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AN INTRINSIC PROOF OF AN EXTENSION OF ITÔ'S ISOMETRY FOR ANTICIPATING STOCHASTIC INTEGRALS

HUI-HSIUNG KUO, PUJAN SHRESTHA, AND SUDIP SINHA*

ABSTRACT. Itô's isometry forms the cornerstone of the definition of Itô's integral and consequently the theory of stochastic calculus. Therefore, for any theory which extends Itô's theory, it is important to know if the isometry holds. In this paper, we use probabilistic arguments to demonstrate that the extension of the isometry formula contains an extra term for the anticipating stochastic integral defined by Ayed and Kuo. We give examples to illustrate the usage of this formula and to show that the extra term can be positive or negative.

1. Introduction

Let $B_t, t \geq 0$, be a Brownian motion and $[a,b]$ a fixed interval with $a \geq 0$. Suppose f and ϕ are continuous functions on $\mathbb{R}$. In [1] the following anticipating stochastic integral is defined as

$$\int_a^b f(B_t)\phi(B_b - B_t)\, dB_t = \lim_{\|\Delta_n\| \to 0} \sum_{i=1}^{n} f(B_{t_{i-1}})\phi(B_b - B_{t_i})\Delta B_i \tag{1.1}$$

provided that the limit exists in probability. Here $\Delta_n = \{a = t_0, t_1, t_2, \ldots, t_n = b\}$ is a partition of $[a,b]$ and $\Delta B_i = B_{t_i} - B_{t_{i-1}}$. Note that when $\phi \equiv 1$ this stochastic integral is an Itô integral (see Theorem 5.3.3 in [6].) It is proved in Theorem 3.1 [8] that when f and ϕ are C^1-functions we have the equality:

$$\begin{aligned}\mathbb{E}\left[\left(\int_a^b f(B_t)\phi(B_b - B_t)\, dB_t\right)^2\right] &= \int_a^b \mathbb{E}\Big[f(B_t)^2\phi(B_b - B_t)^2\Big]\, dt \\ &\quad + 2\int_a^b \int_a^t \mathbb{E}\Big[f(B_s)\phi'(B_b - B_s)f'(B_t)\phi(B_b - B_t)\Big]\, ds\, dt, \end{aligned} \tag{1.2}$$

provided that the integrals in the right-hand side exist. In particular, when $\phi \equiv 1$, the equality in equation (1.2) is the well-known Itô isometry.

We need to point out that the proof of equation (1.2) in [8] is too lengthy and involves rather tedious computations by using the binomial expansion. Moreover,

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it does not exhibit the crucial feature of the new theory of stochastic integration [1, 2, 3, 4, 7, 9, 10], namely, the evaluation points for the counterpart $\phi(B_b - B_t)$ are the right-endpoints t_i of the subintervals $[t_{i-1}, t_i]$ of the partition Δ_n. The main purpose of this paper is to give an intrinsic proof of the formula in equation (1.2), in fact, for the more general case.

The formula in equation (1.2) is motivated by Theorem 13.16 in the book [5]. The white noise approach to anticipating stochastic integral is given by the Hitsuda–Skorokhod integral

$$\int_a^b \partial_t^* \Phi(t)\ dt,$$

see Chapter 13 in [5] for detail. Here $\Phi(t)$ is a Brownian functional and ∂_t^* is the adjoint of the white noise differential operator ∂_t (see page 107 [5]). Under a certain condition on $\Phi(t)$ Theorem 13.16 in [5]) asserts that the white noise integral $\int_a^b \partial_t^* \Phi(t)\ dt$ is a Hitsuda–Skorokhod integral and we have

$$\mathbb{E}\left[\left(\int_a^b \partial_t^* \Phi(t)\ dt\right)^2\right] = \int_a^b \mathbb{E}\Big[\Phi(t)^2\Big]dt + \int_a^b \int_a^b \mathbb{E}\Big\{\big[\partial_t \Phi(s)\big]\big[\partial_s \Phi(t)\big]\Big\}\ ds\ dt. \tag{1.3}$$

We also need to mention an informal expression from white noise analysis:

$$\partial_t B_s = \begin{cases} 1, & \text{if } t < s; \\ 0, & \text{if } t > s. \end{cases} \tag{1.4}$$

To check this expression, use the last line on page 103 and the Brownian motion representation B_t on page 254 of the book [5].

Now, let f and ϕ be C^1-functions on $\mathbb{R}$ and consider the Brownian functional

$$\Phi(t) = f(B_t)\phi(B_b - B_t), \quad a \le t \le b. \tag{1.5}$$

Suppose $s < t$. Then by the chain rule and equation (1.4), we have

$$\begin{aligned} \partial_t \Phi(s) &= f'(B_s)\big(\partial_t B_s\big)\phi(B_b - B_s) \\ &\quad + f(B_s)\phi'(B_b - B_s)\big(\partial_t(B_b - B_s)\big) \\ &= f(B_s)\phi'(B_b - B_s). \end{aligned} \tag{1.6}$$

Similarly, for $s < t$, we have

$$\begin{aligned} \partial_s \Phi(t) &= f'(B_t)\big(\partial_s B_t\big)\phi(B_b - B_t) \\ &\quad + f(B_t)\phi'(B_b - B_t)\big(\partial_s(B_b - B_t)\big) \\ &= f'(B_t)\phi(B_b - B_t). \end{aligned} \tag{1.7}$$

Putting equations (1.3), (1.5), (1.6), (1.7) together, we get

$$\begin{aligned} \mathbb{E}\left[\left(\int_a^b \partial_t^*\Big(f(B_t)\phi(B_b - B_t)\Big)dt\right)^2\right] &= \int_a^b \mathbb{E}\Big[f(B_t)^2\phi(B_b - B_t)^2\Big]dt \\ &\quad + 2\int_a^b \int_a^t \mathbb{E}\Big[f(B_s)\phi'(B_b - B_s)f'(B_t)\phi(B_b - B_t)\Big]\ ds\ dt. \end{aligned} \tag{1.8}$$

But for the Brownian functional $\Phi(t) = f(B_t)\phi(B_b - B_t)$, its Hitsuda–Skorokhod integral turns out to be the same as the new stochastic integral in equation (1.1), namely,

$$\int_a^b \partial_t^* \Big(f(B_t)\phi(B_b - B_t) \Big)\, dt = \int_a^b f(B_t)\phi(B_b - B_t)\, dB_t. \tag{1.9}$$

Obviously, equations (1.8) and (1.9) yield equation (1.2). This completes the explanation why the formula in equation (1.2) is motivated by Theorem 13.16 in the book [5].

In this paper, we give an intrinsic proof of the formula (1.2) using the definition of the anticipating stochastic integral within a purely probabilistic framework. The definition of the integral using left and right endpoint evaluation of the adapted and instantly independent parts forms the backbone of the proof. Even though we used white-noise distribution theory as a motivation, our proof does not rely on it.

The paper is organized as follows. In Section 2, we briefly illustrate the new anticipating stochastic integral with a few examples. In Section 3, we prove an extension of Itô's isometry for the anticipating stochastic integral, the main result and technique of this paper. We include numerous examples to demonstrate the usage and implication of the theorem.

2. The Anticipating Stochastic Integral

From here on, let $B_t, t \geq 0$, be a fixed Brownian motion and $\{\mathcal{F}_t\}$ is a filtration such that the following conditions are satisfied:

(i) B_t is adapted to $\{\mathcal{F}_t\}$.
(ii) For any $t \leq s$, $B_s - B_t$ and $\mathcal{F}_t$ are independent.

Let $[a, b]$ a fixed interval with $a \geq 0$, and assume $t \in [a, b]$ unless otherwise specified.

A stochastic process $\phi(t)$ is called *instantly independent* with respect to $\{\mathcal{F}_t\}$ if for each $t \in [a, b]$, the random variable $\phi(t)$ and the σ-field $\mathcal{F}_t$ are independent.

We refer to Section 2 of [3] for a detailed definition of the anticipating stochastic integral. In what follows, we highlight the crucial steps in the definition in a concise manner.

Definition 2.1 (Definition 2.3 of [3])**.** The anticipating integral is defined in following three steps:

(1) Suppose $f(t)$ is an $\mathcal{F}_t$-adapted continuous stochastic process and $\phi(t)$ is a continuous stochastic processes that is instantly independent with respect to $\{\mathcal{F}_t\}$. Then the stochastic integral of $\Phi(t) = f(t)\phi(t)$ is defined by

$$\int_a^b f(t)\phi(t)\, dB_t = \lim_{\|\Delta_n\| \to 0} \sum_{j=1}^n f(t_{j-1})\phi(t_j)(B_{t_j} - B_{t_{j-1}}),$$

provided that the limit exists in probability.

(2) For a stochastic process of the form $\Phi(t) = \sum_{i=1}^{n} f_i(t)\phi_i(t)$, where $f_i(t)$ and $\phi_i(t)$ are given as in step (1), the stochastic integral is defined by

$$\int_a^b \Phi(t)\ dB_t = \sum_{i=1}^{n} \int_a^b f_i(t)\phi_i(t)\ dB_t.$$

(3) Let $\Phi(t)$ be a stochastic process such that there is a sequence $(\Phi_n(t))_{n=1}^{\infty}$ of stochastic processes of the form in step (2) satisfying
   (a) $\int_a^b |\Phi_n(t) - \Phi(t)|^2\ dt \to 0$ almost surely as $n \to \infty$, and
   (b) $\int_a^b \Phi_n(t)\ dB_t$ converges in probability as $n \to \infty$.

   Then the stochastic integral of $\Phi(t)$ is defined by

$$\int_a^b \Phi(t)\ dB_t = \lim_{n\to\infty} \int_a^b \Phi_n(t)\ dB_t \quad \text{in probability.}$$

This integral is well defined, as demonstrated in Lemma 2.1 of [3]. The *key idea* of the anticipating stochastic integral is to decompose the integrand into a sum of products of adapted and instantly independent parts. Then we use the left-endpoints of subintervals to evaluate the adapted parts and the right-endpoints of subintervals to evaluate the instantly independent parts. We demonstrate this intristic nature of the anticipating stochastic integral with an example.

Example 2.2 (Example 2.4 of [3])**.** Consider $\int_a^b B_b\ dB_t$. Since the integrand B_b is not $\{\mathcal{F}_t\}$-adapted, the integral is not defined within the Itô theory. Note that B_b can be decomposed as

$$B_b = B_t + (B_b - B_t), \quad a \le t \le b,$$

where the first term B_t is $\{\mathcal{F}_t\}$-adapted and the second term $B_b - B_t$ is instantly independent with respect to $\{\mathcal{F}_t\}$. Thus we have decomposed the anticipating integrand into a sum of an adapted stochastic process and an instantly independent stochastic process. Then we use the definition of the integral 2.1 to get

$$\begin{aligned}
\int_a^b B_b\ dB_t &= \int_a^b \left[B_t + (B_b - B_t)\right]\ dB_t \\
&= \lim_{\|\Delta_n\|\to 0} \sum_{j=1}^{n} \left[B_{t_{j-1}} + \left(B_b - B_{t_j}\right)\right] (B_{t_j} - B_{t_{j-1}}) \\
&= \lim_{\|\Delta_n\|\to 0} \sum_{j=1}^{n} \left[B_b - \left(B_{t_j} - B_{t_{j-1}}\right)\right] (B_{t_j} - B_{t_{j-1}}) \\
&= B_b \lim_{\|\Delta_n\|\to 0} \sum_{j=1}^{n} (B_{t_j} - B_{t_{j-1}}) - \lim_{\|\Delta_n\|\to 0} \sum_{j=1}^{n} (B_{t_j} - B_{t_{j-1}})^2 \\
&= B_b\,(B_b - B_a) - (b - a),
\end{aligned}$$

where, in the last equality, we have used the quadratic variation of a Brownian motion.

We note that this is a true extension of Itô's stochastic integral since for the case when the integrand is purely adapted, the anticipating integral reduces to Itô's stochastic integral.

Now, Itô's isometry is an important property in the Itô theory of stochastic integration. Do we have a similar result for the anticipating integral? The next section addresses this question.

3. Extension of Itô's Isometry

Recall that $\{B_t, t \geq 0\}$ is a fixed Brownian motion and $[a, b]$ a fixed interval with $a \geq 0$. In the proof of Theorem 3.1 below, we shall use the σ-fields

$$\begin{aligned} \mathcal{F}_s &= \sigma\{B_u;\ a \leq u \leq s\}, \qquad a \leq s \leq b, \\ \mathcal{G}^{(t)} &= \sigma\{B_b - B_v;\ t \leq v \leq b\}, \qquad a \leq t \leq b, \\ \mathcal{H}_s^{(t)} &= \sigma\left(\mathcal{F}_s \cup \mathcal{G}^{(t)}\right), \qquad a \leq s \leq t \leq b. \end{aligned}$$

We call $\{\mathcal{F}_s : s \in [a, b]\}$ the *forward*-filtration and $\{\mathcal{G}^{(t)} : t \in [a, b]\}$ the *backward*- or *counter*-filtration generated by the Brownian motion. Taking the conditional expectation judiciously with respect to the σ-field $\mathcal{H}_s^{(t)}$ plays a crucial part in the proof of the following main theorem.

Theorem 3.1. *Suppose $f, \phi \in C^1(\mathbb{R})$ such that $f(B_t)\phi(B_b - B_t), f(B_t)\phi'(B_b - B_t), f'(B_t)\phi(B_b - B_t) \in L^2([a, b] \times \Omega)$. Then*

$$\mathbb{E}\left[\left(\int_a^b f(B_t)\phi(B_b - B_t)\ dB_t\right)^2\right] = \int_a^b \mathbb{E}\Big[f(B_t)^2\phi(B_b - B_t)^2\Big]\ dt$$
$$+ 2\int_a^b \int_a^t \mathbb{E}\Big[f(B_s)\phi'(B_b - B_s)f'(B_t)\phi(B_b - B_t)\Big]\ ds\ dt. \tag{3.1}$$

Remark 3.2. For the right-hand side of (3.1) to be well-defined, we need the well-definedness of the two integrals. For the first integral, we directly see that the integral exists if $f(B_t)\phi(B_b - B_t) \in L^2([a, b] \times \Omega)$. For conciseness, we write $f_t = f(B_t)$, $\phi_t = \phi(B_b - B_t)$, and similarly their corresponding derivatives. Using this notation, for the second integral, we can use the Schwarz inequality to get

$$\begin{aligned} &\int_a^b \int_a^t \mathbb{E}\Big[f_s\phi_s' f_t'\phi_t\Big] ds\ dt \\ &\leq \int_a^b \int_a^t \left(\mathbb{E}\left[|f_s\phi_s'|^2\right]\right)^{\frac{1}{2}} \left(\mathbb{E}\left[|f_t'\phi_t|^2\right]\right)^{\frac{1}{2}} ds\ dt \\ &\leq \int_a^b \left(\mathbb{E}\left[|f_s\phi_s'|^2\right]\right)^{\frac{1}{2}} ds \int_a^b \left(\mathbb{E}\left[|f_t'\phi_t|^2\right]\right)^{\frac{1}{2}} dt \\ &\leq (b-a)\left(\int_a^b \mathbb{E}\left[|f_s\phi_s'|^2\right] ds\right)^{\frac{1}{2}} \left(\int_a^b \mathbb{E}\left[|f_t'\phi_t|^2\right] dt\right)^{\frac{1}{2}}. \end{aligned}$$

Combining these results, we see that a sufficient condition for the second integral to exist is $f(B_t)\phi'(B_b - B_t), f'(B_t)\phi(B_b - B_t) \in L^2([a, b] \times \Omega)$.

Remark 3.3. In the proof of Itô's isometry, one typically takes conditional expectation with respect to the σ-field $\mathcal{F}_s$ in a simple manner. On the other hand, our proof requires conditioning with respect to the σ-field $\mathcal{H}_s^{(t)}$ in a very specific manner.

Proof. For notational convenience, let

$$\begin{aligned}
\Delta B_k &= B_{t_k} - B_{t_{k-1}}, \\
\Delta t_k &= t_k - t_{k-1}, \\
f_{k-1} &= f(B_{t_{k-1}}), \\
\phi_k &= \phi(B_b - B_{t_k}).
\end{aligned}$$

Then by the definition of the anticipating stochastic integral, we get

$$\int_a^b f(B_t)\phi(B_b - B_t)\, dB_t = \lim_{\|\Delta_n\|\to 0} \sum_{i=1}^n f_{i-1}\phi_i \Delta B_i.$$

By taking a subsequence, if necessary, we may assume that the convergence is in $L^2(\Omega)$. Therefore,

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_a^b f(B_t)\phi(B_b - B_t)\, dB_t\right)^2\right] \\
&= \lim_{\|\Delta_n\|\to 0} \sum_{i=1}^n \sum_{j=1}^n \mathbb{E}\left[f_{i-1}\phi_i f_{j-1}\phi_j \Delta B_i \Delta B_j\right] \\
&= \lim_{\|\Delta_n\|\to 0} \sum_{i=1}^n \mathbb{E}\left[f_{i-1}^2 \phi_i^2 \left(\Delta B_i\right)^2\right] + 2 \lim_{\|\Delta_n\|\to 0} \sum_{j=1}^n \sum_{i=1}^{j-1} \mathbb{E}\left[f_{i-1}\phi_i f_{j-1}\phi_j \Delta B_i \Delta B_j\right] \\
&=: D_0 + 2D_1,
\end{aligned}$$

where we separated the sum into diagonal and off-diagonal elements in the penultimate step and used the symmetry of $i < j$ and $i > j$.

First we focus on the diagonal elements. Note that ΔB_i is independent of both $\mathcal{F}_{t_{i-1}}$ and $\mathcal{G}^{(t_i)}$. Moreover, f_{i-1} is $\mathcal{F}_{t_{i-1}}$-measurable and independent of $\mathcal{G}^{(t_i)}$. Similarly ϕ_i is $\mathcal{G}^{(t_i)}$-measurable and independent of $\mathcal{F}_{t_{i-1}}$. Therefore, by taking conditional expectation with respect to $\mathcal{F}_{t_{i-1}}$, we get

$$\begin{aligned}
\mathbb{E}\left[f_{i-1}^2 \phi_i^2 \left(\Delta B_i\right)^2\right] &= \mathbb{E}\left[\mathbb{E}\left(f_{i-1}^2 \phi_i^2 \left(\Delta B_i\right)^2 \mid \mathcal{F}_{t_{i-1}}\right)\right] \\
&= \mathbb{E}\left[f_{i-1}^2\ \mathbb{E}\left(\phi_i^2 \left(\Delta B_i\right)^2 \mid \mathcal{F}_{t_{i-1}}\right)\right] \\
&= \mathbb{E}\left[f_{i-1}^2\right] \mathbb{E}\left[\phi_i^2 \left(\Delta B_i\right)^2\right].
\end{aligned}$$

Similarly, taking conditional expectation with respect to $\mathcal{G}^{(t_i)}$ gives us

$$\begin{aligned}\mathbb{E}\left[\phi_i^2\,(\Delta B_i)^2\right] &= \mathbb{E}\left[\mathbb{E}\left(\phi_i^2\,(\Delta B_i)^2 \mid \mathcal{G}^{(t_i)}\right)\right]\\ &= \mathbb{E}\left[\phi_i^2\;\mathbb{E}\left((\Delta B_i)^2 \mid \mathcal{G}^{(t_i)}\right)\right]\\ &= \mathbb{E}\left[\phi_i^2\right]\mathbb{E}\left[(\Delta B_i)^2\right].\end{aligned}$$

Putting it all together along with the fact that $\mathbb{E}\left[(\Delta B_i)^2\right] = \Delta t_i$, we get

$$\mathbb{E}\left[f_{i-1}^2\phi_i^2\,(\Delta B_i)^2\right] = \mathbb{E}\left[f_{i-1}^2\right]\mathbb{E}\left[\phi_i^2\right]\Delta t_i = \mathbb{E}\left[f(B_t)^2\phi(B_b-B_t)^2\right]\Delta t_i,$$

where we used the independence of increments of Brownian motion in the last equality. Summing over i and taking limits, we get

$$D_0 = \int_a^b \mathbb{E}\left[f(B_t)^2\phi(B_b-B_t)^2\right]\,dt. \tag{3.2}$$

The method for the off-diagonal elements is not so direct, and we highlight the key tricks.

Trick 1: Note that ΔB_i is independent of both $\mathcal{F}_{t_{i-1}}$ and $\mathcal{G}^{(t_i)}$, and is therefore independent of $\mathcal{H}_{t_{i-1}}^{(t_i)}$. So conditioning with respect to $\mathcal{H}_{t_{i-1}}^{(t_i)}$ gives us

$$\begin{aligned}\mathbb{E}\left(\Delta B_i \mid \mathcal{H}_{t_{i-1}}^{(t_i)}\right) &= \mathbb{E}\left[\Delta B_i\right] = 0,\\ \mathbb{E}\left((\Delta B_i)^2 \mid \mathcal{H}_{t_{i-1}}^{(t_i)}\right) &= \mathbb{E}\left[(\Delta B_i)^2\right] = \Delta t_i.\end{aligned}$$

Trick 2: Consider $B_b - B_{t_i} - \Delta B_j = (B_b - B_{t_j}) + (B_{t_{j-1}} - B_{t_i})$. Since $B_b - B_{t_j}$ is $\mathcal{G}^{(t_j)}$-measurable and $B_{t_{j-1}} - B_{t_i}$ is $\mathcal{F}_{t_{j-1}}$-measurable, the sum $B_b - B_{t_i} - \Delta B_j$ is $\mathcal{H}_{t_{j-1}}^{(t_j)}$-measurable. By continuity of ϕ, we see that $\phi(B_b - B_{t_i} - \Delta B_j)$ is also $\mathcal{H}_{t_{j-1}}^{(t_j)}$-measurable. This allows us to conclude that

$$\begin{aligned}&\mathbb{E}\left[f_{i-1}\phi(B_b-B_{t_i}-\Delta B_j)f_{j-1}\phi_j\;\Delta B_i\;\Delta B_j\right]\\ &= \mathbb{E}\left[\mathbb{E}\left(f_{i-1}\phi(B_b-B_{t_i}-\Delta B_j)f_{j-1}\phi_j\;\Delta B_i\;\Delta B_j \mid \mathcal{H}_{t_{j-1}}^{(t_j)}\right)\right]\\ &= \mathbb{E}\left[f_{i-1}\phi(B_b-B_{t_i}-\Delta B_j)f_{j-1}\phi_j\;\Delta B_i\;\mathbb{E}\left(\Delta B_j \mid \mathcal{H}_{t_{j-1}}^{(t_j)}\right)\right]\\ &= 0.\end{aligned} \tag{3.3}$$

Therefore, subtracting $\mathbb{E}\left[f_{i-1}\phi(B_b-B_{t_i}-\Delta B_j)f_{j-1}\phi_j\;\Delta B_i\;\Delta B_j\right]$ from the term $\mathbb{E}\left[f_{i-1}\phi_i f_{j-1}\phi_j\Delta B_i\Delta B_j\right]$ does not change anything. This allows us to remove the dependence of ϕ_i on $\{B_t : t \in (t_{j-1}, t_j)\}$. This is illustrated in Figure 1 by the dotted region of ϕ_i.

Trick 3: Using the assumption $\phi \in C^1(\mathbb{R})$ and considering the fact that B_t is continuous and so $\Delta B_j \to 0$ as $\|\Delta_n\| \to 0$, we can approximate

$$\phi(B_b - B_{t_i}) - \phi(B_b - B_{t_i} - \Delta B_j) \simeq \phi'(B_b - B_{t_i} - \Delta B_j)\;\Delta B_j. \tag{3.4}$$

For brevity, we write $\Phi_{ij} = \phi'(B_b - B_{t_i} - \Delta B_j)$, $i < j$. Note that Φ_{ij} is $\mathcal{H}_{t_{i-1}}^{(t_i)}$-measurable.

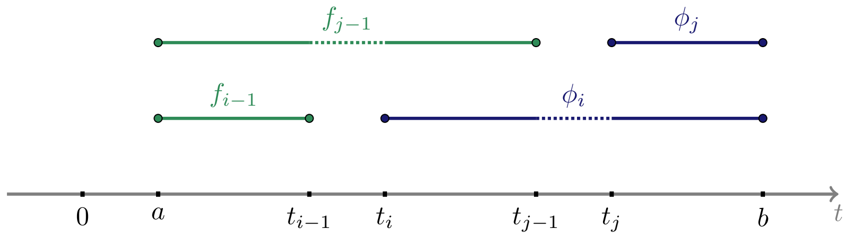

FIGURE 1. A t-dependence plot of the various processes. The dotted regions are removed.

Putting these together, we see that

$$\begin{aligned}
&\mathbb{E}\left[f_{i-1}\phi_i f_{j-1}\phi_j\ \Delta B_i\ \Delta B_j\right] \\
&= \mathbb{E}\left[f_{i-1}\left(\phi(B_b - B_{t_i}) - \phi(B_b - B_{t_i} - \Delta B_j)\right) f_{j-1}\phi_j\ \Delta B_i\ \Delta B_j\right] \\
&\simeq \mathbb{E}\left[f_{i-1}\Phi_{ij} f_{j-1}\phi_j\ \Delta B_i\ (\Delta B_j)^2\right] \\
&= \mathbb{E}\left[\mathbb{E}\left(f_{i-1}\Phi_{ij} f_{j-1}\phi_j\ \Delta B_i\ (\Delta B_j)^2 \mid \mathcal{H}^{(t_j)}_{t_{j-1}}\right)\right] \\
&= \mathbb{E}\left[f_{i-1}\Phi_{ij} f_{j-1}\phi_j\ \Delta B_i\ \mathbb{E}(\Delta B_j)^2\right] \\
&= \mathbb{E}\left[f_{i-1}\Phi_{ij} f_{j-1}\phi_j\ \Delta B_i\right]\Delta t_j.
\end{aligned} \tag{3.5}$$

We repeat Trick 2 on $f(B_{t_{j-1}} - \Delta B_i)$ just as we did for $\phi(B_b - B_{t_i} - \Delta B_j)$ to derive (3.3). This allows us to remove the dependence of f_{j-1} on $\{B_t : t \in (t_{i-1}, t_i)\}$. This is illustrated in Figure 1 by the dotted region of f_{j-1}. Therefore,

$$\mathbb{E}\left[f_{i-1}\Phi_{ij} f_{j-1}\phi_j\ \Delta B_i\ \Delta B_j\right] = 0, \tag{3.6}$$

where we used the tower property with respect to the σ-field $\mathcal{H}^{(t_i)}_{t_{i-1}}$ in this case. As before, we get

$$f(B_{t_{j-1}}) - f(B_{t_{j-1}} - \Delta B_i) \simeq f'(B_{t_{j-1}} - \Delta B_i)\ \Delta B_i. \tag{3.7}$$

Continuing from (3.5),

$$\begin{aligned}
&\mathbb{E}\left[f_{i-1}\phi_i f_{j-1}\phi_j\ \Delta B_i\ \Delta B_j\right] \\
&= \mathbb{E}\left[f_{i-1}\Phi_{ij} f_{j-1}\phi_j\ \Delta B_i\right]\Delta t_j \\
&= \mathbb{E}\left[f_{i-1}\Phi_{ij}\left(f(B_{t_{j-1}}) - f(B_{t_{j-1}} - \Delta B_i)\right)\phi_j\ \Delta B_i\right]\Delta t_j \\
&\simeq \mathbb{E}\left[f_{i-1}\Phi_{ij} f'(B_{t_{j-1}} - \Delta B_i)\phi_j\ (\Delta B_i)^2\right]\Delta t_j \\
&= \mathbb{E}\left[\mathbb{E}\left(f_{i-1}\Phi_{ij} f'(B_{t_{j-1}} - \Delta B_i)\phi_j\ (\Delta B_i)^2 \mid \mathcal{H}^{(t_i)}_{t_{i-1}}\right)\right]\Delta t_j \\
&= \mathbb{E}\left[f_{i-1}\Phi_{ij} f'(B_{t_{j-1}} - \Delta B_i)\phi_j \mathbb{E}\left((\Delta B_i)^2 \mid \mathcal{H}^{(t_i)}_{t_{i-1}}\right)\right]\Delta t_j \\
&= \mathbb{E}\left[f_{i-1}\Phi_{ij} f'(B_{t_{j-1}} - \Delta B_i)\phi_j\right]\ \Delta t_i\ \Delta t_j.
\end{aligned} \tag{3.8}$$

By the continuity of B_t, we see that as $\|\Delta_n\| \to 0$, so does ΔB_i and ΔB_j. Moreover, by the continuity of f' and ϕ', we can conclude that as $\|\Delta_n\| \to 0$,

$$\begin{aligned} f'(B_{t_{i-1}} - \Delta B_i) &\longrightarrow f'(B_{t_{j-1}}) = f'_{j-1}, \\ \Phi_{ij} = \phi'(B_b - B_{t_i} - \Delta B_j) &\longrightarrow \phi'(B_b - B_{t_i}) = \phi'_i. \end{aligned}$$

Finally, summing up (3.8) over $i < j$ and taking limit, we get

$$D_1 = \int_a^b \int_a^t \mathbb{E}\left[f(B_s)\phi'(B_b - B_s)f'(B_t)\phi(B_b - B_t)\right] \, ds \, dt. \tag{3.9}$$

This concludes the proof. □

One of the features of Theorem 3.1 is that the result enables us to evaluate the second moment of these anticipating integrals without having to evaluate the integral itself. This is advantageous as explicitly evaluating the integral via the definition can be very complicated and, in fact, is impossible in general. We demonstrate this particular feature in the next example.

Example 3.4. Apply Theorem 3.1 to the case with $f(x) = x$ and $\phi(y) = y$. Then we have

$$\begin{aligned} &\mathbb{E}\left[\left(\int_a^b B_t(B_b - B_t) \, dB_t\right)^2\right] \\ &= \int_a^b \mathbb{E}\left[B_t^2(B_b - B_t)^2\right] \, dt + 2\int_a^b \int_a^t \mathbb{E}\left[B_s(B_b - B_t)\right] \, ds \, dt \\ &= \int_a^b \mathbb{E}\left[B_t^2\right]\mathbb{E}\left[(B_b - B_t)^2\right] \, dt + 2\int_a^b \int_a^t \mathbb{E}\left[B_s\right]\mathbb{E}\left[B_b - B_t\right] \, ds \, dt \\ &= \int_a^b t(b-t) \, dt \\ &= \frac{1}{6}(b^3 - 3a^2b + 2a^3). \end{aligned} \tag{3.10}$$

On the other hand, let us evaluate the stochastic integral $\int_a^b B_t(B_b - B_t) \, dB_t$ and then use it to compute its second moment. By equation (2.5) in [2], we have

$$\int_0^t B_s(B_T - B_s) \, dB_s = \frac{1}{2}B_T(B_t^2 - t) - \frac{1}{3}B_t^3, \quad 0 \le t \le T,$$

which immediately yields the following stochastic integral

$$\int_a^b B_t(B_b - B_t) \, dB_t = \frac{1}{2}B_b\Big[(B_b^2 - B_a^2) - (b-a)\Big] - \frac{1}{3}\left(B_b^3 - B_a^3\right).$$

For brevity, we write $\Delta_B = B_b - B_a$, so $B_b = (B_b - B_a) + B_a = \Delta_B + B_a$. Performing algebraic simplification, we get

$$\int_a^b B_t(B_b - B_t) \, dB_t = \frac{1}{6}\left(\Delta_B^3 + 3B_a\Delta_B^2 - 3(b-a)\Delta_B - 3(b-a)B_a\right).$$

Note that B_a and Δ_B are independent with $B_a \sim N(0,a)$ and $\Delta_B \sim N(0,b-a)$. Therefore, any odd moment of either of B_a or Δ_B is zero. Using this, we get

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_a^b B_t(B_b-B_t)\, dB_t\right)^2\right] \\
&= \frac{1}{36}\,\mathbb{E}\left[\Delta_B^6 + 9B_a^2\Delta_B^4 + 9(b-a)^2\Delta_B^2 + 9(b-a)^2 B_a^2\right. \\
&\qquad\qquad \left. -6(b-a)\Delta_B^4 - 18(b-a)B_a^2\Delta_B^2\right] \\
&= \frac{1}{6}\left(b^3 - 3a^2 b + 2a^3\right),
\end{aligned}$$

which is exactly what we obtained in equation (3.10). But, obviously, here the computation is more tedious and complicated.

Now, we can use the same arguments as those in the proof of Theorem 3.1 to get the following general theorem.

Theorem 3.5. *Let* $\Phi(x,y) \in C^1(\mathbb{R}^2)$ *and assume that*

$$\Phi(B_t, B_b - B_t), \Phi_x(B_t, B_b - B_t),\ \Phi_y(B_t, B_b - B_t) \in L^2([a,b]\times\Omega).$$

Then

$$\begin{aligned}
\mathbb{E}\left[\left(\int_a^b \Phi(B_t, B_b - B_t)\, dB_t\right)^2\right] &= \int_a^b \mathbb{E}\left[\Phi(B_t, B_b - B_t)^2\right]\, dt \\
&+ 2\int_a^b\int_a^t \mathbb{E}\Big[\Phi_y(B_s, B_b - B_s)\ \Phi_x(B_t, B_b - B_t)\Big]\, ds\, dt. \qquad (3.11)
\end{aligned}$$

We can further extend this result to the product of two integrals.

Theorem 3.6. *Let* $\Phi(x,y), \Psi(x,y) \in C^1(\mathbb{R}^2)$ *and assume that*

(1) $\Phi(B_t, B_b - B_t), \Phi_x(B_t, B_b - B_t),\ \Phi_y(B_t, B_b - B_t) \in L^2([a,b]\times\Omega)$*, and*
(2) $\Psi(B_t, B_b - B_t), \Psi_x(B_t, B_b - B_t),\ \Psi_y(B_t, B_b - B_t) \in L^2([a,b]\times\Omega)$.

Then

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_a^b \Phi(B_t, B_b - B_t)\, dB_t\right)\left(\int_a^b \Psi(B_t, B_b - B_t)\, dB_t\right)\right] \\
&= \int_a^b \mathbb{E}\Big[\Phi(B_t, B_b - B_t)\Psi(B_t, B_b - B_t)\Big]\, dt \\
&\quad + \int_a^b\int_a^t \mathbb{E}\Big[\Phi_y(B_s, B_b - B_s)\Psi_x(B_t, B_b - B_t) \\
&\qquad\qquad + \Phi_x(B_t, B_b - B_t)\Psi_y(B_t, B_b - B_t)\Big]\, ds\, dt. \qquad (3.12)
\end{aligned}$$

Proof. For this proof, we write

$$\begin{aligned}
F(t) &= \Phi(B_t, B_b - B_t), \\
G(t) &= \Psi(B_t, B_b - B_t), \\
H(t) &= F(t) + G(t).
\end{aligned}$$

Moreover, for brevity, we write $F_x(t) = \Phi_x(B_t, B_b - B_t)$, $F_y(t) = \Phi_y(B_t, B_b - B_t)$ and corresponding notations for $G(t)$ and $H(t)$.

From the definition of $H(t)$, we see that

$$\begin{aligned}\mathbb{E}\left[\left(\int_a^b H(t)\ dB_t\right)^2\right] &= \mathbb{E}\left[\left(\int_a^b F(t)\ dB_t + \int_a^b G(t)\ dB_t\right)^2\right] \\ &= \mathbb{E}\left[\left(\int_a^b F(t)\ dB_t\right)^2\right] + \mathbb{E}\left[\left(\int_a^b G(t)\ dB_t\right)^2\right] \\ &\quad + 2\ \mathbb{E}\left[\left(\int_a^b F(t)\ dB_t\right)\left(\int_a^b G(t)\ dB_t\right)\right].\end{aligned}$$

Now, applying Theorem 3.5 for $F(t)$, we get

$$\mathbb{E}\left[\left(\int_a^b F(t)\ dB_t\right)^2\right] = \int_a^b \mathbb{E}\left[F(t)^2\right]\ dt + 2\int_a^b \int_a^t \mathbb{E}\left[F_y(s)\ F_x(t)\right]\ ds\ dt.$$

We can obtain a similar equality for $G(t)$. Putting all this together, we get

$$\begin{aligned}\mathbb{E}\left[\left(\int_a^b H(t)\ dB_t\right)^2\right] &= \int_a^b \mathbb{E}\left[F(t)^2\right]\ dt + 2\int_a^b \int_a^t \mathbb{E}\left[F_y(s)\ F_x(t)\right]\ ds\ dt \\ &\quad + \int_a^b \mathbb{E}\left[G(t)^2\right]\ dt + 2\int_a^b \int_a^t \mathbb{E}\left[G_y(s)\ G_x(t)\right]\ ds\ dt \\ &\quad + 2\ \mathbb{E}\left[\left(\int_a^b F(t)\ dB_t\right)\left(\int_a^b G(t)\ dB_t\right)\right]. \end{aligned} \tag{3.13}$$

On the other hand, first applying Theorem 3.5 and then using the definition of $H(t)$, we get

$$\begin{aligned}&\mathbb{E}\left[\left(\int_a^b H(t)\ dB_t\right)^2\right] \\ &= \int_a^b \mathbb{E}\left[H(t)^2\right] dt + 2\int_a^b \int_a^t \mathbb{E}\left[H_y(s)\ H_x(t)\right]\ ds\ dt \\ &= \int_a^b \mathbb{E}\left[F(t)^2\right] dt + \int_a^b \mathbb{E}\left[G(t)^2\right] dt + 2\int_a^b \mathbb{E}\left[F(t)G(t)\right] dt \\ &\quad + 2\int_a^b \int_a^t \mathbb{E}\left[(F_y(s) + G_y(s))(F_x(t) + G_x(t))\right]\ ds\ dt \\ &= \int_a^b \mathbb{E}\left[F(t)^2\right] dt + \int_a^b \mathbb{E}\left[G(t)^2\right] dt + 2\int_a^b \mathbb{E}\left[F(t)G(t)\right] dt \\ &\quad + 2\int_a^b \int_a^t \mathbb{E}\Big[F_y(s)F_x(t) + F_y(s)G_x(t) + G_y(s)F_x(t) + G_y(s)G_x(t)\Big]\ ds\ dt. \end{aligned} \tag{3.14}$$

Finally, equations (3.13) and (3.14) imply that

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_a^b F(t)\ dB_t\right)\left(\int_a^b G(t)\ dB_t\right)\right] \\
&= \int_a^b \mathbb{E}\left[F(t)G(t)\right]\ dt + \int_a^b \int_a^t \mathbb{E}\left[F_y(s)G_x(t) + G_y(s)F_x(t)\right]\ ds\ dt,
\end{aligned}$$

which is exactly the desired result. □

If $\Phi(x,y) = f(x)$ and $\Psi(x,y) = \phi(y)$, we have $\Phi_y \equiv 0$ and $\Psi_x \equiv 0$. Therefore, we obtain the following corollary.

Corollary 3.7. *Let $f, \phi \in C^1(\mathbb{R})$ and assume that*

(1) $f(B_t), \phi(B_b - B_t) \in L^2([a,b]\times\Omega)$, and
(2) $f'(B_t), \phi'(B_b - B_t) \in L^2([a,b]\times\Omega)$.

Then

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_a^b f(B_t)\ dB_t\right)\left(\int_a^b \phi(B_b - B_t)\ dB_t\right)\right] \\
&= \int_a^b \mathbb{E}\Big[f(B_t)\phi(B_b - B_t)\Big]\ dt + \int_a^b \int_a^t \mathbb{E}\Big[\phi'(B_b - B_s)f'(B_t)\Big] ds\ dt.
\end{aligned}$$

Similar to Theorem 3.1, Corollary 3.7 enables us to evaluate the covariance between anticipating and adapted integrals without explicitly calculating the integral itself. This is illustrated in the following example.

Example 3.8. Let $f(x) = x$ and $\phi(y) = y$. Using Corollary 3.7, we get

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_a^b B_t\ dB_t\right)\left(\int_a^b (B_b - B_t)\ dB_t\right)\right] \\
&= \int_a^b \mathbb{E}\left[B_t(B_b - B_t)\right]\ dt + \int_a^b \int_a^t \mathbb{E}\left[1\right] ds\ dt \\
&= \int_a^b \mathbb{E}\left[B_t\right]\mathbb{E}\left[B_b - B_t\right]\ dt + \int_a^b (t-a)\ dt \\
&= \frac{1}{2}(b-a)^2.
\end{aligned}$$

Finally, we want to point our that the double integral in equation (3.11) can be regarded as a correction term when we extend Itô's theory to anticipating stochastic integration. This correction term can be positive or negative, as illustrated in the next example.

Example 3.9. Consider the case $\Phi(x,y) = px + y$ in Theorem 3.5, where $p \in \mathbb{R}$. Then $\Phi_x = p$ and $\Phi_y = 1$. Hence we can directly evaluate the double integral in equation (3.11) as

$$2\int_a^b \int_a^t \mathbb{E}\Big[\Phi_y(B_s, B_b - B_s)\ \Phi_x(B_t, B_b - B_t)\Big]\ ds\ dt$$
$$= 2\int_a^b \int_a^t p\ ds\ dt = p(b-a)^2.$$

Therefore, the final term will be positive or negative depending on the sign of p.

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