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Multigrid methods for $H(\text{div})$ in three dimensions with nonoverlapping domain decomposition smoothers

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SUMMARY

We design and analyze $V$-cycle multigrid methods for an $H(\text{div})$ problem discretized by the lowest order Raviart-Thomas hexahedral element. The smoothers in the multigrid methods involve nonoverlapping domain decomposition preconditioners that are based on substructuring. We prove uniform convergence of the $V$-cycle methods on bounded convex hexahedral domains (rectangular boxes). Numerical experiments that support the theory are also presented. Copyright © 2017 John Wiley & Sons, Ltd.

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KEY WORDS: $H(\text{div})$, multigrid method, lowest order Raviart-Thomas hexahedral element, nonoverlapping domain decomposition.
1. INTRODUCTION

Let $\Omega$ be a bounded convex hexahedral domain (rectangular box) in $\mathbb{R}^3$ whose faces are parallel to the coordinate planes, and $H_0(\text{div}; \Omega)$ be the space of square integrable vector fields on $\Omega$ that have square integrable divergences in $\Omega$ and vanishing normal components on $\partial \Omega$ (cf. [12]). In this paper we consider multigrid methods for the following problem: Find $u \in H_0(\text{div}; \Omega)$ such that

$$a(u, v) = (f, v) \quad \forall \ v \in H_0(\text{div}; \Omega),$$

(1.1)

where

$$a(w, v) = \alpha(\text{div} w, \text{div} v) + (w, v),$$

(1.2)

and $(\cdot, \cdot)$ is the inner product on $L_2(\Omega)$ (or $[L_2(\Omega)]^3$). We assume that $\alpha$ is positive and $f \in [L_2(\Omega)]^3$.

We will also use the notation

$$a_D(w, v) = \alpha(\text{div} w, \text{div} v)_{L_2(D)} + (w, v)_{L_2(D)}$$

for any subdomain $D$ of $\Omega$.

Unlike the scalar elliptic equation case, multigrid methods for the problem (1.1) with simple smoothers do not work. We need a special treatment for the smoother. In [2–4, 15], an overlapping domain decomposition preconditioner was employed in the construction of the smoother. Our goal is to develop multigrid methods in the same spirit but using instead nonoverlapping domain decomposition preconditioners, which reduce the dimensions of the subproblems that have to be solved. We note that other multigrid methods for $H(\text{div})$ were investigated in [14, 16, 17]. Also, related domain decomposition methods have been applied successfully in [21, 22, 24, 25, 29].

Applications of fast solvers for $H(\text{div})$ problems are discussed for example in [2, 18, 28]. In particular the multigrid method in this paper can be applied to a mixed method for second order partial differential equations based on a first-order system least-squares formulation [2, 10], which is equivalent to our model problem. It can also be used as an effective preconditioner for $H(\text{div})$ problems with variable coefficients. The model problem also arises in Reissner-Mindlin plates [1] and Brinkman equations [27].
The rest of this paper is organized as follows. We present the standard discretization of (1.1) by the lowest order Raviart-Thomas hexahedral element in Section 2 and introduce the $V$-cycle multigrid method in Section 3. We establish stability estimates for the nonoverlapping domain decompositions in Section 4, which is crucial for the convergence analysis carried out in Section 5. Numerical results are presented in Section 6 and we end with some concluding remarks in Section 7.

2. THE DISCRETE PROBLEM

Let $\mathcal{T}_h$ be a hexahedral triangulation of $\Omega$. The lowest order Raviart-Thomas $H(\text{div})$ conforming finite element space [20,23] is denoted by $V_h$. A vector field $v$ belongs to $V_h$ if and only if it belongs to $H_0(\text{div}; \Omega)$ and takes the form

$$
\begin{bmatrix}
a_1 \\
a_2 \\
a_3 \\
\end{bmatrix} + 
\begin{bmatrix}
b_1 x_1 \\
b_2 x_2 \\
b_3 x_3 \\
\end{bmatrix}
$$

on each hexahedral element, where the $a_i$’s and $b_i$’s are constants. On each hexahedral element $T$ the vector field $v$ is determined by the six degrees of freedom defined by

$$
\lambda_{RT}^F(v) := \frac{1}{|F|} \int_F v \cdot n \, dS,
$$

(2.1)

where $F$ is one of the six faces of $T$, $|F|$ is the area of $F$, and $n$ is the unit outer normal.

The discrete problem for (1.1) is to find $u_h \in V_h$ such that

$$
a(u_h, v) = \int_{\Omega} f \cdot v \, dx \quad \forall v \in V_h.
$$

(2.2)

In the multigrid approach we solve (2.2) on a sequence of triangulations $\mathcal{T}_0, \mathcal{T}_1, \ldots$, where $\mathcal{T}_0$ is an initial triangulation of $\Omega$ by hexahedral elements and $\mathcal{T}_k$ ($k \geq 1$) is obtained from $\mathcal{T}_{k-1}$ by uniform subdivision. We will denote the lowest order Raviart-Thomas finite element space associated with $\mathcal{T}_k$ by $V_k$. The $k$-th level discrete problem is to find $u_k \in V_k$ such that

$$
a(u_k, v) = (f, v) \quad \forall v \in V_k.
$$
Let $A_k : V_k \rightarrow V'_k$ be defined by

$$
\langle A_k w, v \rangle = a(w, v) \quad \forall v, w \in V_k,
$$

(2.3)

where $\langle \cdot, \cdot \rangle$ is the canonical bilinear form on $V'_k \times V_k$. We can then rewrite the $k$-th level discrete problem as

$$
A_k u_k = f_k,
$$

(2.4)

where $f_k \in V'_k$ is defined by

$$
\langle f_k, v \rangle = (f, v) \quad \forall v \in V_k.
$$

Multigrid methods are optimal order iterative methods for equations of the form

$$
A_k z = g
$$

(2.5)

that includes (2.4) as a special case.

3. V-CYCLE MULTIGRID METHODS

Since the finite element spaces are nested, we can take the coarse-to-fine operator $I_{k-1}^k : V_{k-1} \rightarrow V_k$ to be the natural injection. The fine-to-coarse operator $I_{k-1}^k : V'_k \rightarrow V'_{k-1}$ is then defined by

$$
\langle I_{k-1}^k \ell, v \rangle = \langle \ell, I_{k-1}^k v \rangle \quad \forall \ell \in V'_k, v \in V_{k-1}.
$$

(3.1)

The output $MG(k, g, z_0, m)$ of the $k$th level (symmetric) multigrid V-cycle algorithm for (2.5), with initial guess $z_0 \in V_k$ and $m$ smoothing steps, is defined by the following recursive steps:

For $k = 0$, the output is obtained from a direct method:

$$
MG(0, g, z_0, m) = A_0^{-1} g.
$$

For $k \geq 1$, we set

$$
z_l = z_{l-1} + M_{k-1}^{-1} (g - A_k z_{l-1}) \quad \text{for } 1 \leq l \leq m,
$$

$$
\overline{g} = I_{k-1}^k (g - A_k z_m),
$$

$$
z_{m+1} = z_m + I_{k-1}^k MG(k-1, \overline{g}, 0, m).$$
\[ z_l = z_{l-1} + M_k^{-1} (g - A_k z_{l-1}) \quad \text{for } m + 2 \leq l \leq 2m + 1. \]

The output of \( MG(k, g, z_0, m) \) is \( z_{2m+1} \).

**Remark 3.1**

Given \( \ell \in V'_k \), we will construct \( M_k \) so that the cost of computing \( M_k^{-1} \ell \) is proportional to \( n_k \), where \( n_k \) is the dimension of \( V_k \). Therefore the overall cost for computing \( MG(k, g, z_0, m) \) is also \( O(n_k) \).

We will use a smoother of the form

\[ z_{\text{new}} = z_{\text{old}} + M_k^{-1} (g - A_k z_{\text{old}}) \]  

(3.2)

for the equation (2.5), where \( M_k^{-1} : V'_k \rightarrow V_k \) is a nonoverlapping domain decomposition smoother defined below.

To conform with standard terminology in domain decomposition, in the rest of this section we will denote \( T_{k-1} \) by \( T_H \) and \( T_k \) by \( T_h \). (Thus each element in \( T_H \) is partitioned into eight elements in \( T_h \)). The spaces \( V_{k-1} \) and \( V_k \) are denoted by \( V_H \) and \( V_h \) respectively. The smoother \( M_k^{-1} \) in (3.2) is denoted by \( M_h^{-1} \) here. It is constructed by substructuring. We introduce two types of smoothers: face based and edge based.

For each element \( T \in T_H \), we define the twelve dimensional subspace \( V^T_h \) of \( V_h \) by

\[ V^T_h = \{ v \in V_h : v = 0 \text{ on } \Omega \setminus T \}. \]  

(3.3)

The natural injection from \( V^T_h \) into \( V_h \) is denoted by \( J_T \) and the operator \( A_T : V^T_h \rightarrow (V^T_h)' \) is defined by

\[ \langle A_T w, v \rangle = a(w, v) \quad \forall v, w \in V^T_h. \]  

(3.4)

We note that \( A_T \) is just a submatrix of \( A \) associated with \( T \).

### 3.1. Face-based Smoothers

We first consider a face based smoother. Let \( F_H \) be the set of the interior faces of the triangulation \( T_H \). Given any \( F \in F_H \) that is the common face of two elements \( T^+_F \) and \( T^-_F \) in \( T_H \), we define the...
four dimensional subspace $V^F_h$ of $V_h$ by

$$V^F_h = \{ v \in V_h : v = 0 \text{ on } \Omega \setminus (T^+ \cup T^-) \quad \text{and} \quad a(v, w) = 0 \quad \forall \ w \in (V^T_h + V^T_h) \}. \quad (3.5)$$

Here, $V^T_h$ (resp. $V^T_h$) is defined by (3.3) with respect to $T^+$ (resp. $T^-$).

We note that each $F \in \mathcal{F}_h$ consists of four faces in $\mathcal{F}_h$, denoted by $f_1, f_2, f_3$ and $f_4$. We consider the basis functions $\phi^F_i$ associated with $f_i$ with the properties that (i) $\phi^F_i \cdot n = 1$ (ii) $a(\phi^F_i, w) = 0, \forall w \in (V^T_h + V^T_h)$. The basis functions can be easily constructed by solving local Dirichlet problems. The space $V^F_h$ is spanned by the four basis functions.

The natural injection from $V^F_h$ into $V_h$ is denoted by $J_F$ and the operator $A_F : V^F_h \rightarrow (V^F_h)'$ is defined by

$$\langle A_F w, v \rangle = a(w, v) \quad \forall \ v, w \in V^F_h. \quad (3.6)$$

The natural injection $J_F$ is implemented by representing the basis constructed from local Dirichlet problems in terms of the standard global basis. The operator $A_F$ can be implemented by a Galerkin product using $J_F$ and $A$. Note that (3.5) implies $v \in V^F_h$ is determined by its degrees of freedom on $F$ and

$$a_T^+(v, v) \leq a_T^+(w, w) \quad (3.7)$$

if $w \in V_h$ has the same degrees of freedom as $v$ on $\partial T^+ \cup \partial T^-$. The subspaces associated with the elements and interior faces of $\mathcal{T}_h$ form a direct sum decomposition of $V_h$:

$$V_h = \sum_{T \in \mathcal{T}_h} V^T_h + \sum_{F \in \mathcal{F}_h} V^F_h, \quad (3.8)$$

and the smoother $M^{-1}_{F,h}$ is given by

$$M^{-1}_{F,h} = \eta_F (\sum_{T \in \mathcal{T}_h} J_T A_T^{-1} J_T^T + \sum_{F \in \mathcal{F}_h} J_F A_F^{-1} J_F^T), \quad (3.9)$$

where $\eta_F$ is a damping factor and $J_T^T : V^T_h \rightarrow (V^T_h)'$ (resp. $J_F^T : V^F_h \rightarrow (V^F_h)'$) is the transpose of $J_T$ (resp. $J_F$) with respect to the canonical bilinear forms.
3.2. Edge-based smoothers

We now construct an edge based smoother. Let \( \mathcal{E}_H \) be the set of interior edges of the triangulation \( T_H \). Given any \( E \in \mathcal{E}_H \), there are four elements in \( T_H \), \( T^1_E, T^2_E, T^3_E, \) and \( T^4_E \), and four faces in \( F_H \), \( F^1_E, F^2_E, F^3_E, \) and \( F^4_E \), that are sharing the edge \( E \). We define the sixteen dimensional subspace \( V^E_h \) of \( V_h \) by

\[
V^E_h = \{ v \in V_h : v = 0 \text{ on } \Omega \setminus \left( \bigcup_{i=1}^4 T^i_E \right), \quad v \cdot n = 0 \text{ on } \left( \bigcup_{i=1}^4 \partial T^i_E \right) \setminus \left( \bigcup_{i=1}^4 F^i_E \right), \quad a(v,w) = 0 \quad \forall w \in \left( V^1_h + V^2_h + V^3_h + V^4_h \right) \}
\]  

(3.10)

We note that \( v \in V^E_h \) is determined by its degrees of freedom on \( \bigcup_{i=1}^4 F^i_E \) and \( a_{T^i_E}(v,v) \leq a_{T^i_E}(w,w) \) if \( w \) has the same degrees of freedom as \( v \) on \( \partial T^i_E, i = 1, 2, 3, 4 \).

The spaces associated with the elements and interior edges of \( T_H \) form a decomposition of \( V_h \):

\[
V_h = \sum_{T \in T_H} V^T_h + \sum_{E \in \mathcal{E}_H} V^E_h.
\]  

(3.12)

Let \( J_E : V^E_h \rightarrow V_h \) be the natural injection and the operator \( A_E : V^E_h \rightarrow (V^E_h)' \) be defined by

\[
\langle A_E w, v \rangle = a(w,v) \quad \forall v, w \in V^E_h.
\]  

(3.13)

We note that the constructions of \( J_E \) and \( A_E \) are similar to the constructions of \( J_F \) and \( A_F \) for the face based method.

The edge based smoother \( M^{-1}_{E,h} \) is then defined by

\[
M^{-1}_{E,h} = \eta_E \left( \sum_{T \in T_H} J_T A_T^{-1} J_T^T + \sum_{E \in \mathcal{E}_H} J_E A_E^{-1} J_E^T \right),
\]  

(3.14)

where \( \eta_E \) is a damping factor and \( J_E^T : V'_h \rightarrow (V^E_h)' \) is the transpose of \( J_E \).

We note that the domain decomposition smoother \( M^{-1}_{F,h} \) is a one-level version of the preconditioner in [28]. The smoothers \( M^{-1}_{F,h} \) and \( M^{-1}_{E,h} \) are clearly symmetric. They are also positive-definite because of (3.8) and (3.12) (cf. [9,26]). By a standard coloring argument in [26, Chapter 2], we also have

the spectral radius of \( M^{-1}_{F,h} A_h \leq 1 \) and the spectral radius of \( M^{-1}_{E,h} A_h \leq 1 \)  

(3.15)
if \( \eta_F \leq 1/11 \) and \( \eta_E \leq 1/33 \), which are assumed to be the case from now on. We note that for any \( F \in \mathcal{F}_H \) there are two elements \( T^+_F \) and \( T^-_F \) that share \( F \) as a common face. There are at most 11 faces in \( \mathcal{F}_H \) that have overlap with \( \partial T^+_F \cup \partial T^-_F \). Hence, we have an upper bound 1/11 for \( \eta_F \). Similarly we have an upper bound 1/33 for \( \eta_E \).

4. STABILITY ESTIMATES FOR THE DOMAIN DECOMPOSITIONS

In this Section we will establish stability estimates for the domain decompositions that are crucial for the convergence analysis in Section 5. As in Section 3, we will write \( T_{k-1} \) and \( T_k \) as \( T_H \) and \( T_h \) respectively. The Ritz projection operator from \( V_h \) to \( V_H \) with respect to \( a(\cdot, \cdot) \) is denoted by \( P_H \) and the identity operator on \( V_h \) is denoted by \( I \).

We begin with three technical lemmas. The first one concerns an extension result for the trace space of the lowest order Raviart-Thomas finite element. For \( T \in \mathcal{T}_H \), let \( S_h(\partial T) \) be the space of piecewise constant functions on \( \partial T \) with respect to the twenty four faces of \( \mathcal{T}_h \) that are part of \( \partial T \), and \( S_{0;h}(\partial T) \) be the subspace of \( S_h(\partial T) \) whose members have zero mean, i.e., \( \int_{\partial T} \mu dS = 0 \) if \( \mu \in S_{0;h}(\partial T) \).

We will also need the lowest order Nédélec edge element space \( N_h \) (cf. [20]) for the convergence analysis. Let \( H_0(\mathbf{curl}; \Omega) \) be the space of square integrable vector fields with square integrable curl in \( \Omega \) and vanishing tangential components along \( \partial \Omega \) (cf. [12]). A vector field \( \mathbf{q} \) belongs to \( N_h \) if and only if it belongs to \( H_0(\mathbf{curl}; \Omega) \) and on each hexahedral element has the form

\[
\begin{bmatrix}
a_1 + a_2 x_2 + a_3 x_3 + a_4 x_2 x_3 \\
b_1 + b_2 x_3 + b_3 x_1 + b_4 x_3 x_1 \\
c_1 + c_2 x_1 + c_3 x_2 + c_4 x_1 x_2
\end{bmatrix},
\]

where the \( a_i \)'s, \( b_i \)'s and \( c_i \)'s are constants. On each hexahedral element \( \mathbf{q} \) is determined by the twelve degrees of freedom defined by

\[
\lambda_{ND}^N (\mathbf{q}) := \frac{1}{|E|} \int_E \mathbf{q} \cdot \mathbf{t} \, ds, \quad (4.1)
\]
where \( E \) is one of the twelve edges of the element, \(|E|\) is the length of \( E \), and \( t \) is a unit vector tangential to \( E \).

The following result, which will be used in the derivation of the stability estimate for the face based domain decomposition, is a direct consequence of [28, Lemma 4.3] and a scaling argument.

**Lemma 4.1**

Given any \( T \in V_H \) and \( \mu \in S_{0,h}(\partial T) \), there exists \( \zeta \in V_h(T) \) such that (i) \( \zeta \cdot n = \mu \) on \( \partial T \), (ii) \( \text{div} \, \zeta = 0 \), and (iii) there exists a positive constant \( C \) independent of \( h \) such that

\[
\|\zeta\|_{L^2(T)}^2 \leq C h \|\mu\|_{L^2(\partial T)}^2.
\]  

(4.2)

The second lemma, which will also be used in the derivation of the stability estimate for the face based domain decompositon, is obtained by a direct calculation.

**Lemma 4.2**

Let \( T \) be an element of \( \mathcal{T}_H \), \( F_1 \) and \( F_2 \) be two opposite faces of \( T \), and \( n_1 \) and \( n_2 \) be the unit normals of \( F_1 \) and \( F_2 \) pointing towards the outside of \( T \). Define \( u \in V_h(T) \) by the properties that (i) \( u \cdot n_1 = 1 \) on \( F_1 \), (ii) \( u \cdot n_2 = -1 \) on \( F_2 \), (iii) the normal component of \( u \) vanishes on the other faces of \( T \), and (iv) \( u \) is orthogonal to \( V_{h}^{T} \) with respect to \( (\cdot, \cdot)_{L^2(T)} \). Then we have \( \text{div} \, u \neq 0 \).

The following result on the orthogonal complement of the coarse space is from [4, Proposition 4.3]. It is crucial for the stability estimates for both domain decompositions.

**Lemma 4.3**

For any \( v \in (I - P_H)V_h \), there exist \( w \in V_h \) and \( q \in N_h \) such that

\[
v = w + \text{curl} \, q,
\]  

(4.3)

and

\[
\|w\|_{L^2(\Omega)}^2 + \|\text{curl} \, q\|_{L^2(\Omega)}^2 = \|v\|_{L^2(\Omega)}^2,
\]  

(4.4)

\[
\alpha \|w\|_{L^2(\Omega)}^2 \leq CH^2 \alpha (v, v),
\]  

(4.5)

\[
\|q\|_{L^2(\Omega)}^2 \leq CH^2 \|v\|_{L^2(\Omega)}^2,
\]  

(4.6)
where the positive constant $C$ is independent of $h$.

We are now ready to establish stability estimates for the decompositions (3.8) and (3.12).

4.1. A Stability Estimate for the Face Based Domain Decomposition

We have the following result for the face based domain decomposition.

**Lemma 4.4**

There exists a positive constant $C_{F,\dagger}$, independent of $\alpha$, $h$, and the number of elements in $T_H$, such that the unique decomposition

$$v = \sum_{T \in T_H} v_T + \sum_{F \in F_H} v_F$$

of any $v \in (I - P_H)V_h$ satisfies the estimate

$$\sum_{T \in T_H} a(v_T, v_T) + \sum_{F \in F_H} a(v_F, v_F) \leq C_{F,\dagger} a(v, v).$$

**Proof**

Let $v \in (I - P_H)V_h$ be arbitrary. We will prove (4.7) by treating the two vector fields $w$ and $\text{curl} \ q$ in the decomposition (4.3) separately.

We first consider $w$, which has a unique decomposition

$$w = \sum_{T \in T_H} w_T + \sum_{F \in F_H} w_F,$$

where $w_T \in V_h^T$ and $w_F \in V_h^F$.

Since the subdomain spaces $V_h^T$ are mutually orthogonal and they are also orthogonal to all the face spaces $V_h^F$ with respect to the bilinear form $a(\cdot, \cdot)$, we deduce from (4.5), (4.8), and a standard inverse estimate that

$$\sum_{T \in T_H} a(w_T, w_T) = a(\sum_{T \in T_H} w_T, \sum_{T \in T_H} w_T)$$

$$\leq a(w, w)$$

$$= \sum_{T \in T_H} (\alpha \| \text{div} \ w \|^2_{L^2(T)} + \| w \|^2_{L^2(T)})$$

$$\leq \sum_{T \in T_H} (\alpha C h^{-2} ||w||^2_{L^2(T)} + ||w||^2_{L^2(T)}) \leq C a(v, v).$$


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Given any $F \in \mathcal{F}_H$ shared by the two hexahedral elements $T^\pm_F \in \mathcal{T}_H$, we define $\tilde{w}_F \in V_h$ by the conditions that (i) $\tilde{w}_F$ and $w$ share the same degrees of freedom associated with the four faces of $T_h$ that are subsets of $F$ and (ii) all the other degrees of freedom of $\tilde{w}_F$ equal 0. Then $\tilde{w}_F$ vanishes on $\Omega \setminus (T^-_F \cup T^+_F)$ and hence,

$$a(w_F, w_F) \leq a(\tilde{w}_F, \tilde{w}_F) \tag{4.10}$$

by (3.7). Moreover, the following estimate

$$\|\tilde{w}_F\|_{L^2(T^\pm_F)} \leq C \|w\|_{L^2(T^\pm_F)} \tag{4.11}$$

can be proved by a standard scaling argument.

Let $T_F$ be the set $\{T^-_F, T^+_F\}$. Combining (4.4), (4.5), (4.10), (4.11) and a standard inverse estimate, we find

$$\sum_{F \in \mathcal{F}_H} a(w_F, w_F) \leq \sum_{F \in \mathcal{F}_H} a(\tilde{w}_F, \tilde{w}_F)$$

$$= \sum_{F \in \mathcal{F}_H} \sum_{T \in T_F} \left[ \alpha \|\text{div} \tilde{w}_F\|^2_{L^2(T)} + \|\tilde{w}_F\|^2_{L^2(T)} \right]$$

$$\leq C \sum_{F \in \mathcal{F}_H} \sum_{T \in T_F} \left[ \alpha h^{-2} \|w\|^2_{L^2(T)} + \|w\|^2_{L^2(T)} \right] \leq Ca(v, v),$$

which together with (4.9) implies

$$\sum_{T \in \mathcal{T}_H} a(w_T, w_T) + \sum_{F \in \mathcal{F}_H} a(w_F, w_F) \leq Ca(v, v). \tag{4.12}$$

Next we consider $c = \text{curl} q$, which has a unique decomposition

$$c = \sum_{T \in \mathcal{T}_H} c_T + \sum_{F \in \mathcal{F}_H} c_F, \tag{4.13}$$

where $c_T \in V^T_h$ and $c_F \in V^F_h$.

As in the case of $w$, we have, by (4.4),

$$\sum_{T \in \mathcal{T}_H} a(c_T, c_T) \leq a(c, c) = \|c\|^2_{L^2(\Omega)} \leq \|v\|^2_{L^2(\Omega)} \leq a(v, v). \tag{4.14}$$

We now consider $c_F$, where the estimate involving $\text{div} c_F$ is more delicate. Let $F \in \mathcal{F}_H$ be the common face of $T^\pm_F \in \mathcal{T}_H$ and $n_F$ be the unit normal of $F$ pointing from $T^-_F$ to $T^+_F$. Then we have
\( c_F \cdot n_F = \text{curl} \ q \cdot n_F \). Let \( c_{F,H} \in V_h^F \) such that \( c_{F,H} \cdot n_F \) is a constant on \( F \) and

\[
\int_F \left( c_F - c_{F,H} \right) \cdot n_F \, dS = 0.
\]

According to Lemma 4.1 and the definition of \( c_F \) and \( c_{F,H} \), there exist \( \zeta_\pm \in V_h(T^\pm) \) such that \( \zeta_\pm \) has the same degrees of freedom as \( c_F - c_{F,H} \) on \( \partial T^\pm_F \), \( \text{div} \zeta_\pm = 0 \), and

\[
\| \zeta \|^2_{L_2(T^\pm_F)} \leq Ch \| (c_F - c_{F,H}) \cdot n_F \|^2_{L_2(F)} \leq Ch^{-1} \| q \|^2_{L_2(F)}.
\]  \hspace{1cm} (4.15)

It then follows from (3.7) and (4.15) that

\[
a(c_F - c_{F,H}, c_F - c_{F,H}) \leq a(\zeta, \zeta) = \| \zeta \|^2_{L_2(T^\pm_F)} + \| \zeta \|^2_{L_2(T^\pm_F)} \leq Ch^{-1} \| q \|^2_{L_2(F)}
\]  \hspace{1cm} (4.16)

and it only remains to estimate \( a(c_{F,H}, c_{F,H}) \).

Let \( F' \) be the face of \( T^-_F \) opposite \( F \) and \( n_{F'} \) be the outward pointing unit normal of \( F' \) with respect to \( T^-_F \). We note that \( n_F \) is the outward pointing unit normal of \( F \) with respect to \( T^-_F \). Let \( \tilde{c}_{F,H} \in V_h(T^-_F) \) be defined by the following properties: (i) \( \tilde{c}_{F,H} \cdot n_F = c_{F,H} \cdot n_F \) on \( F \), (ii) \( \tilde{c}_{F,H} \cdot n_{F'} = -c_{F,H} \cdot n_F \) on \( F' \), (iii) the normal component of \( \tilde{c}_{F,H} \) vanishes on the other faces of \( T^-_F \), and

\[
a_{T^-_F}(\tilde{c}_{F,H}, z) = 0 \quad \forall z \in V_h^{T^-_F}.
\]

Then, by the same arguments that led to (4.16), we have

\[
a_{T^-_F}(\tilde{c}_{F,H}, \tilde{c}_{F,H}) \leq Ch^{-1} \| q \|^2_{L_2(F)}.
\]  \hspace{1cm} (4.17)

Finally we compare \( c_{F,H} \) and \( \tilde{c}_{F,H} \). Since \( \| \tilde{c}_{F,H} \|^2_{L_2(T^-_F)} = 0 \) if and only if \( c_{F,H} = 0 \), we have

\[
\| c_{F,H} \|^2_{L_2(T^-_F)} \leq C \| \tilde{c}_{F,H} \|^2_{L_2(T^-_F)}
\]  \hspace{1cm} (4.18)

by a scaling argument. Moreover, it follows from Lemma 4.2 that \( \text{div} \tilde{c}_{F,H} = 0 \) if and only if \( c_{F,H} = 0 \). Hence we have, by a scaling argument,

\[
\| \text{div} c_{F,H} \|^2_{L_2(T^-_F)} \leq C \| \text{div} \tilde{c}_{F,H} \|^2_{L_2(T^-_F)}.
\]  \hspace{1cm} (4.19)

It follows from (4.17)–(4.19) that

\[
a_{T^-_F}(c_{F,H}, c_{F,H}) = \alpha \| \text{div} c_{F,H} \|^2_{L_2(T^-_F)} + \| c_{F,H} \|^2_{L_2(T^-_F)} \leq C a_{T^-_F}(\tilde{c}_{F,H}, \tilde{c}_{F,H}) \leq Ch^{-1} \| q \|^2_{L_2(F)}
\]
and hence

$$a(c_{F,H}, c_{F,H}) \leq C h^{-1} \|q\|_{L_2(F)}^2$$

(4.20)

by symmetry. Combining (4.6), (4.16), (4.20) and scaling, we have

$$\sum_{F \in \mathcal{F}_H} a(c_{F,H}, c_{F,H}) \leq C h^{-1} \sum_{F \in \mathcal{F}_H} \|q\|_{L_2(F)}^2 \leq C h^{-2} \|q\|_{L_2(\Omega)}^2 \leq C a(v, v).$$

(4.21)

It follows from the estimates (4.14) and (4.21) that

$$\sum_{T \in \mathcal{T}_H} a(c_T, c_T) + \sum_{F \in \mathcal{F}_H} a(c_{F,H}, c_{F,H}) \leq C a(v, v),$$

which together with (4.12) implies (4.7), since $v_T = w_T + c_T$ and $v_F = w_F + c_F$. \qed

4.2. A Stability Estimate for the Edge Based Domain Decomposition

We have the following result for the edge based domain decomposition.

Lemma 4.5

For any $v \in (I - P_H) V_h$, there exists a decomposition

$$v = \sum_{T \in \mathcal{T}_H} v_T + \sum_{E \in \mathcal{E}_H} v_E$$

and a constant $C_{E,1}$, independent of $\alpha, h$ and the number of elements in $\mathcal{T}_H$, such that

$$\sum_{T \in \mathcal{T}_H} a(v_T, v_T) + \sum_{E \in \mathcal{E}_H} a(v_E, v_E) \leq C_{E,1} a(v, v).$$

(4.22)

Proof

We will follow the approach for Lemma 4.4 and treat $w$ and $\text{curl} \, q$ in (4.3) separately.

For each $E \in \mathcal{E}_H$, there are four coarse faces $F^1_E, F^2_E, F^3_E$, and $F^4_E (\in \mathcal{F}_H)$ that are sharing $E$. Let $N_E^i$ be the number of coarse edges in $\mathcal{E}_H$ that are parts of $\partial F^i_E$, for $i = 1, 2, 3, 4$. We construct $w_E \in V^E_h$ by

$$w_E \cdot n = \frac{1}{N_E^i} w \cdot n \text{ on } F^i_E, \quad i = 1, 2, 3, 4.$$  

(4.23)

Since $w$ and $\sum_{E \in \mathcal{E}_H} w_E$ have identical degrees of freedom on the faces of $V_H$ by (4.23), $w - \sum_{E \in \mathcal{E}_H} w_E$ belongs to $\sum_{T \in \mathcal{T}_H} V^T_h$ and hence

$$w = \sum_{T \in \mathcal{T}_H} w_T + \sum_{E \in \mathcal{E}_H} w_E$$

(4.24)
for unique vector fields \( w_T \in V_h^T \).

Using the same arguments in the derivation for (4.9), we obtain

\[
\sum_{T \in \mathcal{T}_h} a(w_T, w_T) \leq Ca(v, v). \tag{4.25}
\]

Recall that each \( E \in \mathcal{E}_h \) is shared by four hexahedral elements \( T^i_E \in \mathcal{T}_h \), \( i = 1, 2, 3, 4 \). Let \( \tilde{w}_E \in V_h \) be defined by the conditions that (i) \( \tilde{w}_E \) and \( w_E \) share the same degrees of freedom associated with the sixteen faces of \( T_h \) that are subsets of \( \bigcup_{i=1}^4 F^i_E \), and (ii) all the other degrees of freedom of \( \tilde{w}_E \) equal zero. Then \( \tilde{w}_E \) vanishes on \( \Omega \setminus \left( \bigcup_{i=1}^4 T^i_E \right) \) and hence,

\[
a(w_E, w_E) \leq a(\tilde{w}_E, \tilde{w}_E) \tag{4.26}
\]

by (3.11). Moreover, by a standard scaling argument, we have

\[
\|\tilde{w}_E\|_{L_2(T^i_E)} \leq C \|w\|_{L_2(T^i_E)}, \quad i = 1, 2, 3, 4. \tag{4.27}
\]

Let \( \mathcal{T}_E \) be the set \( \{ T^1_E, T^2_E, T^3_E, T^4_E \} \). Putting (4.4), (4.5), (4.26) and (4.27) together, we have

\[
\sum_{E \in \mathcal{E}_h} a(w_E, w_E) \leq \sum_{E \in \mathcal{E}_h} a(\tilde{w}_E, \tilde{w}_E) = \sum_{E \in \mathcal{E}_h} \sum_{T \in \mathcal{T}_E} \left[ \alpha \|\text{div} \; \tilde{w}_E\|_{L_2(T)}^2 + \|\tilde{w}_E\|_{L_2(T)}^2 \right] \leq \sum_{E \in \mathcal{E}_h} \sum_{T \in \mathcal{T}_E} \left[ \alpha h^{-2} \|\tilde{w}_E\|_{L_2(T)}^2 + \|\tilde{w}_E\|_{L_2(T)}^2 \right] \leq \sum_{E \in \mathcal{E}_h} \sum_{T \in \mathcal{T}_E} C \left[ \alpha h^{-2} \|w\|_{L_2(T)}^2 + \|w\|_{L_2(T)}^2 \right] \leq Ca(v, v),
\]

which together with (4.25) implies

\[
\sum_{T \in \mathcal{T}_h} a(w_T, w_T) + \sum_{E \in \mathcal{E}_h} a(w_E, w_E) \leq Ca(v, v). \tag{4.28}
\]

Next we consider \( c = \text{curl} \; q \). We can write

\[
q = \sum_{e \in \mathcal{E}_h} q_e, \tag{4.29}
\]

where

\[
\int_e q_e \cdot t_e \, ds = \int_e q \cdot t_e \, ds \tag{4.30}
\]
and \( \lambda^{N_D}_e(q_e) = 0 \) for all \( e' \in \mathcal{E}_h \setminus \{e\} \). Here, \( \mathcal{E}_h \) is the set of interior edges of the triangulation \( T_h \).

Let \( \tilde{c}_E \in V_h \) be defined by

\[
\tilde{c}_E = \text{curl} \left( \sum_{e \in \mathcal{E}_h, \, e \subset E} q_e + \sum_{i=1}^{4} \frac{1}{N_E} \sum_{e \in \mathcal{E}_h, \, e \subset (F_E \setminus \partial F_E)} q_e \right).
\] (4.31)

Note that, by a standard inverse estimate, we have

\[
\|\tilde{c}_E\|_{L^2(T)}^2 \leq C h^{-2} \|q\|_{L^2(T)}^2 \quad \text{for any } T \in \mathcal{T}_h.
\] (4.32)

We then construct \( c_E \in V^E_h \) so that

\[
c_E \cdot n = \tilde{c}_E \cdot n \quad \text{on } F^i_E, \quad i = 1, 2, 3, 4.
\] (4.33)

Since \( c \) and \( \sum_{E \in \mathcal{E}_h} c_E \) have the same degrees of freedom on the faces of \( V_h \) by (4.31), we have

\[
c = \sum_{T \in \mathcal{T}_h} c_T + \sum_{E \in \mathcal{E}_h} c_E
\] (4.34)

for unique vector fields \( c_T \in V^T_h \).

Again the estimate (4.14) holds for \( c_T \). Moreover, by the construction of \( c_E \) and (3.11), we have the estimate

\[
a(c_E, c_E) \leq a(\tilde{c}_E, \tilde{c}_E) = \sum_{T \in \mathcal{T}_h} \|\tilde{c}_E\|^2_{L^2(T)}.
\] (4.35)

Hence, by (4.6), (4.32) and (4.35), we have

\[
\sum_{E \in \mathcal{T}_E} a(c_E, c_E) \leq C h^{-2} \|q\|^2_{L^2(\Omega)} \leq C \|v\|^2_{L^2(\Omega)} \leq C a(v, v).
\] (4.36)

It follows from the estimates (4.14) and (4.36) that

\[
\sum_{T \in \mathcal{T}_H} a(c_T, c_T) + \sum_{E \in \mathcal{E}_H} a(c_E, c_E) \leq C a(v, v),
\]

which together with (4.28) implies (4.22) for the decomposition with \( v_T = w_T + c_T \) and \( v_E = w_E + c_E \). \( \square \)
5. CONVERGENCE ANALYSIS

Let $E_k : V_k \rightarrow V_k$ be the error propagation operator for the $V$-cycle multigrid algorithm with $m$ smoothing steps. Then $E_0 = 0$ and we have a well-known recursive relation [7, 13, 19]

$$E_k = R_k^m (I_{d_k} - I_{k-1}^k P_{k-1}^k) R_k^m + R_k^m (T_{k-1}^k E_{k-1} P_{k-1}^k) R_k^m \quad \text{for } k \geq 1. \quad (5.1)$$

Here the operator $R_k : V_k \rightarrow V_k$ is defined by

$$R_k = Id_k - M_k^{-1} A_k, \quad (5.2)$$

where $Id_k$ is the identity operator on $V_k$, and $P_{k-1}^k : V_k \rightarrow V_{k-1}$ is the Ritz projection operator defined by

$$a(P_{k-1}^k w, v) = a(w, T_{k-1}^k v) \quad \forall w \in V_k, \ v \in V_{k-1}. \quad (5.3)$$

Note that $R_k$ is symmetric with respect to the inner product $a(\cdot, \cdot)$. Hence it follows immediately from (5.1) that $E_k$ is symmetric positive semi-definite with respect to $a(\cdot, \cdot)$.

We will follow the approach in [6] (cf. also [9, Chapter 6]). We begin with a smoothing property.

**Lemma 5.1**

We have

$$a((Id_k - R_k) R_k^m v, R_k^m v) \leq \frac{1}{2m^2} a((Id_k - R_k^2m) v, v) \quad \forall v \in V_k, \ k \geq 1. \quad (5.4)$$

**Proof**

Let $v \in V_k$ be arbitrary. Since $R_k$ is symmetric with respect to the inner product $a(\cdot, \cdot)$, it follows from (3.15) and the spectral theorem that

$$a((Id_k - R_k) R_k^j v, v) \leq a((Id_k - R_k) R_k^j v, v) \quad \forall v \in V_k, \ 0 \leq j \leq l,$$

and hence

$$(2m)a((Id_k - R_k) R_k^m v, R_k^m v) = (2m)a((Id_k - R_k^2m) v, v)$$

$$\leq \sum_{j=0}^{2m-1} a((Id_k - R_k) R_k^j v, v) = a((Id_k - R_k^2m) v, v).$$

$\square$
Next we derive approximation properties.

**Lemma 5.2**

We have

\[
\langle M_F, k^{-1} w, (Id_k - I_{k-1}^k P_{k-1})^2 w \rangle \leq \frac{C_{F, \dagger}}{\eta_F} a(w, w)
\]

and

\[
\langle M_E, k^{-1} w, (Id_k - I_{k-1}^k P_{k-1})^2 w \rangle \leq \frac{C_{E, \dagger}}{\eta_E} a(w, w)
\]

for all \( w \in V_k \) and \( k \geq 1 \).

**Proof**

Since \( M^{-1} \) defined by (3.9) is an additive Schwarz preconditioner, we can apply the theory of additive Schwarz preconditioners [5, 9, 11] to conclude that, for any \( v \in V_h \),

\[
\langle M_F, k v, v \rangle = \eta_F^{-1} \left( \sum_{T \in T_H} a(v_T, v_T) + \sum_{F \in F_H} a(v_F, v_F) \right),
\]

(5.4)

where \( v = \sum_{T \in T_H} v_T + \sum_{F \in F_H} v_F \) is the unique decomposition of \( v \) according to (3.8). The approximation property for \( M_F, k \) then follows from Lemma 4.4 and (5.4), with \( v = (Id_k - I_{k-1}^k P_{k-1}) w \).

Similarly we have, for any \( v \in V_h \),

\[
\langle M_E, k v, v \rangle = \eta_E^{-1} \left( \inf_{v_T = \sum_{T \in T_H} v_T + \sum_{E \in E_H} v_E} \left( \sum_{T \in T_H} a(v_T, v_T) + \sum_{E \in E_H} a(v_E, v_E) \right) \right).
\]

(5.5)

The approximation property for \( M_E, k \) follows from Lemma 4.5 and (5.5), with \( v = (Id_k - I_{k-1}^k P_{k-1}) w \).

We can derive another approximation property using Lemma 5.2 and duality.

**Lemma 5.3**

We have

\[
a((Id_k - I_{k-1}^k P_{k-1}) w, (Id_k - I_{k-1}^k P_{k-1})^2 w) \leq \frac{C_1}{\eta} a((Id_k - R_k) w, w) \quad \forall w \in V_k, k \geq 1,
\]

where \( C_1 = C_{F, \dagger} \) (resp. \( C_{E, \dagger} \)) and \( \eta = \eta_F \) (resp. \( \eta_E \)) if \( M_k = M_{F, k} \) (resp. \( M_{E, k} \)).

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Proof

Let \( \| \cdot \|_a = \sqrt{a(\cdot, \cdot)} \) be the energy norm. By duality, we have

\[
\| (\text{Id}_k - I_{k-1}^k P_k^{k-1}) w \|_a = \max_{v \in V_k} \frac{a((\text{Id}_k - I_{k-1}^k P_k^{k-1}) w, v)}{\| v \|_a} \quad (5.6)
\]

We can estimate the numerator on the right-hand side of (5.6) by (5.3), Lemma 5.2 and the Cauchy-Schwarz inequality as follows:

\[
a((\text{Id}_k - I_{k-1}^k P_k^{k-1}) w, v) = a(\text{Id}_k - I_{k-1}^k P_k^{k-1}) w, (\text{Id}_k - I_{k-1}^k P_k^{k-1}) v) \\
\leq \left\langle \left( M_k(M_k^{-1} A_k) w, (M_k^{-1} A_k) w \right) \right\rangle^{\frac{1}{2}} \left\langle \left( M_k((\text{Id}_k - I_{k-1}^k P_k^{k-1}) v, (\text{Id}_k - I_{k-1}^k P_k^{k-1}) v) \right) \right\rangle^{\frac{1}{2}} \\
= a(M_k^{-1} A_k w, w)^{\frac{1}{2}} \left( \frac{C_F}{\eta} \right)^{\frac{1}{2}} a(v, v)^{\frac{1}{2}} \\
= a((\text{Id}_k - R_k) w, w)^{\frac{1}{2}} \left( \frac{C_F}{\eta} \right)^{\frac{1}{2}} \| v \|_a.
\]

The lemma follows from this estimate and (5.6). \(\square\)

We can now establish the uniform convergence of the V-cycle methods.

**Theorem 5.4**

We have

\[
\| E_k v \|_a \leq \left( \frac{C_F}{\eta} \right) \left( \frac{C_F}{\eta} \right) + 2m \| v \|_a \quad \forall v \in V_k, k \geq 1,
\]

where \( C_F = C_{F,1} \) (resp. \( C_{E,1} \)) and \( \eta = \eta_F \) (resp. \( \eta_E \)) if \( M_k = M_{F,k} \) (resp. \( M_{E,k} \)).

**Proof**

Since \( E_k \) is symmetric positive semi-definite, it suffices to show that

\[
a(E_k v, v) \leq \frac{C_*}{C_* + 2m} a(v, v) \quad \forall v \in V_k, k \geq 1, \quad (5.7)
\]

where \( C_* = C_F/\eta \).

We will prove (5.7) by mathematical induction. The case \( k = 0 \) is trivial since \( E_0 = 0 \). Assume that (5.7) holds for \( k - 1 \) and let \( \delta = C_*/(C_* + 2m) \). It follows from (5.1), (5.3), the induction hypothesis, Lemma 5.1 and Lemma 5.3 that

\[
a(E_k v, v) = a(R_{k}^m (\text{Id}_k - I_{k-1}^k P_k^{k-1} + I_{k-1}^k E_{k-1} P_k^{k-1}) R_{k}^m w, v)
\]
≤ a((Id_k - I_{k-1}^k P_k^{k-1}) R_k^m v, (Id_k - I_{k-1}^k P_k^{k-1}) R_k^m v) + \delta a(P_k^{k-1} R_k^m v, P_k^{k-1} R_k^m v)
= (1 - \delta)a((Id_k - I_{k-1}^k P_k^{k-1}) R_k^m v, (Id_k - I_{k-1}^k P_k^{k-1}) R_k^m v) + \delta a(R_k^m v, R_k^m v)
≤ (1 - \delta)C_a((Id_k - R_k) R_k^m v, R_k^m v) + \delta a(R_k^m v, R_k^m v)
≤ (1 - \delta)\frac{C_*}{2m}a((Id_k - R_k^m v, v) + \delta a(R_k^m v, R_k^m v) = \delta a(v, v).

Since the constants $C_{F,\dagger}, \eta_{F,\dagger}, C_{E,\dagger}, \eta_{E}$ are independent of $k$ and $\alpha$, Theorem 5.4 implies that the $V$-cycle method converges uniformly on all levels and for any $\alpha > 0$.

6. NUMERICAL RESULTS

In the first experiment we consider (1.1) on the unit cube $\Omega = (0, 1)^3$ with $\alpha = 1$. We apply multigrid algorithms with the smoothers introduced in Section 3. The damping factors $\eta_F$ and $\eta_E$ are taken to be $1/11$ and $1/33$, respectively. The initial triangulation $T_0$ consists of eight identical cubes and we compute the contraction numbers of the $k^{th}$ level $V$-cycle multigrid method for $k = 1, \ldots, 5$ and for $m$ smoothing steps, where $m = 1, \ldots, 6$. We report the contraction numbers obtained by computing the largest eigenvalue of the error propagation operators. The results are presented in Table I. The uniform convergence of the $V$-cycle multigrid methods for $m \geq 1$ is clearly observed. The contraction numbers of the computationally more intensive edge based algorithm are also smaller.

In the second numerical experiment we compute the contraction numbers of the $V$-cycle multigrid methods on level five, for $\alpha = 0.01, 0.1, 1, 10, 100$ and $m = 1, \ldots, 6$. The results are depicted by the log-log plots in Figure 1. It is observed that both methods are robust with respect to $\alpha$. The contraction numbers for the edge based algorithm for different values of $\alpha$ are in fact indistinguishable.

Finally, we compare the multigrid method based on nonoverlapping smoothers with the method based on overlapping smoothers (where the amount of overlap consists of one layer of elements). To do so, we use the multigrid methods as preconditioners for the preconditioned conjugate gradient
Table I. Contraction numbers of the $V$-cycle multigrid method for the unit cube

<table>
<thead>
<tr>
<th>$m = 1$</th>
<th>$m = 2$</th>
<th>$m = 3$</th>
<th>$m = 4$</th>
<th>$m = 5$</th>
<th>$m = 6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Face based smoother ($M^{-1}_{F,h}$)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$k = 1$</td>
<td>9.1e-1</td>
<td>8.3e-1</td>
<td>7.1e-1</td>
<td>5.0e-1</td>
<td>3.1e-1</td>
</tr>
<tr>
<td>$k = 2$</td>
<td>9.2e-1</td>
<td>8.7e-1</td>
<td>7.9e-1</td>
<td>6.3e-1</td>
<td>5.0e-1</td>
</tr>
<tr>
<td>$k = 3$</td>
<td>9.3e-1</td>
<td>9.0e-1</td>
<td>8.4e-1</td>
<td>7.4e-1</td>
<td>6.3e-1</td>
</tr>
<tr>
<td>$k = 4$</td>
<td>9.4e-1</td>
<td>9.1e-1</td>
<td>8.7e-1</td>
<td>8.0e-1</td>
<td>7.1e-1</td>
</tr>
<tr>
<td>$k = 5$</td>
<td>9.4e-1</td>
<td>9.2e-1</td>
<td>8.8e-1</td>
<td>8.2e-1</td>
<td>7.5e-1</td>
</tr>
<tr>
<td>Edge based smoother ($M^{-1}_{E,h}$)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$k = 1$</td>
<td>9.1e-1</td>
<td>8.3e-1</td>
<td>6.9e-1</td>
<td>4.8e-1</td>
<td>2.3e-1</td>
</tr>
<tr>
<td>$k = 2$</td>
<td>9.0e-1</td>
<td>8.2e-1</td>
<td>6.8e-1</td>
<td>4.6e-1</td>
<td>2.1e-1</td>
</tr>
<tr>
<td>$k = 3$</td>
<td>9.0e-1</td>
<td>8.2e-1</td>
<td>6.8e-1</td>
<td>4.6e-1</td>
<td>2.1e-1</td>
</tr>
<tr>
<td>$k = 4$</td>
<td>9.0e-1</td>
<td>8.2e-1</td>
<td>6.7e-1</td>
<td>4.6e-1</td>
<td>2.1e-1</td>
</tr>
<tr>
<td>$k = 5$</td>
<td>9.0e-1</td>
<td>8.1e-1</td>
<td>6.7e-1</td>
<td>4.6e-1</td>
<td>2.1e-1</td>
</tr>
</tbody>
</table>

method. We use $V$-cycle multigrids on level five with $m$ smoothing steps, where $m = 1, \ldots, 6$. The iterations are stopped when the $l^2$–norm of the residuals have been reduced by a factor of $10^{-6}$. The results are displayed in Table II. It is observed that comparing to the multigrid method based on the overlapping smoother, on average the multigrid method that uses the face based smoother is 1.19 times faster and the multigrid method that uses the edge based smoother is 1.35 times faster.

7. CONCLUDING REMARKS

We have developed new $V$-cycle multigrid methods for an $H(div)$ problem discretized by the lowest order Raviart-Thomas hexahedral element. The smoothers for the multigrid methods involve nonoverlapping domain decomposition preconditioners that are based on substructuring. The subproblems that need to be solved for the preconditioners involve $12 \times 12$ sparse matrices and $4 \times 4$ full matrices (face based) or $16 \times 16$ full matrices (edge based), while preconditioners based on overlapping domain decomposition would require the solution of $36 \times 36$ sparse systems.
A MULTIGRID METHOD FOR $H(DIV)$

Figure 1. Decay of contraction numbers at level 5 for different values of $\alpha$

Table II. CPU time and iteration counts for face based, edge based, and overlapping smoothers

<table>
<thead>
<tr>
<th>$m$</th>
<th>face based</th>
<th>edge based</th>
<th>overlapping</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>131.52(30)</td>
<td>128.98(17)</td>
<td>151.26(14)</td>
</tr>
<tr>
<td>2</td>
<td>120.75(20)</td>
<td>108.34(11)</td>
<td>144.43(10)</td>
</tr>
<tr>
<td>3</td>
<td>121.11(16)</td>
<td>109.03(9)</td>
<td>146.13(9)</td>
</tr>
<tr>
<td>4</td>
<td>120.84(13)</td>
<td>101.86(7)</td>
<td>161.87(7)</td>
</tr>
<tr>
<td>5</td>
<td>129.69(12)</td>
<td>115.24(7)</td>
<td>153.06(6)</td>
</tr>
<tr>
<td>6</td>
<td>137.52(11)</td>
<td>114.33(6)</td>
<td>154.33(6)</td>
</tr>
</tbody>
</table>

We have proved the uniform convergence of the $V$-cycle multigrid method for convex hexahedral domains (rectangular boxes), which is confirmed by numerical experiments. It is possible to

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extend the result to general convex polyhedral domains by using the lowest order Raviart-Thomas tetrahedral element. The result can also be extended to problems where $H_0(\text{div}; \Omega)$ is replaced by $H(\text{div}; \Omega)$ since the analog of Lemma 4.3 is available for $H(\text{div}; \Omega)$ [4].

Numerical results indicate that the $V$-cycle multigrid method is uniformly convergent on nonconvex domains. Also, even though the current theory does not cover the coefficients with jumps, numerical experiments show that the smoothers are robust. For more detail, see [8].

Finally we note that the techniques developed in this paper can be applied to multigrid methods for $H(\text{div})$ problems that are based on other domain decomposition preconditioners [21, 22, 24, 25, 29].

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