The Periodic Hall Effect in Metals at Low Temperatures and High Magnetic Fields.

Edward George Grimsal Jr
Louisiana State University and Agricultural & Mechanical College

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THE PERIODIC HALL EFFECT
IN METALS AT LOW TEMPERATURES AND HIGH MAGNETIC FIELDS

A Thesis

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
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in
The Department of Physics

by
Edward George Grimsal, Jr.
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ABSTRACT

A preliminary discussion is given of the existence, in certain effects in a magnetic field, of components which are quasi-periodic in the reciprocal of the magnetic field. Three of the more important effects so far discovered to have this property are the magnetic susceptibility, the conductivity, and the Hall effect. The dependence of each of these effects on the number of free electrons is discussed. An expression is then derived, predicting oscillations in the number of a small but special group of electrons. With the aid of approximate formulae for the above effects, expressions are obtained for the oscillations in each effect. The relative magnitude and phase of the oscillations in each effect is discussed and compared with experiment. Qualitatively, the agreement is good, and some semi quantitative agreement with experiment is obtained. The Hall effect data obtained in this laboratory are analyzed by this theory, with good internal agreement. An attempt is made to improve the formula used for the Hall effect, but although two new expressions are obtained in special cases, no better agreement with experimental results is obtained.
CHAPTER 1

INTRODUCTION

The electron theory of metals, a prominent part of the theory of the solid state, attempts to relate many of the thermal, electric, magnetic, and electromagnetic properties of metals to the presence of free or quasi-free electrons in the metals. Properties such as electrical and thermal conductivity, magnetic susceptibility, the galvanomagnetic effects, specific heat, the electrothermal effects, and some optical properties, to name a few, are attributed to the presence of a degenerate Fermi gas of electrons in the metal, and to the possible interactions of these electrons with the lattice.

It would then seem reasonable, in a case where anomalous behaviour is found in one of the above properties for a certain metal, to investigate others of these properties for analogous effects. Hence the discovery by Schubnikow and de Haas (1930a,b) of departures, at low temperature, from the expected monotonic increase with magnetic field of the magnetoresistance of bismuth led de Haas and van Alphen (1930a,b, 1932a,b, to investigate the field dependence of the magnetic susceptibility of that metal. Appreciable oscillations, later found to be quasi-periodic in the reciprocal of the magnetic field, were noted at liquid hydrogen temperatures and were even more marked at liquid helium temperatures. The
effect was systematically investigated by Schoenberg and Uadin (1936), and especially by Schoenberg (1939), stimulated by the development of a theory for this effect through the successive contributions of Peierls (1933), Blackman (1938), and Landau (1939).

The anomalies in the magnetoresistance were further investigated by de Haas, Blom and Schubnikow (1935); and other investigators found similar, albeit much weaker effects in the magnetoresistance of zinc (Nakimovitch, 1942), and tin (Borovik, 1949). However, until 1947, it was assumed that the oscillations in the magnetic susceptibility of bismuth were a particular property of that metal. In that year, Marcus (1947) found that zinc showed oscillations in its low temperature magnetic susceptibility. The subsequent work of Mackinnon (1949), and especially of Syacriak and Robinson (1949), served to confirm the close relationship between the effects in bismuth and in zinc. Since then, oscillations in the magnetic susceptibility have been found in a number of metals (Schoenberg, 1949) (Berlincourt, 1954d).

In the meantime, several new approaches to the theory of the susceptibility oscillations were advanced by Humer (1943), Sondheimer and Wilson (1951), and Dingle (1952a,b,c, 1953); Akheiser (1939a), considered the effect of spin in the case of free electrons, and Robinson (1950) modified the Peierls-Blackman-Landau theory in such a way as to obtain better agreement with the observed temperature variation of
the oscillations. Also theories of the periodic variations in the magnetoresistance, based on the early work of Titeica (1935), were advanced by Akheiser (1939b) and Davydov and Pomeranchuk (1940).

The definitive experimental work of Alers and Webber (1953) on the magnetoresistance of bismuth, coupled with the investigations of Berlincourt (1953) on the magnetic susceptibility of the same crystals, confirmed the intimate relation between the two effects. The similarity was again confirmed in the case of graphite by Berlincourt and Logan (1954) and later extended by Berlincourt (1954b) to include the field oscillations in the Hall effect. This periodic Hall effect was first indicated in bismuth by the work of Gerritsen and de Haas (1940) and clearly demonstrated in the work of Reynolds, Leinhardt and Hemstreet (1954). These disclosures were closely followed by the reports of field oscillations in the thermoelectric effect in bismuth (Steele and Babiskin, 1954) and in the thermal conductivity of bismuth (Babiskin and Steele, 1954) which apparently have the same periodicity in the reciprocal of the magnetic field as the three effects named above.

Thus it would seem reasonable to search for some property of the conduction electrons on which the above named effects all depend. From their dependence on this property it should be possible to draw some conclusions as to the relative magnitudes and the phases of the oscillations in these effects. At the time this investigation was
begun, the existence of oscillations had been definitely shown only in the case of the magnetic susceptibility and the magnetoresistance, although they were expected in the Hall effect. Because of the known dependence of these three effects on the "number of free electrons", it was decided to investigate the possibility of finding oscillations in this quantity. In Chapter II we shall see that indeed we can expect appreciable oscillations in the number of a small but special group of electrons (Grimsal and Levinger, 1953; Levinger and Grimsal, 1954). In Chapter III, we shall discuss the experimental results with the aid of the calculations of Chapter II and the very simplest formulas for the above effects in terms of the number of electrons. Chapter IV will contain a further discussion on the general problem of the magnetoresistance and Hall effect at high magnetic fields and low temperatures.
THE OSCILLATIONS IN THE NUMBER OF CARRIERS

The Peierls-Blackman-Landau theory of the oscillating magnetic susceptibility is based on the fact that in the presence of a magnetic field the motion of the free or conduction electrons in the plane normal to the magnetic field is quantized with energy \( \omega^{\star} \), where \( \omega \) is the Larmor angular velocity, \( eH/mc \). Moreover, the degeneracy of each level is found to be proportional to \( H \). It would not seem amiss to give a simple derivation of these results here. More rigorous quantum mechanical derivations may be found in Landau (1930) or Dingle (1952a).

For free electrons with a magnetic field applied in the z direction, the motion parallel to the field is unaffected. The energy due to this motion is still

\[
E_h = \frac{p_z^2}{2m} = \frac{\hbar^2 k_z^2}{2m}
\]

(1)

Since \( R = L/L_z \), where \( L \) is the length of the normalizing box (or the metal crystal) in the z direction, we may have

\[
2n_{max} = 2k_z L_z/2\pi = (2mE_h/h)^2 L_z/2\pi
\]

states with energy less than \( E_h \) (the doubling occurring because of the two possible orientations of the electron spin). The motion perpendicular to the field will be circular with a frequency \( eH/2\pi mc \). This type of motion can be considered as the resultant of two simple harmonic oscillators at right angles. According to
quantum theory, the energy of a simple harmonic oscillator is 
\((\omega+\frac{1}{2})\hbar \omega\) where \(\omega\) is an integer. Substituting the value of
the frequency given above, we find that the energy associated
with this motion is \(\beta H\langle \ell + \frac{1}{2} \rangle\), where \(\beta\) is the doubled Bohr
magnetron, \(e\hbar/mc\). (Another proof of this dependence will be
found in Chapter IV.) Letting \(k_x^2 + k_y^2 = k^2\), we then have
\[ E_{\ell} = \frac{\hbar^2 k^2}{2m} = \beta H(\ell + \frac{1}{2}) \quad (2) \]
Hence
\[ 2k_x k_y = (2m/\hbar^2)\beta HS \]
or
\[ k_x k_y = eH/\hbar c \]
for \(\delta l = 1\). The area in phase space associated with each
level will be
\[ (2\pi k_x k_y)S = 2\pi eH D/\hbar c \]
where \(D\) is the cross sectional area of the metal. The
number of states for each level is then twice the area in
phase space divided by \(4\pi^2\) or
\[ \frac{eHD}{\pi^2 \hbar c} \quad (3) \]
Thus the number of states with quantum number \(l\) and with the
energy due to motion in the \(z\) direction less than \(E_{\parallel}\) is
\[ N_l = (eHV/\pi^2\hbar^2)(2mE_l)^{1/2} \quad (4) \]
where \(V = LD\) is the volume in coordinate space. Since
\(E = E_{\parallel} + E_\perp\), we may replace \(E_{\parallel}\) by \(E - \beta H(\ell + \frac{1}{2})\). The total
number of states per unit volume with energy less than \(E\) is
then
where the summation is to be carried out over all values of \( \ell \) which give a positive value to the quantity in the brackets.

For convenience in later calculations we wish to change to the dimensionless variable \( \epsilon = E/\beta H \) and write this equation in the form

\[
Z = A (\beta H)^{3/2} \sum E \{ e^{-(E+\frac{1}{2})^2/3} \}
\]

where \( A = (2m)^{3/2}/(2\pi^2 k^3) \)

Proceeding from this result, Blackman (1938) calculated the free energy per unit volume, \( F \), and the number of electrons per unit volume, \( N \), from the following formulae.

\[
F - NE_0 = -kT \int_0^\infty dE \frac{dZ}{dE} \log \left[ 1 + \exp \left( \frac{E-E_0}{kT} \right) \right]
\]

\[
N = \int_0^\infty dE \frac{dZ}{dE} \left\{ 1 + \exp \left( \frac{E-E_0}{kT} \right) \right\}^{-1}
\]

The reciprocal of the quantity in curly brackets in equation (5) is the Fermi-Dirac distribution function, and will be represented by \( g(E, E_0, kT) \). It should also be noted that if we take the partial derivative of the log term in equation (7) with respect to \( E \), we will obtain \( (kT)^{-1} g(E, E_0, kT) \).

Both expressions can be integrated by parts, giving

\[
F - NE_0 = -\int_0^\infty dE Z(EH) g(E, E_0, kT) - \left[ Z \log g + \exp \left( \frac{E-E_0}{kT} \right) \right]_0^\infty
\]

\[
N = (1/kT) \int_0^\infty dE Z(EH) \frac{dZ}{dE} + \frac{1}{kT} \left[ \frac{Z}{1 + \exp \left( \frac{E-E_0}{kT} \right)} \right]_0^\infty
\]
The bracketed terms vanish at the limits, and a second integra-
tion by parts of the expression for $F - NE_0$ leaves us with the
result

$$F - NE_0 = \frac{1}{kT} \int_{0}^{\infty} dE \left[ \int \frac{dE}{dE_{0}} g(E_{0}, kT) \right]$$  \hspace{1cm} (9)

$$N = \frac{1}{kT} \int_{0}^{\infty} dE \ z(EH) \frac{d}{dE} g(E_{0}, kT)$$  \hspace{1cm} (10)

Blackman then integrated the above expressions in the special
case of zero temperature, where $(1/kT) dE/dE$ becomes the Dirac
delta function $\delta(E - E_0)$. The magnetic susceptibility, $\chi$, is then found from the expression (Seitz, 1949, p. 580)

$$\chi = -\frac{1}{H} \frac{dF}{dH}$$  \hspace{1cm} (11)

using equation (9) and equation (10), and either assuming
fixed $N$ with $E_0$ varying, or letting $N$ be fixed and $E_0$ the
parameter that varies. The two expressions thus obtained
for $\chi$ are similar (Schoenberg, 1949; Dingle, 1952a).

We shall illustrate Blackman's method by obtaining
the expression for $N(E, H)$ assuming $E_0$ constant. Substituting
equation (6) into equation (10), we obtain

$$N = A (\beta H)^{3/2} \int_{0}^{\infty} dE \ y(E) \theta^{-1} \frac{d}{d\theta} g(E, \theta)$$  \hspace{1cm} (12)

where $\theta = kT/\beta H$ and $y(E) = \frac{\pi}{2} \left[ E - (E + 1/2)^{3/2} \right]$. For zero tempera-
ture, i.e., $\theta = 0$, we obtain

$$N = A (\beta H)^{3/2} y(E_0) = A E_0^{3/2} e_0^{-3/2} y(E_0)$$
Figure 1 is a plot of $\xi^{3/2} \psi(\xi)$ against $\xi$. Thus for the absolute zero of temperature and for low values of $\xi$, we may expect appreciable field oscillations in the number of carriers, if $E_0$ is held fixed. The oscillations are seen to be quasi-periodic in $\xi$ and hence will be quasi-periodic in the reciprocal of the magnetic field. For large values of $\xi$, the summation can be replaced by an integration, the oscillations disappear, and $\xi^{3/2} \psi(\xi)$ will approach the value $2/3$ as might be expected from an examination of the curve in Figure 1. In this case $N$ will be given by

$$N(\xi, H) = \frac{(2mE_0)^{3/2}}{2\pi^{1/2}h^3} \xi^{3/2} \psi(\xi)$$

(13)

where $\psi(\xi)$ is the usual expression relating $N$ and $E_0$ at the absolute zero of temperature.

That such oscillations in $N$ should exist can be shown rather simply by a two-dimensional model, i.e., a model neglecting the degeneracy due to motion in the $z$ direction. We notice that both the energy and the degeneracy of the rotational levels are proportional to the magnetic field. Consider then the case where the magnetic field is so large that the energy for the lowest level is above the Fermi energy. Then the number of electrons in the levels induced by the field will be zero. As the applied field is decreased, a point is reached at which the energy of the $L=0$ level is
just equal to the Fermi energy, i.e., $H = 2E_0/B$. A certain
number of electrons can then be accommodated in this level,
which number will decrease as the field is lowered further.
This decrease in the number of electrons will continue until
the energy of the $\ell=1$ level is equal to the Fermi energy,
which occurs when $H = 2E_0/3B$. At this point there will be
a sharp increase in the number of available states, followed
by a further decrease until the field reaches the value
$H = 2E_0/5B$ and so on. Thus the two-dimensional model gives
rise to a quasi-periodic sawtooth pattern. The effect of
the continuous energy spectrum associated with the motion in
the $z$ direction is to round off the corners of the sawtooth,
and the curve of Figure 1 results.

This, of course, is all for the absolute zero of
temperature. Any finite temperature has the effect of
making the Fermi surface indefinite within a range of the
order of $kT$. Hence for $kT$ much larger than the separation
of the discrete levels, i.e., $kT > \beta H$, we would expect little
or no oscillations. Even for $kT$ of the same order of magni
tude as the separation of the levels, the effect should be
considerably diminished. So even at this early stage, from
the considerations of this paragraph and from an examination
of Figure 1, we can see that the periodic effects will be
negligible unless $kT < \beta H < E_0$. Since $k$ is $1.380 \times 10^{-16}$
ergs per degree Kelvin and $\beta$ is $1.855 \times 10^{-20}$ ergs per gauss,
we must have $H/T$ greater than $7.5 \times 10^3$ gauss per degree and
$E_0/H$ of the order of $2 \times 10^{-20}$ erg per gauss. For a field of
10,000 gauss we must then have temperatures of the order of one degree Kelvin, and a Fermi energy of \(2 \times 10^{-16}\) ergs or about \(10^{-4}\) electron volts. Obviously, the average conduction electrons in a metal are not going to fulfill the last condition, and we must look for a special group of electrons.

Moreover, if these electrons have a low effective mass, the value of the doubled Bohr magneton will be raised correspondingly. The method for introducing an anisotropic effective mass in the Bohr magneton was also shown by Blackman. For carriers near the boundary of a Brillouin zone, the energy often can be given as a quadratic function of the crystal momentum \(\hbar k\) by

\[
E = E_b \pm \frac{\hbar^2}{2m} \sum_{\alpha=1}^{3} \alpha \cdot k_{\alpha}^2
\]

where \(E_b\) is the energy at the boundary of the zone under consideration, and the plus and minus signs are used for electrons and holes respectively. This convention will keep the \(\alpha\)'s positive for the holes since for this type of carrier \(E_b\) is greater than \(E\). We wish now to change to new variables \(k'_\alpha\) and \(\chi'_\alpha\) such that

\[
E - E_b = \frac{\hbar^2}{2m} \sum_{\alpha=1}^{3} k'_{\alpha}^2
\]

and yet retain the usual commutation relations,

\[
[k'_x, k'_y] = \frac{ie}{\hbar c} H_z \quad \text{etc.} \quad [k'_x, \chi'] = i \quad \text{etc.}
\]

This is possible if we set

\[
k'_x = \alpha_{1/2} k_x \quad \text{etc.} \quad \chi' = \alpha_{1/2} \chi
\]
Moreover we must let
\[ \beta H' = \beta \left( \alpha_2 \alpha_3 H_N^2 + \alpha_3 \alpha_1 H_{\alpha}^2 + \alpha_1 \alpha_2 H_{\alpha}^3 \right)^{1/2} = \beta^* H \]  
(15)

and
\[ V' = (\alpha_1 \alpha_2 \alpha_3)^{-1/2} V \]  
(16)

The small effective mass and low Fermi energy have a further interesting consequence. In the absence of a magnetic field, the energy density of the levels, i.e., \( \frac{dE}{dE} \) is proportional to the three-halves power of the effective mass times the one-half power of the energy (Seitz, 1940, p. 142). Hence the carriers in question are associated with a band having low energy density of levels. Suppose now that these particular carriers are due to an overlap with a band of much greater energy level density. Then the Fermi level will be kept essentially constant by the "reservoir" action of the latter band, and the conditions for appreciable oscillations in number, after the manner of equation (13), will be filled by the carriers in the first band.

The use of \( \beta^* \) and \( V' \) allow us to take into account the interactions of the electrons with the ideal static lattice. In any real metal, however, there will be impurities and imperfections in the lattice. As has been shown by Robinson (1950) and Dingle (1952b), these have the same effect as raising the temperature, that is to say they introduce an effective temperature. This would seem reasonable since the actual effect of any finite temperature is to spoil the
perfect regularity of the static lattice. To appreciate this effect, we must see how the temperature affects the results already obtained. In this, we will solve equation (10) by a method paralleling that used by Landau (1939) on equation (9).

The summation involved in \( \psi(e) \) is effected by using Poisson's summation formula:

\[
\psi(e) = \sum_{p=-\infty}^{\infty} (-1)^{p} \int_{0}^{e} (e-t)^{1/2} e^{2\pi ipt} dt
\]

The term for \( p = 0 \) integrates simply to \((2/3) e^{3/2}\). Hence the constant term in the expression for the number of carriers per unit volume becomes

\[
N_0 = \frac{2}{3} A (\beta \hbar)^{3/2} \int_{0}^{\infty} e^{3/2} \frac{dg}{de} de
\]

For sufficiently low temperatures this reduces to the result quoted previously, as the limiting value of the zero temperature case for large \( e \):

\[
N_0 = \frac{2}{3} A (\beta \hbar)^{3/2} \varepsilon_0^{3/2} = \frac{1}{3} \frac{(2m\hbar)^{3/2}}{\pi^2 \alpha^3}
\]

For the general term with \( p \neq 0 \) integration by parts gives

---

\[
\int_0^\infty e^{-\sqrt{t}} e^{2\pi i p t} dt = \left[\frac{(e^{-\sqrt{t}})^{1/2}}{2\pi ip}\right]_0^\infty \\
+ \frac{1}{4\pi ip} \int_0^\infty (e^{-\sqrt{t}})^{1/2} e^{2\pi i p t} dt
\]

\[
= -\frac{e^{1/2}}{2\pi ip} + \frac{1}{4\pi ip} \int_0^\infty (e^{-\sqrt{t}})^{1/2} e^{2\pi i p t} dt
\]

The integral can be reduced to a standard form, for the case where \( p > 0 \), by the substitution \( s = 2\pi p (e^{-t}) \), for the case \( p < 0 \), the corresponding substitution is \( s = -2\pi p (e^{-t}) \). Making these substitutions, we obtain for \( p > 0 \),

\[
\int_0^\infty (e^{-\sqrt{t}})^{1/2} e^{2\pi i p t} dt = \frac{1}{1p} e^{2\pi p} \int_0^{2\pi p} (e^{i s})^{1/2} e^{-is} ds
\]

and for \( p < 0 \),

\[
\int_0^\infty (e^{-\sqrt{t}})^{1/2} e^{2\pi i p t} dt = \frac{1}{-1p} e^{2\pi p} \int_0^{2\pi p} (e^{i s})^{-1/2} e^{+is} ds
\]

The integrals in equation (20) and equation (21) can be given in terms of the Fresnel integrals. The integral in equation (20) may be given as \( C - i S \) and the integral in equation (21) is then \( C + i S \). It is usually assumed that \( \epsilon \) is large enough so that we may use the value for an infinite argument. In this case \( C = S = 1/2 \), and the integrals become \( 1/2 + i/2 = (1/2)e^{i\pi/4} \), the sign being plus or minus according as

---

the sign of the exponential is plus or minus. Thus the terms for \( p > 0 \) make a contribution to \( \psi(\epsilon) \) of

\[
\sum_{p=1}^{\infty} (-1)^{p} \left\{ -\frac{\epsilon^{1/2}}{2\pi ip} + \frac{1}{4^{1/2}\pi ip^{3/2}} \exp\left[i(2\pi pe - \pi/4)\right] \right. \tag{22}
\]

and the terms for \( p < 0 \) make a contribution of

\[
\sum_{p=-\infty}^{0} (-1)^{p} \left\{ -\frac{\epsilon^{1/2}}{2\pi ip} + \frac{1}{4^{1/2}\pi ip^{3/2}} \exp\left[i(2\pi pe + \pi/4)\right] \right. \tag{23}
\]

if we rewrite equation (23) in such a way as to make \( p > 0 \), we obtain

\[
\sum_{p=1}^{\infty} (-1)^{p} \left\{ -\frac{\epsilon^{1/2}}{2\pi ip} + \frac{1}{4^{1/2}\pi ip^{3/2}} \exp\left[i(2\pi pe - \pi/4)\right] \right. \tag{23a}
\]

combining equations (17), (19), (22), and (23a), and substituting in equation (12)

\[
N = N_0 - A(\beta H)^{3/2} \left\{ \sum_{p=1}^{\infty} (-1)^{p} \frac{1}{4^{1/2}\pi ip^{3/2}} \times \left[ \int_{0}^{\alpha \epsilon} \exp\left[i(2\pi pe - \pi/4)\right] \frac{de}{e} - \int_{0}^{\alpha \epsilon} \exp\left[-i(2\pi pe + \pi/4)\right] \frac{de}{e} \right] \right\} \tag{24}
\]

in evaluating the integrals appearing in equation (24), we consider the expression for \( \frac{d\Phi}{de} \).

\[
\frac{d\Phi}{de} = \frac{d}{de} \frac{1}{1 + \exp[-(e-\epsilon)\beta]} = \frac{1}{\beta} \frac{\exp[(e-\epsilon)\beta]}{(1 + \exp[(e-\epsilon)\beta])^2}
\]

\[
= -\frac{1}{\beta} \frac{1}{(1 + e^{-\beta})(1 + e^{\beta})}
\]

where \( \beta = \left[ (e-\epsilon)\beta \right] \). This, however, can be written as

\[
\frac{d\Phi}{de} = \frac{4}{\beta} \frac{1}{\cos^2(\beta /2)}
\]
Let us then write the quantity appearing in the square brackets in equation (24) as
\[ \exp \left[ i \left( 2\pi p e_0 - \frac{i\pi}{4} \right) \right] \int_{e_0/\theta}^{\infty} e^{2\pi ip\theta x} \left[ \cos \left( i x / 2 \right) \right]^{-2} \]
\[ \exp \left[ -i \left( 2\pi p e_0 - \frac{i\pi}{4} \right) \right] \int_{e_0/\theta}^{\infty} e^{2\pi ip\theta x} \left[ \cos \left( i x / 2 \right) \right]^{-2} \]

(25)

Since \( e_0/\theta = E_0/kT \) is much greater than one, we replace the lower limits on these integrals by minus infinity. Both integrals now have the form of Fourier integrals, and each one is equal to \( px/\sinh px \) where \( x = 2\pi^2 \theta = 2\pi^2 kT/\beta H \).

Substituting these in equation (24), we finally obtain for the number of carriers as a function of temperature and applied magnetic field,
\[ N = N_0 + A(\beta H)^{3/2} \sum_{p=1}^{\infty} \frac{(-i)^p}{2^{2p} \pi^{3/2}} \frac{px}{\sinh px} \sin \left( 2\pi p e_0 - \frac{i\pi}{4} \right) \]

(26)

As before, the effect of the lattice is taken into consideration by the use of \( \beta^* \) and \( \gamma^1 \). In some cases (Schoenberg, 1949) it will be found that more than one group of electrons contribute to the effects considered, and we will then have to sum over the various \( \beta^* \). In most cases of interest it will be sufficient to retain only the first term of the summation in equation (26). Moreover, since \( x \) is small \( 2\pi^2 kT \ll \beta^* H \), we may replace

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3G. H. Campbell and R. M. Foster, *Fourier Integrals* (no publisher, 1942) Eq. 607.3, Table 1.
Making these two approximations, and substituting for $A$ from page 7, we find

$$N = N_0 - \frac{3\lambda^2}{2\pi^2 n} \exp\left(-\frac{2\pi^2 kT}{\beta\hbar}\right) \chi(\beta\hbar) \frac{3\lambda}{\pi} \sin\left(\frac{2\pi E_0}{3\lambda^2 \hbar^2} - \frac{\pi}{4}\right)$$

(27)

The above calculation of $N$, similar mathematically to Landau's calculation of the free energy per unit volume, involves no new physical assumptions over the work of Blackman previously cited. The main assumption here is that the energy of the carriers in the absence of a magnetic field can be represented by an equation like equation (14), and that the presence of the magnetic field has little effect on the zone structure of the metal and does not cause a significant number of transitions between the bands. Our result for the oscillatory term in $N$ is identical with that of Single (1952a), obtained by a somewhat different method.

Two mathematical assumptions have been made, however, in this extension to finite temperatures. The second, that of replacing the lower limit of the integrals in equation (27) by minus infinity, is largely a matter of convenience. Moreover as we shall see, this approximation is rather a good one, since $E_0/k$ is $410^\circ K$ for bismuth and even higher for other metals showing the oscillating effects (Schoenberg, 1947). The first assumption made in this derivation is not

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4 For a criticism of this assumption, see Adams (1953).
so easy to support and the failure of this assumption leads to much graver consequences. It will be remembered that in evaluating the Fresnel integrals, we replaced the upper limit by infinity. This resulted in an amplitude for $C-iS$ or $-\pi/4$. Considering that, in many cases, the value of the argument may be as low as one, the amplitude of $C-iS$ can run from about $-\pi/6$ to $-\pi/3$. This value will oscillate as $\varepsilon$ varies, approaching $-\pi/4$ as $\varepsilon$ approaches infinity. For values of $\varepsilon$ less than $1/4$ the amplitude will approach zero monotonically as $\varepsilon$ approaches zero. The effects due to this assumption, introduced by Landau, are usually ignored, although Dingle (1952a) and Robinson (1950) use asymptotic expansions for the Fresnel integral which partly take this into account.

It might be thought that the effect of the electron spin should be included in the calculations. Neglecting exchange and correlation effects, the spin can be very simply included in the theory by means of the formula

$$Z_{\text{spin}}(E, H) = \frac{1}{2} \left[ Z(E + \frac{\beta H}{2}, H) + Z(E - \frac{\beta H}{2}, H) \right]$$

where $\beta$ is here the true Bohr magneton. A moment's consideration will show that this will give us directly

$$N_{\text{spin}}(E, H) = \frac{1}{2} \left[ N(E + \frac{\beta H}{2}, H) + N(E - \frac{\beta H}{2}, H) \right]$$

(28)

Thus for the case of free electrons, or electrons with an effective mass of about one electron mass unit, $\psi(E)$ will become $\sum_{E - \theta^2}$, or there will be a phase change of $\theta^2$ in all of the effective mass of about one electron mass unit, $\psi(E)$ will become $\sum_{E - \theta^2}$, or there will be a phase change of $\theta^2$ in all of the
ood terms of equation (26) which will result in the coefficient of all terms being positive. For our special electrons, however, \( \beta \ll \beta^* \) and the splitting of the energy levels due to spin will be negligible in comparison to the separation of the levels. To prove this, we note that under the inclusion of spin, the sine term in equation (26) becomes

\[
\sin \left( 2\pi \frac{E_0 + \frac{\beta H}{\beta^*}}{\beta^*} \right)
\]

which upon expansion becomes

\[
\sin(2\pi \frac{E_0}{\beta^*} + \frac{\beta H}{\beta^*}) \cos(2\pi \frac{E_0}{\beta^*}) \pm \cos(2\pi \frac{E_0}{\beta^*} - \frac{\beta H}{\beta^*}) \sin(2\pi \frac{E_0}{\beta^*})
\]

(29)

For \( \beta \ll \beta^* \), the second term is negligible, and so the effect of spin is to multiply each term in the summation in equation (26) by the factor \( \cos(2\pi \frac{E_0}{\beta^*}) \) which in all cases of interest can be replaced by one.

Thus granting the validity of our initial assumptions, i.e., those of Blackman and Landau plus the assumption that \( E_0 \) is held constant by some mechanism such as described on page 13, equation (26) will be valid for all cases of interest, except for the question of the phase of the oscillations. Since we will be primarily interested in comparing the phases of the oscillations in the different effects due to the oscillations in \( N \), we shall henceforth beg this question.

The extent of the validity of the approximation involved in neglecting all but the first term of the series in equation (26) is shown in Figures 2 and 3. Here are plotted, for the
case where $\beta^*$ is equal to $2.4 \times 10^{-18}$ ergs per gauss and $E_0$ equals $2.9 \times 10^{-14}$ ergs, the first term and the sum of the first two terms in the expression for $N$, for temperatures of 1.3 and 4.2 degrees Kelvin respectively. Figure 4 is a plot of equation (27) for bismuth when the magnetic field is at right angles to both the principal axis and the binary axis. In this case (Schoenberg, 1949), the Fermi energy is $2.9 \times 10^{-14}$ ergs and three special groups of carriers are present, one with an effective Bohr magneton of $2.4 \times 10^{-18}$ and two with effective Bohr magnetons of $1.2 \times 10^{-18}$ erg per gauss.
CHAPTER III

THE PERIODIC EFFECTS

The first attempts to explain many of the electrical and thermal properties of metals, for example the conductivities and the transverse effects in magnetic fields, were made on the basis of the kinetic theory of gases and the Boltzmann transport equation, as applied to the electrons in the metals. At first, the electrons were treated as if they were perfectly free, except inasmuch as they were confined to the metal by a potential at the walls. It was assumed that there were \(zn\) electrons per unit volume, where \(z\) was the valency of the metal and \(n\) the number of atoms per unit volume, and that the electron gas so formed obeyed Maxwell-Boltzmann statistics. The various successes and failures of this approach are well known (see Seitz, 1940). The application of quantum mechanics to the problem of electrons in metals allowed the revision of the theory in such a way as to give much better agreement with experiment. Besides necessitating the introduction of Fermi-Dirac statistics, the results of a quantum mechanical investigation showed that the assumption of \(zn\) free electrons was a poor approximation, except for certain cases in the alkali metals. Also for the multivalent metals, and particularly for the so-called semi-metals like bismuth and graphite, the energy is

\footnote{\textit{cf.} Chapter IV}
not a simple function of the momentum. In these cases, it was then necessary to introduce the concept of effective mass in the hope that the energy could be represented by an equation of the form of equation (14). Even this energy dependence is too complicated to fit easily within the framework of the transport equation, and one had to be satisfied with assuming spherical energy surfaces, i.e., one assumed the reciprocal mass tensor was diagonal with

\[ \alpha_{11} = \alpha_{22} = \alpha_{33} \]

For many effects, such as the electrical and thermal conductivities in the absence of a magnetic field, the one band model, where \( E = \left( h^2 / 2m \right) \kappa^2 \) is sufficient for an understanding of all phenomena observed. In the final analysis, this is due to the fact that the conductivities depend essentially on the square of the gradient in \( k \) space of the energy. Other phenomena, such as the Hall effect, are found to depend on the first power of the second derivative of the energy with respect to the wave number \( k \). In these cases, the differential geometry of the surfaces of constant energy becomes most important. Since, at the present state of the theory of metals, it is impossible to give a detailed analysis of the shape of the surfaces of constant energy, it becomes necessary to consider the simplest model which allows for both positive and negative second derivatives. This model is the isotropic two band model, where in the first band

\[ E = \frac{h^2}{2m} \alpha_1 k^2 \] (30)
and for the second, or inverted, band,

\[ E = h^2 \frac{m^*}{2m} \omega x^2 \]  

(31)

in this case, \( \omega x^2 \) is some suitable average over the surfaces of positive curvature, and \( \omega x^2 \) is the average for surfaces of negative curvature. Such a model will almost certainly be far from the actual physical situation, but it should reproduce the general trend for many effects that we might wish to consider. Anisotropic effects cannot, of course, be understood on the basis of this model, and even in isotropic effects, the parameters obtained cannot be regarded as an exact description for all of the carriers concerned.

If we assume that the mean free time between collisions is a constant, \( 2\tau \), for each of the bands, we obtain the following formulae for the conductivity, \( \sigma \), and the Hall coefficient, \( R \) (Wilson, 1953):

\[ \sigma = (N_1 \alpha_1 + N_2 \alpha_2) \frac{e^2 \tau}{m} \]  

(32)

\[ R = -\frac{1}{eC} \left( \frac{N_1 \alpha_1^2 - N_2 \alpha_2^2}{(N_1 \alpha_1 + N_2 \alpha_2)^2} \right) \]  

(33)

where \( N_1 \) and \( N_2 \) are the number of electrons per unit volume in the first band and the number of holes in the second band respectively.

Although the usual derivation of these formulae is by the use of the transport equation with appropriate modifications for quantum mechanical effects, the author has found a semi-classical treatment which leads to the same results. Quantum effects are introduced only indirectly through the use of effective masses as the ratio of the applied force
to the acceleration. Moreover, the absence of integrals
over the Fermi surface permits the use of anisotropic ef-
flective masses. For simplicity we shall treat only the
isotropic case here, indicating the changes anisotropic
effective masses would have.

Assuming that $\mathbf{n}_x H$, the product of the drift velocity
of the electrons in the $y$ direction and the magnetic field
in the $z$ direction, can be neglected, the force in the
direction is that due to the electric field $E_x$. This gives
rise to an acceleration of $-eE_x (\alpha_y/m)$, and an average drift
velocity of $-eE_x (\alpha_y/m)$. Similarly, the average velocity
of the holes is $eE_x (\alpha_y/m)\mathbf{v}$. The current will then be

$$j_x = n_x (-e) n_x + n_x e E_x$$

or

$$j_x = (n_x \alpha + n_z \alpha_2) \frac{e^2 c}{m} E_x$$

Thus the electrical conductivity, $j_x/E_x$, is as given by
equation (32).

The acceleration in the $y$ direction will be

$$-eE_y (\alpha_y/m) + e(n_1 H/c) (\alpha_y/m)$$

for the electrons and

$$eE_y (\alpha_z/m) - e(n_2 H/c) (\alpha_z/m)$$

for the holes. Thus

$$j_y = (n_x \alpha + n_z \alpha_2) \frac{e^2 c}{m} E_y + (n_x \alpha_1 - n_z \alpha_2) \frac{e^2 c}{m} H$$

If the Hall current $j_y = 0$, we find

$$E_y = - \frac{n_x \alpha_1 - n_z \alpha_2}{(n_x \alpha + n_z \alpha_2)} \frac{e}{m} \frac{E_x H}{c}$$

Substituting for $E_x$ from equation (35), this becomes,
The Hall coefficient, $\frac{E_y}{B H}$, is then that given by equation (33). The extension to anisotropic effective masses is straightforward, and gives

$$R = -\frac{1}{eC} \frac{N_1a_1^2 - N_2a_2^2}{(N_1a_1 + N_2a_2)(N_1a_1 + N_2a_2)}$$

(39)

Obviously, this derivation is no better or worse, as regards necessary assumptions, than the usual quantum mechanical derivation. We are limited by the assumptions that must be made to put the problem in a soluble form. For example, we have had to assume an collision time which is a constant for both bands. Although it is true in special cases that $\tau$ is independent of velocity in either band (Mott and Jones, 1936, p. 236), there is no reason to expect it is the same for both bands. Nor have we considered the possibility of transitions between the bands. The approximation that $\nu_y H$ is negligible limits the validity of the results to very low fields. If we remove this assumption but limit ourselves to the isotropic case, the Hall coefficient becomes

$$R = -\frac{1}{eC} \frac{N_1a_1^2}{1 + (\omega_i^2)^2} - \frac{N_2a_2^2}{1 + (\omega_f^2)^2}$$

(40)

where $\omega_i = \alpha_i \left( eH/mc \right)$ and $\omega_f = \alpha_f \left( eH/mc \right)$. It is seen that
equation (36) is actually the zero field approximation to the Hall coefficient. In the case where \( R \) is given by equation (40), the conductivity becomes

\[
\sigma = \left[ \frac{m}{\hbar} \left( \frac{N \alpha_1}{1 + (\omega_1 T)^2} + \frac{N \alpha_2}{1 + (\omega_2 T)^2} \right)^2 - \frac{N \alpha_1^2}{1 + (\omega_1 T)^2} \right] \frac{e^2}{m} \tag{41}
\]

where \( K \) is a function of the number of electrons in each band, their effective masses, and the mean free time between collisions. This shows that the Hall coefficient should increase in absolute value with increasing magnetic field.

It should be emphasized that the various parameters which characterize the carriers in the preceding equations have little meaning outside of the equations themselves. At best, the effective masses are only averages over the Fermi surface, and assumptions such as the equality of \( \tau \) for the two bands will be reflected in the values for the masses. Thus, it is impossible at present to obtain detailed information about the Fermi surface from the usual measurements of the magnetoresistance and the Hall effect.

Yet, although the special carriers of Chapter II may occupy only a portion of one band, we may still obtain some information about them from magnetoresistance and Hall effect measurements. For a first approximation, these
carriers are treated as if they filled a special band. The
effects due to all other carriers are then ascribed to one
other band. This approach may well be far from the true
physical picture, but in view of the present state of the
theory of metals, the approach is useful enough to warrant
some consideration.

We shall use only the zero field approximation to the
conductivity, equation (32), and the hall coefficient,
equation (33). The advantages of equation (37) at this point
are slight, and would be outweighed by the disadvantages
attendant upon a dilution of accuracy. Next, we shall assume
that either \( N_1 \), the number of electrons in the first band
per unit volume, or \( N_2 \), the number of holes in the second
band per unit volume, show the oscillations of equation (27).
That is to say, in the limit the number of carriers will vary
with the magnetic field according to the formula

\[
N = N_0 - \eta \sin \left( \frac{2 \pi}{\hbar} \left( E_0/\beta^*H \right) - \pi/4 \right)
\]

(43)

where \( \eta \) is a slowly varying function of the magnetic field.

The conductivity depends directly on \( N_1 \) and \( N_2 \),
having a maximum when they are a maximum and a minimum when
they are a minimum. The conductivity will then be a maximum
when \( 2 \pi (E_0/\beta^*H) = \pi/4 \) is \( 3 \pi/4 \) \( \eta \) \( \eta \eta \) \( \eta \)

\[
2 \pi \left( E_0/\beta^*H \right) = \pi/4 + 2 \eta \pi.
\]

The minimum will occur when

\[
2 \pi \left( E_0/\beta^*H \right) = \pi/4 + 2 \eta \pi.
\]

The hall coefficient varies as

\[ -N_1 + \eta \] N_2 \]. Since, if \( N_2 \) is the oscillating quantity,
the oscillations in the hall coefficient will be in phase
with the field oscillations in the conductivity. If it is
the electrons whose number, \( N_i \), oscillates appreciably, the oscillations in the Hall coefficient will be \( \pi \) out of phase with those in the conductivity, the maxima occurring at
\[
2\pi \left( E_0/E^*H \right) = \left( 3\pi/4 \right) + 2\pi \text{ and the minima at }
2\pi \left( E_0/E^*H \right) = \left( 7\pi/4 \right) + 2\pi.
\]

In the case of the magnetic susceptibility, Dingle has pointed out that, except for very large magnetic fields, the susceptibility oscillations will be given by the Landau (\( \gamma \)\( \phi \)) formula, whether it is \( N \) or \( E_0 \) that varies. This result is based on the fact that if the number of electrons is considered to vary, the susceptibility is not given by the usual formula
\[
\chi = -\frac{1}{H} \left( \frac{\partial F}{\partial H} \right)
\]
but by
\[
\chi = -\frac{1}{H} \frac{\partial (F - N E_0)}{\partial H}
\]
in this case, the susceptibility contains an oscillating term
\[
-A_{\chi} \sin \left[ 2\pi \left( E_0/E^*H \right) - \pi/4 \right]
\]
where \( A_{\chi} \) is a slowly varying function of \( H \). Thus we should expect the oscillations in the magnetic susceptibility to be in phase with those in the conductivity.

However, if the reservoir contains the same type of carriers as the special band, the total number of carriers of that type will remain constant. The free energy will change as the carriers move from one system of energy levels to the other, and the change should contain a term depending directly on the change in the number of carriers. The
susceptibility would then contain a term proportional to $-\frac{\partial N}{\partial H}$, or

$$-\cos\left(2\pi \frac{E_0}{\beta^*H} - \frac{\pi}{4}\right)$$

(46)

Hence the magnetic susceptibility will be a maximum at

$$2\pi \left(\frac{E_0}{\beta^*H}\right) = \frac{5\pi}{4} + 2\pi m$$

To summarize: if the special carriers are electrons, the field oscillations in the conductivity and Hall coefficient will be $\pi$ out of phase (we consider all curves to be plotted against $1/H$); if the special carriers are holes the field oscillations in the conductivity and the Hall coefficient will be in phase. In either case, the oscillations in the magnetic susceptibility will either lead those in the conductivity by $\pi/2$ or be exactly in phase with them, depending on the nature of the reservoir, that is, on whether we use equation (45) or equation (46). Two points about signs should be noted. First, by minimum we mean the most negative value or least positive value. Second, the Hall coefficient is so defined that it has a negative value for predominately electron conduction, and the sign of the magnetic susceptibility is negative for diamagnetic materials. In these cases then, a minimum corresponds to a maximum in the absolute value.

The above results have been derived on the basis of questionable formulae for the effects considered. In particular, our assertions about the phase of the magnetic susceptibility oscillations should be thoroughly examined;
and the expressions used for the conductivity and the Hall coefficient certainly are not exact for the anisotropic, high field, low temperature region we wish to consider. However, in any extension to this region, some general features might well be preserved, and the above discussion would then be not without some significance. For example, equation (32) does not predict the extreme magnetoresistive effect observed in bismuth, but it still might account for the field oscillations in this quantity.

Figure 5 shows two early measurements made by Reynolds on bismuth. The two lower curves are unpublished plots against magnet current of the magnetoresistance of a particular sample of bismuth with the trigonal axis tilted about 25° from the magnetic field. The upper curves show the Hall probe voltages (negative) for the same orientation. The two sets of points correspond to two orientations of the magnetic field, differing by 180°. The actual Hall voltage and magnetoresistance are taken as the average of the two curves, thus cancelling the effect due to slight misalignments of the probes (Reynolds, Leinharut and Hemstreet, 1954). It will be noted that a maximum in the magnetoresistance accompanies each minimum in the Hall voltage. (It must be remembered here that a peak in the curves shown for the Hall voltage actually corresponds to a minimum because the Hall voltage is negative for bismuth.) Since a maximum in the resistance corresponds to a minimum in the conductivity, these results would indicate that our special carriers are
Magnet Current

Resistance (arbitrary units)

Hall Voltage

Figure 5
holes. This is in agreement with the conclusions of Mott and Jones (1936), who attribute the magnetic susceptibility of bismuth perpendicular to the trigonal axis (for which direction of $\mathbf{H}$, our oscillations are most marked) to holes. The results of Schoenberg and Suain (1936) would seem to contradict the above argument, since they found that the addition of small amounts of lead or tin (both elements with a valence of four) lower the Fermi energy of the special carriers in bismuth (valence 5), while the addition of tellurium (with 6 valence electrons) raises the Fermi energy. On the surface, this would seem to indicate that our carriers are electrons. However, the addition of antimony, which belongs to the same column of the periodic table as bismuth, also lowers the Fermi energy. The answer to this contradiction can only come from further experimental and theoretical work on the periodic effects and on the zone structure of bismuth.

In the case of graphite, Berlincourt (1954a) has obtained data on all three effects discussed above. Here the field oscillations in the conductivity lag those in the susceptibility by $90^\circ$. So it would seem that the reasoning leading to equation (45) may well be correct. The oscillations in the Hall coefficient seem to be about $135^\circ$ out of phase with those in the conductivity. Berlincourt has stated, however, that the experimental uncertainties could be of the order of $45^\circ$ for the phase difference between the two effects. Thus the upper limit on the phase difference is $180^\circ$, 
which would indicate that the special carriers in graphite are electrons.

Some indication of the relative magnitudes to be expected for the various effects can be obtained from the analysis of this chapter. Figure 5 of the last chapter shows us that we should not expect the number of our carriers to oscillate by more than a few per cent. Since the conductivity is the sum of two positive terms, each involving the first power of the number of one particular type of carrier, the oscillations in the conductivity will at best be small. The oscillations in the magnetic susceptibility are contained in a term which involves the derivative of the number of special carriers with respect to $H$. This term is then a purely oscillatory term, and the relative magnitude of the oscillations will depend on the magnitude of the other contributions to the susceptibility. The relative magnitude of the oscillations in the Hall effect also may vary over a wide range for different metals, since the Hall coefficient involves the difference of two terms, one essentially a constant and the other varying directly as the number of special carriers.

If the oscillations in the Hall effect are to have any appreciable magnitude we must have

$$\frac{N_{1}}{\rho_{1}^{2}} \approx \frac{N_{2}}{\rho_{2}^{2}}$$

(47)

Assuming that the special carriers are holes (a similar argument could be made if they are electrons) we also know that
This tells us that

\[ N_i \alpha_i \gg N_0 \alpha_2 \]  

and

\[ N_i \gg N_0 \]  

According to equation (49), the second term appearing in the squared quantity in the denominator of equation (33) can be neglected. Then

\[ R = -\frac{1}{e c} \frac{N_i \alpha_i^2 - N_0 \alpha_2^2}{(N_i \alpha_i)^2} \]  

If \( N_0 \) varies according to equation (27), we see that the Hall coefficient has the form

\[ R = R_0 - R_1 H^{1/2} \exp[-2\pi^2 k T/\beta^* H] \sin(2\pi E_0/\beta^* H - \pi/4) \]  

where \( R_0 \) and \( R_1 \) are constants.

Figure 6 shows the Hall coefficient as a function of \( H^{-1} \) at various temperatures (Triantos, 1974). The data is for the magnetic field along the trigonal axis and the current along a binary axis. It will be seen that an oscillating term of the nature of the term in equation (52) is present, but that there is a marked increase with increasing magnetic field that is not predicted by equation (52). (That is, the magnitude of \( R_1 \) increases with \( H_0 \).) To obtain this, we would have to use equation (41) as a starting point. We
small content ourselves here with a discussion of the oscillating term.

The amplitude of the oscillating term is

\[ A = R \sqrt{\frac{\hbar}{2}} T \exp\left[-2\pi^2 r_0 T / \beta^* \hbar \right] \]  

(53)

Taking the logarithm of this we obtain

\[ \frac{T}{H} = -\frac{\beta^*}{2\pi^2} \log \frac{A}{\sqrt{\hbar}} + \text{const.} \]  

(54)

A graph of \( T/H \) against \( \log( A / \sqrt{\hbar} \cdot T) \) should then result in a straight line which would enable us to determine \( \beta^* \), the effective doubled Bohr magneton for this orientation. When this was done, it was found that the points clustered around a straight line, but that definite deviations from this line did exist. However, if the value \( T+0.5 \) was used in place of \( T \), the curve of figure 7 was obtained in which the points fell on a straight line. This temperature correction, as mentioned earlier, is necessitated by the spreading of the energy levels due to collisions of the electrons with imperfections in the lattice. The value of the effective doubled Bohr magneton so obtained is \( 3 \times 10^{-19} \) erg/ gauss. This corresponds to an effective mass of 0.06.

Moreover, the absolute value of the oscillating term is equal to the amplitude when

\[ 2\pi E_0 / \beta^* \equiv 3\pi / 2 + \pi n \]  

(55)

being \( \pm A \) for even integers, and \( -A \) for odd integers. So if we plot, against the integers, the values of \( 1/H \) for which
\( \ln\left(\frac{A}{(T+0.5)H^{1/2}}\right) \)

\( \frac{(T+0.5)/H \times 10^4}{(\degree K/\text{Gauss})} \)

- ○ 1.36\(^\circ\)K
- ● 1.82\(^\circ\)
- △ 2.21\(^\circ\)
- ▲ 2.59\(^\circ\)
- □ 3.15\(^\circ\)

Graph showing the logarithmic relationship between the normalized amplitude and the inverse square root of the field strength, adjusted for temperature.
the curve is tangent to its envelope, we will be able to
determine $E_0/\beta^*$. The results of this procedure are shown
in Figure 3, and the value of $E_0$ so obtained is $2 \times 10^{-14}$
erg, corresponding to a degeneracy temperature of $140^\circ K$.

The value obtained from magnetic susceptibility
measurements for the Fermi energy of the special carriers in
bismuth is $2.9 \times 10^{-14}$ ergs (Schoenberg, 1934). The magneto-
resistance measurements of Aiers and Webber (1953) also
give the same value. The discrepancy between these results
and that for the Hall effect is much too large to be attributed
to experimental error, and may arise from some special
properties of the bismuth crystal used in the Hall effect
measurements. The necessity of using the same crystal and
the same orientation for all measurements is clearly shown
here. As mentioned earlier, this has been done by
Berlincourt (1954a) for graphite, where equal periods were
found in all three effects. However, he reported that for
the oscillations in the Hall coefficient, the amplitude at
constant temperature seemed to be proportional to
$H^{-5/2}\exp(c0nt/H)$ instead of $H^{1/2}\exp(consti/H)$ as
predicted here. His data cover the range from 3 to 25
kilogauss, in contrast to the range from 4 to 8 kilogauss
used in the measurements on the Hall effect in bismuth. It
well may be that the fields involved in the measurements of
Berlincourt are too high for the use of the simple formula
of equation (33).

In any case, the straight lines obtained in Figures
ODD INTEGERS - MAX
EVEN INTEGERS - MIN

$\frac{1}{H} \times 10^4 \text{(GAUSS$^{-1}$)}$

$N$

1  2  3  4  5  6  7  8  9
7 and 8 are convincing arguments of the validity of equation (52), at least as a first approximation to the Hall coefficient at low fields. The relative phases of the various oscillatory effects also have been explained by the model which led to equation (52). The question of a better model for the Hall effect will be taken up in the next chapter.
CHAPTER IV

A FURTHER INVESTIGATION OF THE HALL EFFECT

In chapter 3, it was seen that our model of the oscillating number of electrons, when combined with almost the very simplest of formulae for the Hall effect and magnetoresistance, has appreciable success in explaining the field and temperature dependence of the periodic part of these effects. However, the model falls far short in predicting the exact shape of the curve against $1/H$ for either effect. In this chapter we will discuss the possibility of obtaining better agreement between experiment and theory.

The history of attempts to explain the various transport phenomena in metals extends back to the end of the nineteenth century (Riecke, 1893). The general form of the procedure used today was introduced by Lorentz (1904). It is assumed that a distribution function, $f(N,T)$, exists and is constant in time. Then

$$\frac{df}{dt} = 0$$ (56)

Now $f$ may vary with time in two independent ways: 1) a drift variation may exist, which is to say that the electrons move about from place to place in the metal and may also be accelerated by external fields, 2) a variation due to collisions with the atoms of the metal may exist. For equilibrium, these two variations must be equal in magnitude.
\[
\left( \frac{\partial f}{\partial t} \right)_{coll} = \left( \frac{\partial f}{\partial t} \right)_{drift}
\]

It should be noted that the presence of only one distribution function implies we are considering only one band at the moment. The model can be simply extended to two bands if we wish to neglect the possibility of transitions between the bands.

It can be shown that

\[
\left( \frac{\partial f}{\partial t} \right) = -\frac{e}{m} \left[ E + \frac{\mathbf{v} \times \mathbf{H}}{c} \right] \mathbf{grad} f + \mathbf{v} \cdot \mathbf{grad} f
\]

where \( \mathbf{v} = \sum \frac{\partial \mathbf{f}}{\partial \mathbf{k}} \).

and we take the scalar product of this vector with the expression in brackets. However, equation (58) is valid only for times short compared to the Larmor period \( 2\pi mc/\mathbf{e} \mathbf{H} \) (Wilson, 1953, p. 52). This limitation was also present in our work in chapter 5, where we considered the velocity as staying constant over the mean free time between collisions.

For free electrons \( 2\pi mc/\mathbf{e} \) is of the order of \( 10^{-7} \) gauss-seconds, but for the special carriers in bismuth, the low effective mass reduces the value to about \( 10^{-9} \). For the normal conduction electrons in most metals, \( \mathbf{v} \) is of the order of \( 10^{-12} \) seconds, so we may deal with fields up to about \( 10^4 \) gauss; but the mean free path in bismuth is about twenty times that of a true metal like silver (Pippard and Chambers, 1952; Sondheimer, 1952), so the mean free time is correspondingly
longer, limiting the validity of the above expression to very low fields in the case of bismuth. We shall, however, beg this question and turn to the problem of the rate of change of $\xi$ due to collisions.

The simplest assumption to make concerning $\frac{\partial f}{\partial t}_{\text{coll}}$ is that it obeys a law of the form

$$\frac{\partial f}{\partial t}_{\text{coll}} = \frac{f - f_0}{\tau(\mathbf{x}, t)}$$

(59)

where $f_0$ is the distribution function in the absence of applied fields and temperature gradients. The function $\tau(\mathbf{x}, t)$ is called the time of relaxation. It would be suspected that this function is closely related to the mean free time introduced in the last chapter, and we shall take the liberty of using the same symbol in defining both.

The introduction of a time of relaxation considerably simplifies the solution of equation (57), but we are then left with the unknown $\mathcal{C}$ in the integral equations for the electrical conductivity, thermal conductivity, etc. Finding the time of relaxation turns out to be a problem of solving an integro-differential equation. This equation can be solved only in a few simple cases. Moreover, in the general case it is extremely doubtful if a time of relaxation even exists (Wilson, 1953), i.e., that $\frac{\partial f}{\partial t}_{\text{coll}}$ can be put in the form of equation (59).

Assuming, however, the existence of a time of relaxation, general expressions can be obtained for the Hall effect (Sondheimer, 1948) and the conductivity in a magnetic...
field (Sondheimer and Wilson, 1947) for an isotropic two band model. The times of relaxation for the two bands are eliminated from the results by considering the expression for the partial conductivities due to each of the bands in the absence of a magnetic field

\[ \sigma_1 = \frac{N_1 e^2 \tau_1}{m/\alpha_1} \]
\[ \sigma_2 = \frac{N_2 e^2 \tau_2}{m/\alpha_2} \]  \hspace{1cm} (60)

It might then be hoped that the resulting expressions would have a more general validity. For arbitrary values of the magnetic field, they obtain,

\[ R = \frac{\frac{\sigma_1^2 - \sigma_2^2}{N_1} + \left( \frac{H}{ec} \right)^2 \left( \frac{N_1 - N_1 \tau_1}{M_1 N_2} \sigma_1^2 \sigma_2^2 \right)}{\sigma_1 + \sigma_2 + \left( \frac{H}{ec} \right)^2 \left( \frac{N_1 - N_1 \tau_1}{M_1 N_2} \sigma_1^2 \sigma_2^2 \right)} \]  \hspace{1cm} (61)

\[ \zeta = \frac{\left( \sigma_1 + \sigma_2 \right)^2 + \left( \frac{H}{ec} \right)^2 \left( \frac{N_1 - N_1 \tau_1}{M_1 N_2} \sigma_1^2 \sigma_2^2 \right)}{\sigma_1 + \sigma_2 + \left( \frac{H}{ec} \right)^2 \left( \frac{\sigma_1^2 + \sigma_2^2}{N_1 N_2} \right) \sigma_1 \sigma_2} \]  \hspace{1cm} (62)

Slightly more general expressions have been obtained by Chambers (1952a) using a thermodynamical type of argument. Chambers' results reduce to those of Sondheimer and Wilson under the assumption that the Hall coefficient in low magnetic fields is \( \pm (Ne)^{-1} \). For the zero field case, the above expressions reduce to the same value as we obtained in chapter 3, although the complete expressions are not the same. However, these results bring us no nearer our goal of relating the observed effects to the constants of the metal, although we can again see how we might expect an increase in the magnitude of the Hall coefficient as the applied magnetic field increases.
A variational method has been developed by Kohler (1949) and Sondheimer (1950) for the solution of equation (57) in the case of applied electric fields and temperature gradients, but in its present form the theory cannot be simply extended to the case where $H \neq 0$. However, we have already pointed out the number of serious objections to the above approach, two of these objections being the failure to consider the possibility of quantized orbits in large magnetic fields ($2\pi mc/eH \ll \gamma$) and the possibility of transitions between the bands.

The method used by Davydov and Pomeranchuk (1940) for the calculation of the magnetoresistance of bismuth avoids the above failures of the Lorentz approach of solving the Boltzmann transport equation. The first part of the work of Davydov and Pomeranchuk will be covered here in somewhat more detail than is found in the original paper. An attempt will be made here to extend the method to the calculation of the Hall coefficient. The second part of their method will only be outlined, and reasons will be given for the failure of this approach to apply to the Hall effect.

Davydov and Pomeranchuk use the approximation of tight-binding to solve the Schrödinger equation for the electrons of a metal in transverse electric and magnetic fields. In this approximation the wave functions are given by

$$\Psi(x) = \sum_m a_m u(x-L_m)$$
where $\psi_{m}(r)$ is the wave function for an electron in the valence shell of the $m$th atom neglecting the interaction with all other atoms of the crystal. Since $\psi_{m}$ is a function of $m$ it can also be considered as a function of $r_{m}$ or just of $x$. It can be shown that $\psi(x)$ will then obey an equation of the Schrödinger form

$$\frac{1}{2m} \sum_{j=1}^{3} \Delta_{j} \psi_{j} \psi_{j} = (E - E_{b} - e\mathbf{F} \cdot \mathbf{A}) \psi(x)$$  \hspace{2cm} (64)

where $\mathbf{F}$ is the applied electric field and $\Delta_{j} = \left( \frac{\partial}{\partial x_{j}} + \frac{e}{c} A_{j} \right)$.

We shall take the $x$ direction as the direction of the applied electric field and the $z$ direction as the direction of the magnetic field, using for the vector potential, $\mathbf{A}$, the form $-\frac{1}{2} \mathbf{E} \cdot \mathbf{B}$, $0$. For the sake of simplicity we shall assume all of the off-diagonal components of the reciprocal mass tensor are zero. (Actually this problem can be solved in the same manner, giving rise to the same values for the current density in the $x$ and $y$ directions, if we allow a finite value for $\alpha_{23} = \alpha_{32}$, but we need $\alpha_{12} = \alpha_{21} = 0$.) Under these assumptions, equation (64) becomes

$$\alpha_{1} \partial_{xx} - 2 \alpha_{1} \partial_{xy} - \alpha_{2} \partial_{yx} - \alpha_{3} \partial_{yy} + 2 \alpha_{2} \partial_{xy} - \alpha_{3} \partial_{yy} = \left( -2m/\hbar^{2} \right)(E - E_{b} - e\mathbf{F}) \partial_{x}$$  \hspace{2cm} (65)

where letter subscripts indicate differentiation with respect to the variable shown, and $\mathbf{F} = e\mathbf{H}/2\hbar \mathbf{c}$. As a first step, we notice that the equation is separable in $z$ and let $\psi(x, y, z) = e^{ikz} b(x, y)$. Then, to perform the separation
Performing these substitutions, we obtain the equation for \( f \):
\[
\alpha_1 f_{xx} + \alpha_2 f_{yy} + 4\alpha_4 \phi_x f_y - 4\alpha_6 \phi_y f - \frac{2m}{\hbar^2} E \phi f = \frac{2m}{\hbar^2} (E - E_b - \frac{\hbar^2}{2m} \alpha_2 E^2) f
\]
(66)

Since \( \phi \) no longer appears explicitly, we can separate this by the substitution \( f(x, y) = e^{-i\xi(x)} c(x) \), and the equation to be solved for \( c \) is
\[
\frac{\partial^2 c}{\partial x^2} - \left[ 4\alpha_4 \phi_x^2 - 4\alpha_6 \phi_y^2 \right] c = -\frac{2m}{\hbar^2} \left[ E - E_b - \frac{\hbar^2}{2m} (\alpha_2 E^2 + \alpha_4 E^2) \right] c
\]
(67)

If we write the quantity in square brackets as \( C^2 (\alpha_2 - \alpha_4) = C^2 \) then
\[
\frac{\partial^2 c}{\partial x^2} + \frac{2m}{\hbar^2} (E - E_b - 4\alpha_4 \phi_x^2) c = \frac{2m}{\hbar^2} C^2 c
\]
or
\[
\chi_0 = \frac{\hbar^2}{2m} - \frac{mcE}{4\alpha_4 \phi_x^2} \phi_x^2
\]
(68)

Subtracting \( C^2 \chi_0^2 \) from each side of equation (67)
\[
\frac{\partial^2 c}{\partial x^2} - C^2 (\alpha_2 - \alpha_4) c = -\frac{2m}{\hbar^2} \left[ E - E_b - \frac{\hbar^2}{2m} (\alpha_2 E^2 + \alpha_4 E^2) \right] c - C^2 \chi_0^2 c
\]
(69)

We now make the substitutions \( \xi = C^{1/2} (\chi - \chi_0) \) and \( f(x) = \phi(\xi) \).

The equation for \( \phi \) is then
\[
\frac{\phi''}{\phi} + (\lambda - \xi^2) = 0
\]
(70)

which is the well-known equation for the simple harmonic oscillator (Mott and Sneddon, 1948, p. 50). Here \( \lambda \) is
The solutions to equation (70) are bounded only if \( \chi = 2n+1 \), in this case being given by \( \Phi(\xi) = \Phi_n(\xi) = \exp\left(-\frac{\xi^2}{2}\right) H_n(\xi) \), where \( H_n \) is the Hermite polynomial of order \( n \). Thus our "wave function", \( \psi \), varies as

\[
e^{-\frac{\xi^2}{2}} \exp\left(\frac{i2n+1}{4}(x-x_0)\right) e^{ik\xi} \tag{72}
\]

The eigenvalues for the energy can be found from equation (71)

\[
E = E_b + \frac{\hbar^2}{2m} (\alpha_0^2 + \alpha_2^2) - \frac{\alpha_0^2 \hbar^2}{2m} \chi^2 + \frac{\alpha_2 \hbar^2}{2m} (2n+1) \tag{73}
\]

in particular, the last term becomes

\[
E_n = \frac{\hbar^2}{\beta^2 \alpha_2^2 m} H(n\frac{1}{2}) = \beta^2 H(n\frac{1}{2}) \tag{74}
\]

If we neglect terms of the order of \( (E/H)^2 \), the energy will be given by

\[
E = E_b + \frac{\hbar^2}{2m} \alpha_0^2 + \beta^2 H(n\frac{1}{2}) + e\chi \tag{75}
\]

in the case \( E = 0 \), we have here an exact derivation of the statement made in chapter 11 about the energy levels in a magnetic field.

The derivation can be simplified somewhat by combining the two substitutions used to make the equation separable in \( y \), and letting \( b(x, y) = \exp\left[i\beta(x-x_0)y\right] H(\chi) \). We then choose \( \chi_0 \) such that the terms in \( \chi \) vanish, i.e., we let
The wave function is then given by
\[ \chi_0 = \frac{meE}{2\alpha_0^2k^2x_0^2z} \] (76)
and the eigenvalues for the energy are
\[ E = E_b + \frac{k^2}{2m}\alpha_0^2k^2 + \beta^*H(n+\frac{1}{2}) + e\varepsilon\frac{x_0^2}{4} \] (78)

The approach of Davydov and Pomeranchuk necessitates the use of equation (72). Since equation (77) for the wave function leads to the same results in the cases of interest to us, we shall use the latter. The current due to the electrons will be found from (Wilson, 1953, p. 51)
\[ J_y = -e\varepsilon_0 \left\{ \frac{k}{2\alpha_0} \left[ \frac{1}{2}\chi(yz) \alpha^* a + \frac{e}{mc} \left( -\frac{Hx}{2} \right) a^* a \right] \right\} \] (79)

It is seen that the only terms that can contribute to the first term in the current are the ones containing imaginary exponentials. Thus
\[ J_x = -e\varepsilon_0 \left\{ \frac{k}{2\alpha_0} \left[ \frac{1}{2}\chi(yz) \alpha^* a + \frac{e}{mc} \left( -\frac{Hx}{2} \right) a^* a \right] \right\} \] (80)

which is identically zero since \( \gamma = e\hbar /2mc \). The conductivity is therefore zero in the presence of a transverse magnetic field if there are no collisions. This is the same result one obtains classically. On the other hand,
\[ J_y = -e\varepsilon_0 \left\{ \frac{k}{2\alpha_0} \left[ \frac{1}{2}\chi(yz) \alpha^* a + \frac{e}{mc} \left( \frac{Hx}{2} \right) a^* a \right] \right\} \] (81)

For the simple harmonic oscillator wave functions,
\[ \int_{-\infty}^{\infty} dx \chi \varphi^2(x) = 0 \]

It is then simple to integrate equation (81) over all space obtaining for the average value of the current in the direction,

\[ j_y = \frac{\alpha e \hbar}{m} \chi \int \sigma \, dx \, dy \, dz = \frac{e^2 E}{2\hbar \chi} N \]

(82)

if we substitute for \( \chi \) and remember that \( \int \sigma \, dx \, dy \, dz = N \), the number of free electrons per unit volume. Substituting for \( \gamma \), this reduces to

\[ j_y = N e c \frac{E}{H} \]

(83)

if the coordinate system is rotated \( \gamma 0^\circ \),

\[ j_x = -N e c \frac{E_x}{H} \]

(84)

where \( E_y \) is the field in the direction, or

\[ E_y = -\frac{1}{N e c} j x H \]

(85)

Thus if we neglect collisions, as has been done here, the magnetoresistance of an anisotropic one band model is infinite and the Hall coefficient is

\[ R = \frac{1}{N(-e)c} \]

(86)

In a private communication to Chambers (1972a), Sondheimer has predicted that the Hall coefficient for very large magnetic fields should be just this value. Thus we have the phenomenon of the Hall coefficient being \(-1/N e c\) both for
high temperatures and very low magnetic fields \((\nu \approx 0)\) and for the zero of temperature (i.e., no collisions) or infinite magnetic fields \((\nu \rightarrow \infty)\).

Some question might be raised about calling the coefficient of \(E/H\) in equation (84) the reciprocal of the Hall coefficient, since the calculation involved postulating the electric field and then determining the current. Physically, however, it should make no difference whether the current is set up or the field is applied first. It is also hard to see from this approach how the introduction of the collisions of the electrons with the lattice could affect our results, since there is no average motion of the electrons in the \(y\) direction which could be affected by the collisions. As long as \(j_x\) stays constant, \(E_y\) should stay constant unless there is a variation in \(N\).

For two bands, the conductivity is still zero, and the Hall coefficient is found from

\[
j_x = j_1 x + j_2 y = + (N_1 e + N_2 e_2) \frac{E_y}{H}
\]

or in the usual case considered \((e_1 = -e, e_2 = e)\)

\[
E_y = \frac{-1}{(N_1 - N_2)e} j_x H
\]

Certainly this expression will not help us in our attempt to explain the behaviour of the Hall effect in bismuth. In the usual two band model \(N_1 - N_2\) is always a constant, and if we assume, as does Jones (1936) that \(N_1 = N_2\) we get an infinite Hall coefficient. Davydov and Rumerchuk, under the same
assumptions, conclude that the Hall effect is zero. Their reasoning is based on equation (83). They point out that with a field in the $x$ direction, if $N_1 = N_2$, $j_{y1} = -j_{y2}$; hence the two Hall currents cancel. Their reasoning seems open to objection since under the conditions of equation (83), $j_x$ is identically zero and the Hall coefficient cannot be defined. Their result does seem to point out that our model so far has not taken into account the possibility of transitions between the bands.

The remainder of Davydov and Pomeranchuk proceeds as follows. Under certain simplifying assumptions, the wave functions are calculated in the presence of scattering centers of potential $U(x-Lm)$. The transition probabilities are found from these wave functions and a distribution function is defined, after the manner of Titeica (1935). When a transition occurs such that the quantum number $\chi_0$ is changed to $\chi_2$, the electron moves a distance ($\chi_2 - \chi_0$) in the direction of the electric field. Multiplying this distance by the electronic charge and the transition probability, and then averaging over all of the states gives the current in the $x$ direction. It is found that there are three contributions to the current, corresponding to transitions in each of the bands and between the bands. Oscillations can occur in the contribution from one of the bands (assumed to be electrons) and in the contribution due to the transitions. The maximum current flows when $E_0/\hbar^*H = n + 1/2$. This should be compared with our result which gives a maximum in the conductivity for
and the experimental results of Alers and Webber (1953) which give conductivity maxima for $E_0/\beta^*H = n + 7/8$; and the experimental results of Alers and Webber (1953) which give conductivity maxima for $E_0/\beta^*H = n + 7/8$; inclining experimental error, this could agree with our $n + 7/8$; but not with Davydov and Pomeranchuk's $n + 1/2$.

The obvious extension of this procedure to the study of the Hall effect would be to introduce a term $-eE_ya(\kappa)$ into equation (65). The equation would then be solved with an attempt to obtain a quantum parameter $\gamma_0$ similar to the parameter $\chi_0$; i.e., $\gamma_0$ would be the $y$-coordinate of the center of gravity of the wave packet of the electron. The probability of a collision, changing the quantum number $\gamma_0$ to $\gamma_0^2$, would then be calculated. From this, the current in the $y$-direction could be found as a function of the parameters $E_x, E_y$, and $H$. Setting $f_y$ equal to zero, we could then solve for $E_y$ in terms of the other parameters.

This approach, however, did not prove feasible. The introduction of an additional term in $y$ into equation (65) made the ensuing separation impossible, even in the case of an isotropic effective mass for the electron.

Nor can we expect any help from the other approaches to the oscillations in the magnetic susceptibility. The work of Aumer (1948) and Sondheimer and Wilson (1951) is based on a direct calculation of the partition function by methods of complex variable theory. Steele also has given a number theoretic calculation of $Z(E, H)$ which allows one to calculate $F - NE_0$ and hence the magnetic susceptibility, with the addition that surface effects can be included. Each of
those treatments gives essentially the same results as the
first work of Landau. Zilberman (1951) has advanced a theory
of the oscillations in the magnetic susceptibility based on
interactions between the electrons themselves. This paper,
and the paper by Rumer (1952) on the oscillations in the
magnetoresistance, are in Russian and unavailable to the
author at present, so they have not been reviewed.

The necessity for an even more general theory than
any of those above, even for the magnetic susceptibility, has
been shown by Berlincourt (1954a) in his work on cadmium.
The orientation dependence of the period and magnitude of the
oscillations in the susceptibility of cadmium cannot be under­
stood on the basis of any model so far proposed, although the
periodic oscillations in the magnetoresistance are similar to
those in zinc. (Zinc shows the expected orientation de­
pendence in both the magnetoresistance and magnetic sus­
ceptibility.) The basis for this new theory will probably
arise from a better understanding of the nature of the
electron in a periodic lattice.

The Peierls theory, as applied here to the calculation
of the number of free electrons of a very special type, can
be used with suitable formulae for the magnetic susceptibility,
the conductivity, and the Hall coefficient to obtain an under­
standing of the relative magnitudes and phases of the oscil­
lations in these effects. Exact numerical agreement with
experiment has not been obtained, due to the approximate
nature of the formulae used for the effects in terms of the
number of electrons. The qualitative description contained here could probably be improved by a further study of the low temperature high magnetic field behaviour of these effects and by further study of the behaviour of electrons in metals, particularly in multivalent metals. It is hoped that the investigations reported here will be of some use to others in this field in determining what course of investigation shows most promise for the future. In the course of this investigation, two new expressions have been obtained for the Hall effect. One of these, valid when the Larmor period is much greater than the time of relaxation, is based on a classical approach, using quantum mechanical results only in the assumption of an effective mass. The value of this expression

\[ R = -\frac{1}{ec} \frac{N_{x+y} - N_{x-y}}{(N_{x+y} + N_{x-y})(N_{x+y} + N_{x-y})} \]

lies in the extension, impossible in the usual quantum mechanical approach, to the case of an anisotropic effective mass. The second expression, for the Hall effect in a one band model, shows that in the limit where the Larmor period is very much greater than the time of relaxation, the expression for the Hall coefficient is the same as that given for the high-temperature, vanishing field case. Neither of these expressions have been applicable to our problem but would be useful in other cases.
APPENDIX

LIST OF COMMON SYMBOLS

$\alpha y$ Components of the reciprocal mass tensor

$$\alpha y = \frac{2m}{h^2} \frac{\partial^2 E}{\partial k_i \partial k_j}$$

$\beta$ The doubled Bohr magneton. When the effective mass is introduced, this becomes $\beta^*$. $\beta^* = \alpha (e\hbar/mc)$

$e$ Reduced energy, $e = E/\beta^*H$.

$e_0$ Reduced Fermi energy.

$\theta$ Reduced temperature, $\theta = kT/\beta^*H$.

$\sigma$ Conductivity.

$\tau$ One half of the mean free time between collisions (Chapter 3), or the time of relaxation (Chapter 4).

$\omega$ Larmor angular velocity $e\hbar/mc$. If effective masses are introduced $\omega = \omega_0$.

$\chi$ Magnetic susceptibility.

$E$ Energy.

$E_0$ Fermi energy.

$E$ Electric field.

$e$ The absolute value of the charge of the electron.

$g$ The Fermi distribution function, $g(E; E_0, kT) = \left\{1 + e^{(E - E_0)/kT}\right\}^{-1}$.

$\kappa$ This symbol was used both for the wave number (sometimes appearing as $k$ or $k_i$) and for Boltzmann's constant.

$N$ The number per unit volume of carriers of type $i$.

$R$ The Hall coefficient, confined so as to be negative for electronic conduction.

$Z$ The number of carriers per unit volume with energy less than $E$ in a magnetic field $H = Z(E_0, H)$.
BIBLIOGRAPHY

Babiskin, J. see Steele and Babiskin 1954.
Babiskin, J. see Steele and Babiskin 1954.
Berlincourt, T. G. 1953 Phys. rev. 84, 1277.
1954a Private communication.
1954b Phys. rev. 84, 1172.
& Logan, J. A. 1934 Phys. rev. 73, 348.
Bouma, J. W. see de Haas, Blom & Schaubnikow 1935.
See Pippard and Chambers 1952
Davis, Jr., L. 1939 Phys. Rev. 56, 93.
de Haas, W. J. & van Alphen, P. M. 1930a Leiden comm. 208d.
1930b Leiden comm. 212a.
1932a Leiden comm. 220d.
1932b Leiden comm. 225b.
de Haas, W. J., Blom, J. W. & Schaubnikow, L. 1935
Leiden comm. 237
de Haas, N. J. see Schottnikov and de Haas 1930.
    see Gerritsen and de Haas 1940.


Frank, N. H. see Sommerfeld and Frank 1931.


Grimsai, E. G. & Levinger, J. S. 1933 Third International
    Conference on Low Temperature Physics.

Grimsai, E. G. see Levinger & Grimsai 1934.

Hemstreet, H. W. see Reynolds, Leinharat & Hemstreet 1954.


    See Mott and Jones 1936.

    1949 Z. Phys. 142, 679


    1939 Private communication to Schoenberg (1939).

Leinharat, T. E. see Reynolds, Leinharat & Hemstreet 1954.


Levinger, J. S. see Grimsai and Levinger 1953.

Logan, J. K. see Berlincourt and Logan 1954.


Onsager, L. 1952 Phil. Mag. 42, 1006.


Peierls, R. 1933a & Phys. 80, 763.

1933b Z. Phys. 81, 186.


Romanchuk, I. see Levyov & Romanchuk 1946.


see Sycamore and Robinson 1949.


Schubnikow, L. see De Haas, Wm & Schubnikow 1935.


& Badiskin, V. V. 1973 Phys. Rev. 74, 1374.

see Badiskin and Sondheimer 1974.


Udini, M. E. see Schoenborn and Udini 1936.

van Alphen, P. M. see de Haas and van Alphen 1930a,b; 1932a,b.

Webber, R. T. see Niels and Webber 1933.


1953 Theory of Metals, Second Edition
Cambridge: Cambridge University Press

see Sondheimer & Wilson 1947, 1951.


Edward George Grimsal was born in 1927, in Detroit, Michigan. He attended public schools in Michigan, graduating from Mary Free Bed School in Grand Rapids, Michigan, in 1944. After receiving a Bachelor of Arts degree in chemistry from Western Michigan College in 1948, he entered Graduate School at Iowa State College, where he received a Master of Science degree in physics. After teaching a year at Western Michigan College, he entered the Graduate School of Louisiana State University in 1951, and is now a candidate for the degree of Doctor of Philosophy in the Department of Physics.
EXAMINATION AND THESIS REPORT

Candidate:        E. G. Grunfeld
Major Field:      Physics
Title of Thesis:  The Periodic Hall Effect

Approved:

J. S. Levinson
Major Professor and Chairman

Richard M. Darrell
Dean of the Graduate School

EXAMINING COMMITTEE:

G. Wendel
Vincent E. Parker
George W. Morris
Joseph M. Reynolds

Date of Examination:

11-17-54