A Balancing Domain Decomposition by Constraints Preconditioner for a $C^0$ Interior Penalty Method

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1 Introduction

Consider the following weak formulation of a fourth order problem on a bounded polygonal domain $\Omega$ in $\mathbb{R}^2$:
Find $u \in H^2_0(\Omega)$ such that
\[
\int_{\Omega} \nabla^2 u : \nabla^2 v \, dx = \int_{\Omega} f v \, dx \quad \forall v \in H^2_0(\Omega),
\]
where $f \in L^2(\Omega)$, and $\nabla^2 v : \nabla^2 w = \sum_{i,j=1}^{2} (\partial^2 v / \partial x_i \partial x_j)(\partial^2 w / \partial x_i \partial x_j)$ is the inner product of the Hessian matrices of $v$ and $w$.

For simplicity, let $\mathcal{T}_h$ be a quasi-uniform triangulation of $\Omega$ consisting of rectangles and take $V_h \subset H^1_0(\Omega)$ to be the $Q_2$ Lagrange finite element space associated with $\mathcal{T}_h$. Then the model problem (1) can be discretized by the following $C^0$ interior penalty Galerkin method [7, 3]:
Find $u_h \in V_h$ such that
\[
a_h(u_h, v) = \int_{\Omega} f v \, dx \quad v \in V_h,
\]

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where
\[
a_h(v, w) = \sum_{D \in \mathcal{T}_h} \int_T \nabla^2 v : \nabla^2 w \, dx + \sum_{e \in \mathcal{E}_h} \eta |e| \left[ \left( \frac{\partial^2 v}{\partial n^2} \right) \left[ \frac{\partial w}{\partial n} \right] + \left( \frac{\partial^2 w}{\partial n^2} \right) \left[ \frac{\partial v}{\partial n} \right] \right] ds.
\]

Here \( \eta \) is a positive penalty parameter, \( \mathcal{E}_h \) is the set of edges of \( \mathcal{T}_h \), and \( |e| \) is the length of the edge \( e \). The jump \( [ \cdot ] \) and the average \( \{ \cdot \} \) are defined as follows.

Let \( n_e \) be the unit normal chosen according to a reference direction shown in Fig. 1. If \( e \) is an interior edge of \( \mathcal{T}_h \) shared by two elements \( D_- \) and \( D_+ \), we define on \( e \),
\[
\left[ \frac{\partial v}{\partial n} \right] = \frac{\partial v_+}{\partial n_e} - \frac{\partial v_-}{\partial n_e} \quad \text{and} \quad \left\{ \frac{\partial^2 v}{\partial n^2} \right\} = \frac{1}{2} \left( \frac{\partial^2 v_+}{\partial n_e^2} + \frac{\partial^2 v_-}{\partial n_e^2} \right),
\]
where \( v_{\pm} = v|_{D_{\pm}} \). On an edge of \( \mathcal{T}_h \) along \( \partial \Omega \), we define
\[
\left[ \frac{\partial v}{\partial n} \right] = \pm \frac{\partial v}{\partial n_e} \quad \text{and} \quad \left\{ \frac{\partial^2 v}{\partial n^2} \right\} = \frac{\partial^2 v}{\partial n_e^2},
\]
in which the negative sign is chosen if \( n_e \) points towards the outside of \( \Omega \), and the positive sign otherwise.

It is noted that for \( \eta > 0 \) sufficiently large (Lemma 6 in [3]), there exist positive constants \( C_1 \) and \( C_2 \) independent of \( h \) such that
\[
C_1 a_h(v, v) \leq |v|^2_{H^2(\Omega, \mathcal{T}_h)} \leq C_2 a_h(v, v) \quad \forall v \in V_h,
\]
where
\[
|v|^2_{H^2(\Omega, \mathcal{T}_h)} = \sum_{D \in \mathcal{T}_h} |v|^2_{H^2(D)} + \sum_{e \in \mathcal{E}_h} \frac{1}{|e|} \left\| \left[ \frac{\partial v}{\partial n} \right] \right\|_{L^2(e)}^2.
\]

Compared with classical finite element methods for fourth order problems, \( C^0 \) interior penalty methods have many advantages [3, 5, 7]. However, due to the nature of fourth order problems, the condition number of the discrete problem resulting from \( C^0 \) interior penalty methods grows at the rate of \( h^{-4} \) [8]. Thus a good precon-
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...conditioner is essential for solving the discrete problem efficiently and accurately. In this paper, we develop a nonoverlapping domain decomposition preconditioner for $C^0$ interior penalty methods that is based on the balancing domain decomposition by constraints (BDDC) approach [6, 4, 1].

The rest of the paper is organized as follows. In Section 2 we introduce the subspace decomposition. We then design a BDDC preconditioner for the reduced problem in Section 3, followed by condition number estimates in Section 4. Finally, we report numerical results in Section 5 that illustrate the performance of the proposed preconditioner and corroborate the theoretical estimates.

2 A Subspace Decomposition

We begin with a nonoverlapping domain decomposition of $\Omega$ consisting of rectangular (open) subdomains $\Omega_1, \Omega_2, \cdots, \Omega_J$ aligned with $\mathcal{T}_h$ such that $\partial \Omega_j \cap \partial \Omega_\ell = \emptyset$, a vertex, or an edge, if $j \neq \ell$.

We assume the subdomains are shape regular and denote the typical diameter of the subdomains by $H$. Let $\Gamma = (\bigcup_{j=1}^J \partial \Omega_j) \setminus \partial \Omega$ be the interface of the subdomains, and $\mathcal{E}_h, \Gamma$ be the subset of $\mathcal{E}_h$ containing the edges on $\Gamma$.

Since the condition that the normal derivative of $v$ vanishes on $\Gamma$ is implicit in terms of the standard degrees of freedom (dofs) of the $Q_2$ finite element, it is more convenient to use the modified $Q_2$ finite element space (Fig. 2) as $V_h$. Details of the modified $Q_2$ finite element space can be found in [5].

![Fig. 2](image-url) (a) A nonoverlapping decomposition of $\Omega$ into $\Omega_1, \cdots, \Omega_J$ and a triangulation of the subdomain $\Omega_j$. (b) Dofs of $V_h|\Omega_j$. (c) Reference directions for the first order and mixed derivatives.

First of all, we decompose $V_h$ into two subspaces

$$V_h = V_{h,C} \oplus V_{h,D},$$

where

$$V_{h,C} = \left\{ v \in V_h : \left[ \frac{\partial v}{\partial n} \right] = 0 \text{ on the edges in } \mathcal{E}_h \text{ that are subsets of } \bigcup_{j=1}^J \partial \Omega_j \right\}$$

and
\[ V_{h,D} = \left\{ v \in V_h : \left\{ \frac{\partial v}{\partial n} \right\} = 0 \text{ on edges in } \partial_h, \text{ and } v \text{ vanishes at all interior nodes of each subdomain} \right\}. \]

Let \( A_h : V_h \to V_h^\prime \) be the symmetric positive definite (SPD) operator defined by

\[ \langle A_h v, w \rangle = a_h(v, w) \quad \forall v, w \in V_h, \]

where \( \langle \cdot, \cdot \rangle \) is the canonical bilinear form between a vector space and its dual. Similarly, we define \( A_{h,C} : V_{h,C} \to V_{h,C}^\prime \) and \( A_{h,D} : V_{h,D} \to V_{h,D}^\prime \) by

\[ \langle A_{h,C} v, w \rangle = a_h(v, w) \quad \forall v, w \in V_{h,C} \quad \text{and} \quad \langle A_{h,D} v, w \rangle = a_h(v, w) \quad \forall v, w \in V_{h,D}. \]

Then we have the following lemma.

**Lemma 1.** For any \( v \in V_h \), there is a unique decomposition \( v = v_C + v_D \), where \( v_C \in V_{h,C} \) and \( v_D \in V_{h,D} \). In addition, it holds that

\[ \langle A_h v, v \rangle \approx \langle A_{h,C} v_C, v_C \rangle + \langle A_{h,D} v_D, v_D \rangle \quad \forall v \in V_h. \]

**Remark 1.** Since the subspace \( V_{h,D} \) only contains dofs on the boundary of subdomains, the size of the matrix \( A_{h,D}^{-1} \) is of order \( J/h \). We can implement the solve \( A_{h,D}^{-1} \) directly. Therefore, it is crucial to have an efficient preconditioner for \( A_{h,C} \).

Because functions in \( V_{h,C} \) have continuous normal derivatives on the edges in \( \partial_h \) and vanishing normal derivatives on \( \partial \Omega \), it is easy to observe that

\[ a_h(v, w) = \sum_{j=1}^J a_{h,j}(v_j, w_j) \quad \forall v, w \in V_{h,C}, \]

where \( v_j = v|_{\Omega_j}, w_j = w|_{\Omega_j} \), and \( a_{h,j}(\cdot, \cdot) \) is the analog of \( a_h(\cdot, \cdot) \) defined on elements and interior edges of \( \Omega_j \). Note that \( a_{h,j}(\cdot, \cdot) \) is a localized bilinear form.

Next we define

\[
V_{h,C}(\Omega \setminus \Gamma) = \left\{ v \in V_{h,C} : v \text{ has its vanishing derivatives up to order } 1 \text{ on } \Gamma \right\}
\]

\[
V_{h,C}(\Gamma) = \left\{ v \in V_{h,C} : a_h(v, w) = 0, \forall w \in V_{h,C}(\Omega \setminus \Gamma) \right\}.
\]

Functions in \( V_{h,C}(\Gamma) \) are referred to as discrete biharmonic functions. They are uniquely determined by the dofs associated with \( \Gamma \).

For any \( v_C \in V_{h,C} \), there is a unique decomposition \( v_C = v_{C,\Omega \setminus \Gamma} + v_{C,\Gamma} \), where \( v_{C,\Omega \setminus \Gamma} \in V_{h,C}(\Omega \setminus \Gamma) \) and \( v_{C,\Gamma} \in V_{h,C}(\Gamma) \). Furthermore, let \( A_{h,C,\Omega \setminus \Gamma} : V_{h,C}(\Omega \setminus \Gamma) \to V_{h,C}(\Omega \setminus \Gamma)' \) and \( S_h : V_{h,C}(\Gamma) \to V_{h,C}(\Gamma)' \) be SPD operators defined by

\[ \langle A_{h,C,\Omega \setminus \Gamma} v, w \rangle = a_h(v, w) \quad \forall v, w \in V_{h,C}(\Omega \setminus \Gamma), \]

\[ \langle S_h v, w \rangle = a_h(v, w) \quad \forall v, w \in V_{h,C}(\Gamma), \]
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then it holds that for all $v_C \in V_{h,C}$ with $v_C = v_{C,\Omega} + v_{C,\Gamma}$,

$$
\langle A_{h,C}v_C, v_C \rangle = \langle A_{h,C,\Omega\setminus\Gamma}v_C, v_C, \Omega \setminus \Gamma \rangle + \langle S_h v_{C,\Gamma}, v_{C,\Gamma} \rangle.
$$

Remark 2. It is noted that $A_{h,C,\Omega\setminus\Gamma}^{-1}$ can be implemented by solving the localized biharmonic problems on each subdomain in parallel. Hence, a preconditioner for $S_h^{-1}$ needs to be constructed.

3 A BDDC Preconditioner

In this section a preconditioner for the Schur complement $S_h$ is constructed by the BDDC methodology.

Let $V_{h,C,j}, 1 \leq j \leq J$ be the restriction of $V_{h,C}$ on the subdomain $\Omega_j$. We define $H_j$, the space of local discrete biharmonic functions, by

$$
H_j = \{ v \in V_{h,C,j} : a_{h,j}(v, w) = 0 \quad \forall w \in V_{h,C}(\Omega_j) \},
$$

where $V_{h,C}(\Omega_j)$ is the subspace of $V_{h,C,j}$ whose members vanish up to order 1 on $\partial \Omega_j$. The space $H_0$ is then defined by gluing the spaces $H_j$ together at the cross points such that

$$
H_0 = \{ v \in L^2(\Omega) : v|_{\Omega_j} \in H_j \text{ and } v \text{ has continuous dofs at subdomain corners} \}.
$$

We equip $H_0$ with the bilinear form:

$$
da^C_h(v, w) = \sum_{1 \leq j \leq J} a_{h,j}(v_j, w_j) \quad \forall v, w \in H_0,
$$

where $v_j = v|_{\Omega_j}$ and $w_j = w|_{\Omega_j}$.

Next we introduce a decomposition of $H_0$,

$$
H_0 = \tilde{H} \oplus H_0^0
$$

where

$$
\tilde{H} = \{ v \in H_0 : \text{the dofs of } v \text{ vanish at the corners of the subdomains } \Omega_1, \ldots, \Omega_j \},
$$

$$
H_0^0 = \{ v \in H_0 : d^C_h(v, w) = 0 \quad \forall w \in \tilde{H} \}.
$$

Let $\tilde{H}_j$ be the restriction of $\tilde{H}$ on $\Omega_j$. We then define SPD operators $S_0 : H_0^0 \longrightarrow \tilde{H}_0^0$ and $S_j : \tilde{H}_j \longrightarrow \tilde{H}_j^j$ by

$$
\langle S_0 v, w \rangle = d^C_h(v, w) \quad \forall v, w \in H_0^0 \quad \text{and} \quad \langle S_j v, w \rangle = a_{h,j}(v, w) \quad \forall v, w \in \tilde{H}_j.
$$
Now the BDDC preconditioner $B_{\text{BDDC}}$ for $S_h$ is given by

$$B_{\text{BDDC}} = (P_T I_0) S_h^{-1} (P_T I_0)' + \sum_{j=1}^{J} (P_T E_j) S_j^{-1} (P_T E_j)',$$

where $I_0 : \mathcal{H}_0 \rightarrow \mathcal{H}_0$ is the natural injection, $E_j : \mathcal{H}_j \rightarrow \mathcal{H}_0$ is the trivial extension, and $P_T : \mathcal{H}_0 \rightarrow V_{h,C}$ is a projection defined by averaging such that for all $v \in \mathcal{H}_0$, $P_T v$ is continuous on $\Gamma$ up to order 1.

**Remark 3.** A preconditioner $B : V_h' \rightarrow V_h$ for $A_h$ can then be constructed as follows:

$$B = I_D A_D^{-1} I_D + I_{h,C,\Omega} I_{h,C,\Omega}^{-1} I_{h,C,\Omega} I_{h,C,\Omega} + I_{\Gamma} B_{\text{BDDC}} I_{\Gamma},$$

where $I_D : V_h, I_{h,C,\Omega,\Gamma} : V_{h,C}(\Omega \setminus \Gamma) \rightarrow V_h$, and $I_{\Gamma} : V_{h,C}(\Gamma) \rightarrow V_h$ are natural injections.

### 4 Condition Number Estimates

In this section we present the condition number estimates of $B_{\text{BDDC}} S_h$. Let us begin by noting that

$$V_{h,C}(\Gamma) = P_T I_0 \mathcal{H}_0 + \sum_{j=1}^{J} P_T E_j \mathcal{H}_j.$$  

Then it follows from the theory of additive Schwarz preconditioners (see for example [10, 11, 9, 2]) that the eigenvalues of $B_{\text{BDDC}} S_h$ are positive, and the extreme eigenvalues of $B_{\text{BDDC}} S_h$ are characteristic by the following formulas

$$\lambda_{\text{min}} (B_{\text{BDDC}} S_h) = \min_{v \in V_{h,C}(\Gamma)} \frac{\langle S_h v, v \rangle}{\min_{v \neq 0 \in V_{h,C}(\Gamma)} \langle S_h v_0, v_0 \rangle + \sum_{j=1}^{J} \langle S_j \hat{v}_j, \hat{v}_j \rangle},$$

$$\lambda_{\text{max}} (B_{\text{BDDC}} S_h) = \max_{v \in V_{h,C}(\Gamma)} \frac{\langle S_h v, v \rangle}{\min_{v \neq 0 \in V_{h,C}(\Gamma)} \langle S_h v_0, v_0 \rangle + \sum_{j=1}^{J} \langle S_j \hat{v}_j, \hat{v}_j \rangle},$$

from which we can establish a lower bound for the minimum eigenvalue of $B_{\text{BDDC}} S_h$, an upper bound for the maximum eigenvalue of $B_{\text{BDDC}} S_h$, and then an estimate on the condition number of $B_{\text{BDDC}} S_h$.

**Theorem 1.** It holds that $\lambda_{\text{min}} (B_{\text{BDDC}} S_h) \geq 1$ and $\lambda_{\text{max}} (B_{\text{BDDC}} S_h) \leq (1 + \ln(H/h))^2/C$, which imply
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\[ \kappa(B_{\text{BDDC}}S_h) = \frac{\lambda_{\text{min}}(B_{\text{BDDC}}S_h)}{\lambda_{\text{max}}(B_{\text{BDDC}}S_h)} \leq C(1 + \ln(H/h))^2, \]

where the positive constant $C$ is independent of $h, H,$ and $J$.

5 Numerical Results

In this section we present some numerical results to illustrate the performance of the preconditioners $B_{\text{BDDC}}$ and $B$. We consider our model problem (1) on the unit square $(0,1) \times (0,1)$. By taking the penalty parameter $\eta$ in $a_h(\cdot, \cdot)$ and $a_{h,j}(\cdot, \cdot)$ to be 5, we compute the maximum eigenvalue, the minimum eigenvalue, and the condition number of the systems $B_{\text{BDDC}}S_h$ and $BA_h$ for different values of $H$ and $h$.

The eigenvalues and condition numbers of $B_{\text{BDDC}}S_h$ and $BA_h$ for 16 subdomains are presented in Tables 1 and 2, respectively. They confirm our theoretical estimates. In addition, the corresponding condition numbers of $A_h$ are provided in Table 2.

Moreover, to illustrate the practical performance of the preconditioner, we present in Table 3 the number of iterations required to reduce the relative residual error by a factor of $10^{-6}$ for the preconditioned system and the un-preconditioned system, from which we can observe the dramatic improvement in efficiency due to the preconditioner, especially as $h$ gets smaller.

Table 1: Eigenvalues and condition numbers of $B_{\text{BDDC}}S_h$ for $H = 1/4$ (J = 16 subdomains )

<table>
<thead>
<tr>
<th>$h$</th>
<th>$\lambda_{\text{max}}(B_{\text{BDDC}}S_h)$</th>
<th>$\lambda_{\text{min}}(B_{\text{BDDC}}S_h)$</th>
<th>$\kappa(B_{\text{BDDC}}S_h)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h=1/8$</td>
<td>3.6073</td>
<td>1.0000</td>
<td>3.6073</td>
</tr>
<tr>
<td>$h=1/12$</td>
<td>2.9197</td>
<td>1.0000</td>
<td>2.9197</td>
</tr>
<tr>
<td>$h=1/16$</td>
<td>3.0908</td>
<td>1.0000</td>
<td>3.0908</td>
</tr>
<tr>
<td>$h=1/20$</td>
<td>3.2756</td>
<td>1.0000</td>
<td>3.2756</td>
</tr>
<tr>
<td>$h=1/24$</td>
<td>3.4535</td>
<td>1.0000</td>
<td>3.4535</td>
</tr>
</tbody>
</table>

Table 2: Eigenvalues and condition numbers of $BA_h$, and condition numbers of $A_h$ for $H = 1/4$ (J = 16 subdomains )

<table>
<thead>
<tr>
<th>$h$</th>
<th>$\lambda_{\text{max}}(BA_h)$</th>
<th>$\lambda_{\text{min}}(BA_h)$</th>
<th>$\kappa(BA_h)$</th>
<th>$\kappa(A_h)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h=1/8$</td>
<td>4.0705</td>
<td>0.2148</td>
<td>18.9490</td>
<td>1.1064e+03</td>
</tr>
<tr>
<td>$h=1/12$</td>
<td>3.4107</td>
<td>0.2507</td>
<td>13.6054</td>
<td>1.3426e+04</td>
</tr>
<tr>
<td>$h=1/16$</td>
<td>3.4866</td>
<td>0.2578</td>
<td>13.5244</td>
<td>6.1689e+04</td>
</tr>
<tr>
<td>$h=1/20$</td>
<td>3.5947</td>
<td>0.2590</td>
<td>13.8787</td>
<td>1.8215e+05</td>
</tr>
<tr>
<td>$h=1/24$</td>
<td>3.7123</td>
<td>0.2593</td>
<td>14.3181</td>
<td>4.2288e+05</td>
</tr>
</tbody>
</table>
Table 3 Number of iterations for reducing the relative residual error by a factor of $10^{-6}$ for $H = 1/4$ ( $J = 16$ subdomains )

<table>
<thead>
<tr>
<th>$h$</th>
<th>$N_{iter}(A_{h}x = b)$</th>
<th>$N_{iter}(BA_{h}x = Bb)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/8</td>
<td>95</td>
<td>27</td>
</tr>
<tr>
<td>1/12</td>
<td>235</td>
<td>23</td>
</tr>
<tr>
<td>1/16</td>
<td>434</td>
<td>23</td>
</tr>
<tr>
<td>1/20</td>
<td>704</td>
<td>23</td>
</tr>
<tr>
<td>1/24</td>
<td>1026</td>
<td>23</td>
</tr>
</tbody>
</table>

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