_t, Y_t) mentioned in the first paragraph, we have $\mathbb{E}(X_tY_t) = t\rho_t$, $\mathrm{Var}(X_tY_t/t^2) = (1+\rho_t^2)/t^2 \le 2/t^2$, $\mathbb{E}(X_tY_t)^2/t^4 = [(1+\rho_t^2)t^2 + \rho_t^2t^2]/t^4 \le 3t^{-2}$ and for $s < t$,

$$\begin{aligned}\mathbb{E}\left(\frac{X_tY_tX_sY_s}{t^2s^2}\right) &= \frac{1}{t^2s^2}\{\mathbb{E}[(X_t-X_s)(Y_t-Y_s)X_sY_s] + \mathbb{E}[(Y_t-Y_s)X_s^2Y_s] \\ &\qquad + \mathbb{E}[(X_t-X_s)X_sY_s^2] + \mathbb{E}[X_s^2Y_s^2]\} \\ &\le \frac{C_1(t-s)s + C_2(t-s)^{1/2}s^{3/2} + C_3s^2}{t^2s^2} \\ &\le C\left(\frac{1}{ts} + \frac{\sqrt{1-(s/t)}}{t^{3/2}s^{1/2}} + \frac{1}{t^2}\right) \le \frac{C}{s^2}, \text{ since } 1 \le s < t.\end{aligned} \tag{3.1}$$

Here C_1, C_2, C_3 and C are generic finite positive constants.

In order to get the inequality in (3.1), we first note that X_t and Y_t may be written as

$$X_t = \rho_tY_t + \sqrt{1-\rho_t^2}X_t^{(1)},\ Y_t = \rho_tX_t + \sqrt{1-\rho_t^2}Y_t^{(1)}, \tag{3.2}$$

where $\{X_t^{(1)}\}$ and $\{Y_t^{(1)}\}$ are mean zero Brownian motions with $\{X_t^{(1)}\}$ independent of $\{Y_t\}$ and $\{Y_t^{(1)}\}$ independent of $\{X_t\}$. Using (3.2) we may write

$$\begin{aligned}(X_t - X_s)X_s =&(\rho_t\rho_s)(Y_t-Y_s)Y_s + (\rho_t-\rho_s)Y_s^2 \\ &+ \sqrt{1-\rho_t^2}(X_t^{(1)} - X_s^{(1)})Y_s + \left[\sqrt{1-\rho_t^2} - \sqrt{1-\rho_s^2}\right]X_s^{(1)}Y_s.\end{aligned} \tag{3.3}$$

so that

$$\begin{aligned}(X_t-X_s)(Y_t-Y_s)X_sY_s \le&(Y_t-Y_s)^2Y_s^2 + 2|Y_t-Y_s||Y_s|^3 \\ &+ 2|X_t^{(1)} - X_s^{(1)}||Y_t-Y_s|Y_s^2 + 2|X_s^{(1)}||Y_t-Y_s|Y_s^2.\end{aligned}$$

Hence, using the fact that for a Brownian motion $\{B_u\}$, $\mathbb{E}|B_u|^\alpha \le Ku^{\alpha/2}$,

$$\mathbb{E}[(X_t-X_s)(Y_t-Y_s)X_sY_s] \le C_1(t-s)s + C_2\sqrt{(t-s)s^3}.$$

Using (3.3) again, we will get

$$\mathbb{E}[(X_t-X_s)X_sY_s^2] \le C_2\sqrt{(t-s)s^3}$$

and similarly

$$\mathbb{E}[(Y_t-Y_s)X_s^2Y_s] \le C_2\sqrt{(t-s)s^3}.$$

We also have $\mathbb{E}(X_s^2Y_s^2) \le 4s^2$. Putting all these together we get the inequality in (3.1). This completes Remark 3.4.

We denote $S_a = \sum_{n=1}^\infty n^{-a}$ for $a > 0$; $S_a < \infty$ for $a > 1$. Although the primary focus of this subsection and the main result is for the case $p > q = 1/2$, we have the following result for $p = q = 0$.

Lemma 3.5. *$\mathrm{Var}(\hat{\gamma}_i^{0,0}) \to 0$, $\mathrm{Var}([\hat{\sigma}_i^{x,0,0}]^2) \to 0$ and $\mathrm{Var}([\hat{\sigma}_i^{y,0,0}]^2) \to 0$, for each i, as $T \to \infty$.*

Proof.

$$\begin{aligned}\mathrm{Var}(\hat{\gamma}_i^{0,0}) &\le \mathbb{E}(\hat{\gamma}_i^{0,0})^2 \\ &= \frac{1}{(T-1)^2}\sum_{\substack{j=1\\ j\neq i}}^{T}\frac{\mathbb{E}[(X_i-X_j)(Y_i-Y_j)]^2}{(i-j)^4} \qquad (3.4)\\ &+ \frac{1}{(T-1)^2}\sum_{\substack{j=1\\ j\neq i}}^{T}\sum_{\substack{k=1\\ k\neq i,j}}^{T}\frac{\mathbb{E}[(X_i-X_j)(Y_i-Y_j)(X_i-X_k)(Y_i-Y_k)]}{(i-j)^2(i-k)^2}\end{aligned}$$

Using one of the facts mentioned in Remark 3.4 in the inequality below,

$$\begin{aligned}1^{st}\text{ term } &= \frac{1}{(T-1)^2}\left\{\sum_{j=1}^{i-1}\frac{\mathbb{E}(X_{i-j}Y_{i-j})^2}{(i-j)^4}+\sum_{j=i+1}^{T}\frac{\mathbb{E}(X_{j-i}Y_{j-i})^2}{(j-i)^4}\right\}\\ &\le \frac{CS_2}{(T-1)^2} = O(T^{-2}),\end{aligned}$$

where C is a finite positive constant and we also used the property $X_{u+h}-X_u \overset{d}{=} X_h$ for Brownian motion. As k ranges within the inner sum in the second term in (3.4), there are L values for which $(i-j)^2 < (i-k)^2$, for some fixed L in $\{1,\dots,T\}$, and for the remaining $T-L-2$ values $(i-j)^2 > (i-k)^2$. So, applying another fact mentioned in Remark 3.4, we get,

$$2^{nd}\text{ term } \le \frac{1}{(T-1)^2}\sum_{\substack{j=1\\ j\neq i}}^{T}\left\{\frac{L}{(i-j)^2}+C_1S_2\right\} \le \frac{(C_2+C_1T)S_2}{(T-1)^2} = O(T^{-1}),$$

where C_1 and C_2 are finite constants. Putting this together, we get $\mathrm{Var}(\hat{\gamma}_i^{0,0}) \to 0$ as $T\to\infty$. Using similar arguments it is easy to prove the remaining statements in the lemma. □

Lemma 3.6. $\mathrm{Var}(\hat{\gamma}_i^{q,p}) \to 0$*,* $\mathrm{Var}([\hat{\sigma}_i^{x,q,p}]^2) \to 0$ *and* $\mathrm{Var}([\hat{\sigma}_i^{y,q,p}]^2) \to 0$*, for each i and for each q and p with $0 < q \le 1/2$, $p > 1/2$, as $T\to\infty$.*

Proof.

$$\begin{aligned}\mathrm{Var}(\hat{\gamma}_i^{q,p}) \le{}& \frac{1}{(T-1)^2}\sum_{\substack{j=1\\ j\neq i}}^{T}\frac{\mathbb{E}[(j^qX_i-X_jj^{-p})(j^qY_i-Y_jj^{-p})]^2}{(i-j)^4} \qquad (3.5)\\ &+\frac{1}{(T-1)^2}\sum_{\substack{j=1\\ j\neq i}}^{T}\sum_{\substack{k=1\\ k\neq i,j}}^{T}\mathbb{E}\left[\frac{(j^qX_i-X_jj^{-p})(j^qY_i-Y_jj^{-p})}{(i-j)^2}\right.\\ &\qquad\left.\times\frac{(k^qX_i-X_kk^{-p})(k^qY_i-Y_kk^{-p})}{(i-k)^2}\right]\end{aligned}$$

and

$$\begin{aligned}[(j^q X_i - X_j j^{-p})(j^q Y_i - Y_j j^{-p})]^2 \\ &= [j^{2q}(X_i - X_j)^2 + 2X_j(X_i - X_j)j^q(j^q - j^{-p}) + X_j^2(j^q - j^{-p})^2] \\ &\quad \times [j^{2q}(Y_i - Y_j)^2 + 2Y_j(Y_i - Y_j)j^q(j^q - j^{-p}) + Y_j^2(j^q - j^{-p})^2].\end{aligned}$$

Expanding and taking expectations, many terms become zero by applying properties of Brownian motion. With the remaining non-zero terms, the first term in (3.5) becomes

$$\frac{1}{(T-1)^2}\sum_{\substack{j=1\\ j\neq i}}^{T}(j-i)^{-4}\left\{\mathbb{E}[j^{4q}(X_j - X_i)^2(Y_j - Y_i)^2]\right. \tag{3.6}$$

$$+\ \mathbb{E}[j^{2q}(X_j - X_i)^2]\mathbb{E}[Y_j^2(j^q - j^{-p})^2] \tag{3.7}$$

$$+\ 4\mathbb{E}[X_jY_j]\mathbb{E}[(X_j - X_i)(Y_j - Y_i)j^{2q}(j^q - j^{-p})^2] \tag{3.8}$$

$$+\ \mathbb{E}[X_j^2]\mathbb{E}[(Y_j - Y_i)^2 j^{2q}(j^q - j^{-p})^2] \tag{3.9}$$

$$\left.+\ \mathbb{E}[X_j^2Y_j^2(j^q - j^{-p})^4]\right\}. \tag{3.10}$$

We will now show that the product of $(j-i)^{-4}$ and each expectation term in (3.6) to (3.10) can be written as $C_1 + C_2T$, for some finite constants C_1 and C_2 so that this product when multiplied by $(T-1)^{-2}$ is $O(T^{-1})$. Using Remark 3.4, for $j > i$,

$$\begin{aligned}\frac{j^{4q}\mathbb{E}[(X_j - X_i)^2(Y_j - Y_i)^2]}{(j-i)^4} &\leq \frac{3j^{4q}}{(j-i)^2} < \frac{3j^2}{(j-i)^2} \text{ if } 4q \leq 2;\ \text{i.e. if } q \leq 1/2 \\ &\leq 3\left(1 + \frac{2i}{(j-i)} + \frac{i^2}{(j-i)^2}\right) < 12,\ \text{if } j > 2i.\end{aligned} \tag{3.11}$$

Since the first $2i$ terms in the sum in (3.6) is a constant, it is easy to see based on (3.11) that the whole sum can be bounded by $C_1 + C_2T$, for some finite constants C_1 and C_2.

$$\frac{j^{2q}(j^q - j^{-p})^2\mathbb{E}(X_j - X_i)^2\mathbb{E}(Y_j^2)}{(j-i)^4} < \frac{j^{2q+1}(j^{2q}+1)}{(j-i)^3}$$

can be bounded by a constant if $4q + 1 \leq 3$; i.e. if $q \leq 1/2$. Hence the sum corresponding to (3.7) can be expressed as $C_1 + C_2T$. Similar reasoning can be applied to sums in (3.8), (3.9) and (3.10) so that the first term in (3.5) is $O(T^{-1})$.

Expanding the second term in (3.5) gives 16 terms. The double-sum ranging over j and k for each of these 16 terms can be expressed as $[C_1S_a + C_2]S_bT$ for some $a > 1$ and $b > 1$, so that after multiplication with $(T-1)^{-2}$ each term becomes $O(T^{-1})$. For example, the last term will be

$$\frac{\mathbb{E}(X_jY_jj^{-2p}X_kY_kk^{-2p})}{(j-i)^2(k-i)^2} \leq \frac{j^{-2p+2}k^{-2p+2}}{(j-i)^2(k-i)^2} \times \frac{3}{j^2},\ \text{for } k > j > i.$$

$$\text{The sum over } k \text{ of the above terms} < \frac{j^{-2p}}{(j-i)^2}(C + S_{2p}),$$

$$\text{and further sum over } j < [C_2 + C_1S_{2p}]S_{2p+2}T.$$

Proceeding similarly with other terms, the 2^{nd} term in (3.5) is also seen to be $O(T^{-1})$ and hence $\mathrm{Var}(\hat{\gamma}_i^{q,p}) \to 0$ as $T \to \infty$. Using similar arguments the remaining statements in the lemma can also be proved. □

Lemma 3.7. *For each* $t \in \{1, \ldots, n(=T)\}$,

$$Q_t = \mathbb{E}(\hat{\gamma}_t^{q,p}) / \sqrt{\mathbb{E}([\hat{\sigma}_t^{x,q,p}]^2)\mathbb{E}([\hat{\sigma}_t^{y,q,p}]^2)} \to \rho_t$$

as $T \to \infty$, *if* $p > q \geq 1/2$.

Proof.

$$\begin{aligned}(T-1)\mathbb{E}(\hat{\gamma}_t^{q,p}) &= t\rho_t \sum_{\substack{s=1\\s\neq t}}^{T} \frac{s^{2q}}{(s-t)^2} + \sum_{\substack{s=1\\s\neq t}}^{T} \frac{\rho_s s^{-2p+1}}{(s-t)^2} \\ &\quad - 2\left[\sum_{s=1}^{t-1} \frac{\rho_s s^{-p+q+1}}{(t-s)^2} + \sum_{s=t+1}^{T} \frac{t\rho_t s^{-p+q}}{(s-t)^2}\right] \\ &= t\rho_t \sum_{\substack{s=1\\s\neq t}}^{T} \frac{s^{2q}}{(s-t)^2} + \sum_{\substack{s=1\\s\neq t}}^{T} \frac{\rho_s s^{-2p+1}}{(s-t)^2} \\ &\quad - 2\sum_{\substack{s=1\\s\neq t}}^{T} \frac{\rho_s s^{-p+q+1}}{(s-t)^2} + 2\sum_{s=t+1}^{T} \frac{s^{-p+q}(s\rho_s - t\rho_t)}{(s-t)^2}.\end{aligned}$$

Similarly,

$$\begin{aligned}(T-1)\mathbb{E}([\hat{\sigma}_t^{x,p}]^2) = (T-1)\mathbb{E}([\hat{\sigma}_t^{y,p}]^2) &= t\sum_{\substack{s=1\\s\neq t}}^{T} \frac{s^{2q}}{(s-t)^2} + \sum_{\substack{s=1\\s\neq t}}^{T} \frac{s^{-2p+1}}{(s-t)^2} \\ &\quad - 2\sum_{\substack{s=1\\s\neq t}}^{T} \frac{s^{-p+q+1}}{(s-t)^2} + 2\sum_{s=t+1}^{T} \frac{s^{-p+q}(s-t)}{(s-t)^2}.\end{aligned}$$

Hence Q_t can be written as a ratio

$$Q_t = \frac{t\rho_t + K_T^{-1}A_{1,T} - 2K_T^{-1}A_{2,T} + 2K_T^{-1}A_{3,T}}{t + K_T^{-1}B_{1,T} - 2K_T^{-1}B_{2,T} + 2K_T^{-1}B_{3,T}}, \tag{3.12}$$

where

$$K_T = \sum_{\substack{s=1\\s\neq t}}^{T} \frac{s^{2q}}{(s-t)^2},\ A_{1,T} = \sum_{\substack{s=1\\s\neq t}}^{T} \frac{\rho_s s^{-2p+1}}{(s-t)^2},$$

$$A_{2,T} = \sum_{\substack{s=1\\s\neq t}}^{T} \frac{\rho_s s^{-p+q+1}}{(s-t)^2},\ A_{3,T} = \sum_{s=t+1}^{T} \frac{s^{-p+q}(s\rho_s - t\rho_t)}{(s-t)^2},$$

$$B_{1,T} = \sum_{\substack{s=1\\s\neq t}}^{T} \frac{s^{-2p+1}}{(s-t)^2},\ B_{2,T} = \sum_{\substack{s=1\\s\neq t}}^{T} \frac{s^{-p+q+1}}{(s-t)^2},\ B_{3,T} = \sum_{s=t+1}^{T} \frac{s^{-p+q}(s-t)}{(s-t)^2}.$$

K_T diverges as $T \to \infty$ for $2q \geq 1$, and all other sums ($A_{1,T}$, $A_{2,T}$, $A_{3,T}$, $B_{1,T}$, $B_{2,T}$ and $B_{3,T}$) converge to finite constants in passage to the limit when $p > q \geq 1/2$. Thus by (3.12), $Q_t \to \rho_t$ as $T \to \infty$ as desired. □

Proof of Theorem 3.1: Note that the only value of q for which *both* Lemma 3.6 and Lemma 3.7 hold, is $q = 1/2$. Using Lemma 3.6, Chebychev's inequality, Slutsky's lemma, Continuous mapping theorem [8] and part (iii) of Theorem 2.7 in [8], p.10, we get $\hat{\rho}_u^{q,p} \to \mathbb{E}(\hat{\gamma}_u^{q,p})/\sqrt{\mathbb{E}([\hat{\sigma}_u^{x,q,p}]^2)\mathbb{E}([\hat{\sigma}_u^{y,q,p}]^2)}$ in probability for each u and $p > q = 1/2$; this fact combined with Lemma 3.7 proves the theorem.

3.2. Weak consistency of dynamic correlation estimators in the Geometric Brownian Motion case. The main results in this subsection are the following two theorems. The strategy for proofs in this subsection is the same as that used in the Brownian motion case; only the details in the calculations differ.

Theorem 3.8. *$\hat{\rho}_t^{a,b,c} \to \rho_t$ in probability for each t, $c > a > 0$ and $b > a + 10$, as $T \to \infty$, where $\hat{\rho}_t^{a,b,c} = \hat{\gamma}_t/(\hat{\sigma}_{t,W}\hat{\sigma}_{t,U})$ with $\hat{\gamma}_t$ given in (2.2) and $\hat{\sigma}_{t,W}^2$, $\hat{\sigma}_{t,U}^2$ the corresponding estimates of the variances.*

Theorem 3.9. *$\hat{\rho}_t^{a,b,c} \to \rho_t$ in probability for each t, $b > 15$ and $c > a > 0$, as $T \to \infty$, where $\hat{\rho}_t^{a,b,c} = \hat{\gamma}_t/(\hat{\sigma}_{t,W}\hat{\sigma}_{t,U})$ with $\hat{\gamma}_t$ given in (2.3) and $\hat{\sigma}_{t,W}^2$, $\hat{\sigma}_{t,U}^2$ the corresponding estimates of the variances.*

We first consider Theorem 3.8 and begin with a lemma similar to Lemma 3.6.

Lemma 3.10. *For $\hat{\gamma}_t$ given in (2.2), and the corresponding estimates of the variances $\hat{\sigma}_{t,W}^2$ and $\hat{\sigma}_{t,U}^2$, $\mathrm{Var}(\hat{\gamma}_t) \to 0$, $\mathrm{Var}([\hat{\sigma}_{t,W}]^2) \to 0$ and $\mathrm{Var}([\hat{\sigma}_{t,U}]^2) \to 0$, for each t and for $c > a > 0$ and $b > a + 10$, as $T \to \infty$.*

Based on (2.2)

$$\begin{aligned}
\hat{\gamma}_t = \frac{1}{e^{c\sigma^2 T}} \sum_{k=1}^{T} \Big\{ &\Big[e^{\frac{-b\sigma^2 k}{2}} \Big(e^{\sigma W_k} - e^{\frac{\sigma^2 k}{2}} \Big) - e^{\frac{a\sigma^2 k}{2}} \Big(e^{\sigma W_t} - e^{\frac{\sigma^2 t}{2}} \Big) \Big] \\
&\times \Big[e^{\frac{-b\sigma^2 k}{2}} \Big(e^{\sigma U_k} - e^{\frac{\sigma^2 k}{2}} \Big) - e^{\frac{a\sigma^2 k}{2}} \Big(e^{\sigma U_t} - e^{\frac{\sigma^2 t}{2}} \Big) \Big] \Big\} \\
&= G_1 - G_2 - G_3 + G_4,
\end{aligned}$$

where

$$\begin{aligned}
G_1 &= e^{-c\sigma^2 T} \sum_{k=1}^{T} \Big[e^{-b\sigma^2 k} \Big(e^{\sigma W_k} - e^{\frac{\sigma^2 k}{2}} \Big) \Big(e^{\sigma U_k} - e^{\frac{\sigma^2 k}{2}} \Big) \Big], \\
G_2 &= e^{-c\sigma^2 T} \sum_{k=1}^{T} \Big[e^{(a-b)\sigma^2 k/2} \Big(e^{\sigma W_k} - e^{\frac{\sigma^2 k}{2}} \Big) \Big(e^{\sigma U_t} - e^{\frac{\sigma^2 t}{2}} \Big) \Big], \\
G_3 &= e^{-c\sigma^2 T} \sum_{k=1}^{T} \Big[e^{(a-b)\sigma^2 k/2} \Big(e^{\sigma W_t} - e^{\frac{\sigma^2 t}{2}} \Big) \Big(e^{\sigma U_k} - e^{\frac{\sigma^2 k}{2}} \Big) \Big], \\
G_4 &= e^{-c\sigma^2 T} \sum_{k=1}^{T} \Big[e^{a\sigma^2 k} \Big(e^{\sigma W_t} - e^{\frac{\sigma^2 t}{2}} \Big) \Big(e^{\sigma U_t} - e^{\frac{\sigma^2 t}{2}} \Big) \Big].
\end{aligned}$$

In order to show that $\mathrm{Var}(\hat{\gamma}_t) \to 0$, it suffices to show that $\mathbb{E}(\hat{\gamma}_t^2) \to 0$. Now

$$\begin{aligned}\hat{\gamma}_t^2 &= G_1^2 + G_2^2 + G_3^2 + G_4^2 - 2G_1G_2 \\ &\quad - 2G_1G_3 + 2G_1G_4 + 2G_2G_3 - 2G_2G_4 - 2G_3G_4.\end{aligned} \tag{3.13}$$

We will show that the expectation of each of the 10 terms in the right-hand side of (3.13) converges to zero as $T \to \infty$ for the range of a, b and c assumed in the statement of Lemma 3.10, thereby showing that $\mathbb{E}(\hat{\gamma}_t^2) \to 0$, as $T \to \infty$. The proof is lengthy but the calculations involved are very routine and repetitious. Hence we relegate the proof of Lemma 3.10 to the appendix. Next we state and prove a lemma similar to Lemma 3.7.

Lemma 3.11. *For $\hat{\gamma}_t$ given in (2.2), and the corresponding estimates of the variances $\hat{\sigma}_{t,W}^2$ and $\hat{\sigma}_{t,U}^2$, $\mathbb{E}(\hat{\gamma}_t)/\sqrt{\mathbb{E}(\hat{\sigma}_{t,W}^2)\mathbb{E}(\hat{\sigma}_{t,U}^2)} \to \rho_t$, as $T \to \infty$, if $b > a > 0$.*

Proof. $e^{c\sigma^2 T}\hat{\gamma}_t = \sum_{k=1}^{t-1}\{\cdot\} + \sum_{k=t}^{T}\{\cdot\}$ where

$$\begin{aligned}\{\cdot\} &= \left[e^{-b\sigma^2 k/2}(e^{\sigma W_k} - e^{\sigma^2 k/2}) - e^{a\sigma^2 k/2}(e^{\sigma W_t} - e^{\sigma^2 t/2})\right] \\ &\quad \times \left[e^{-b\sigma^2 k/2}(e^{\sigma U_k} - e^{\sigma^2 k/2}) - e^{a\sigma^2 k/2}(e^{\sigma U_t} - e^{\sigma^2 t/2})\right] \\ &= e^{-b\sigma^2 k}\left[e^{\sigma(W_k+U_k)} - e^{\sigma^2 k/2}e^{\sigma W_k} - e^{\sigma^2 k/2}e^{\sigma U_k} + e^{\sigma^2 k}\right] \\ &\quad - e^{(a-b)\sigma^2 k/2}\left[e^{\sigma(W_k+U_t)} - e^{\sigma^2 k/2}e^{\sigma U_t} - e^{\sigma^2 t/2}e^{\sigma W_k} + e^{\sigma^2(k+t)/2}\right] \\ &\quad - e^{(a-b)\sigma^2 k/2}\left[e^{\sigma(U_k+W_t)} - e^{\sigma^2 t/2}e^{\sigma U_k} - e^{\sigma^2 k/2}e^{\sigma W_t} + e^{\sigma^2(k+t)/2}\right] \\ &\quad + e^{a\sigma^2 k}\left[e^{\sigma(W_t+U_t)} - e^{\sigma^2 t/2}e^{\sigma W_t} - e^{\sigma^2 t/2}e^{\sigma U_t} + e^{\sigma^2 t}\right].\end{aligned}$$

We recall

$$\begin{aligned}\mathbb{E}(e^{\sigma(W_k+U_k)}) &= e^{\sigma^2 k}[1 + \rho_k(e^{\sigma^2 k} - 1)], \\ \mathbb{E}(e^{\sigma(W_t+U_t)}) &= e^{\sigma^2 t}[1 + \rho_t(e^{\sigma^2 t} - 1)], \\ W_k &= \rho_k U_k + \sqrt{1-\rho_k^2}M_k^{(1)} \\ \mathbb{E}(e^{\sigma(W_k+U_t)}) &= \mathbb{E}\left(\exp\{\sigma[\rho_k U_k + U_t + \sqrt{1-\rho_k^2}M_k^{(1)}]\}\right)\end{aligned}$$

so that when $k \geq t$,

$$\begin{aligned}\mathbb{E}(e^{\sigma(W_k+U_t)}) &= \mathbb{E}(e^{\sigma\rho_k(U_k-U_t)})\mathbb{E}(e^{\sigma(1+\rho_k)U_t})\mathbb{E}(e^{\sigma\sqrt{1-\rho_k^2}M_k^{(1)}}) \\ &= e^{\frac{\sigma^2}{2}[\rho_k^2(k-t)+(1+\rho_k)^2 t+(1-\rho_k^2)k]} = e^{\sigma^2(k+t)/2}e^{\sigma^2\rho_k t} \leq e^{\sigma^2 k/2}e^{3\sigma^2 t/2}.\end{aligned}$$

The same relationship as above holds for $\mathbb{E}(e^{\sigma(U_k+W_t)})$ also. When $k \leq t$,

$$\begin{aligned}\mathbb{E}(e^{\sigma(W_k+U_t)}) &= \mathbb{E}(e^{\sigma\rho_k(U_t-U_k)})\mathbb{E}(e^{\sigma(1+\rho_k)U_k})\mathbb{E}(e^{\sigma\sqrt{1-\rho_k^2}M_k^{(1)}}) \\ &= e^{\frac{\sigma^2}{2}[\rho_k^2(t-k)+(1+\rho_k)^2 t+(1-\rho_k^2)k]} = e^{\sigma^2(k+t)/2}e^{\sigma^2\rho_k k} \leq e^{\sigma^2 t/2}e^{3\sigma^2 k/2};\end{aligned}$$

again the same relationship holding for $\mathbb{E}(e^{\sigma(U_k+W_t)})$ also. Hence

$$\begin{aligned}
e^{c\sigma^2 T}\mathbb{E}(\hat{\gamma}_t) &= \sum_{k=1}^{T} e^{-b\sigma^2 k}\{e^{\sigma^2 k}[1+\rho_k(e^{\sigma^2 k}-1)] - e^{\sigma^2 k}\} \\
&+ \sum_{k=1}^{T} e^{a\sigma^2 k}\{e^{\sigma^2 t}[1+\rho_t(e^{\sigma^2 t}-1)] - e^{\sigma^2 t}\} \\
&- 2\sum_{k=1}^{t-1} e^{(a-b)\sigma^2 k/2}\{e^{\sigma^2(k+t)/2}[e^{\sigma^2\rho_k k}-1]\} \\
&- 2\sum_{k=t}^{T} e^{(a-b)\sigma^2 k/2}\{e^{\sigma^2(k+t)/2}[e^{\sigma^2\rho_k t}-1]\} \\
&= \left\{\sum_{k=1}^{T}\rho_k[e^{(2-b)\sigma^2 k} - e^{(1-b)\sigma^2 k}]\right\} + \rho_t(e^{\sigma^2 t}-1)e^{(a+t)\sigma^2}\left[\frac{e^{a\sigma^2 T}-1}{e^{a\sigma^2}-1}\right] \\
&\quad - 2\sum_{k=1}^{t-1} e^{(a-b)\sigma^2 k/2}\{e^{\sigma^2(k+t)/2}[e^{\sigma^2\rho_k k} - e^{\sigma^2\rho_k t}]\} \\
&\quad - 2\sum_{k=1}^{T} e^{(a-b)\sigma^2 k/2}\{e^{\sigma^2(k+t)/2}[e^{\sigma^2\rho_k t}-1]\} \\
&= A_T + D_T\rho_t - 2B_t - 2C_T,
\end{aligned}$$

where B_t is a finite sum of $(t-1)$ terms that does not depend on T, A_T and C_T converge to finite constants as $T \to \infty$ since we assume $b > a$, and

$$D_T = e^{(a+t)\sigma^2}(e^{\sigma^2 t}-1)\left[\frac{e^{a\sigma^2 T}-1}{e^{a\sigma^2}-1}\right] \text{ diverges as } T \to \infty.$$

Similarly, $e^{c\sigma^2 T}\hat{\sigma}^2_{t,W} = \sum_{k=1}^{t-1}\{\cdot\} + \sum_{k=t}^{T}\{\cdot\}$ where

$$\begin{aligned}
\{\cdot\} &= \left[e^{-b\sigma^2 k/2}(e^{\sigma W_k} - e^{\sigma^2 k/2}) - e^{a\sigma^2 k/2}(e^{\sigma W_t} - e^{\sigma^2 t/2})\right]^2 \\
&= e^{-b\sigma^2 k}(e^{\sigma W_k} - e^{\sigma^2 k/2})^2 + e^{a\sigma^2 k}(e^{\sigma W_t} - e^{\sigma^2 t/2})^2 \\
&\quad - 2e^{(a-b)\sigma^2 k/2}(e^{\sigma W_k} - e^{\sigma^2 k/2})(e^{\sigma W_t} - e^{\sigma^2 t/2}).
\end{aligned}$$

We have $\mathbb{E}(e^{\sigma W_k} - e^{\sigma^2 k/2})^2 = \mathbb{E}[e^{2\sigma W_k} - 2e^{\sigma W_k}e^{\sigma^2 k/2} + e^{\sigma^2 k}] = e^{2\sigma^2 k} - e^{\sigma^2 k} = e^{\sigma^2 k}(e^{\sigma^2 k}-1)$ and $\mathbb{E}(e^{\sigma W_t} - e^{\sigma^2 t/2})^2 = e^{\sigma^2 t}(e^{\sigma^2 t}-1)$.

When $k \geq t$,

$$\mathbb{E}(e^{\sigma(W_k+W_t)}) = \mathbb{E}(e^{\sigma(W_k-W_t)})\mathbb{E}(e^{2\sigma W_t}) = \exp\{\frac{\sigma^2}{2}(k-t) + 2\sigma^2 t\} = e^{\sigma^2(k+t)/2}e^{\sigma^2 t},$$

and when $k \leq t$,

$$\mathbb{E}(e^{\sigma(W_k+W_t)}) = e^{\sigma^2(k+t)/2}e^{\sigma^2 k}.$$

Putting it all together,

$$\begin{aligned} e^{c\sigma^2 T}\mathbb{E}(\hat{\sigma}^2_{t,W}) &= \sum_{k=1}^{T}[e^{(2-b)\sigma^2 k} - e^{(1-b)\sigma^2 k}] + e^{\sigma^2 t}(e^{\sigma^2 t}-1)e^{a\sigma^2}\left[\frac{e^{a\sigma^2 T}-1}{e^{a\sigma^2}-1}\right] \\ &\quad - 2\sum_{k=1}^{t-1} e^{(a-b)\sigma^2 k/2} e^{\sigma^2(k+t)/2}[e^{\sigma^2 k} - e^{\sigma^2 t}] \\ &\quad - 2e^{\sigma^2 t/2}(e^{\sigma^2 t}-1)\sum_{k=1}^{T} e^{\sigma^2 k(a-b+1)/2}. \end{aligned}$$

Hence $e^{c\sigma^2 T}\mathbb{E}(\hat{\sigma}^2_{t,W})$ can be written as $E_T + D_T - 2F_t - 2G_T$, where D_T is same as the one given further above, F_t is a finite sum that does not depend on T and E_T and G_T converges to finite constants as $T \to \infty$. The same result holds for $\mathbb{E}(\hat{\sigma}^2_{t,U})$ also. Hence

$$\begin{aligned} \mathbb{E}(\hat{\gamma}_t)/\sqrt{\mathbb{E}(\hat{\sigma}^2_{t,W})\mathbb{E}(\hat{\sigma}^2_{t,U})} &= \frac{\rho_t + A_T/D_T - 2B_t/D_T - 2C_T/D_T}{1 + E_T/D_T - 2F_t/D_T - 2G_T/D_T} \\ &\to \rho_t, \text{ as } T \to \infty, \end{aligned}$$

since D_T diverges as $T \to \infty$, B_t and F_t are finite terms that do not depend on T and A_T, C_T, E_T and G_T are all finite constants in the passage to the limit. This proves the lemma. □

Lemmas 3.10 and 3.11 suffices to prove Theorem 3.8. Similarly lemmas 3.12 and 3.13 given below will suffice to prove Theorem 3.9.

Lemma 3.12. *For $\hat{\gamma}_t$ given in (2.3), and the corresponding estimates of the variances $\hat{\sigma}^2_{t,W}$ and $\hat{\sigma}^2_{t,U}$, $\mathrm{Var}(\hat{\gamma}_t) \to 0$, $\mathrm{Var}([\hat{\sigma}_{t,W}]^2) \to 0$ and $\mathrm{Var}([\hat{\sigma}_{t,U}]^2) \to 0$, for each t and for $c > a > 0$ and $b > 15$, as $T \to \infty$.*

Proof.

$$\begin{aligned} \hat{\gamma}_t^2 = e^{-2c\sigma^2 T}\sum_{k=1}^{T}\sum_{j=1}^{T}\Big\{ & e^{a\sigma^2(k+j)}(e^{\sigma W_t} - e^{\sigma^2 t/2})^2(e^{\sigma U_t} - e^{\sigma^2 t/2})^2 \\ & - e^{a\sigma^2 k}e^{-b\sigma^2 j}(e^{\sigma W_t} - e^{\sigma^2 t/2})(e^{\sigma U_t} - e^{\sigma^2 t/2})(e^{\sigma W_j} - e^{\sigma^2 j/2})(e^{\sigma U_j} - e^{\sigma^2 j/2}) \\ & - e^{a\sigma^2 j}e^{-b\sigma^2 k}(e^{\sigma W_k} - e^{\sigma^2 k/2})(e^{\sigma U_k} - e^{\sigma^2 k/2})(e^{\sigma W_t} - e^{\sigma^2 t/2})(e^{\sigma U_t} - e^{\sigma^2 t/2}) \\ & + e^{-b\sigma^2(k+j)}(e^{\sigma W_k} - e^{\sigma^2 k/2})(e^{\sigma U_k} - e^{\sigma^2 k/2})(e^{\sigma W_j} - e^{\sigma^2 j/2})(e^{\sigma U_j} - e^{\sigma^2 j/2})\Big\} \end{aligned}$$

It is easy to see that when taking expectations, the first term converges to zero as $T \to \infty$ if $c > a$. The expectations of the second and third terms are the same. We exhibit only the calculation for the third term. Within the third term if we expand

$$(e^{\sigma W_k} - e^{\sigma^2 k/2})(e^{\sigma U_k} - e^{\sigma^2 k/2})(e^{\sigma W_t} - e^{\sigma^2 t/2})(e^{\sigma U_t} - e^{\sigma^2 t/2})$$

there will be 16 terms. We exhibit only the calculation for the term containing $e^{\sigma(W_k+U_k+W_t+U_t)}$ since similar calculations apply to the remaining 15 terms.

Recalling again $W_k = \rho_k U_k + \sqrt{1-\rho_k^2} M_k^{(1)}$ we have

$$\begin{aligned} W_k + U_k + W_t + U_t &\leq (1+\rho_k)U_k + \sqrt{1-\rho_k^2} M_k^{(1)} + (1+\rho_t)U_t + \sqrt{1-\rho_t^2} M_t^{(1)} \\ &\leq 2|U_k| + |M_k^{(1)}| + 2|U_t| + |M_t^{(1)}| \end{aligned}$$

so that when $k \geq t$,

$$\begin{aligned} &W_k + U_k + W_t + U_t \leq 2|U_k - U_t| + 4|U_t| + |M_k^{(1)} - M_t^{(1)}| + 2|M_t^{(1)}| \\ &\mathbb{E}(e^{\sigma(W_k+U_k+W_t+U_t)}) \leq C \exp\{\frac{\sigma^2}{2}[4(k-t) + 16t + (k-t) + 4t]\} \leq Ce^{5\sigma^2 k/2} \end{aligned}$$

and when $k \leq t$,

$$\begin{aligned} &W_k + U_k + W_t + U_t \leq 2|U_t - U_k| + 4|U_k| + |M_t^{(1)} - M_k^{(1)}| + 2|M_k^{(1)}| \\ &\mathbb{E}(e^{\sigma(W_k+U_k+W_t+U_t)}) \leq C \exp\{\frac{\sigma^2}{2}[4(t-k) + 16k + (t-k) + 4k]\} \leq Ce^{15\sigma^2 k/2}. \end{aligned}$$

Hence the expectation of the term containing $e^{\sigma(W_k+U_k+W_t+U_t)}$ in the third term is

$$\begin{aligned} &\leq Ce^{-2c\sigma^2 T} \sum_{k=1}^{T} \sum_{j=1}^{T} e^{a\sigma^2 j} e^{-b\sigma^2 k} e^{15\sigma^2 k/2} \\ &= Ce^{-2c\sigma^2 T} e^{a\sigma^2} \left[\frac{e^{a\sigma^2 T} - 1}{e^{a\sigma^2} - 1}\right] e^{(7.5-b)\sigma^2} \left[\frac{1 - e^{(7.5-b)\sigma^2 T}}{1 - e^{(7.5-b)\sigma^2}}\right] \\ &\leq C[e^{(a-2c)\sigma^2 T} - e^{-2c\sigma^2 T}][1 - e^{(7.5-b)\sigma^2 T}] \\ &\to 0, \end{aligned}$$

as $T \to \infty$, if $c > a/2 > 0$ and $b > 15/2$. Finally we exhibit the calculation for the first term in the expansion of the fourth term of $\hat{\gamma}_t^2$; the corresponding expectation is

$$\begin{aligned} &= e^{-2c\sigma^2 T} \sum_{k=1}^{T} \sum_{j=1}^{T} e^{-b\sigma^2(k+j)} \mathbb{E}(e^{\sigma(W_k+U_k+W_j+U_j)}) \\ &\leq Ce^{-2c\sigma^2 T} \sum_{k=1}^{T} \sum_{j=1}^{T} e^{(15-b)\sigma^2(k+j)} \\ &\to 0, \end{aligned}$$

as $T \to \infty$, if $c > 0$ and $b > 15$. The calculations for the remaining terms within the expansion of the fourth term are similar. □

Lemma 3.13. *For $\hat{\gamma}_t$ given in (2.3), and the corresponding estimates of the variances $\hat{\sigma}_{t,W}^2$ and $\hat{\sigma}_{t,U}^2$, $\mathbb{E}(\hat{\gamma}_t)/\sqrt{\mathbb{E}(\hat{\sigma}_{t,W}^2)\mathbb{E}(\hat{\sigma}_{t,U}^2)} \to \rho_t$, as $T \to \infty$, if $b > 2$ and $a > 0$.*

Proof.

$$e^{c\sigma^2 T}\hat{\gamma}_t = \sum_{k=1}^{T}\Big\{e^{a\sigma^2 k}[(e^{\sigma W_t} - e^{\sigma^2 t/2})(e^{\sigma U_t} - e^{\sigma^2 t/2})] \\ -e^{-b\sigma^2 k}[(e^{\sigma W_k} - e^{\sigma^2 k/2})(e^{\sigma U_k} - e^{\sigma^2 k/2})]\Big\}$$

$$\begin{aligned} e^{c\sigma^2 T}\mathbb{E}(\hat{\gamma}_t) &= \sum_{k=1}^{T}\Big\{e^{a\sigma^2 k}\left[e^{\sigma^2 t}[1+\rho_t(e^{\sigma^2 t}-1)] - e^{\sigma^2 t}\right] \\ &\qquad -e^{-b\sigma^2 k}\left[e^{\sigma^2 k}[1+\rho_k(e^{\sigma^2 k}-1)] - e^{\sigma^2 k}\right]\Big\} \\ &= \rho_t e^{\sigma^2 t}(e^{\sigma^2 t}-1)e^{a\sigma^2}\left[\frac{e^{a\sigma^2 T}-1}{e^{a\sigma^2}-1}\right] - \sum_{k=1}^{T}\Big\{e^{(1-b)\sigma^2 k}\rho_k(e^{\sigma^2 k}-1)\Big\} \\ &= D_T\rho_t - A_T, \end{aligned}$$

where D_T diverges and A_T converges if $b > 2$ as $T \to \infty$.

$$e^{c\sigma^2 T}\hat{\sigma}^2_{t,W} = \sum_{k=1}^{T}\Big\{e^{a\sigma^2 k}[(e^{\sigma W_t} - e^{\sigma^2 t/2})]^2 - e^{-b\sigma^2 k}[(e^{\sigma W_k} - e^{\sigma^2 k/2})]^2\Big\}$$

$$\begin{aligned} e^{c\sigma^2 T}\mathbb{E}(\hat{\sigma}^2_{t,W}) &= \sum_{k=1}^{T}\Big\{e^{a\sigma^2 k}e^{\sigma^2 t}(e^{\sigma^2 t}-1) - e^{(1-b)\sigma^2 k}(e^{\sigma^2 k}-1)\Big\} \\ &= D_T\rho_t - B_T, \end{aligned}$$

where B_T converges if $b > 2$. The same result holds for $e^{c\sigma^2 T}\mathbb{E}(\hat{\sigma}^2_{t,W})$ also. Hence we have

$$\begin{aligned} \mathbb{E}(\hat{\gamma}_t)/\sqrt{\mathbb{E}(\hat{\sigma}^2_{t,W})\mathbb{E}(\hat{\sigma}^2_{t,U})} &= \frac{D_T\rho_t - A_T}{D_T - B_T} \\ &= \frac{\rho_t - (A_T/D_T)}{1 - (B_T/D_T)} \\ &\to \rho_t, \end{aligned}$$

as $T \to \infty$, if $a > 0$ and $b > 2$. This proves the lemma. □

4. Conclusions and Discussion

In this paper we presented estimators for dynamic correlation between a pair of correlated Brownian motions and separately estimators for dynamic correlation between a pair of correlated Geometric Brownian motions. The main thrust of this paper was in showing the weak consistency of all the estimators presented.

The type of estimators that we presented here are generalizations of estimators that we introduced in an earlier work [4]. The special case that we considered in [4] had shown good empirical performance in a few simulation scenarios. One reason for the good empirical performance of the special case could be additional steps such as smoothing based on median filtering and the arctan function that we applied to the estimators. We conjecture that such extra processing steps may lead to reduction of bias empirically even for estimators which exhibit asymptotic bias theoretically. We also note here that the empirical study conducted in [4] did not include Brownian motions or Geometric Brownian motions.

The generalizations that we considered in this paper are based on inserting certain "weights" within the estimator: v^{-p} and v^q in the Brownian motion case, and $e^{a\sigma^2 k/2}$ and $e^{-b\sigma^2 k/2}$ in the Geometric Brownian motion case. When considering the estimator for correlation ρ_t at a time point t, such terms up-weight the contribution of the time series value at t and down-weight the values corresponding to other time points. Such down-weighting gets intensified for time series values further away from the currently considered time point t. In other words, the contribution of values further away, towards estimating the current correlation, is smaller compared to the contribution of values at nearby time points. In this sense, although the generalized estimator is global in nature, the emphasis is placed on local information. The special case (i.e. $p = q = 0$ and $a = b = 0$) estimators give equal emphasis to values even much further away. Such equal emphasis is not intuitively very appealing and could also be the reason for theoretical asymptotic bias for the special case estimators.

The asymptotics considered in this paper is based on the sample size T going to infinity. Another type of asymptotics that could be of interest (e.g. in high-frequency time series framework) is when the "mesh-size" goes to zero. Because of the scaling properties of Brownian motion, reducing the mesh-size is effectively same as increasing the sample size, roughly speaking. Hence, at least in the Brownian motion case, it may be possible to obtain consistency results theoretically, as the mesh-size goes to zero.

Assessment of empirical performance of generalized versions introduced in this paper will be of interest for future work. Currently, we are working on a program to see whether these types of estimators may be obtained (and weak consistency proved for) in other models such as pairs of fractional Brownian motions, autoregressive (e.g. AR(1)) or moving average (e.g. MA(1)) processes. We plan to conduct the empirical assessment after the estimators in the above-mentioned program are hopefully worked out theoretically. The 'hyperparameters' p, q, a, b and c occurring in the estimators presented in this paper may be considered as tuning parameters, similar to considering the 'window-size' as a tuning parameter in the dynamic correlation estimation approach based on sliding windows. Computational approaches such as cross-validation may be utilized in obtaining the optimal hyperparameters. The empirical study that we will conduct in future will help to provide a better understanding on those topics as well.

Appendix (Proof of Lemma 3.10)

We provide the proof of Lemma 3.10 for the convenience of the reader.

Proof. Based on equation (2.2)

$$\begin{aligned}\hat{\gamma}_t = \frac{1}{e^{c\sigma^2 T}} \sum_{k=1}^{T} \Big\{ &\Big[e^{\frac{-b\sigma^2 k}{2}} \left(e^{\sigma W_k} - e^{\frac{\sigma^2 k}{2}} \right) - e^{\frac{a\sigma^2 k}{2}} \left(e^{\sigma W_t} - e^{\frac{\sigma^2 t}{2}} \right) \Big] \\ &\times \Big[e^{\frac{-b\sigma^2 k}{2}} \left(e^{\sigma U_k} - e^{\frac{\sigma^2 k}{2}} \right) - e^{\frac{a\sigma^2 k}{2}} \left(e^{\sigma U_t} - e^{\frac{\sigma^2 t}{2}} \right) \Big] \Big\} \\ &= G_1 - G_2 - G_3 + G_4,\end{aligned}$$

where

$$G_1 = e^{-c\sigma^2 T} \sum_{k=1}^{T} \left[e^{-b\sigma^2 k} \left(e^{\sigma W_k} - e^{\frac{\sigma^2 k}{2}} \right) \left(e^{\sigma U_k} - e^{\frac{\sigma^2 k}{2}} \right) \right],$$

$$G_2 = e^{-c\sigma^2 T} \sum_{k=1}^{T} \left[e^{(a-b)\sigma^2 k/2} \left(e^{\sigma W_k} - e^{\frac{\sigma^2 k}{2}} \right) \left(e^{\sigma U_t} - e^{\frac{\sigma^2 t}{2}} \right) \right],$$

$$G_3 = e^{-c\sigma^2 T} \sum_{k=1}^{T} \left[e^{(a-b)\sigma^2 k/2} \left(e^{\sigma W_t} - e^{\frac{\sigma^2 t}{2}} \right) \left(e^{\sigma U_k} - e^{\frac{\sigma^2 k}{2}} \right) \right],$$

$$G_4 = e^{-c\sigma^2 T} \sum_{k=1}^{T} \left[e^{a\sigma^2 k} \left(e^{\sigma W_t} - e^{\frac{\sigma^2 t}{2}} \right) \left(e^{\sigma U_t} - e^{\frac{\sigma^2 t}{2}} \right) \right].$$

In order to show that $\mathrm{Var}(\hat{\gamma}_t) \to 0$, it suffices to show that $\mathbb{E}(\hat{\gamma}_t^2) \to 0$.

$$\begin{aligned}\hat{\gamma}_t^2 = \big[&G_1^2 + G_2^2 + G_3^2 + G_4^2 \\ &-2G_1G_2 - 2G_1G_3 + 2G_1G_4 + 2G_2G_3 - 2G_2G_4 - 2G_3G_4 \big].\end{aligned} \tag{4.1}$$

We will show that the expectation of each of the 10 terms in the right-hand side of (4.1) converges to zero as $T \to \infty$ for the range of a, b and c assumed in the statement of Lemma 3.10, thereby showing that $\mathbb{E}(\hat{\gamma}_t^2) \to 0,$ as $T \to \infty$.

$$\begin{aligned}G_1^2 = e^{-2c\sigma^2 T} \Bigg\{ &\sum_{k=1}^{T} \sum_{j=1}^{T} e^{-b\sigma^2 (k+j)} e^{\sigma (W_k + U_k + W_j + U_j)} \\ &- \sum_{k=1}^{T} \sum_{j=1}^{T} e^{-b\sigma^2 (k+j)} e^{\frac{\sigma^2 j}{2}} e^{\sigma (W_k + U_k + W_j)} - \sum_{k=1}^{T} \sum_{j=1}^{T} e^{-b\sigma^2 (k+j)} e^{\frac{\sigma^2 j}{2}} e^{\sigma (W_k + U_k + U_j)} \\ &+ \sum_{k=1}^{T} \sum_{j=1}^{T} e^{-b\sigma^2 (k+j)} e^{\frac{\sigma^2 j}{2}} e^{\sigma (W_k + U_k)} - \sum_{k=1}^{T} \sum_{j=1}^{T} e^{-b\sigma^2 (k+j)} e^{\frac{\sigma^2 k}{2}} e^{\sigma (W_k + W_j + U_j)} \\ &+ \sum_{k=1}^{T} \sum_{j=1}^{T} e^{-b\sigma^2 (k+j)} e^{\frac{\sigma^2 (k+j)}{2}} e^{\sigma (W_k + W_j)} + \sum_{k=1}^{T} \sum_{j=1}^{T} e^{-b\sigma^2 (k+j)} e^{\frac{\sigma^2 (k+j)}{2}} e^{\sigma (W_k + U_j)} \\ &- \sum_{k=1}^{T} \sum_{j=1}^{T} e^{-b\sigma^2 (k+j)} e^{\frac{\sigma^2 (k+2j)}{2}} e^{\sigma W_k} - \sum_{k=1}^{T} \sum_{j=1}^{T} e^{-b\sigma^2 (k+j)} e^{\frac{\sigma^2 k}{2}} e^{\sigma (U_k + W_j + U_j)}\end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=1}^{T}\sum_{j=1}^{T} e^{-b\sigma^2(k+j)}e^{\frac{\sigma^2(k+j)}{2}}e^{\sigma(U_k+W_j)} + \sum_{k=1}^{T}\sum_{j=1}^{T} e^{-b\sigma^2(k+j)}e^{\frac{\sigma^2(k+j)}{2}}e^{\sigma(U_k+U_j)} \\
&-\sum_{k=1}^{T}\sum_{j=1}^{T} e^{-b\sigma^2(k+j)}e^{\frac{\sigma^2(k+2j)}{2}}e^{\sigma U_k} + \sum_{k=1}^{T}\sum_{j=1}^{T} e^{-b\sigma^2(k+j)}e^{\sigma^2 k}e^{\sigma(W_j+U_j)} \\
&-\sum_{k=1}^{T}\sum_{j=1}^{T} e^{-b\sigma^2(k+j)}e^{\frac{\sigma^2(2k+j)}{2}}e^{\sigma W_j} - \sum_{k=1}^{T}\sum_{j=1}^{T} e^{-b\sigma^2(k+j)}e^{\frac{\sigma^2(2k+j)}{2}}e^{\sigma U_j} \\
&+\sum_{k=1}^{T}\sum_{j=1}^{T} e^{-b\sigma^2(k+j)}e^{\sigma^2(k+j)}\Bigg\}
\end{aligned}
$$

We label the 16 terms in the right-hand side of the equation for G_1^2 as G_1^2.term1, G_1^2.term2, ..., G_1^2.term16, in the same order as they appear above, and we will show the expectation of each of these terms to converge to zero as $T \to \infty$. We start with the last term since it is the easiest.

$$
\begin{aligned}
\mathbb{E}(G_1^2.\text{term16}) &= e^{-2c\sigma^2 T}\sum_{k=1}^{T}\sum_{j=1}^{T} e^{-b\sigma^2(k+j)}e^{\sigma^2(k+j)} \\
&= e^{-2c\sigma^2 T}e^{2(1-b)\sigma^2}\left[\frac{1-e^{(1-b)\sigma^2 T}}{1-e^{(1-b)\sigma^2}}\right]^2 \\
&\to 0 \text{ as } T \to \infty, \text{ if } c > 0 \text{ and } b > 1.
\end{aligned}
$$

Next we show that the expectation of G_1^2.term8 converges to zero. The same proof holds for the terms G_1^2.term12, G_1^2.term14 and G_1^2.term15 also.

$$
\begin{aligned}
\mathbb{E}(G_1^2.\text{term8}) &= e^{-2c\sigma^2 T}\sum_{k=1}^{T}\sum_{j=1}^{T} e^{-b\sigma^2(k+j)}e^{\frac{\sigma^2(k+2j)}{2}}\mathbb{E}(e^{\sigma(W_k)}) \\
&= e^{-2c\sigma^2 T}\sum_{k=1}^{T}\sum_{j=1}^{T} e^{(1-b)\sigma^2(k+j)} \\
&= e^{-2c\sigma^2 T}e^{2(1-b)\sigma^2}\left[\frac{1-e^{2(1-b)\sigma^2 T}}{1-e^{2(1-b)\sigma^2}}\right]^2 \\
&\to 0 \text{ as } T \to \infty, \text{ if } c > 0 \text{ and } b > 1.
\end{aligned}
$$

We will use the following inequality frequently in the remaining parts of the proof of Lemma 3.10.

$$
\mathbb{E}(e^{\sigma|W_t|}) \le 2e^{\sigma^2 t/2}. \tag{4.2}
$$

The inequality in (4.2) can be obtained by direct calculation as follows.

$$\begin{aligned}
\mathbb{E}(e^{\sigma|W_t|}) &= \frac{1}{\sqrt{2\pi t}}\left\{\int_0^\infty e^{(\sigma x - x^2/2t)}dx + \int_{-\infty}^0 e^{-(\sigma x + x^2/2t)}dx\right\} \\
&= 2e^{\sigma^2 t/2}\left(\frac{1}{\sqrt{2\pi t}}\int_0^\infty e^{-(x-t\sigma)^2/2t}dx\right) \\
&= 2e^{\sigma^2 t/2}(1-\Phi(-\sigma\sqrt{t})) \le 2e^{\sigma^2 t/2},
\end{aligned}$$

where Φ denotes the standard normal distribution function.

Recall that we may write $W_k = \rho_k U_k + \sqrt{1-\rho_k^2}M_k^{(1)}$, where $M_k^{(1)}$ is a Brownian motion process independent of U_k. With this, we have

$$\begin{aligned}
W_k + U_k &= (1+\rho_k)U_k + \sqrt{1-\rho_k^2}M_k^{(1)} \le 2|U_k| + |M_k^{(1)}| \\
\mathbb{E}(e^{\sigma(W_k+U_k)}) &\le Ce^{2\sigma^2 k}e^{\sigma^2 k/2} = Ce^{5\sigma^2 k/2} \\
\mathbb{E}(G_1^2.\text{term4}) &\le Ce^{-2c\sigma^2 T}\sum_{k=1}^{T}\sum_{j=1}^{T} e^{-b\sigma^2(k+j)}e^{\frac{\sigma^2 j}{2}}e^{\frac{5\sigma^2 k}{2}} \\
&= Ce^{-2c\sigma^2 T}e^{(3.5-b)\sigma^2}\left[\frac{1-e^{(1-b)\sigma^2 T}}{1-e^{(1-b)\sigma^2}}\right]\left[\frac{1-e^{(2.5-b)\sigma^2 T}}{1-e^{(2.5-b)\sigma^2}}\right] \\
&\to 0 \text{ as } T\to\infty, \text{ if } c>0 \text{ and } b>2.5.
\end{aligned}$$

Here and below C denotes a generic positive finite constant ($0 < C < \infty$). The same type of calculation for $\mathbb{E}(G_1^2.\text{term13})$ also.

When $k \ge j$,

$$\begin{aligned}
\mathbb{E}(e^{W_k+U_j}) &= \mathbb{E}(e^{\sigma[\rho_k U_k + U_j + \sqrt{1-\rho_k^2}M_k^{(1)}]}) \le \mathbb{E}(e^{\sigma[|U_k|+|U_j|+|M_k^{(1)}|]}) \\
&= \mathbb{E}(e^{\sigma[|U_k|+|U_j|]})\mathbb{E}(e^{\sigma[|M_k^{(1)}|]}) \le \mathbb{E}(e^{\sigma[|U_k-U_j|]})\mathbb{E}(e^{2\sigma|U_j|})\mathbb{E}(e^{\sigma[|M_k^{(1)}|]}) \\
&\le Ce^{\frac{\sigma^2}{2}(k-j)}e^{2\sigma^2 j}e^{\frac{\sigma^2}{2}k} \le Ce^{\frac{3\sigma^2}{2}(k+j)}.
\end{aligned}$$

Because of the symmetry in the upper bound, the above inequality holds for $k \le j$ as well. Hence

$$\begin{aligned}
\mathbb{E}(G_1^2.\text{term7}) &\le e^{-2c\sigma^2 T}\sum_{k=1}^{T}\sum_{j=1}^{T} e^{-b\sigma^2(k+j)+\frac{\sigma^2}{2}\sigma^2(k+j)+\frac{3\sigma^2}{2}\sigma^2(k+j)} \\
&= e^{-2c\sigma^2 T}\sum_{k=1}^{T}\sum_{j=1}^{T} e^{(2-b)\sigma^2(k+j)} \\
&= e^{-2c\sigma^2 T}e^{2(2-b)\sigma^2}\left[\frac{1-e^{(2-b)\sigma^2 T}}{1-e^{(2-b)\sigma^2}}\right]^2 \\
&\to 0 \text{ as } T\to\infty, \text{ if } c>0 \text{ and } b>2.
\end{aligned}$$

A similar calculation holds for $\mathbb{E}(G_1^2.\text{term10})$ also.

$$\mathbb{E}(e^{\sigma(U_k+U_j)}) = \mathbb{E}(e^{\sigma(U_k-U_j)})\mathbb{E}(e^{2\sigma U_j}) = e^{\frac{\sigma^2}{2}(k-j)}e^{2\sigma^2 j} \le e^{\frac{3\sigma^2}{2}(k+j)},$$

when $k \geq j$, but the same holds true for $k \leq j$ also because of symmetry. Hence

$$\begin{aligned}
\mathbb{E}(G_1^2.\text{term11}) &= e^{-2c\sigma^2 T} \sum_{k=1}^{T} \sum_{j=1}^{T} e^{-b\sigma^2(k+j)} e^{\frac{\sigma^2}{2}(k+j)} \mathbb{E}(e^{\sigma(U_k+U_j)}) \\
&\leq e^{-2c\sigma^2 T} \sum_{k=1}^{T} \sum_{j=1}^{T} e^{-b\sigma^2(k+j)} e^{2\sigma^2(k+j)} \\
&= e^{-2c\sigma^2 T} e^{2(2-b)\sigma^2} \left[\frac{1-e^{(2-b)\sigma^2 T}}{1-e^{(2-b)\sigma^2}} \right]^2 \\
&\to 0 \text{ as } T \to \infty, \text{ if } c > 0 \text{ and } b > 2.
\end{aligned}$$

Same calculation works for G_1^2.term6 also. So far, we have shown convergence to zero of the expectation the terms 4,6,7,8,10,11,12,13,14,15 and 16 of G_1^2. It remains to show the convergence for terms 1,2,3,5 and 9. The calculations for terms 2,3,5 and 9 are similar. So, we will exhibit just the calculation for G_1^2.term5 and then for G_1^2.term1.

$$\begin{aligned}
W_k + W_j + U_j &= [\rho_k U_k + \sqrt{1-\rho_k^2} M_k^{(1)}] + [(1+\rho_j)U_j + \sqrt{1-\rho_j^2} M_j^{(1)}] \\
&\leq |U_k| + |M_k^{(1)}| + 2|U_j| + |M_j^{(1)}| \\
&\leq |U_k - U_j| + 3|U_j| + |M_k^{(1)} - M_j^{(1)}| + 2|M_j^{(1)}| \text{ when } k \geq j.
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}(e^{\sigma(W_k+W_j+U_j)}) &\leq \mathbb{E}(e^{\sigma|U_k-U_j|}) \mathbb{E}(e^{3\sigma|U_j|}) \mathbb{E}(e^{\sigma|M_k^{(1)}-M_j^{(1)}|}) \mathbb{E}(e^{2\sigma|M_j^{(1)}|}) \\
&\leq C e^{\frac{\sigma^2}{2}(k-j)} e^{\frac{9\sigma^2}{2}j} e^{\frac{\sigma^2}{2}(k-j)} e^{2\sigma^2 j} = C e^{\sigma^2 k} e^{\frac{11\sigma^2}{2}j} \leq C e^{\frac{11\sigma^2}{2}(k+j)}.
\end{aligned}$$

We will get the same upper bound for $k \leq j$ also because of symmetry. Hence,

$$\begin{aligned}
\mathbb{E}(G_1^2.\text{term5}) &\leq C e^{-2c\sigma^2 T} \sum_{k=1}^{T} \sum_{j=1}^{T} e^{-b\sigma^2(k+j)} e^{\frac{\sigma^2}{2}k} e^{\frac{11\sigma^2}{2}(k+j)} \\
&= C e^{-2c\sigma^2 T} e^{(11.5-2b)\sigma^2} \left[\frac{1-e^{(5.5-b)\sigma^2 T}}{1-e^{(5.5-b)\sigma^2}} \right] \left[\frac{1-e^{(6.5-b)\sigma^2 T}}{1-e^{(6.5-b)\sigma^2}} \right] \\
&\to 0 \text{ as } T \to \infty, \text{ if } c > 0 \text{ and } b > 6.5.
\end{aligned}$$

As mentioned above, the calculations for G_1^2 terms 2,3 and 9 are similar to that of term 5. Thus, it remains to show the calculation for G_1^2.term1 to complete the proof for G_1^2.

$$\begin{aligned}
W_k + U_k + W_j + U_j &= [(1+\rho_k)U_k + \sqrt{1-\rho_k^2} M_k^{(1)}] \\
&\quad + [(1+\rho_j)U_j + \sqrt{1-\rho_j^2} M_j^{(1)}] \\
&\leq 2|U_k| + |M_k^{(1)}| + 2|U_j| + |M_j^{(1)}| \\
&\leq 2|U_k - U_j| + 4|U_j| + |M_k^{(1)} - M_j^{(1)}| + 2|M_j^{(1)}|.
\end{aligned}$$

$$\mathbb{E}(e^{\sigma(W_k+U_k+W_j+U_j)}) \le Ce^{2\sigma^2(k-j)}e^{8\sigma^2 j}e^{\frac{\sigma^2}{2}(k-j)}e^{2\sigma^2 j} \le Ce^{\frac{15\sigma^2}{2}(k+j)}.$$

The above inequality was obtained assuming $k \ge j$ but the same upper bounds holds for $k \le j$ as well. Hence

$$\begin{aligned}\mathbb{E}(G_1^2.\text{term1}) &\le e^{-2c\sigma^2 T}\sum_{k=1}^{T}\sum_{j=1}^{T} e^{(7.5-b)\sigma^2(k+j)} \\ &= e^{-2c\sigma^2 T}e^{(7.5-b)\sigma^2}\left[\frac{1-e^{(7.5-b)\sigma^2 T}}{1-e^{(7.5-b)\sigma^2}}\right]^2 \\ &\to 0 \text{ as } T\to\infty, \text{ if } c>0 \text{ and } b>7.5.\end{aligned}$$

Putting all the calculations together for the 16 terms involved in G_1^2, we see that

$$\mathbb{E}(G_1^2) \to 0 \text{ as } T\to\infty, \text{ if } c>0 \text{ and } b>7.5.$$

Now we move onto the calculations for G_2^2, G_3^2 and G_4^2. We will do the calculation for G_4^2 first as it is relatively easy. Also, since $\mathbb{E}(G_2^2) = \mathbb{E}(G_3^2)$, we will exhibit only the calculation for $\mathbb{E}(G_2^2)$.

$$\begin{aligned}\mathbb{E}(G_4^2) &\le e^{-2c\sigma^2 T}\sum_{k=1}^{T}\sum_{j=1}^{T} e^{a\sigma^2(k+j)}\mathbb{E}\left\{(e^{\sigma W_t - e^{\sigma^2 t/2}})^2(e^{\sigma U_t - e^{\sigma^2 t/2}})^2\right\} \\ &\le Ce^{-2c\sigma^2 T}\sum_{k=1}^{T}\sum_{j=1}^{T} e^{a\sigma^2(k+j)} \\ &= C\left[\frac{e^{a\sigma^2}}{1-e^{a\sigma^2}}\right]^2\left[e^{-c\sigma^2 T} - e^{(a-c)\sigma^2 T}\right]^2 \\ &\to 0 \text{ as } T\to\infty, \text{ if } c>\max\{a,0\}.\end{aligned}$$

$$\begin{aligned}G_2^2 = e^{-2c\sigma^2 T}\sum_{k=1}^{T}\sum_{j=1}^{T} e^{(a-b)\frac{\sigma^2}{2}(k+j)}\Big\{&e^{\sigma(W_k+W_j+2U_t)} - 2e^{\frac{\sigma^2}{2}t}e^{\sigma(W_k+W_j+U_t)} \\ &+ e^{\sigma^2 t}e^{\sigma(W_k+W_j)} - e^{\frac{\sigma^2}{2}j}e^{\sigma(W_k+2U_t)} \\ &+ 2e^{\frac{\sigma^2}{2}(j+t)}e^{\sigma(W_k+U_t)} - e^{\sigma^2 t}e^{\frac{\sigma^2}{2}j}e^{\sigma W_k} \\ &- e^{\frac{\sigma^2}{2}k}e^{\sigma(W_j+2U_t)} + 2e^{\frac{\sigma^2}{2}(k+t)}e^{\sigma(W_j+U_t)} \\ &- e^{\sigma^2 t}e^{\frac{\sigma^2}{2}k}e^{\sigma W_j} + e^{\frac{\sigma^2}{2}(k+j)}(e^{\sigma U_t} - e^{\sigma^2 t/2})^2\Big\}\end{aligned}$$

$$\begin{aligned}\mathbb{E}(G_2^2.(\text{last term})) &= e^{-2c\sigma^2 T}\sum_{k=1}^{T}\sum_{j=1}^{T} e^{(a-b)\frac{\sigma^2}{2}(k+j)}e^{\frac{\sigma^2}{2}(k+j)}\mathbb{E}(e^{\sigma U_t} - e^{\sigma^2 t/2})^2 \\ &\le Ce^{-2c\sigma^2 T}\sum_{k=1}^{T}\sum_{j=1}^{T} e^{(a-b)\frac{\sigma^2}{2}(k+j)}e^{\frac{\sigma^2}{2}(k+j)}\end{aligned}$$

$$= Ce^{-2c\sigma^2 T} e^{(a-b+1)\sigma^2} \left[\frac{1 - e^{(a-b+1)\frac{\sigma^2}{2}T}}{1 - e^{(a-b+1)\frac{\sigma^2}{2}}} \right]^2$$
$$\to 0, \text{ as } T \to \infty, \text{ if } c > 0 \text{ and } b > a + 1.$$

Same proof holds for G_2^2.term3, G_2^2.term6 and G_2^2.term9.

For G_2^2.term4, we start by noting that

$$W_k + 2U_t \leq |U_k| + |M_k^{(1)}| + 2|U_t| \leq |U_k - U_t| + 3|U_t| + |M_k^{(1)}|.$$

$$\mathbb{E}(e^{\sigma(W_k+2U_t)}) \leq Ce^{\frac{\sigma^2}{2}(k-t)} e^{\frac{9\sigma^2}{2}t} e^{\frac{\sigma^2}{2}k} \leq Ce^{\sigma^2 k}, \text{ when } k \geq t,$$

$$W_k + 2U_t \leq 2|U_t - U_k| + 3|U_k| + |M_k^{(1)}|,$$

$$\mathbb{E}(e^{\sigma(W_k+2U_t)}) \leq Ce^{2\sigma^2(t-k)} e^{\frac{9\sigma^2}{2}k} e^{\frac{\sigma^2}{2}k} \leq Ce^{3\sigma^2 k}, \text{ when } k \leq t.$$

Combining both cases, $k \geq t$ and $k \leq t$, we get $\mathbb{E}(e^{\sigma(W_k+2U_t)}) \leq Ce^{3\sigma^2 k}$. Hence

$$\mathbb{E}(G_2^2.\text{term4}) \leq e^{-2c\sigma^2 T} \sum_{k=1}^{T} \sum_{j=1}^{T} e^{(a-b)\frac{\sigma^2}{2}(k+j)} e^{\frac{\sigma^2}{2}j} e^{3\sigma^2 k}$$
$$\leq e^{-2c\sigma^2 T} \sum_{k=1}^{T} \sum_{j=1}^{T} e^{(6+a-b)\frac{\sigma^2}{2}(k+j)}$$
$$\leq e^{-2c\sigma^2 T} e^{(6+a-b)\sigma^2} \left[\frac{1 - e^{(6+a-b)\frac{\sigma^2}{2}T}}{1 - e^{(6+a-b)\frac{\sigma^2}{2}}} \right]^2$$
$$\to 0, \text{ as } T \to \infty, \text{ if } c > 0 \text{ and } b > a + 6.$$

Same type of calculations as above works for $\mathbb{E}(G_2^2$.term7) also.

For G_2^2.term5, when $k \geq t$, $W_k + U_t \leq |U_k - U_t| + 2|U_t| + |M_k^{(1)}|$, so that

$$\mathbb{E}(e^{\sigma(W_k+U_t)}) \leq C \exp\{\frac{\sigma^2}{2}[(k-t) + 4t + k]\} \leq Ce^{\sigma^2 k}$$

and when $k \leq t$, $W_k + U_t \leq |U_t - U_k| + 2|U_k| + |M_k^{(1)}|$, so that

$$\mathbb{E}(e^{\sigma(W_k+U_t)}) \leq C \exp\{\frac{\sigma^2}{2}[(t-k) + 4k + k]\} \leq Ce^{\sigma^2 k}.$$

Thus for any t, k, $\mathbb{E}(e^{\sigma(W_k+U_t)}) \leq Ce^{\sigma^2 k}$. Hence

$$\mathbb{E}(G_2^2.\text{term5}) \leq e^{-2c\sigma^2 T} \sum_{k=1}^{T} \sum_{j=1}^{T} e^{(a-b)\frac{\sigma^2}{2}(k+j)} e^{\frac{\sigma^2}{2}j} e^{2\sigma^2 k}$$
$$\leq Ce^{-2c\sigma^2 T} e^{(4+a-b)\sigma^2} \left[\frac{1 - e^{(4+a-b)\sigma^2 T/2}}{1 - e^{(4+a-b)\sigma^2/2}} \right]^2$$
$$\to 0, \text{ as } T \to \infty, \text{ if } c > 0 \text{ and } b > a + 4.$$

Same type of calculations as above works for $\mathbb{E}(G_2^2$.term8) also.

For G_2^2.term2, we consider four cases: $\{k \geq t, j \geq t\}$, $\{k \geq t, j \leq t\}$, $\{k \leq t, j \geq t\}$ and $\{k \leq t, j \leq t\}$.

Case 1a: $\{k \geq j \geq t\}$.

$$\begin{aligned} W_k + W_j + U_t &\leq |U_k| + |M_k^{(1)}| + |U_j| + |M_j^{(1)}| + |U_t| \\ &\leq |U_k - U_j| + 2|U_j| + |U_t| + |M_k^{(1)} - M_j^{(1)}| + 2|M_j| \\ &\leq |U_k - U_j| + 2|U_j - U_t| + 3|U_t| + |M_k^{(1)} - M_j^{(1)}| + 2|M_j|, \end{aligned}$$

$$\mathbb{E}(e^{\sigma(W_k+W_j+U_t)}) \leq C\exp\{\frac{\sigma^2}{2}[(k-j)+4(j-t)+9t+(k-j)+4j]\} \leq Ce^{\sigma^2(k+3j)}.$$

Case 1b: $\{j \geq k \geq t\}$. The result for this case can be obtained by interchanging k and j in the result for Case 1a.

$$\mathbb{E}(e^{\sigma(W_k+W_j+U_t)}) \leq Ce^{\sigma^2(3k+j)}.$$

Case 2: $\{k \geq t \geq j\}$.

$$\begin{aligned} W_k + W_j + U_t &\leq |U_k - U_t| + 2|U_t| + |U_j| + |M_k^{(1)} - M_t^{(1)}| + |M_t^{(1)}| + |M_j^{(1)}| \\ &\leq |U_k - U_t| + 2|U_t - U_j| + 3|U_j| + |M_k^{(1)} - M_t^{(1)}| \\ &\quad + |M_t^{(1)} - M_j^{(1)}| + 2|M_j^{(1)}|, \end{aligned}$$

$$\begin{aligned} \mathbb{E}(e^{\sigma(W_k+W_j+U_t)}) &\leq C\exp\{\frac{\sigma^2}{2}[(k-t)+4(t-j)+9j+(k-t)+(t-j)+4j]\} \\ &\leq Ce^{\sigma^2(k+4j)}. \end{aligned} \tag{4.3}$$

Case 3: $\{j \geq t \geq k\}$. The result for this case can be obtained by interchanging k and j in the result for Case 2.

$$\mathbb{E}(e^{\sigma(W_k+W_j+U_t)}) \leq Ce^{\sigma^2(4k+j)}.$$

Case 4a: $\{k \leq j \leq t\}$.

$$W_k + W_j + U_t \leq 3|U_k| + 2|M_k^{(1)}| + 2|U_j - U_k| + |M_j^{(1)} - M_k^{(1)}| + |U_t - U_j|.$$

$$\mathbb{E}(e^{\sigma(W_k+W_j+U_t)}) \leq C\exp\{\frac{\sigma^2}{2}[9k+4k+4(j-k)+(j-k)+(t-j)]\} \leq Ce^{\sigma^2(4k+2j)}.$$

Case 4b: $\{j \leq k \leq t\}$. By interchanging k and j in the result for Case 4a, we get

$$\mathbb{E}(e^{\sigma(W_k+W_j+U_t)}) \leq Ce^{\sigma^2(2k+4j)}.$$

For all cases combined we see that

$$\mathbb{E}(e^{\sigma(W_k+W_j+U_t)}) \leq Ce^{4\sigma^2(k+j)}.$$

Hence

$$\begin{aligned}\mathbb{E}(G_2^2.\text{term2}) &\le e^{-2c\sigma^2 T}\sum_{k=1}^{T}\sum_{j=1}^{T} e^{(a-b)\frac{\sigma^2}{2}(k+j)}e^{4\sigma^2(k+j)}\\ &\le Ce^{-2c\sigma^2 T}e^{(8+a-b)\sigma^2}\left[\frac{1-e^{(8+a-b)\sigma^2 T/2}}{1-e^{(8+a-b)\sigma^2/2}}\right]^2\\ &\to 0,\ \text{as } T\\ &\to \infty,\ \text{if } c>0 \text{ and } b>a+8.\end{aligned}$$

For G_2^2.term1 also we again deal with four cases.

<u>Case 1a</u>: $\{k \ge j \ge t\}$.

$$\begin{aligned}W_k + W_j + 2U_t &\le |U_k| + |M_k^{(1)}| + |U_j| + |M_j^{(1)}| + 2|U_t|\\ &\le |U_k - U_j| + 2|U_j| + 2|U_t| + |M_k^{(1)} - M_j^{(1)}| + 2|M_j|\\ &\le |U_k - U_j| + 2|U_j - U_t| + 4|U_t| + |M_k^{(1)} - M_j^{(1)}| + 2|M_j|,\end{aligned}$$

$$\mathbb{E}(e^{\sigma(W_k+W_j+2U_t)}) \le C\exp\{\frac{\sigma^2}{2}[(k-j)+4(j-t)+16t+(k-j)+4j]\} \le Ce^{\frac{\sigma^2}{2}(2k+6j)}.$$

<u>Case 1b</u>: $\{j \ge k \ge t\}$. By interchanging k and j in the result for Case 1a, we get

$$\mathbb{E}(e^{\sigma(W_k+W_j+2U_t)}) \le Ce^{\frac{\sigma^2}{2}(6k+2j)}.$$

<u>Case 2</u>: $\{k \ge t, j \le t\}$.

$$W_k+W_j+2U_t \le |U_k-U_t|+3|U_t-U_j|+4|U_j|+|M_k^{(1)}-M_t^{(1)}|+|M_t^{(1)}-M_j^{(1)}|+2|M_j^{(1)}|$$

$$\mathbb{E}(e^{\sigma(W_k+W_j+2U_t)}) \le Ce^{\frac{\sigma^2}{2}(2k+10j)}.$$

<u>Case 3</u>: $\{k \le t, j \ge t\}$. By interchanging k and j in Case 2, we get,

$$\mathbb{E}(e^{\sigma(W_k+W_j+2U_t)}) \le Ce^{\frac{\sigma^2}{2}(10k+2j)}.$$

<u>Case 4a</u>: $\{k \le j \le t\}$.

$$W_k + W_j + 2U_t \le 2|U_t - U_j| + 3|U_j - U_k| + 4|U_k| + |M_j^{(1)} - M_k^{(1)}| + 2|M_k^{(1)}|$$

$$\mathbb{E}(e^{\sigma(W_k+W_j+2U_t)}) \le Ce^{\frac{\sigma^2}{2}(10k+6j)}.$$

<u>Case 4b</u>:

$$\mathbb{E}(e^{\sigma(W_k+W_j+2U_t)}) \le Ce^{\frac{\sigma^2}{2}(6k+10j)}.$$

Combining all cases,

$$\begin{aligned}\mathbb{E}(G_2^2.\text{term1}) &\le e^{-2c\sigma^2 T}\sum_{k=1}^{T}\sum_{j=1}^{T} e^{(a-b)\frac{\sigma^2}{2}(k+j)}e^{5\sigma^2(k+j)}\\ &\le Ce^{-2c\sigma^2 T}e^{(10+a-b)\sigma^2}\left[\frac{1-e^{(10+a-b)\sigma^2 T/2}}{1-e^{(10+a-b)\sigma^2/2}}\right]^2\\ &\to 0,\ \text{as } T\to\infty,\ \text{if } c>0 \text{ and } b>a+10.\end{aligned}$$

For the remaining terms $G_1G_2, G_1G_3, \ldots, G_3G_4$ we work out the calculations for only the first term in each of them. The conditions on a, b and c required for convergence of the first term in each product, will ensure the convergence for the remaining terms in each as well. We start with G_1G_2.

$$G_1G_2 = e^{-2c\sigma^2 T}\sum_{k=1}^{T}\sum_{j=1}^{T} e^{-b\sigma^2 k} e^{(a-b)\frac{\sigma^2}{2}j}\Big[(e^{\sigma W_k} - e^{\frac{\sigma^2 k}{2}})(e^{\sigma U_k} - e^{\frac{\sigma^2 k}{2}}) \times (e^{\sigma W_j} - e^{\frac{\sigma^2 j}{2}})(e^{\sigma U_t} - e^{\frac{\sigma^2 t}{2}})\Big]$$

$$G_1G_2.\text{term1} = e^{-2c\sigma^2 T}\sum_{k=1}^{T}\sum_{j=1}^{T} e^{-b\sigma^2 k} e^{(a-b)\frac{\sigma^2}{2}j}\Big[e^{\sigma(W_k+U_k+W_j+U_t)}\Big].$$

$$\begin{aligned} W_k + U_k + W_j + U_t &= (1+\rho_k)U_k + \sqrt{1-\rho_k^2}M_k^{(1)} + \rho_j U_j + \sqrt{1-\rho_j^2}M_j^{(1)} + U_t \\ &\le 2|U_k| + |M_k^{(1)}| + |U_j| + |M_j^{(1)}| + |U_t| \end{aligned}$$

<u>Case 1a</u>: $\{k \ge j \ge t\}$.

$$W_k + U_k + W_j + U_t \le 2|U_k - U_j| + 3|U_j - U_t| + 4|U_t| + |M_k^{(1)} - M_j^{(1)}| + 2|M_j^{(1)}|.$$

$$\mathbb{E}(e^{\sigma(W_k+U_k+W_j+U_t)}) \le Ce^{\frac{\sigma^2}{2}(5k+8j)}.$$

<u>Case 1b</u>: $\{j \ge k \ge t\}$.

$$W_k + U_k + W_j + U_t \le 2|U_j - U_k| + 3|U_k - U_t| + 4|U_t| + |M_j^{(1)} - M_k^{(1)}| + 2|M_k^{(1)}|.$$

$$\mathbb{E}(e^{\sigma(W_k+U_k+W_j+U_t)}) \le Ce^{\frac{\sigma^2}{2}(11k+2j)}.$$

<u>Case 2</u>: $\{k \ge t \ge j\}$.

$$W_k + U_k + W_j + U_t \le 2|U_k - U_t| + 3|U_t - U_j| + 4|U_j| + |M_k^{(1)} - M_j^{(1)}| + 2|M_j^{(1)}|.$$

$$\mathbb{E}(e^{\sigma(W_k+U_k+W_j+U_t)}) \le Ce^{\frac{\sigma^2}{2}(5k+10j)}.$$

<u>Case 3</u>: $\{j \ge t \ge k\}$.

$$W_k + U_k + W_j + U_t \le |U_j - U_t| + 2|U_t - U_k| + 4|U_k| + |M_j^{(1)} - M_k^{(1)}| + 2|M_k^{(1)}|.$$

$$\mathbb{E}(e^{\sigma(W_k+U_k+W_j+U_t)}) \le Ce^{\frac{\sigma^2}{2}(15k+2j)}.$$

<u>Case 4a</u>: $\{t \ge k \ge j\}$.

$$W_k + U_k + W_j + U_t \le |U_t - U_k| + 3|U_k - U_j| + 4|U_j| + |M_k^{(1)} - M_j^{(1)}| + 2|M_j^{(1)}|.$$

$$\mathbb{E}(e^{\sigma(W_k+U_k+W_j+U_t)}) \le Ce^{\frac{\sigma^2}{2}(9k+10j)}.$$

<u>Case 4b</u>: $\{t \ge j \ge k\}$.

$$W_k + U_k + W_j + U_t \le |U_t - U_j| + 2|U_j - U_k| + 4|U_k| + |M_j^{(1)} - M_k^{(1)}| + 2|M_k^{(1)}|.$$

$$\mathbb{E}(e^{\sigma(W_k+U_k+W_j+U_t)}) \le Ce^{\frac{\sigma^2}{2}(15k+4j)}.$$

Combining all cases, we get $\mathbb{E}(e^{\sigma(W_k+U_k+W_j+U_t)}) \leq Ce^{\frac{\sigma^2}{2}(15k+10j)}$.

$$\begin{aligned}
\mathbb{E}(G_1G_2.\text{term1}) &\leq Ce^{-2c\sigma^2 T}\sum_{k=1}^{T} e^{-b\sigma^2 k}e^{15k\frac{\sigma^2}{2}}\left\{\sum_{j=1}^{T} e^{(a-b)\frac{\sigma^2}{2}j}e^{10j\frac{\sigma^2}{2}}\right\}\\
&\leq Ce^{-2c\sigma^2 T}e^{(25+a-3b)\frac{\sigma^2}{2}}\\
&\quad\times\left[\frac{1-e^{(10+a-b)\sigma^2 T/2}}{1-e^{(10+a-b)\sigma^2/2}}\right]\left[\frac{1-e^{(15-2b)\sigma^2 T/2}}{1-e^{(15-2b)\sigma^2/2}}\right]\\
&\to 0,\ \text{as } T\to\infty,\ \text{if } c>0\\
&\quad\text{and } b>\max\{a+10, 15/2\} = a+10 \text{ (since } a>0).
\end{aligned}$$

We skip the proof for G_1G_3, since it is very similar to that of G_1G_2 (and the results are the same). We move onto G_1G_4.

$$G_1G_4 = e^{-2c\sigma^2 T}\sum_{k=1}^{T}\sum_{j=1}^{T} e^{-b\sigma^2 k}e^{a\sigma^2 j}\Big[(e^{\sigma W_k}-e^{\frac{\sigma^2 k}{2}})(e^{\sigma U_k}-e^{\frac{\sigma^2 k}{2}})$$

$$(e^{\sigma W_t}-e^{\frac{\sigma^2 t}{2}})(e^{\sigma U_t}-e^{\frac{\sigma^2 t}{2}})\Big]$$

$$G_1G_4.\text{term1} = e^{-2c\sigma^2 T}\sum_{k=1}^{T}\sum_{j=1}^{T} e^{-b\sigma^2 k}e^{a\sigma^2 j}\left[e^{\sigma(W_k+U_k+W_t+U_t)}\right].$$

$$W_k+U_k+W_t+U_t \leq 2|U_k| + |M_k^{(1)}| + 2|U_t| + |M_t^{(1)}|.$$

When $k\geq t$,

$$W_k+U_k+W_t+U_t \leq 2|U_k-U_t| + 4|U_t| + |M_k^{(1)} - M_t^{(1)}| + 2|M_t^{(1)}|$$

and when $k\leq t$,

$$W_k+U_k+W_t+U_t \leq 2|U_t-U_k| + 4|U_k| + |M_t^{(1)} - M_k^{(1)}| + 2|M_k^{(1)}|.$$

In the first case, we will get $\mathbb{E}(e^{\sigma(W_k+U_k+W_t+U_t)}) \leq Ce^{\frac{5k\sigma^2}{2}}$ and in the second case we will get $\mathbb{E}(e^{\sigma(W_k+U_k+W_t+U_t)}) \leq Ce^{\frac{15k\sigma^2}{2}}$ and so combining both cases we see that $\mathbb{E}(e^{\sigma(W_k+U_k+W_t+U_t)}) \leq Ce^{\frac{15k\sigma^2}{2}}$. Hence,

$$\begin{aligned}
\mathbb{E}(G_1G_4.\text{term1}) &\leq Ce^{-2c\sigma^2 T}e^{a\sigma^2}\left[\frac{1-e^{a\sigma^2 T}}{1-e^{a\sigma^2}}\right]e^{(15-2b)\frac{\sigma^2}{2}}\left[\frac{1-e^{(15-2b)\sigma^2 T/2}}{1-e^{(15-2b)\sigma^2/2}}\right]\\
&= Ce^{(15-2b)\frac{\sigma^2}{2}}\left(\frac{e^{a\sigma^2}}{1-e^{a\sigma^2}}\right)\\
&\quad\times\left[\frac{1-e^{(15-2b)\sigma^2 T/2}}{1-e^{(15-2b)\sigma^2/2}}\right][e^{-2c\sigma^2 T} - e^{(a-2c)\sigma^2 T}]\\
&\to 0,\ \text{as } T\to\infty,\ \text{if } b>15/2\\
&\quad\text{and } c>\max\{0, a/2\} = a/2 \text{ (since } a>0).
\end{aligned}$$

$$G_2G_3 = e^{-2c\sigma^2 T}\sum_{k=1}^{T}\sum_{j=1}^{T} e^{(a-b)\frac{\sigma^2}{2}(k+j)}\Big[(e^{\sigma W_k}-e^{\frac{\sigma^2 k}{2}})$$

$$\times(e^{\sigma U_t}-e^{\frac{\sigma^2 t}{2}})(e^{\sigma W_t}-e^{\frac{\sigma^2 t}{2}})(e^{\sigma U_j}-e^{\frac{\sigma^2 j}{2}})\Big]$$

$$G_2G_3.\text{term1} = e^{-2c\sigma^2 T}\sum_{k=1}^{T}\sum_{j=1}^{T} e^{(a-b)\frac{\sigma^2}{2}(k+j)}\Big[e^{\sigma(W_k+U_j+W_t+U_t)}\Big].$$

$$W_k+U_j+W_t+U_t \le |U_k|+|M_k^{(1)}|+|U_j|+2|U_t|+|M_t^{(1)}|$$

<u>Case 1a</u>: $\{k \ge j \ge t\}$.

$$W_k+U_j+W_t+U_t \le |U_k-U_j|+2|U_j-U_t|+4|U_t|+|M_k^{(1)}-M_t^{(1)}|+2|M_t^{(1)}|.$$

$$\mathbb{E}(e^{\sigma(W_k+U_j+W_t+U_t)}) \le Ce^{\frac{\sigma^2}{2}(2k+3j)}.$$

<u>Case 1b</u>: $\{j \ge k \ge t\}$.

$$W_k+U_j+W_t+U_t \le |U_j-U_k|+2|U_k-U_t|+4|U_t|+|M_k^{(1)}-M_t^{(1)}|+2|M_t^{(1)}|.$$

$$\mathbb{E}(e^{\sigma(W_k+U_j+W_t+U_t)}) \le Ce^{\frac{\sigma^2}{2}(4k+j)}.$$

<u>Case 2</u>: $\{k \ge t \ge j\}$.

$$W_k+U_j+W_t+U_t \le |U_k-U_j|+3|U_t-U_j|+4|U_j|+|M_k^{(1)}-M_t^{(1)}|+2|M_t^{(1)}|.$$

$$\mathbb{E}(e^{\sigma(W_k+U_j+W_t+U_t)}) \le Ce^{\frac{\sigma^2}{2}(2k+7j)}.$$

<u>Case 3</u>: $\{j \ge t \ge k\}$.

$$W_k+U_j+W_t+U_t \le |U_j-U_t|+3|U_t-U_k|+4|U_k|+|M_t^{(1)}-M_k^{(1)}|+2|M_k^{(1)}|.$$

$$\mathbb{E}(e^{\sigma(W_k+U_j+W_t+U_t)}) \le Ce^{\frac{\sigma^2}{2}(10k+j)}.$$

<u>Case 4a</u>: $\{k \le j \le t\}$.

$$W_k+U_j+W_t+U_t \le 2|U_t-U_j|+3|U_j-U_k|+4|U_k|+|M_t^{(1)}-M_k^{(1)}|+2|M_k^{(1)}|.$$

$$\mathbb{E}(e^{\sigma(W_k+U_j+W_t+U_t)}) \le Ce^{\frac{\sigma^2}{2}(10k+5j)}.$$

<u>Case 4b</u>: $\{j \le k \le t\}$.

$$W_k+U_j+W_t+U_t \le 2|U_t-U_k|+3|U_k-U_j|+4|U_j|+|M_t^{(1)}-M_k^{(1)}|+2|M_k^{(1)}|.$$

$$\mathbb{E}(e^{\sigma(W_k+U_j+W_t+U_t)}) \le Ce^{\frac{\sigma^2}{2}(8k+7j)}.$$

Overall, the following upper bound is satisfied in all the cases:

$$\mathbb{E}(e^{\sigma(W_k+U_j+W_t+U_t)}) \le Ce^{\frac{\sigma^2}{2}[10(k+j)]}.$$

Hence we have,

$$\begin{aligned}\mathbb{E}(G_2G_3.\text{term1}) &\le Ce^{-2c\sigma^2 T}\sum_{k=1}^{T}\sum_{j=1}^{T} e^{(10+a-b)\frac{\sigma^2}{2}(k+j)}\\ &\le Ce^{-2c\sigma^2 T}e^{(10+a-b)\frac{\sigma^2}{2}}\left[\frac{1-e^{(10+a-b)\frac{\sigma^2}{2}T}}{1-e^{(10+a-b)\frac{\sigma^2}{2}}}\right]^2\\ &\to 0,\ \text{as } T\to\infty,\ \text{if } c>0 \text{ and } b>a+10.\end{aligned}$$

Next we consider G_2G_4.

$$\begin{aligned}G_2G_4 = e^{-2c\sigma^2 T}\sum_{k=1}^{T}\sum_{j=1}^{T} e^{(a-b)\frac{\sigma^2}{2}k}e^{a\frac{\sigma^2}{2}j}\Big[(e^{\sigma W_k}-e^{\frac{\sigma^2 k}{2}})\\ \times(e^{\sigma U_t}-e^{\frac{\sigma^2 t}{2}})(e^{\sigma W_t}-e^{\frac{\sigma^2 t}{2}})(e^{\sigma U_t}-e^{\frac{\sigma^2 t}{2}})\Big]\end{aligned}$$

$$G_2G_4.\text{term1} = e^{-2c\sigma^2 T}\sum_{k=1}^{T}\sum_{j=1}^{T} e^{(a-b)\frac{\sigma^2}{2}k}e^{a\frac{\sigma^2}{2}j}\Big[e^{\sigma(W_k+2U_t+W_t)}\Big].$$

$$W_k+2U_t+W_t \le |U_k|+|M_k^{(1)}|+3|U_t|+|M_t^{(1)}|$$

$k\ge t$:

$$W_k+2U_t+W_t \le |U_k-U_t|+4|U_t|+|M_k^{(1)}-M_t^{(1)}|+2|M_t^{(1)}|.$$

$$\mathbb{E}(e^{\sigma(W_k+2U_t+W_t)}) \le C\exp\{\frac{\sigma^2}{2}[(k-t)+16t+(k-t)+4t]\} \le Ce^{\sigma^2 k}.$$

$t\ge k$:

$$W_k+2U_t+W_t \le 3|U_t-U_k|+4|U_k|+|M_t^{(1)}-M_k^{(1)}|+2|M_k^{(1)}|.$$

$$\mathbb{E}(e^{\sigma(W_k+2U_t+W_t)}) \le C\exp\{\frac{\sigma^2}{2}[9(t-k)+16k+(t-k)+4k]\} \le Ce^{5\sigma^2 k}.$$

Combining the the two cases $k\ge t$ and $k\le t$, we get

$$\mathbb{E}(e^{\sigma(W_k+2U_t+W_t)}) \le Ce^{5\sigma^2 k}$$

so that

$$\begin{aligned}\mathbb{E}(G_2G_4.\text{term1}) &\le Ce^{-2c\sigma^2 T}\sum_{j=1}^{T} e^{a\sigma^2 j}\sum_{k=1}^{T} e^{(10+a-b)\frac{\sigma^2}{2}k}\\ &\le Ce^{-2c\sigma^2 T}e^{(10+a-b)\frac{\sigma^2}{2}}\left[\frac{1-e^{(10+a-b)\frac{\sigma^2}{2}T}}{1-e^{(10+a-b)\frac{\sigma^2}{2}}}\right]e^{a\sigma^2}\left[\frac{1-e^{a\sigma^2 T}}{1-e^{a\sigma^2}}\right]\\ &\to 0,\ \text{as } T\to\infty,\ \text{if } b>a+10\\ &\quad \text{and } c>\max\{a/2,0\}(=a/2,\ \text{since } a>0).\end{aligned}$$

Proof and results for G_3G_4 are very similar to that of G_2G_4 and so we skip it. Combing through all the terms we see that if $c>a>0$ and $b>a+10$ each of the expectations converges to zero (and hence $\mathbb{E}(\hat{\gamma}_t^2)\to 0$) as $T\to\infty$, thereby proving the lemma. □

Acknowledgment. We are grateful to the anonymous referee for the helpful suggestions. Majnu John would like to acknowledge the partial support for the work based on the following grants from the National Institute of Mental Health awarded to PI: Malhotra, AK (R01MH109508 and R01MH108654); PIs: Kane, J and Robinson, D (R01MH120594); PI: Lencz, T (R01MH117646); PI: Deligiannidis, K (R01MH120313); PI: Barber, A (R21MH122886); PI: Birnbaum, M (R34MH120790) and PI: DeChoudhury, M (R01MH117172).

References

1. Engle, R. F.: *Anticipating Correlations*. Princeton University Press, Princeton, New Jersey, USA, 2009.
2. Gaunt, R.: Variance-Gamma approximation via Stein's method, *Electronic Journal of Probability* **19** (2014), no. 38, 1–33.
3. Gaunt, R.: A note on the distribution of the product of zero mean correlated normal random variables, *Statistica Neerlandica* **73** (2019), no. 2, 176–179.
4. John, M., Wu, Y., Narayan, M., John, A., Ikuta, T., and Ferbinteanu, J.: Estimation of dynamic bivariate correlation using a weighted graph algorithm, *Entropy* **22** (2020), no. 6, article number 617.
5. Lindquist, M. A., Xu, Y., Nebel, M. B., and Caffo, B. S.: Evaluating dynamic bivariate correlations in resting-state fMRI: A comparison study and a new approach, *NeuroImage* **101** (2014), 531–546.
6. Nadarajah, S., and Pogány, T. K.: On the distribution of the product of correlated normal random variables, *C. R. Acad. Sci. Paris, Ser.I* **354** (2015), no. 2, 201–204.
7. Thomakos, D., Klepsch, J., and Politis, D. N.: Model Free Inference on Multivariate Time Series with Conditional Correlations, *Stats* **3** (2020), no. 4, 484–509.
8. van der Vaart, A. W.: *Asymptotic Statistics*. Cambridge University Press, New York, USA, 1998.

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