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Optimal number and location of storage hubs and biogas production reactors in farmlands with allocation of multiple feedstocks

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ABSTRACT

This paper focuses on the problem for designing a logistics system for bio-methane gas (BMG) production. In practice, farm residues such as crop residue, wood residue, and livestock manure are used in reactors as reactants to generate BMG. A multi-residue, multi-hub, multi-reactor location-allocation model is developed to design the logistics of BMG production system. Both the hubs' and reactors' locations, and the residue's distribution plan are investigated to minimize the total construction and logistical cost. The costs of construction, transportation, feedstocks and labor are taken into consideration to reflect the lifecycle cost of the entire undertaking. In this paper, a mixed-integer nonlinear problem is proposed to simulate a BMG production supply chain. In addition to the optimal solution methods, a search-based heuristic was also proposed to determine the locations of hubs and reactors for large instances and along with a proper allocation of residues that are transported from the farms to the hubs to the reactors. Several numerical examples are tested to evaluate the performance of the heuristic as well.

1. Introduction

The major sources of energy in the world are the petroleum products, hydro-electricity, coal and natural gas. According to the USEIA (U.S. Energy Information Administration) statistics [1], about 86.4% of the energy consumed in the world is obtained from fossil fuels. However, there are various concerns related to this heavy reliance on fossil fuels for energy sources. One of the major issues is that they are nonrenewable resources and may be exhausted eventually. In 2013, USEIA predicted that fossil fuels would be exhausted within 50–200 years. Another key concern results from the harmful emission generated by combustion of fossil fuels. Burning fossil fuels is usually accompanied by a significant amount of carbon dioxide emissions which may cause a series of environmental problems [2].

After the inception of the new energy for America plan in 2008, various renewable energy sources, including biogas, wind and hydropower developed rapidly. Bio-methane gas (BMG) is the methane gas produced from biomass sources (that's why the name is so coined); it is characterized as an economic energy and is used as a substitute for conventional sources of energy in many parts of the world. The feedstocks for generating BMG can be farm-based such as crops, woods residues, manure, and livestock. Compared to the locations necessary to produce energy by wind and tides, the location of a biogas

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Nomenclature

Indices

k	index for residue types, $k = 1, \ldots, K$.
K	residue types.
n	index for reactors, $n = 1, \ldots, N$.
N	total number of reactors to build.
z	index of zip codes for a hub, $z = 1, \ldots, Z$.
Z	total number of zip codes in the region.

Parameters

A_{zk}	availability of residue k in the hub in zip code z (truck-load).
C^f	fixed cost to build all the reactors, shared equally to unit time period (dollar).
C_n^v	variable cost to build reactor n ($n = 1, 2, \ldots, N$) (dollars/reactor); i.e., C_n^v is the cost of building reactor n and it incurs depending on the number of reactors demanded.
C^w	unit cost for hiring a worker (dollar /person/truck load).
C_{zk}^t	unit transportation cost of moving residue k from the hub in zip code z to the reactor (dollar/truck-load/mile).
C_{zk}^r	feedstock cost of residue k supplied from the hub in zip code z to the reactor (dollar/truck-load).
D_{kn}	demand for residue k at the nth reactor (truck-load).
w_k	unit workforce required for collecting residue k (persons/truck-load).
W_n	total workforce availability at the nth reactor (persons).
β	percent of residue to be unaccounted for due to natural loss or timely unavailability.
$\mathbf{h}_z$	$\mathbf{h}_z = (x_z^h, y_z^k)$ represents the coordinate of the hub in zip code z.

Decision variables

$\mathbf{r}_n$	$\mathbf{r}_n = (x_n, y_n)$ represents the coordinate of the reactor.
$\mathbf{r}$	$\mathbf{r} = (r_n) \in \mathbb{R}^{2N}$ represents the vector coordinate of the reactors.
α_{zk}^n	number of truck-loads for moving residue k from the hub in zip code z to the nth reactor.
$\boldsymbol{\alpha}^n$	a series of matrices $\boldsymbol{\alpha}^n = [\alpha_{zk}^n]$, $n = 1, \ldots, N$ representing all truck-loads of residue k ($k = 1, \ldots, k$) to be transported from zip code z ($z = 1, \ldots, Z$) to the nth reactor.

Decision measure

$TC(\mathbf{r}, \boldsymbol{\alpha}^1, \ldots, \boldsymbol{\alpha}^n)$	total cost of the entire system (dollars): For $n = 1, \ldots, N$. It is the summation of locating nth reactor at coordinate point $\mathbf{r}_n$ and transporting $\boldsymbol{\alpha}^n$ truck-loads to the nth reactor from the residue collection hubs.

plant is more flexible. Furthermore, BMG can be liquefied for ease of storage and distribution. In a BMG reactor, these feedstocks go through an "anaerobic digester process" to generate BMG. The production process results in a mutually beneficial relationship between the BMG industry and local farms; that is, farms generate income by selling residues and the BMG industry gains a steady supply of low-priced feedstock. BMG has been used as a household energy source for a long time, but the advent of new technology makes it possible for industrial production and distribution of BMG. Europe produced 30 billion kWh (kilowatt-hours) of power in 2010 (RENI, [3]) and there were 78 biogas plants in Japan by the end of 2012 with projected volume of 38 TWh (terawatt-hours) of electricity from biomass in 2013 (*Enerdata*, [4]).

For lack of other commercial sources of energy, many countries rely heavily on traditional fuel source, especially firewood. In order to solve the energy problem in rural areas, several countries initiated production and distribution of several renewable energy technologies. Among these technologies, biogas has been proven to be viable, and it emerged as a promising technology. Biogas is formed by the biological breakdown of organic matter in an oxygen deficient environment (anaerobic system). It is classified as an eco-friendly biofuel. Biogas contains approximately 60% methane and 40% carbon dioxide [5]. It can be used for generating electricity and also as automotive fuel. It has been one of the most successful models for the production of clean, environmentally friendly, cost effective source of energy and has multiple benefits. Improved health, increased crop productivity and saved time are some of the major benefits to the users. It provides economic benefit to the country through reduced deforestation and carbon trading. In addition, by reducing greenhouse gas emissions, the technology helps mitigate global warming and climate change. Thus, biogas is a renewable, sustainable and clean source of energy that provides multiple benefits, locally and globally. This bio-chemical process makes a clean environment from waste products, and contributes to the bio-energy generation and national savings.

While many petrochemical companies (such as Exxon, Mobile and Shell) may not work in support of BMG production because of competitive profit, market share and customer loss, many small utility companies (e.g., DTE Energy in Michigan, Duke Energy in North Carolina, BG&E) and BMG production enterprises (GENeco, BIOMaster, E^3-Environmental & Energy Engineering) may be the major stakeholders in this pursuit in addition to private enterprises (Hilarides Dairy in California and Fair Oak Dairy in Indiana) and federal government (Environmental Protection Agency, Revenue Services, etc.). Further, the general public will be more supportive since the energy cost may be much cheaper and farm owners will be more

interested because they will be able sell their unused waste or feedstocks. The local government encourages BMG production so that it can meet the local demand without much government undertakings such as legislative bureaucracy, preliminary reconnaissance, technical contracting, monetary investment, construction and commissioning. The local government will thus ease its burden for BMG production and such an undertaking will certainly generate revenue for both local and federal governments.

The crop residues, wood residues and animal waste are three major residuals of livestock and agricultural natural products. These wastes can be converted to some viable products such as bio-methane gas and liquefied carbon products that may amount to significant dollar value with respect to proportion of the waste to the original raw-product volume. The problem has two processes: conversion of waste/residues into biomethane gas through *anaerobic process* and then conversion of biomethane gas into liquefied carbon product through *condensation process*. This bio-chemical process, in one way, makes a clean environment from waste products, and in another way, the process contributes to the bio-energy generation and national savings.

With the increasing need for biofuel manufacturing, plant design and production management become vitally important for development of the BMG industry. Optimization of the supply chain can significantly reduce the initial investment and production costs. Great efforts have been made in designing the supply chain for the biofuel production system. Most recently, Scott et al. [6] studied the supply chain design for the UK bioenergy industry and supplied a strategic plan for the decision-making process. Lim et al. [7] optimized the distribution and capacity of plants that generated electricity from biomass in Malaysia to minimize the total cost. Cucek et al. [8] built a mixed-integer linear model for a multi-period bioenergy network.

There are studies related to locating the facility and planning the residue distribution network in the biomass industry. Among them, Zhu and Yao [9] determined the locations of warehouses for a multi-feedstock system, and provided the sizes for the harvesting team and for biomass storage. Jason et al. [10] addressed a model for a system for siting biomass equipment that uses feedstock. Pathumnakul et al. [11] solved a plant location problem for sugar cane loading station, by considering the maturity periods of cane fields. Zhang and Hu [12] designed an operational planning model for a general biofuel plant to investigate the facility's location and operational levels. Li and Hu [13] selected the locations for a fast pyrolysis facility for a biofuel plant to maximize the net present value of total profit over next 10 years. Natarajan et al. [14] presented a case study on the location problem for biodiesel plants in Finland.

Studies of the problems associated with facility location and product distribution have a long history. One famous benchmark problem is called the Weber problem (Friedrich, [15]), which concerns the facility's location-allocation in a continuous space. Recently, Gülpınar et al. [16] established a facility location model with consideration of the stochastic effect and proposed a robust optimization strategy for solving the problem. A beam search heuristic was proposed by Akyüz et al. [17] to solve the multi-commodity, multi-facility Weber problem. Brimberg et al. [18] solved a multi-source location problem using a new local search method that switched the searching process between a continuous model and a discrete counterpart.

Amigun and von Blotrnitz [19] analyzed capacity-cost and location-cost for biogas plants in Africa followed by Liu's [20] geographic approaches to resolve environmental problems in search of the path to sustainability for polluting plant relocation in China. Later Escalante et al. [21] developed a spatial decision support system to evaluate crop residue energy potential by anaerobic digestion. Alejandra Villamar et al. [22] studied sustainable location sites from a GIS (geographic information system) analysis of anaerobic co-digestion plants for the revaluation of agricultural waste. Aitken et al. [23] assessed a life-cycle of macroalgae cultivation and processing for biofuel production while Karschin and Geldermann [24] developed an efficient cogeneration and district heating systems in bioenergy villages using an optimization approach. Amin and Zhang [25], Amin and Baki [26], Ramezani et al. [27] developed some facilities location problem. Lee and Chang [28] and Yagiiz [29] also proposed facilities location algorithm for specific problems. Aydinel et al. [30] optimized an allocation process forest products company while Ghaderi and Jabalameli [31] modeled the budget-constrained dynamic uncapacitated facility location-network design problem. Also, Siegmeier et al. [32] explained different implications of farm biogas production in organic agricultural systems. At the same time, Song et al. [33] showed the development and utilization of bioenergy and exploring the environmental economic benefits. Reis et al. [34] conducted a pilot study on an anaerobic treatment of food waste. Recently, Nikodinoska et al. [35] showed an integrated environmental accounting framework on wood-based bioenergy value chain in mountain urban districts. Roman-Figuerroa et al. [36] studied the bioenergy potential from residue biomass in Araucania Region of Chile while Kristjanpoller et al. [37] assessed the reliability impact on operational effectiveness of a biomethanation plant. Further, Lopez-Diaz et al. [38] determined an optimal location of biorefineries considering sustainable integration with the environment.

Currently, there is a lack of research that focuses on the supply chain design of a BMG production system. Specifically, more information is needed on the reactors' locations and the optimization of the transportation network, which is critical for reducing the operation costs for the BMG production system. When a BMG reactor is built in a certain area, transportation of the agricultural residues are inevitable, because local farms are usually scattered in that general area. Therefore, transportation costs for the residues constitute a significant portion of the daily operation cost of the BMG reactors. Because transportation costs are affected by distance, a proper selection of reactors' locations is essential to reduce the total cost. Therefore, the prime objective of this paper was to model how to minimize the total cost of building and operating a BMG production system by locating the feedstock hubs and BMG reactors optimally and planning the residue distribution network in an economic fashion. A residue flow map for such a BMG production system is depicted in Fig. 1.

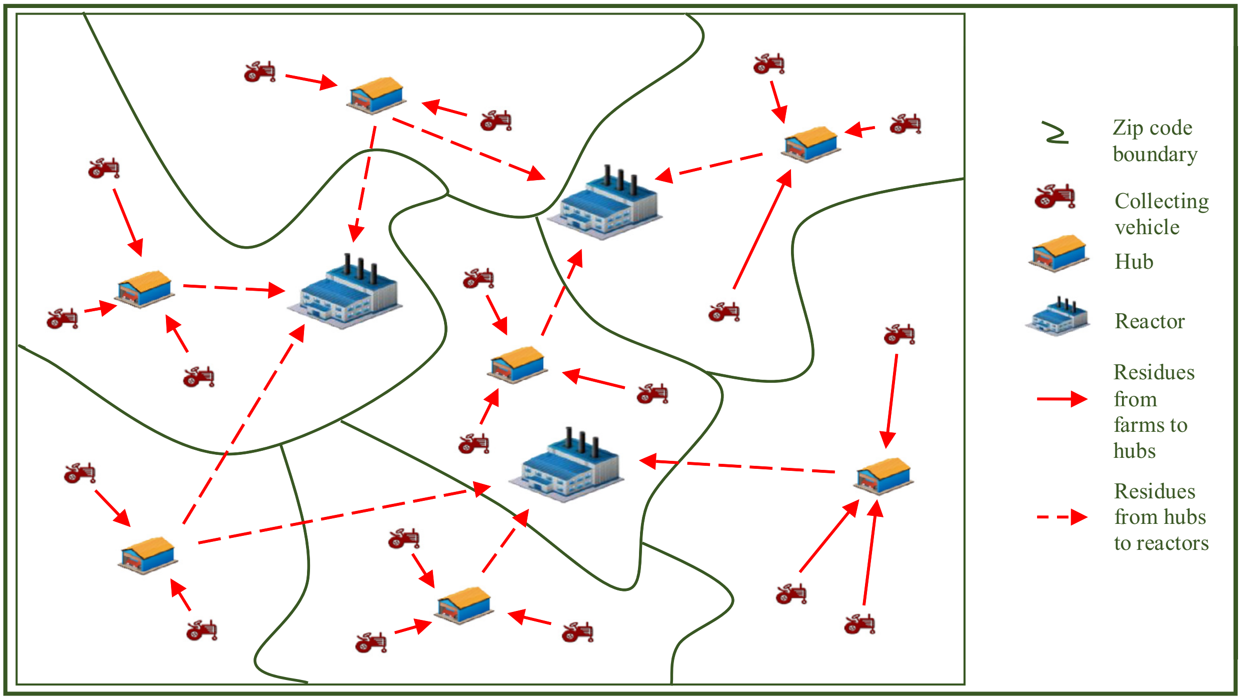

Fig. 1. A residue-flow map within the BMG production system.

This paper models a location-allocation problem for a multi-residue, multi-reactor BMG production system. Raw materials (crop residue, word residues, grasses, animal residues and other bio-materials) are collected from farmlands and are stored in feedstock storage from where the feedstocks are transported to the reactors for BMG production (Fig. 1). In addition, previous studies on the design of the supply chain for the biofuel industry have not reported a sound heuristic that is amenable to solve the mixed-integer nonlinear programming (MINLP) problem. This research builds a model that mathematically describes the supply chain system of a BMG production system and, at the same time, finds an efficient heuristic to solve this complex model.

This paper is organized as follows. Section 2 highlights the description of the BMG problem. Notation, assumption and mathematical models are developed in Section 3. Section 4 illustrates the heuristic designed specifically for the problem. Section 5 provides a small-size numerical example to validate the proposed heuristic step by step. More numerical analyses are presented in Section 6. Section 7 provides concluding remarks.

2. The bio-methane gas production location problem

As indicated earlier, this research problem is primarily categorized as a location-allocation problem for storage hubs and anaerobic reactors in the farmlands to convert the waste into methane gas. The condensation problem to convert the biomethane gas into liquefied carbon product which is another form of the gas product is not considered since this product is the secondary stage of the BMG for commercialization.

Locating anaerobic bioreactors: The factors that affect the decision on determining the number of anaerobic reactors are residues/animal waste available at each zip-code, fixed and variable costs of building these facilities, labor availability in the region, and transportation cost, to mention a few. Also, the GIS generates the data for the optimization model that also consider the roads and railroads for transportation of residues and waste to the hubs and bioreactors. In order to formulate this problem, the system parameters that affect the decision are the number, types and amount of waste/residues in a particular zip code area, the number and size of anaerobic reactors, and the unit labor cost and workforce required for collecting one truck-load of residue. Unit transportation cost of moving residue from zip code to anaerobic reactor, and number of truck-loads of residues moving of residue from a zip code to an anaerobic reactor do also play a role in the decision process. The macro-level transportation scheduling problem for building the *bio-energy fuel reactors* can then be modeled as a mixed-integer nonlinear programming (MINLP) problem with workforce constraint, raw material demand fulfillment constraint, reactor capacity (size) limitation and availability of raw materials in the region. Evaluation of the system parameters such as distance and transportation costs, depend on the GIS information and dynamic market situation. Locating the reactors within a network of roads and railroads is another embedded facility location problem, which makes the whole process lengthy and a multi-stage optimization problem.

An important factor that affects the production of biofuel is the optimal location of a forest biomass-to-biofuel facility and spatially collecting forest biomass which results in significant transportation cost. Several studies report different methodologies to address the biofuel production facility location problem. These studies examined current research status since 2005 and discussed different methods, determined the optimal location or site for a biomass-to-biofuel facility, compared the methods to identify the preferred approach and identified potential gaps for future research. Johnson et al. [39] summarizes major, recent studies that focused on mathematical models to optimally locate a forest biomass-to-biofuel facility. Through analyses of the various methods, including the advantages and disadvantages, this study uncovered preferred approaches for locating an optimal biofuel facility location but it specifically neither does model or formulate any particular problem nor does solve or evaluate any other problem or methods analytically or numerically—it only sums up a synopsis of what are the state-of-the art in hubs and reactor locations. Though the current problem under consideration deals with location and allocation problem for hubs and biogas reactors, unlike Johnson et al. [39], this research addresses how the multi-facility location-allocation problem can be optimally solved to determine the allocation of different feedstocks to these reactors for biogas production and determining the optimum number of such production or reactor facilities in a region of farmlands. While the current problem is a generalized case, Johnson et al. [39] work was a special case of this general problem. The problem is optimized using a mixed-integer programming technique for optimally locating the biofuel sites and allocating the feedstocks to these sites.

Wu et al. [40] studied a similar but not the same problem on location-allocation of hubs and BMG reactors in a region of farmlands. The scope of the paper has been ex*tended* to multi-reactor system*s* capturing more generic aspects of the system as well as the modification of the heuristic *as needed*. In that paper, they optimally located a *single reactor* and allocated the feedstock (forest residues, livestock manure and grass) from each hub to the reactor to minimize the total cost in supplying feedstock materials in a bio-methane production system whereas the current problem focuses on designing a logistics system for BMG production where a more general case of a *multi-residue, multi-hub, multi-reactor* problem is considered to describe the logistics of BMG production system. The current specific problem determines the optimal number and locations of feedstock collection hubs and anaerobic reactors to convert these waste into BMG most economically such that the total cost of transporting or delivering the wastes/residues to anaerobic reactors along with other associated costs is minimized. So, Wu's et al. [40] single-reactor location problem paper is a special case of this *general multi-residue, multi-hub, multi-reactor problem.*

2.1. Bio-methane gas (BMG) location problem

Assume that the local farms are located in a geographical area delineated by the U.S. Zip Code (five-digit number used mainly for postal zoning and the sorting of mails). All farms can generate one or more types of residues, such as crop residue, grass, wood residue, and livestock manure. In each zip code area, there is a hub that collects the residues from the local farms. From the theoretical perspectives, treating the volumes of those materials produced in different farms as relative weights, the hub location within each zip code can be determined optimally using Weber technique (Friedrich 1929) or also known as centroid method where the total transported volume from each farm to the nearest road-point is considered as its weight.

In general, though the farms are fixed in a zip code area, the production and availability of these raw materials depend on the climate, planning for growing the crops and grasses, soil condition of the land, fertilizer, forestation, livestock pasturing, etc. Therefore, the availability of raw materials is to some extent dynamic in nature, not fixed or constant for all seasons. Thus, given the availability of different raw materials in farms, obtained from either local spot reconnaissance or USDA (U.S. Department of Agriculture) geographic information system (GIS), the collection agent can decide on the hub location. USDA GIS data over a reasonable period (maybe, past 3–5 years) are usually taken as an average volume of production that can be used as weight. Once the revised weights (weight $\times$ transportation cost per unit distance $\times$ distance from the farm-to-nearest-road point to the hub) are known to a road (arc), then from the revised-weighted road network within a zip code, the hub location is determined optimally to minimize the total weighted travel distance using the same Weber technique.

The farms and hubs have the configuration such that the farms deliver their available residues to the hubs, and the hubs receive and store all the residues. USDA [41] provides the biogas opportunities roadmap. Residues in the hubs sometime experience natural losses and they are at times unavailable as well.

When a BMG production system is planned in certain area, reactors are needed to produce BMG anaerobically using collected and processed residues from the hubs. Assume that all hubs can supply residues from their entire residue inventory to all reactors and all the reactors can process all types of residues. The best scenario for the construction plan for the BMG production system is that all the available residues in the hubs are transported to and processed by the reactors to maximize BMG output. However, due to some practical limitations, such as the workforce availability [for farm work, the local workforce is used as rural people from distant places are usually reluctant to come], the demand of reactors, and residue availability, the ideal conditions are usually not achievable and at that situation the economic decision on the proper allocation of residues from hubs to reactors becomes more important.

Because the total cost of the supply chain is the major concern for constructing and operating the BMG production system, construction costs, feedstock (residues) costs, transportation costs, and workforce costs are taken into account, because each of them incurs a notable portion of the total cost. The construction cost includes both fixed and variables costs which can be obtained from the construction contract. The residues are transported by trucks and unit transportation cost can be

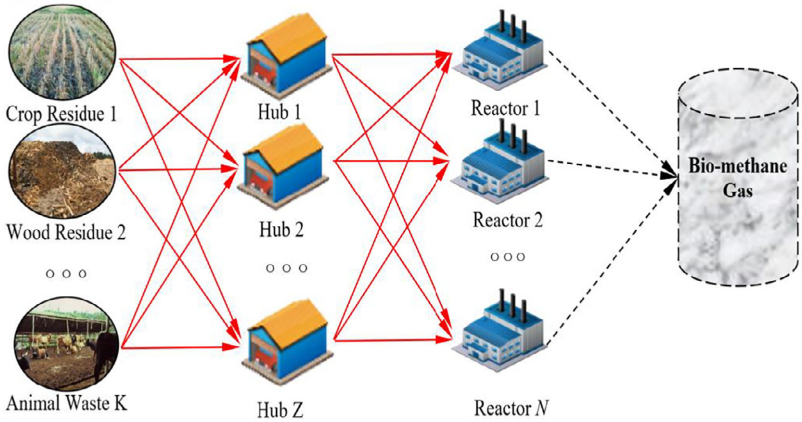

Fig. 2. Residue and bio-gas converting processes.

determined based on fuel, insurance, maintenance, and license costs (Truckers Report, [42]). However, due to the different road conditions in different zip code areas, the unit transportation cost may vary between reactors and hubs. Laborers are hired to collect and load/unload the residues from any area. The unit labor cost and construction cost are assumed to be consistent within the entire area.

This research, thus, focuses on location and allocation problem in the BMG production from the feedstock (crops, woods residues, manure, and livestock) through an anaerobic process. The specific problem aims at determining the optimal *number* and *locations* of feedstock collection hubs and anaerobic reactors to convert these wastes into biomethane gas most economically such that the total cost of transporting or delivering the wastes/residues to anaerobic reactors along with other associated costs is minimized. The network version of this process with hubs and anaerobic reactors is depicted in Fig. 2.

To minimize the total cost for construction and operation, the optimal *locations* for each of the reactors and the optimal *amount of each residue* transported from each hub to the reactors are determined. Various factors such as the reactors' demand requirement, workforce availability and the availability of residues at the hubs are considered as well.

Given the state-of-the-art research into the processes dealing with residues (woods, grass, and livestock manure) to actually produce biogas, this research seems to be the first in the literature that mathematically models the BMG plant location and the supply logistics of biogas feedstock. The major contributions lie in three aspects:

1. The modeling process gives some planning guidance for locating BMG production facilities and allocating feedstocks to these respective BMG facilities that are to be recommended for construction.
2. In addition to the optimal method for finding the optimal solutions, an alternative search-based heuristic is proposed in this research to solve problems with large instances, and it is proved to solve the MINLP problem efficiently and effectively. The numerical examples suggest that *alternative search to convergence heuristic* (ASTCH) can obtain better solutions than LINGO (Linear, Integer, Nonlinear, Global Optimizer).
3. The practical significance of this research lies in two aspects. First, one can notably reduce the planning effort required for the supply chain design of the BMG production system following the proposed systematic methodologies in this paper. Second, the investors can expect a substantial reduction in building and operating costs of the BMG production system by adopting an analytical way to solve a complicated supply chain optimization problem.

3. The BMG problem formulation

The objective of the model is to determine the location for the BMG reactors to which the residues from all hubs can be transported. An optimal plan renders a minimum total cost, which includes construction, feedstock, transportation, and labor cost. Constraints such as labor availability, residue availabilities, and demand of residues at the reactor are considered during the planning process. Before modeling the BMG problem, some important notations and assumptions are stated first besides the ones cited or used locally.

3.1. Assumptions

The assumptions are listed below and they clarify the specific problem on which this research focused, while still preserving the essence of a real-world application as much as possible.

1. The hub locations are determined from the weighted road network.
2. The reactors have the sufficient capacity to process all residues collected from the hubs.

3. There is no loss of residues during the transportation process.
4. Building cost, raw material cost, transportation cost, and labor cost are given.
5. The demand of each residue by each reactor is known.
6. Labor force availability in the region (irrespective of each zip code area) is given.
7. Residues in the hubs experience natural losses and that they are at times unavailable.
8. All residues are not directly supplied to the reactors from the farmlands; they are collected and stored in the hubs. All hubs can supply residues to all reactors and all reactors can process all types of residues.
9. Unit labor cost and construction cost are assumed to be consistent within the entire area.
10. Unit transportation cost may vary between reactors and hubs due to different road conditions in different zip code areas.

3.2. System modeling

The pre-construction survey of residue availability ensures that the feedstock available will be sufficiently enough to meet the existing BMG demand. If BMG demand in the specific area is greater than residue availability—that becomes a new problem. A plant is always constructed with a built-in capability which is usually more than what is expected at the time of construction or in foreseeable future—an industrial organization always keep that in mind and they cannot comply always with the dynamic changes of the situation such as high demand fluctuations. If the demand exceeds, the company has to either expand the capacity of the plant to meet the demand or go by the existing limitation—that is very natural and within the profitable domain as per the planning. The pre-construction survey points out only whether there is enough availability or not; the amount of available residue will not increase by itself if it is not sufficient. In such a situation, if profitable, the BMG company may lease some farmlands/facilities to grow feedstock locally or can import them from distance places if it is economically viable; otherwise the company has to operate within its own built-in capacity limit.

Apparently it may seem that labor is limited by zip code in the assumptions, but it is limited by a reactor depending on which reactor is built at what location and what kind of workforce is available. For farm work, the local workforce is used because rural people from distant places are usually reluctant to come, but that option may be left open for someone who wants to join that workforce from any locality; that's why we use the parameter w_k, the unit workforce required for collecting residue k (persons/truck-load), but not with a zip code indicator z and W_n, the total workforce availability at the nth reactor (persons).

The goal of a system modeling process is to mathematically interpret the total cost of constructing a BMG production system by considering different types of costs and constraints. The components that contribute to major costs are given below:

1. The fixed building cost (C^f) may include the fixed investment, business license fee, and building design fee.
2. The variable cost (C_n^v) for building reactor n incurs on the number of reactors demanded. It is the unit cost to build a reactor when C_n^v (n = 1, 2, ..., N) could be different depending on the location and sizes/capacity of reactors (production demands, volumes, and other capacity specifications).
3. The feedstock cost (C_{zk}^r) of BMG production.
4. The transportation cost (C_{zk}^t) relates to the distance among the hubs and reactors, the road condition, and the capacity of the trucks that delivers the raw material.
5. The cost of hiring laborers to handle the raw material.

Demand of residue at a site (D_{kn}), number of trucks (A_{zk}) with capacity, available labor (W_n), available residue (A_{zk}), and minimum workforce required to collect residue are the resources which are to be obtained from the system. Practical binding constraints are also considered, including the availability of laborers, the demands by reactors, and the natural loss of raw material.

Now, for planar location vectors $\mathbf{a} = (x_a, y_a)$ and $\mathbf{b} = (x_b, y_b)$, the Euclidean distance function can be written as $\|\mathbf{a} - \mathbf{b}\| = \sqrt{(x_a - x_b)^2 + (y_a - y_b)^2}$. If d_{zn} represents the distance from the hub in zip code z to the nth reactor at $\mathbf{r}_n = (x_n, y_n)$, then $d_{zn} = d(\mathbf{r}_n, \mathbf{h}_z) = \sqrt{(x_n - x_z^h)^2 + (y_n - y_z^h)^2}$. Considering all the above factors, the system model for the BMG multi-reactor (BMR) location problem can be built as

Problem BMR:

$$
\begin{aligned}
\operatorname{Min} TC\left(\mathbf{r},\ \boldsymbol{\alpha}^1, \ldots,\ \boldsymbol{\alpha}^N\right) = C^f + \sum_{n=1}^{N} C_n^v + \sum_{z=1}^{Z}\sum_{k=1}^{K}\sum_{n=1}^{N} C_{zk}^r \alpha_{zk}^n \\
+ \sum_{z=1}^{Z}\sum_{k=1}^{K}\sum_{n=1}^{N} d_{zn} C_{zk}^t \alpha_{zk}^n \\
+ C^w \sum_{z=1}^{Z}\sum_{k=1}^{K}\sum_{n=1}^{N} w_k \alpha_{zk}^n
\end{aligned}
\tag{1}
$$

Subject to:

$$\sum_{z=1}^{Z}\sum_{k=1}^{K} w_k\alpha_{zk}^{n} \le W_n, \quad n = 1,\ldots,N \tag{1a}$$

$$\sum_{z=1}^{Z}\alpha_{zk}^{n} \ge D_{kn}, \quad \text{for } k = 1,\ldots K,\ n = 1,\ldots,N \tag{1b}$$

$$\sum_{z=1}^{Z}\alpha_{zk}^{n} \le (1-\beta)\sum_{z=1}^{Z}A_{zk}, \quad \text{for } k = 1,\ldots K,\ n = 1,\ldots,N \tag{1c}$$

$$0 \le \alpha_{zk}^{n} \le A_{zk} \text{ and } \alpha_{zk}^{n} \in \mathbb{Z}, \quad \text{for } n = 1,\ldots,N \tag{1d}$$

The objective function (1) is obtained considering assumptions 1 to 4. It consists of four terms that, from the first to the last, represent the fixed building cost, variable building cost, transportation cost, and labor cost, respectively. Constraint (1a) provides an upper limit to the total workforce for transferring residue k from the hub in zip code z to the nth reactor. Constraint (1b) gives a lower bound on the reactor's demand, for the total truck-loads of residues delivered to the reactor n. Constraint (1c) considers the natural loss of the stocked residue in the hubs and denotes that the total amount of residue k is constrained by its availability. Constraint (1d) states that the amount of residue k from the zth hub to the reactor n is bounded by the availability of residue k in the zth hub and is a non-negative integer.

It can be shown that the model BMR in (1) is a mixed-integer, non-convex, constrained programming problem with respect to its variables $\mathbf{r}$ and $\boldsymbol{\alpha}$. Since d_{zn} is a nonlinear function of coordinates, BMR is basically an MINLP problem for which finding the optimal solution for a large instance is computationally prohibitive. Therefore, it is impractical to enumerate all feasible solutions to obtain the optimal one for a large instance. A compromised methodology is to develop a heuristic such that an optimal/sub-optimal solution can be obtained within a reasonable amount of time. Such a heuristic is thus provided in the next section for solving the BMR problem.

4. Solution procedure

With the expression of Euclidean distance in Eq. (1) and the integer constraint (1d), the BMR location-allocation problem can be considered as a mixed-integer nonlinear programming problem (MINLP). Solving such a problem is a computational burden if the algorithm is not developed properly. Therefore, a heuristic is to be devised in this research with a sound mathematical approach such that the sub-optimal solution is not far from the real optimum.

An ASTCH is proposed in this paper to obtain such a pragmatic result. The ASTCH is designed based on separating two sets of sub-problems in the objective function (1). The motivation behind this heuristic is that function (1) contains two sets of variables, $\mathbf{r}$ and $\boldsymbol{\alpha}$ that influences one another, which causes the function to be non-convex. Decoupling of function (1) in terms of $\mathbf{r}$ and $\boldsymbol{\alpha}$ forms two separate sets of problems that are convex. An alternative search is then applied to each set of functions until the results for both problems are consistent and converge to the optimal/sub-optimal problem.

4.1. Location sub-problem

Before devising the heuristic, the convexity of the two sub-problems from the MINLP (1) is analyzed first. First, a theorem is proposed below to ascertain the concavity property of one of the sub-problems.

Theorem 1. Given $\boldsymbol{\alpha}$, the objective function (1) is always convex in $\mathbf{r}$.

Proof. : see Appendix A. □

Theorem 1 provides one set of convex functions. Although the objective function (1) is non-convex in terms of both $\mathbf{r}$ and$\boldsymbol{\alpha}$, it is convex in $\mathbf{r}$ when $\boldsymbol{\alpha}$ is determined. Therefore, this set of objective functions can be rewritten as in equation (A.1) in Appendix A. The difference of the objective function (1) and (A.1) is that on the left side of the Eq. (1) has both variables $\mathbf{r}$ and $\boldsymbol{\alpha}^n$ while equation (A.1) has the only variable $\mathbf{r}$. Since (A.1) is convex, the optimum occurs at the point where the first-order derivative of (A.1) with respect to $\mathbf{r}$ is zero. That is, setting (A.2) equal to zero yields

$$\mathbf{r}_n = \sum_{z=1}^{Z}\sum_{k=1}^{K}\frac{C_{zk}^{t}\alpha_{zk}^{n}}{d_{zn}}\mathbf{h}_z \Big/ \sum_{z=1}^{Z}\sum_{k=1}^{K}\frac{C_{zk}^{t}\alpha_{zk}^{n}}{d_{zn}}. \tag{2}$$

The solution of Eq. (2) is the location for the sub-problem with a known α. However, obtaining an explicit analytical optimal solution for the nonlinear Eq. (2) is difficult due to its complexity. Therefore, an algorithm is developed below based on the algorithm proposed by Weiszfeld and Pastria [43]. The algorithm with multi-dimensional variables developed here has been extended from Wu et al. [40] and Weiszfeld and Pastria [43], casting a more general situation.

Algorithm A: Optimal Location

Step A-1: Generate a random initial coordinate r_n and denote it as r_n^{curr}.
Step A-2: Update the next coordinate for the reactors by

$$\mathrm{r}_n^{next} = \sum_{z=1}^{Z}\sum_{k=1}^{K}\frac{C_{zk}^t\alpha_{zk}^n}{d_{zn}(\mathrm{r}_n^{curr},\ \mathbf{h}_z)}\mathbf{h}_z \Big/ \sum_{z=1}^{Z}\sum_{k=1}^{K}\frac{C_{zk}^t\alpha_{zk}^n}{d_{zn}(\mathrm{r}_n^{curr}, \mathbf{h}_z)}.$$

Obtain the step size $\Delta = \|\mathrm{r}_n^{next} - \mathrm{r}_n^{curr}\|$, where $\| \bullet \|$indicates the 2-norm of a vector.
Set $\mathrm{r}_n^{next} := \mathrm{r}_n^{curr}$.
Step A-3: If $\Delta >$ set precision, go to Step A-2; otherwise, stop. The optimal location is stored in r_n^{curr} and the optimal total cost is $TC(\mathrm{r}_n^{curr}, \alpha^1, \ldots, \alpha^N)$. ■

Algorithm A solves Eq. (2) numerically and finds the optimal solution with a known allocation $\boldsymbol{\alpha}^n$ for the first set of sub-problems. The second set of sub-problems, which is defined as the reactors' locations $\mathbf{r}_n$ are known and an optimal allocation $\boldsymbol{\alpha}^n$ is required to be solved, which will be addressed next.

4.2. Allocation sub-problem

Before addressing the heuristic, the second set of sub-problems is transferred to a linear programming with a more concise format. The following new problem is provided first, which can be verified by algebraic computation from (1). The purpose of applying a matrix transformation is to simplify the original model (1) with a linear program such that conventional optimization methods such as branch and bound (B&B) can be applied directly to the problem.

Let $T: \mathbb{R}^{Z\times K} \to \mathbb{R}^{ZK}$be a function that transforms a $Z\times K$ matrix to a vector with length ZK by rearranging all columns of the matrix into one column. The unit vector 1_s is a vector of s ones and $\mathbf{I_s}$ is an $s\times s$ identity matrix. Also, the symbol $\otimes$ denotes the Kronecker's matrix-product and the symbol $\circ$ is the same dimensional Hadamard's matrix which is obtained just by multiplying the corresponding elements of the matrices.

If the location of reactor $\mathbf{r}$ is known, the system model (1) can be written in the form of a revised BMR as
Problem BMR-R:

$$\text{Min } TC(\mathbf{V}) = \boldsymbol{B}^T\mathbf{V} + \Theta \tag{3}$$

Subject to:

$$\mathbf{PV} \le \mathbf{Q} \tag{3a}$$

$$\mathbf{V} = (v_i) \in \mathbb{R}^{ZK},\ \ v_i \ge 0 \text{ for } i = 1, \ldots, ZK \tag{3b}$$

where $\mathbf{V} = \begin{pmatrix} T(\boldsymbol{\alpha}^1) \\ \vdots \\ T(\boldsymbol{\alpha}^n) \end{pmatrix} \in \mathbb{R}^{ZKN}$,

$$\begin{aligned}
\mathbf{B} &= 1_n \otimes T\left(\mathbf{C^r}\right) + T(1_k \otimes \mathbf{d}) \circ \left[1_n \otimes T\left(\mathbf{C^t}\right)\right] + C^w 1_n \otimes (\mathbf{w} \otimes 1_z),\\
\mathbf{C^r} &= \left(C_{zk}^r\right) \in \mathbb{R}^{ZK},\\
\mathbf{C^t} &= \left(C_{zk}^t\right) \in \mathbb{R}^{Z\times K},\\
\mathbf{w} &= (w_k) \in \mathbb{R}^K,\\
\mathbf{d} &= (d_{zn}) \in \mathbb{R}^Z,\\
\Theta &= C^f + \sum_{n=1}^{N} C_n^v,\\
\mathbf{P} &= \begin{bmatrix} \mathbf{I}_n \otimes (\mathbf{w} \otimes 1_z)^T \\ -\mathbf{I}_{kn} \otimes \left(1_z^T\right) \\ \mathbf{I}_{kn} \otimes \left(1_z^T\right) \end{bmatrix}, \text{ and}\\
\mathbf{Q} &= \begin{bmatrix} \mathbf{W}_n \\ -T(\mathbf{D}) \\ (1-\beta)1_n \otimes \left(\mathbf{I}_k \otimes \left(1_z^T\right) T(\mathbf{A})\right) \end{bmatrix}.
\end{aligned}$$

Given $\boldsymbol{\alpha}$, Algorithm A provides a numerical solution for the optimal reactor's location $\mathbf{r}$. On the other hand, Problem BMR-R in (3) transforms the problem BMR in (1) into a linear programming problem. The transformation allows the B&B method

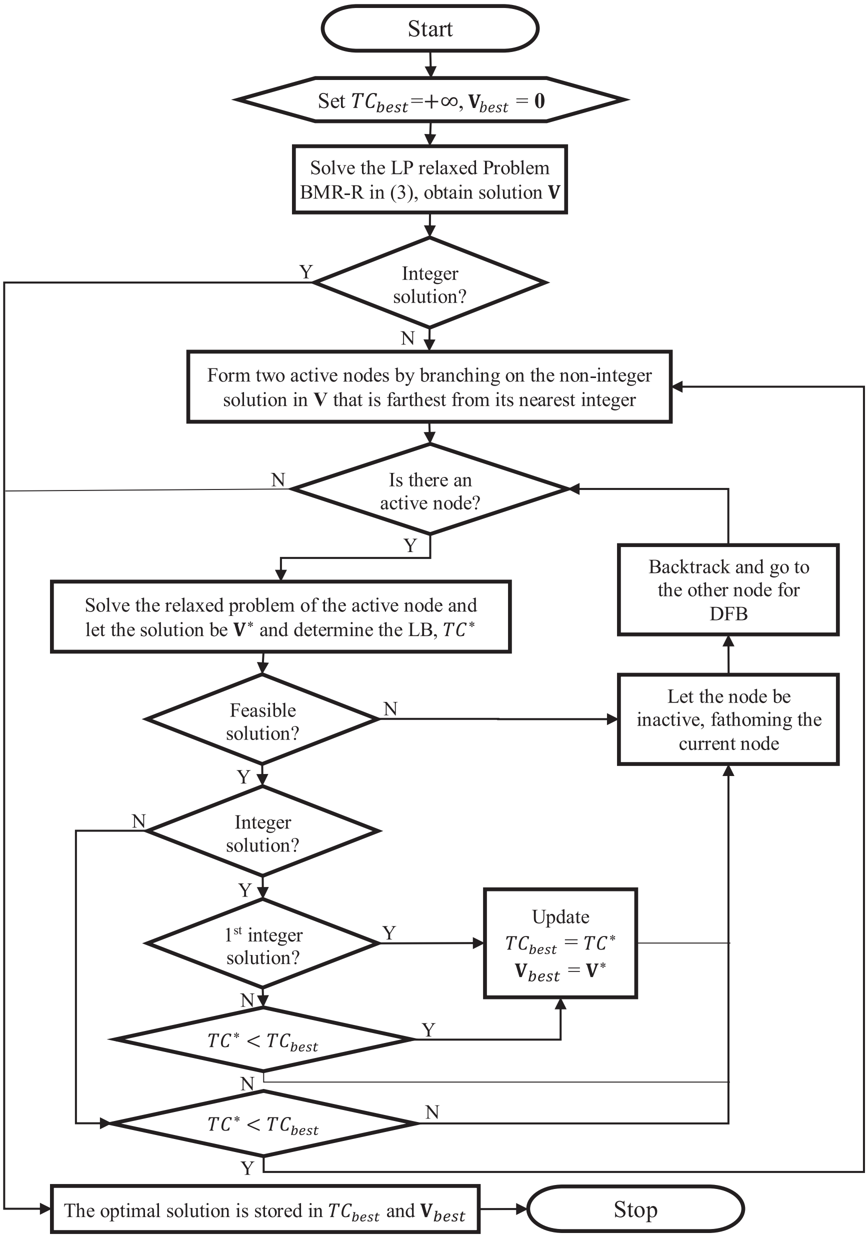

Fig. 3. Flow chart of Algorithm B: branch and bound method.

to be applied successfully to seek an optimal allocation $\boldsymbol{\alpha}$ when the reactor's location $\mathbf{r}$ is known. The optimal solution for each node is obtained by solving the relaxed problem of (3) using the depth-first B&B method. This process of computation by B&B method (Algorithm B) is depicted through a flow chart in Fig. 3. Note that a node is inactive when it is currently being computed, or it has no feasible solution, or its solution is an integer that exceeds the lower bound TC*.

4.3. Heuristic design

The nature of the original model (1) can be considered as two sub-problems being coupled together. The *location* sub-problem is proved convex in Theorem 1 and the global minimum can be obtained using Algorithm A. The *allocation* sub-problem is a constrained linear integer programming problem which is written in the form of (3), and its optimal/sub-optimal solution can be obtained by a B&B method for a reasonable size (say, $Z \times K \times N$ is $50 \times 10 \times 10$). A heuristic that solves the original model (1) consists of alternatively searching the optimal or sub-optimal solution between these two sub-problems. The heuristic runs both Algorithms A and B (Fig. 3) alternatively until the results of both sub-problems converge.

The concept of alternating location and allocation sub-problems was first introduced by Cooper [44]. Here, we try to devise another method, Algorithm C that exploits the special structure of a problem. The difference between the proposed Algorithm C and the heuristic proposed by Cooper [44] lies in the method of solving the allocation sub-problem. In this paper, the allocation is updated by the B&B method, but in Cooper's heuristic, it was achieved by determining if it was possible to allocate the facility to a closer source. The proposed algorithm is given as follows.

Algorithm C (Main Program): Alternative Search to Convergence Heuristic (ASTCH)

Step C-1: Randomly pick $\bar{\mathbf{r}} \in \mathbb{R}^{2N}$ as an initial reactors' locations. Let $TC_{opt} = +\infty$.

Step C-2: (a) Using the reactors' locations $\bar{\mathbf{r}}$, apply Algorithm B and obtain an optimal allocation $\overline{\boldsymbol{\alpha}^n}$, $n=1, \ldots, N$.
(b) Compute $\overline{TC}$ be the objective value.

Step C-3: (a) With the known $\overline{\boldsymbol{\alpha}^n}$, apply Algorithm A and obtain the optimal location $\tilde{\mathbf{r}}$.
(b) Compute the objective value be $\widetilde{TC}$.

Step C-4: Compare $\widetilde{TC}$ with TC_{opt}. If a better solution is found, that is $\widetilde{TC} < TC_{opt}$, replace TC_{opt} with $\widetilde{TC}$, $\boldsymbol{\alpha}^n_{opt}$ with $\overline{\boldsymbol{\alpha}^n}$ for $n=1, \ldots, N$, and $\mathbf{r}_{opt}$ with $\tilde{\mathbf{r}}$.

Step C-5: If $\overline{TC}$ and $\widetilde{TC}$ are not sufficiently close, then $\bar{\mathbf{r}} = \tilde{\mathbf{r}}$ and repeat from Step 2. Otherwise, stop the process and TC_{opt}, $\boldsymbol{\alpha}^n_{opt}$ and $\mathbf{r}_{opt}$ are the solutions. □

The ASTCH is proposed to find the optimal location of reactors and the feedstock distribution plan. The next section provides a numerical example to illustrate the heuristic step by step.

5. An illustrative example

A numerical example of a farmland is provided with 3 hubs ($Z=3$), 2 types of residues ($K=2$), and 2 reactors (N = 2). Other system parameters are as follows: total workforce availability for the reactor, $W=48$ persons, work-force requirement $(w_1, w_2) = (3, 5)$ persons/truck-load, percent of unavailable residue $\beta = 5\%$, demand of residues $\mathbf{D} = [D_{kn}] = \begin{bmatrix} 6 & 5 \\ 6 & 5 \end{bmatrix}$ truck-loads, location of the hubs $[(x^h_1, y^h_1), (x^h_2, y^h_2), (x^h_3, y^h_3)] = [(24.97, 28.07), (48.09, 59.91), (88.08, 2.62)]$. For simplicity, the fixed construction cost C^f and variable construction cost C^v_n are set to 0 and the unit work-force cost $C^w = \$10$/person/truck-load. The unit transportation cost $\mathbf{C}^t = [C^t_{zk}] = \begin{bmatrix} 3.23 & 3.17 \\ 4.93 & 4.43 \\ 3.87 & 4.01 \end{bmatrix}$ dollar/truck-load/mile, feedstock cost $\mathbf{C}^r = [C^r_{zk}] = \begin{bmatrix} 102 & 148 \\ 137 & 114 \\ 125 & 119 \end{bmatrix}$ dollar/truck-load and availability of the residue $\boldsymbol{A} = [A_{zk}] = \begin{bmatrix} 2.5 & 4.5 \\ 4.5 & 3.0 \\ 2.6 & 4.0 \end{bmatrix}$ truck-loads.

From Problem BMR-R, the original model (1) can be transformed to a linear integer problem (3) given that the reactor's location $\mathbf{r}$ is known. Therefore, given the parameters provided above, we can expand and obtain

$$\mathbf{V} = \begin{pmatrix} T(\boldsymbol{\alpha}^1) \\ \vdots \\ T(\boldsymbol{\alpha}^n) \end{pmatrix} = \left[\alpha^1_{11}\ \alpha^1_{21}\ \alpha^1_{31}\ \alpha^1_{12}\ \alpha^1_{22}\ \alpha^1_{32}\ \alpha^2_{11}\ \alpha^2_{21}\ \alpha^2_{31}\ \alpha^2_{12}\ \alpha^2_{22}\ \alpha^2_{32}\ \alpha^3_{11}\ \alpha^3_{21}\ \alpha^3_{31}\ \alpha^3_{12}\ \alpha^3_{22}\ \alpha^3_{32}\right]^T$$

$$\begin{aligned} B &= 1_n \otimes T(\mathbf{C}^r) + T(1_k \otimes \mathbf{d}) \circ \left[1_n \otimes T(\mathbf{C}^t)\right] + C^w 1_n \otimes (\mathbf{w} \otimes 1_z) \\ &= [132 + 3.2d_{11}\ \ 167 + 4.9d_{21}\ \ \ldots\ \ 164 + 4.4d_{22}\ \ 169 + 4.0d_{32}]^T \end{aligned}$$

$$\Theta = C^f + \sum_{n=1}^{N} nC^v = 0$$

$$\mathbf{P} = \begin{bmatrix} \mathbf{I}_n \otimes (\mathbf{w} \otimes 1_z)^T \\ -\mathbf{I}_{kn} \otimes \left(1^T_z\right) \\ \mathbf{I}_{kn} \otimes \left(1^T_z\right) \end{bmatrix} = \begin{bmatrix} 3 & 3 & 3 & 5 & 5 & 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 3 & 3 & 5 & 5 & 5 \\ -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$\mathbf{Q} = \begin{bmatrix} \boldsymbol{W} \\ -\mathbf{D} \\ (1-\beta)\mathbf{I}_k \otimes \left(1^T_z\right)T(\mathbf{A}) \end{bmatrix} = [48.0\ 48.0\ -6.0\ -6.0\ -5.0\ -5.0\ 9.1\ 10.9\ 9.1\ 10.9]^T$$

So, Algorithm C (Main Program) yields:

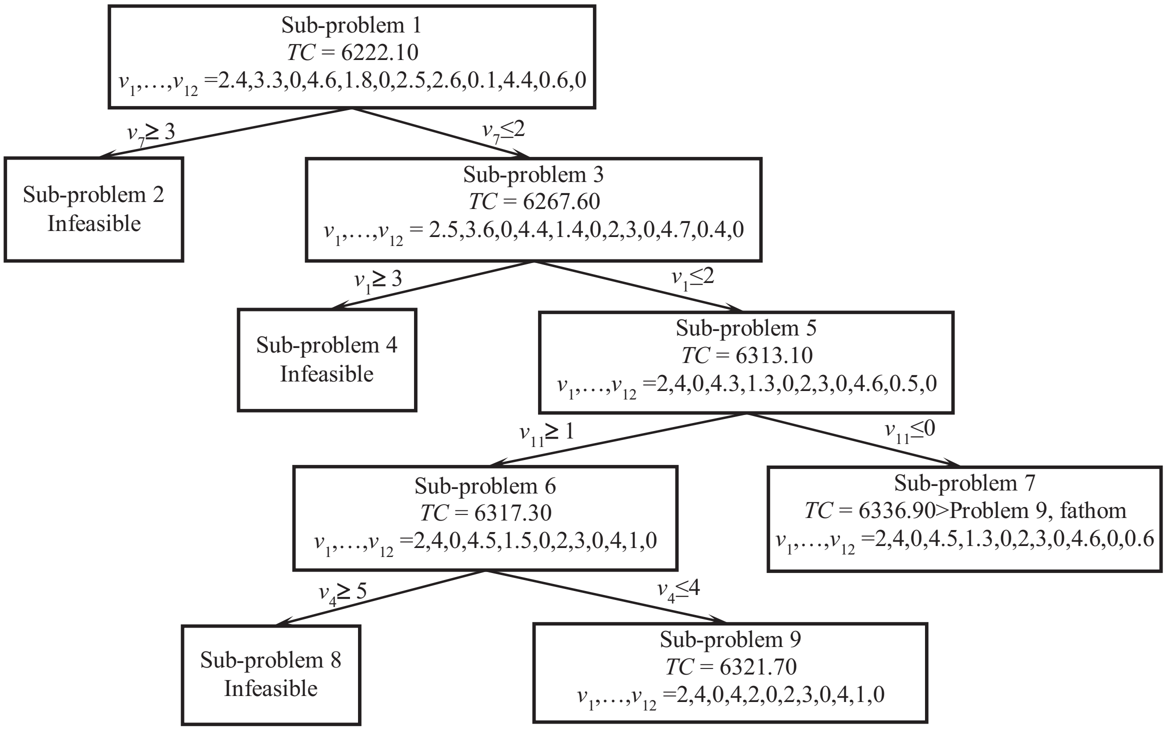

Fig. 4. Branch and bound flow chart for Example 1.

Step C-1: Taking randomly $\bar{\mathbf{r}} = [53.71\ 53.71\ 30.2\ 30.2]^T$ and setting $TC_{opt} = +\ \infty$, the iterative computation is provided below.

Iteration C-1:

Step C-2: Use B&B (Algorithm B) with $\bar{\mathbf{r}} = [53.71\ 53.71\ 30.2\ 30.2]^T$. Because the location of the reactor is known, the distance matrix $\mathbf{d}$ can be obtained as $\mathbf{d} = \begin{bmatrix} 28.82 & 28.82 \\ 30.24 & 30.24 \\ 44.06 & 44.06 \end{bmatrix}$. Therefore, the linear integer program (7) can be solved by the B&B method. The outcome of the steps of the B&B method is shown in Fig. 4. Each rectangular box denotes a node in a tree. The depth-first search procedure is followed to find the next solution, and the visiting/computational order of the nodes of the sub-problem is $1 \to 2 \to 3 \to 4 \to 5 \to 6 \to 8 \to 9 \to 7$. The new constraints added to the sub-problems are shown on the branching arrows. The optimal solution is found in sub-problem 9. Therefore, $\overline{\boldsymbol{\alpha}^1} = \begin{bmatrix} 2 & 4 \\ 4 & 2 \\ 0 & 0 \end{bmatrix}$, $\overline{\boldsymbol{\alpha}^2} = \begin{bmatrix} 2 & 4 \\ 3 & 1 \\ 0 & 0 \end{bmatrix}$ and $\overline{TC} = 6321.70$.

Step C-3: Algorithm A runs with $\overline{\boldsymbol{\alpha}^1}$, and the optimal location $\tilde{\mathbf{r}} = [48.09\ 59.91\ 47.94\ 59.71]^T$, $\widetilde{TC} = 5279.30$.

Step C-4: $\widetilde{TC} < TC_{opt}$; hence, $TC_{opt} := \widetilde{TC} = 5279.30$ with $\boldsymbol{\alpha}^1_{opt} = \begin{bmatrix} 2 & 4 \\ 4 & 2 \\ 0 & 0 \end{bmatrix}$, $\boldsymbol{\alpha}^2_{opt} = \begin{bmatrix} 2 & 4 \\ 3 & 1 \\ 0 & 0 \end{bmatrix}$, $\mathbf{r}_{opt} = \tilde{\mathbf{r}} = [48.09\ 59.91\ 47.94\ 59.71]^T$.

Step C-5: Now, set the optimal criteria to be 0.1. Since $|\widetilde{TC} - \overline{TC}| > 0.1$ (infinitesimal number), then $\bar{\mathbf{r}} = \tilde{\mathbf{r}}$ and go to Step C-2.

Iteration C-2:

Step C-2: Applying Algorithm B with $\bar{\mathbf{r}} = [48.09\ 59.91\ 47.94\ 59.71]^T$, we obtain $\overline{\boldsymbol{\alpha}^1} = \begin{bmatrix} 2 & 3 \\ 4 & 3 \\ 0 & 0 \end{bmatrix}$,

$$\overline{\boldsymbol{\alpha}^2} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \\ 0 & 0 \end{bmatrix} \text{ and } \overline{TC} = 4716.60.$$

Step C-3: Algorithm A yields the optimal location $\tilde{\mathbf{r}} = [48.09\ 59.91\ 48.09\ 59.9]^T$ based on $\overline{\boldsymbol{\alpha}^n}$. Hence, $\widetilde{TC} = 4711.00$.

Step C-4: Since $\widetilde{TC} < TC_{opt}$, $TC_{opt} := \widetilde{TC} = 4711.00$, $\boldsymbol{\alpha}^1_{opt} = \begin{bmatrix} 2 & 3 \\ 4 & 3 \\ 0 & 0 \end{bmatrix}$, $\boldsymbol{\alpha}^2_{opt} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \\ 0 & 0 \end{bmatrix}$, $\mathbf{r}_{opt} = \tilde{\mathbf{r}} = [48.09\ 59.91\ 48.09\ 59.9]^T$.

Step C-5: Here $|\widetilde{TC} - \overline{TC}| > 0.1$, then $\bar{\mathbf{r}} = \tilde{\mathbf{r}}$ and go to Step 2.

Step C-2: Algorithm B with $\bar{\mathbf{r}} = [48.09\ 59.91\ 48.09\ 59.9]^T$ yields $\overline{\boldsymbol{\alpha}^1} = \begin{bmatrix} 2 & 3 \\ 4 & 3 \\ 0 & 0 \end{bmatrix}$, $\overline{\boldsymbol{\alpha}^2} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \\ 0 & 0 \end{bmatrix}$ and $\overline{TC} = 4711.00$.

Step C-3: Likewise Algorithm A yields the optimal location $\tilde{\mathbf{r}} = [48.09\ 59.91\ 48.09\ 59.9]^T$ based on $\overline{\boldsymbol{\alpha}^n}$ when $\widetilde{TC} = 4711.00$.

Step C-4: Since $\widetilde{TC} = TC_{opt}$, then $TC_{opt} := \widetilde{TC} = 4711.00$, $\boldsymbol{\alpha}^1_{opt} = \begin{bmatrix} 2 & 3 \\ 4 & 3 \\ 0 & 0 \end{bmatrix}$, $\boldsymbol{\alpha}^2_{opt} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \\ 0 & 0 \end{bmatrix}$, $\mathbf{r}_{opt} = \tilde{\mathbf{r}} = [48.09\ 59.91\ 48.09\ 59.9]^T$.

Step C-5: Because $\widetilde{TC} = \overline{TC}$, Algorithm C stops, and the optimal solutions are stored in TC_{opt}, $\boldsymbol{\alpha}^n_{opt}$ and $\mathbf{r}_{opt}$. □

Here, a heuristic for the mixed-integer programming problem BMR in (1) is proposed by alternatively searching for a minimum for two sets of sub-problems. An example is provided for demonstrating the heuristic step by step [see the computation above and Fig. 4 for the B&B iteration]. In order to understand general the quality and efficiency of the heuristic, a few more numerical examples are empirically tested in the next section.

6. Empirical test results and discussion

The quality of the proposed heuristic ASTCH is tested in this section using various numerical instances. The parameters of these numerical examples are selected based on real-world applications and most of them are chosen randomly from their statistical distribution to avoid the effect of artificially created numbers. All the numerical experiments were conducted in Matlab using a computer with *Intel® Core* ™ *i7-4750HQ CPU@2.00 GHz*. Without loss of generality, common parameters for all numerical examples were chosen as C^f = \$2000,000, C^v_n = \$200,000 for $n = 1, \ldots, N$, C^w = \$10, β = 0.05. The rest of the parameters were selected randomly such that the values conform to a uniform distribution. The parameter selection process relied on two functions that generate random numbers functions: rand(m, n) and randi([LB, UB], m, n), where m and n are positive integers, and the lower bound (LB) and upper bound (UB) are non-negative real numbers. A uniform generator, rand(m, n), randomly generated an $m \times n$ matrix with each element ranging from 0 to 1 uniformly. The generator randi([LB, UB], m, n) also created an $m \times n$ matrix whose elements are uniformly distributed between the LB and the UB. The parameters for the hubs' locations $\mathbf{x}^h = [x^h_z] \in \mathbb{R}^{\mathbf{Z}}$ and $\mathbf{y}^h = [y^h_z] \in \mathbb{R}^Z$, the residue availability $\mathbf{A} = [A_{zk}] \in \mathbb{R}^{Z\times K}$, and the transportation cost $\mathbf{C}^t = [C^t_{zk}] \in \mathbb{R}^{Z\times K}$ are selected as

$$\begin{aligned} \mathbf{x}^h &= 100\mathrm{rand}(Z, 1), \\ \mathbf{y}^h &= 100\mathrm{rand}(Z, 1), \\ \boldsymbol{A} &= 2 + 3\mathrm{rand}(Z, K), \text{ and} \\ \boldsymbol{C}^t &= 3 + 2\mathrm{rand}(Z, K), \text{ respectively} \end{aligned}$$

The x–y coordinates of the hubs' locations are selected following a uniform distribution within the bound [0100] × [0100] with a function generator, rand(m, n), the residue availability is bounded within 2 to 5 truck-loads for each hub, and the transportation cost for each residue in a hub is between 3 and 5 dollar/truck-load/mile.

The demand for the residues in the reactor $\boldsymbol{D} = [D_{nk}] \in \mathbb{R}^{N\times K}$, feedstock cost $\mathbf{C}^r = [C^r_{zk}] \in \mathbb{R}^{Z\times K}$, and unit workforce $\mathbf{w} = [w_k] \in \mathbb{R}^K$ are set to

$$\begin{aligned} \boldsymbol{D} &= \mathrm{randi}([1.5Z, 2Z], N, K), \\ \boldsymbol{C}^r &= \mathrm{randi}([100, 150], Z, K), \\ \boldsymbol{w} &= \mathrm{randi}([3, 5], K, 1). \end{aligned}$$

The demand for each residue at nth reactor, the feedstock cost, and the unit workforce are random integers. As indicated by the definition of randi([LB, UB], m, n), these three parameters $\boldsymbol{D}$, $\boldsymbol{C}^r$, and $\boldsymbol{w}$ are bounded within 1.5Z to 2Z truck-loads, 100 to 150 dollars/truck-load, and 3 to 5 persons/truck-load, respectively. Finally, the total available workforce W is set empirically to 8ZK.

Numerical examples vary with the number of hubs (Z), types of residues (K), and reactors (N). Only the sets of parameters that result in feasible solutions are tested in the experiment. Table 1 shows 10 sets of numerical results. The number of hubs Z is selected from 3 to 50, the types of residues K increases from 2 to 10, and the number of reactors N is from 2 to 10. Detailed results are obtained from the simulated set of problems as defined by the generator parameters as stated above and also in Wu et al. [40]. The CPU times are recorded in seconds. The last two columns in Table 1 are denoted by the number of iterations "*nIter*" and the number of nodes in the ith iteration ($i = 1,\ldots,nIter$) "$nNod_i$." The number of average nodes is computed as $(\sum_{i=1}^{nIter} nNod_i)/nIter$.

It required 2 to 6 iterations for all sets of problems to converge to an optimal/sub-optimal solution (Table 1). Because the initial guess of the reactors' locations was selected randomly, the efficient convergence of the algorithm to the optimal/sub-optimal solution suggests the effectiveness of the proposed ASTCH. The heuristic guarantees a convergence, a consistency of

Table 1
Comparison of numerical results with LINGO and the proposed heuristic.

Prob.	Prob. size $Z \times K \times N$	LINGO		ASTCH heuristic					
		TC^{min}	CPU time (s)	TC^{min}	CPU time (s)	No. of iterations	Avg. no. of iterations	No. of nodes per iter. (Avg.)	Avg. no. of nodes per iter.
1	$3 \times 2 \times 2$	2,404,653	<1	2,404,653	0.54	3	3	7.6	9.3
		24,056,39	<1	2,405,639	0.54	3		8.4	
		24,034,67	<1	2,403,467	1.24	3		11.5	
		24,058,58	<1	2,405,858	0.60	3		9.6	
2	$3 \times 3 \times 2$	2,406,774	<1	2,406,774	0.64	3	2.5	10.8	19.4
		24,065,13	<1	2,406,513	1.36	2		19.2	
		24,071,85	<1	2,407,185	2.89	3		28.3	
		24,060,11	<1	2,406,011	1.09	2		19.2	
3	$5 \times 3 \times 3$	2,672,248	<1	2,672,248	11.80	5	4.25	116.4	105.2
		26,832,84	<1	2,683,284	11.70	4		114.0	
		26,758,83	<1	2,675,883	15.30	5		101.0	
		26,692,77	<1	2,669,277	8.85	3		89.5	
4	$10 \times 3 \times 3$	2,687,107	<1	2,687,107	8.83	5	4.5	117.6	125.7
		26,881,71	<1	2,686,241	13.38	3		122.4	
		26,871,25	<1	2,685,121	13.21	3		134.4	
		26,887,86	<1	2,687,414	16.33	7		128.4	
5	$10 \times 5 \times 3$	2,709,168	<1	2,708,145	16.24	3	4.75	130.8	130.6
		27,036,16	<1	2,702,951	23.64	5		149.0	
		27,092,46	<1	2,709,106	24.64	5		122.4	
		27,169,18	<1	2,715,741	19.72	6		120.0	
6	$15 \times 5 \times 5$	2,770,456	1	2,768,547	16.02	4	4.5	174.6	193.3
		27,731,90	1	2,772,148	45.96	5		206.6	
		27,708,64	1	2,767,487	66.76	6		218.6	
		27,582,97	1	2,755,214	24.58	3		173.5	
7	$20 \times 5 \times 5$	2,817,602	1	2,815,365	155.10	6	6	391.4	375.7
		28,492,65	1	2,846,548	133.35	5		393.4	
		28,292,66	1	2,827,417	100.80	7		353.0	
		28,254,84	1	2,823,649	90.93	6		364.8	
8	$20 \times 10 \times 8$	2,990,712	1	2,982,011	233.70	3	3.5	718.0	646.6
		29,881,24	1	2,985,475	220.50	3		644.8	
		30,083,94	1	3,006,547	142.95	5		676.8	
		30,121,05	1	3,001,541	196.05	3		546.7	
9	$30 \times 10 \times 8$	3,245,985	1	3,243,514	684.45	5	5	960.5	980.7
		31,547,85	1	3,152,487	298.80	3		1032.6	
		33,214,51	1	3,320,091	591.30	7		1023.8	
		31,254,78	1	3,121,462	623.70	5		906.0	
10	$50 \times 10 \times 10$	3,514,264	2	3,510,021	1092.60	4	4	2871.6	2856.8
		35,478,21	2	3,543,145	1054.35	5		2700.4	
		34,154,74	2	3,412,034	1180.80	3		3352.0	
		36,211,01	2	3,610,851	1243.95	4		2503.2	

$\overline{TC}$ and $\widetilde{TC}$, regardless of the initial condition. The average number of nodes searched varied from 7.7 to 3352 depending on the size of the problem. The average number of nodes for each set of problems remained close to each other, which was caused by the parameters being bounded within a relatively small range. There was a clear tendency that when the size of the problem increased, it led to an increase in the number of visits and increasing visiting to nodes when the B&B method was applied. CPU time also increased with the number of nodes. These observations are consistent with the results from all of the numerical examples.

A conventional way of evaluating the performance of a heuristic is to estimate a lower bound. However, due to the existence of nonlinearity in the fourth term in the model BMR in (1) and its non-convex nature, it is trivial to obtain the lower bound for a MINLP. Therefore, a comparison with the results computed by *LINGO* is conducted to show the quality of the proposed heuristic. LINGO results and CPU times are shown in the 3rd and 4th column in Table 1. By comparing the results of LINGO and ASTCH, the solutions from LINGO were not as good as ASTCH, although LINGO took comparatively less time. ASTCH required a longer CPU time, but it provided better results.

Because LINGO is a commercial software whose algorithm is not revealed to the public, we felt that it would be better to compare the ASTCH results in Table 1 with other open-source heuristics. One popular open-source software is called the OPTI toolbox which provides an interface with *Matlab*. Two general MINLP solvers, BONMIN and NOMAD, in the OPTI toolbox are used to solve the same set of problems of Table 1. The BONMIN solver was developed by Bonami et al. [45] and the NOMAD solver was designed by Le Digabel [46].

Table 2 shows a comparison of the numerical results from BONMIN, NOMAD, and ASTCH. The proposed heuristic always obtained a lower total cost than using the BOMIN and NOMAD solvers (Table 2). However, ASTCH required a longer CPU time when the problem size was larger; for example, ASTCH required much longer CPU time after the third set of problems (see Table 2).

The comparison of ASTCH with LINGO, BOMIN, and NOMAD showed that ASTCH obtained a better solution, but required longer computational time. Further, optimization of the ASTCH code using parallel programming or coding in C may help

Table 2
Comparison among BOMIN, NOMAD, and ASTCH solvers.

Prob.	Prob. size $Z \times K \times N$	BOMIN		NOMAD		ASTCH	
		TC^{min}	CPU time (s)	TC^{min}	CPU time (s)	TC^{min}	CPU time (s)
1	$3 \times 2 \times 2$	2,414,587	1.41	2,406,614	0.29	2,404,653	0.54
		24,101,03	1.07	2,413,611	1.04	2,405,639	0.54
		24,064,87	0.31	2,404,045	3.77	2,403,467	1.24
		24,076,94	1.66	2,406,468	1.74	2,405,858	0.60
2	$3 \times 3 \times 2$	2,408,077	0.36	2,412,846	2.60	2,406,774	0.64
		24,128,07	0.90	2,412,747	2.17	2,406,513	1.36
		24,110,81	1.12	2,412,007	4.12	2,407,185	2.89
		24,086,22	1.04	2,410,409	1.84	2,406,011	1.09
3	$5 \times 3 \times 3$	2,674,396	1.08	2,676,077	4.09	2,672,248	11.80
		26,852,84	2.89	2,685,079	4.56	2,683,284	11.70
		26,762,68	1.31	2,685,399	3.11	2,675,883	15.30
		26,753,94	2.06	2,676,508	1.57	2,669,277	8.85
4	$10 \times 3 \times 3$	2,689,653	1.92	2,692,495	3.78	2,687,107	8.83
		26,880,46	1.75	2,694,976	4.83	2,686,241	13.38
		26,889,16	1.33	2,686,214	3.84	2,685,121	13.21
		26,889,18	1.96	2,690,709	2.77	2,687,414	16.33
5	$10 \times 5 \times 3$	2,710,116	1.89	2,709,854	2.37	2,708,145	16.24
		27,042,94	2.52	2,708,448	2.73	2,702,951	23.64
		27,173,44	2.90	2,717,891	3.72	2,709,106	24.64
		27,187,58	1.55	2,724,377	3.15	2,715,741	19.72
6	$15 \times 5 \times 5$	2,769,018	2.43	2,775,071	3.51	2,768,547	16.02
		27,759,26	5.40	2,775,831	5.15	2,772,148	45.96
		27,683,56	5.86	2,769,280	4.96	2,767,487	66.76
		27,605,75	2.45	2,761,764	2.55	2,755,214	24.58
7	$20 \times 5 \times 5$	2,820,261	12.33	2,816,563	12.61	2,815,365	155.10
		28,511,21	12.12	2,856,036	10.44	2,846,548	133.35
		28,366,89	9.39	2,835,787	7.30	2,827,417	100.80
		28,276,56	7.23	2,826,788	6.23	2,823,649	90.93
8	$20 \times 10 \times 8$	2,988,902	18.73	2,986,716	16.70	2,982,011	233.70
		29,870,76	17.05	2,989,364	16.13	2,985,475	220.50
		30,159,13	12.76	3,013,219	11.31	3,006,547	142.95
		30,098,71	15.21	3,009,186	16.65	3,001,541	196.05
9	$30 \times 10 \times 8$	3,251,944	53.11	3,247,716	48.49	3,243,514	684.45
		31,579,59	23.99	3,159,516	22.03	3,152,487	298.80
		33,213,78	47.38	3,327,861	41.21	3,320,091	591.30
		31,304,01	49.45	3,131,056	44.41	3,121,462	623.70
10	$50 \times 10 \times 10$	3,512,665	84.95	3,518,420	75.41	3,510,021	1092.60
		35,494,79	82.90	3,547,823	71.08	3,543,145	1054.35
		34,157,42	90.95	3,414,572	82.47	3,412,034	1180.80
		36,187,17	96.95	3,620,850	84.02	3,610,851	1243.95

shorten the CPU time of the heuristic. Although ASTCH required a longer time to converge, it converged within a reasonable amount of time relative to others and is thus practical for real-world applications. The results of these comparisons demonstrated a satisfactory performance for ASTCH when applied to the location-allocation problem of the BMG production system.

In the first experiment, each set of problems contains 4 particular numerical examples. Another experiment is conducted by running 30 numerical examples for each problem set in Table 1. The maxima, minima, and average CPU times for solving these problems showed that, even for a large problem in a real-world scenario ($50 \times 10 \times 10$), a reasonable CPU time can be expected for ASTCH, which further conforms its demonstrated efficiency (Table 3).

Two additional columns, the average number of iterations and the average number of nodes per iteration, are added to Table 1. Let the problem size be defined as the value of $Z \times K \times N$. The problem size ($Z \times K \times N$) is used as the x axis in Fig. 4. The average number of iterations and average number of nodes per iteration is shown as the y axes in Fig. 5a. The average CPU time showed in Table 3 is represented by the y axis in Fig. 5b. The average number of nodes per iteration increased appreciably as the problem size increased (Fig. 5a). However, because the average number of iterations decreased when the initial guess of the reactors' locations was close to the optimal solution, a random initial guess yielded an irregular pattern (neither increased nor decreased) of the number of iterations with respect to the problem size (Fig. 5a). The CPU time increased directly with the problem size (Fig. 5b) which is consistent with the expected result.

The model developed and tested here prescribes the number of reactors based on the raw materials information, the demand and other technical constraints. An engineering facility is built with certain capacity limit which is usually more (speculating higher fluctuation in the input parameters such as demand) than the current need. Once the system is commissioned, it can run up to the maximum capacity of the facility (the number of reactors built) and/or reactors, but if the

Table 3
CPU time of numerical results of proposed heuristics.

Problem sets	Prob. size $Z \times K \times N$	MAX CPU time (s)	MIN CPU time (s)	Avg. CPU time (s)
1	$3 \times 2 \times 2$	1.34	0.36	0.67
2	$3 \times 3 \times 2$	3.01	0.38	1.54
3	$5 \times 3 \times 3$	17.64	6.63	14.72
4	$10 \times 3 \times 3$	20.81	8.53	16.97
5	$10 \times 5 \times 3$	27.65	12.96	23.38
6	$15 \times 5 \times 5$	66.76	16.02	38.23
7	$20 \times 5 \times 5$	169.12	89.56	146.73
8	$20 \times 10 \times 8$	233.70	135.98	172.95
9	$30 \times 10 \times 8$	769.86	298.80	589.32
10	$50 \times 10 \times 10$	1432.80	860.01	1218.76

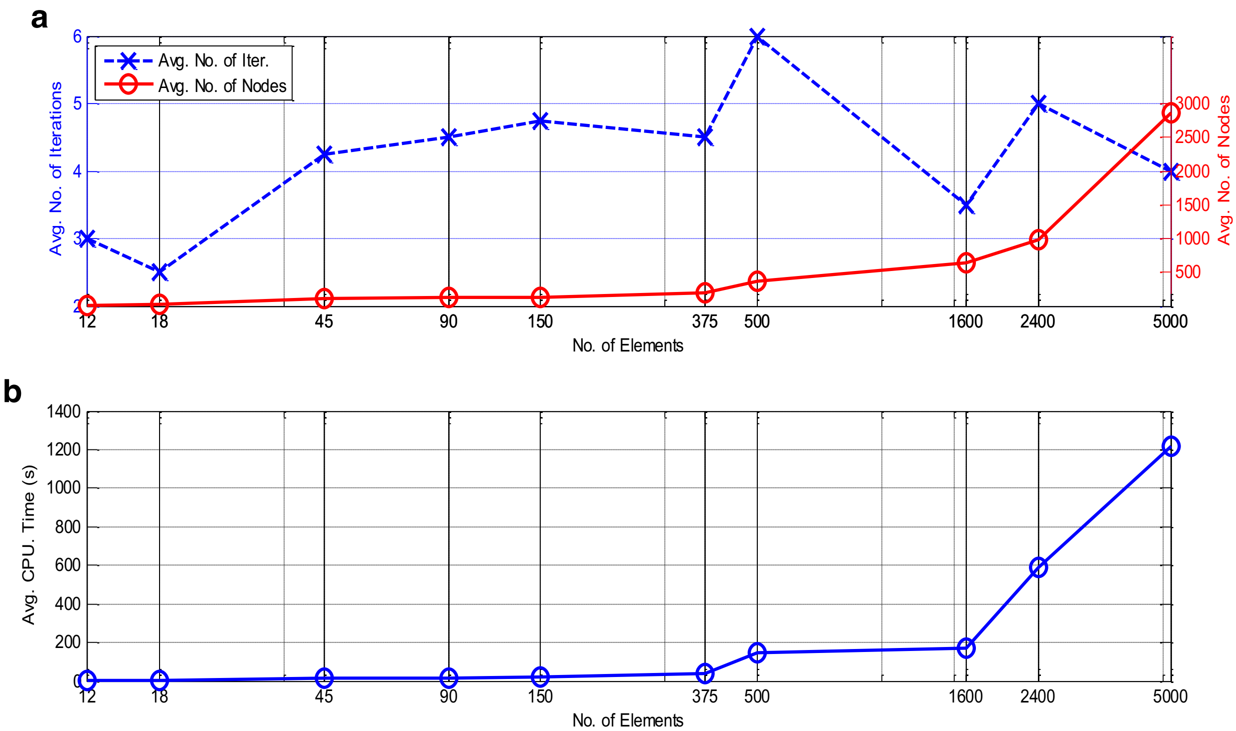

Fig. 5. Average number of iterations, nodes evaluated, and CPU times for different problem sizes.

BMG demand exceeds the current production rate, the system can run up to the capacity limit, meaning it meets the demand beyond its pre-set target. In such a situation, it is a win-win situation whether to build higher capacitated reactor(s) or more number of reactors at higher cost to meet such excess demand or to lose the potential BMG demand by keeping the facility to the bare minimum level at minimal cost. It is a policy decision, not the technical decision. The problem can be modeled with demand uncertainty which is neither captured nor was the scope of this research. To ensure that the demand is less than the residue availability before running the model (or in other words, designing the system), almost all companies would have conducted a preliminary study to assess the availability of minimum level of raw materials so that this information is fed into the design model so as to determine the optimal number of reactors.

However, the supply shortage of raw materials is not immediately expected at the start of application in which case the preliminary survey report is not supportive of the fact. If there is sufficient supply of raw materials, then the system is supposed to run smoothly; but in the event of non-availability, it will certainly affect the performance of the model incurring unproductive operating cost. When the system is about to run with less-than-demand raw materials, some measures such as contingency plan of procuring the materials from other nearby regions should be considered if it is economically viable, or leasing and/or buying of additional farmlands should be considered to grow crops for meeting the demand after commissioning the system.

7. Conclusion

With the increasing needs for renewable energy, BMG production systems developed rapidly due to its overriding advantages in protecting the environment and its returns on investment. Sound planning and efficient management are important in BMG manufacturing, which can significantly reduce the total cost and eventually increase profits. This paper addressed the supply chain optimization problem in constructing and operating a BMG production system. The collected residues from farms were stored in local hubs within the zip code. The collected residues served as reactants for BMG production. A general MINLP was established to describe mathematically the total cost of constructing and operating the BMG production system. An alternating heuristic was proposed to solve the MINLP to obtain the optimal location of the reactors and the amount of residues that needed to be transported from each hub to the reactors. Numerical examples were provided to verify the performance of the heuristic.

This research contributes significantly in the field of BMG production for cheaper energy generation. Unlike the previous studies, this undertaking is unique in the sense that the BMG production supply chain is modeled as a MINLP problem which is solved optimally and new efficient heuristics are developed for solving large instances with reasonably good solution quality. Given the biomass feedstock information is known through USDA-GIS and other government reports, the solution methodology of this problem will be a good stepping stone for commissioning future generation of biogas which will have a significant impact on the private use of energy, growth of new energy enterprises, employment and, overall, the national economy.

The current research models only the feedstock-to-hub-to-reactor supply chain system, i.e., transporting residues from farmlands to multiple hubs to multiple reactors for producing BMG. In practice, after BMG is produced in the reactors, a process takes place in condensers to liquefy the generated BMG to minimize the volume for ease of storage and transportation. The liquefied BMG is then distributed to the end users. Liquefaction and distribution of liquefied BMG increase the number processing stages and the complexity of the supply chain system. Thus, future work may be focused on modeling the inclusion of this embellishment and seeking an effective method for practical solution. Also, flexibility in BMG demand and budgetary constraint can be incorporated in the model using a stochastic programming approach for the biogas production facility.

Appendix A. Proof of Theorem 1

Proof. The objective function (1) is first re-written as a function of $\mathbf{r}$ only

$$\begin{aligned}\mathrm{Min}TC(\mathbf{r}) = C^f + \sum_{n=1}^{N} C_n^v + \sum_{z=1}^{Z}\sum_{k=1}^{K}\sum_{n=1}^{N} C_{zk}^r \alpha_{zk}^n \\ + \sum_{z=1}^{Z}\sum_{k=1}^{K}\sum_{n=1}^{N} d_{zn} C_{zk}^t \alpha_{zk}^n \\ + C^w \sum_{z=1}^{Z}\sum_{k=1}^{K}\sum_{n=1}^{N} w_k \alpha_{zk}^n\end{aligned} \tag{A.1}$$

To show the convexity of equation (A.1), one needs to show that its Hessian matrix is positive semi-definite. To compute its Hessian matrix, the first-order derivative of (A.1) is obtained

$$\frac{\partial TC(\mathbf{r})}{\partial \mathbf{r}_n} = \sum_{z=1}^{Z}\sum_{k=1}^{K} \frac{C_{zk}^t \alpha_{zk}^n}{d_{zn}} (\mathbf{r}_n - \mathbf{h}_z) \tag{A.2}$$

Let $H[\bullet]$ denotes the Hessian matrix of (A.1), that is

$$H[TC(\mathbf{r})] = \mathrm{diag}\begin{pmatrix} \frac{\partial^2 TC(\mathbf{r})}{\partial x_n^2} & \frac{\partial^2 TC(\mathbf{r})}{\partial x_n \partial y_n} \\ \frac{\partial^2 TC(\mathbf{r})}{\partial y_n \partial x_n} & \frac{\partial^2 TC(\mathbf{r})}{\partial y_n^2} \end{pmatrix}, \ n = 1, \ldots, \ N \tag{A.3}$$

where diag($\bullet$) is the block diagonal matrix and

$$\frac{\partial^2 TC(\mathbf{r})}{\partial x_n^2} = \sum_{z=1}^{Z}\sum_{k=1}^{K} \frac{C_{zk}^t \alpha_{zk}^n \left(y_n - y_z^h\right)}{d_{zn}^{3/2}} \tag{A.4}$$

$$\frac{\partial^2 TC(\mathbf{r})}{\partial y_n^2} = \sum_{z=1}^{Z}\sum_{k=1}^{K} \frac{C_{zk}^t \alpha_{zk}^n \left(x_n - x_z^h\right)}{d_{zn}^{3/2}} \tag{A.5}$$

$$\frac{\partial^2 TC(\mathbf{r})}{\partial x_n \partial y_n} = \frac{\partial^2 TC(\mathbf{r})}{\partial y_n \partial x_n} = -\sum_{z=1}^{Z}\sum_{k=1}^{K} \frac{C_{zk}^t \alpha_{zk}^n \left(x_n - x_z^h\right)\ \left(y_n - y_z^h\right)}{d_{zn}^{3/2}} \tag{A.6}$$

The positive semi-definiteness can be shown directly from its definition. Let $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_{2N-1}, \mu_{2N})^T \in \mathbb{R}^{2N}$ be any non-zero column vector, from (A.3) to (A.6)

$$\begin{aligned}
\mu^T\ H[TC(\mathbf{r})]\mu &= \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_{2N} \end{pmatrix}^T \operatorname{diag}\begin{pmatrix} \frac{\partial^2 TC(\mathbf{r})}{\partial x_n^2} & \frac{\partial^2 TC(\mathbf{r})}{\partial x_n \partial y_n} \\ \frac{\partial^2 TC(\mathbf{r})}{\partial y_n \partial x_n} & \frac{\partial^2 TC(\mathbf{r})}{\partial y_n^2} \end{pmatrix} \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_{2N} \end{pmatrix} \\
&= \sum_{n=1}^{N}\left(\frac{\partial^2 TC(\mathbf{r})}{\partial x_n^2}\mu_{2n-1}^2 + 2\frac{\partial^2 TC(\mathbf{r})}{\partial x_n \partial y_n}\mu_{2n-1}\mu_{2n} + \frac{\partial^2 TC(\mathbf{r})}{\partial y_n^2}\mu_{2n}^2 \right) \\
&= \sum_{n=1}^{N}\left(\mu_{2n-1}^2 \sum_{z=1}^{Z}\sum_{k=1}^{K}\frac{C_{zk}^t\alpha_{zk}^n\left(y_n - y_z^h\right)}{d_{zn}^{3/2}} - 2\mu_{2n-1}\mu_{2n}\sum_{z=1}^{Z}\sum_{k=1}^{K}\frac{C_{zk}^t\alpha_{zk}^n\left(x_n - x_z^h\right)\ \left(y_n - y_z^h\right)}{d_{zn}^{3/2}} \right. \\
&\quad \left. + \mu_{2n}^2\sum_{z=1}^{Z}\sum_{k=1}^{K}\frac{C_{zk}^t\alpha_{zk}^n\left(x_n - x_z^h\right)}{d_{zn}^{3/2}} \right) \\
&= \sum_{n=1}^{N}\sum_{z=1}^{Z}\sum_{k=1}^{K}\frac{C_{zk}^t\alpha_{zk}^n}{d_{zn}^{3/2}}\left[\mu_{2n-1}\left(y_n - y_z^h\right) - \mu_{2n}\left(x_n - x_z^h\right)\right]^2 \\
&\geq 0, \forall \mu_1, \mu_2, \ldots, \mu_{2N-1},\ \ \mu_{2N} \neq 0
\end{aligned} \tag{A.7}$$

which indicates that the Hessian matrix for (A.1) is positive semi-definite. Therefore, the sub-problem (A.1) is convex.

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