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SAMPLE PROPERTIES OF RANDOM FIELDS
I: SEPARABILITY AND MEASURABILITY

JÜRGEN POTTHOFF

Abstract. The well-known results about the existence of separable, measurable resp., modifications of stochastic processes (e.g., [4, 5]) are generalized to the case of real valued random fields indexed by a separable, separable and locally convex resp., metric space.

1. Introduction

This is the first in a series of papers in which sample properties of random fields are studied. In the present paper the question of existence of modifications of a random field indexed by a metric space which are separable, measurable resp., is considered. In two other papers continuity [6] and — in case that the index set is an open subset of $\mathbb{R}^m$ — differentiability [7] are addressed.

From the beginning of general theory of stochastic processes an important question has been, how statistical properties of a stochastic process determine analytic properties of its sample paths. The first — and probably most famous — result in this direction is, of course, the celebrated Kolmogorov-Chentsov theorem, of which a preliminary form by Kolmogorov in 1934 has been reported in a paper by Slutsky [8]. (A quite general form of this theorem is given in [6].) A systematic treatment of this type of questions can be found in the books by Doob [4] and by Loève [5] (cf. also [2, 1]).

On the other hand, recently there was a growing interest in random fields, for example within the framework of stochastic partial differential equations. In the present series of papers the author generalizes results in [4, 5] to the case where the underlying index set is a metric space, which seems to be a broad enough setting for most applications.

Let $(\Omega, \mathcal{A}, P)$ be a probability space and let (M, d) be a separable metric space. Throughout this paper we consider real or extended real valued random fields ϕ indexed by M, i.e.,

$$\begin{aligned} \phi : M \times \Omega \quad &\rightarrow \quad \mathbb{R} \text{ or } \overline{\mathbb{R}}, \\ (x, \omega) \quad &\mapsto \quad \phi(\omega, x), \end{aligned}$$

and for every $x \in M$, the mapping $\omega \mapsto \phi(x, \omega)$ from Ω into $\mathbb{R}$ or $\overline{\mathbb{R}}$, is $\mathcal{B}(\mathbb{R})$-$\mathcal{A}$-measurable, $\mathcal{B}(\overline{\mathbb{R}})$-$\mathcal{A}$-measurable respectively. (As it is custom, the second argument of ϕ is often suppressed.)

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The first question addressed in this paper concerns the existence of a modification of ϕ which is separable, where separability is defined in analogy with the case of stochastic processes (cf. [4, 5] and section 2). It turns out that the arguments in [4, 5] can be generalized in a rather straightforward way, and the result is that every random field ϕ as above admits a separable modification. Moreover, if ϕ is continuous in probability, then every countable dense subset of M is a separating set.

Assume that (M, d) is equipped with its Borel σ-algebra $\mathcal{B}(M)$, and that we are given a σ-finite measure μ on $(M, \mathcal{B}(M))$. Similarly as for stochastic processes, a random field is called measurable, if it is measurable as mapping from $M \times \Omega$, equipped with the product σ-algebra, to $\mathbb{R}$ (or $\overline{\mathbb{R}}$). It is called a.e. measurable, if it is measurable when restricted to the complement of a $\mu \otimes P$-null set. The second question considered here is whether a given random field indexed by M has a measurable or a.e. measurable modification. This problem necessitates more serious modifications of the arguments found in [4, 5]. The key is the existence of an appropriate partition of unity in case that (M, d) is in addition locally compact, cf. [3]. With this additional assumption on (M, d) it is proved in section 3, that the continuity in probability of the random field is enough to guarantee the existence of an a.e. measurable modification.

2. Separability

In this section, we assume throughout that (M, d) is a separable metric space, and ϕ is a real valued random field indexed by M defined on the probability space $(\Omega, \mathcal{A}, P)$. We are interested in the question of existence of a separable modification of ϕ. Most of this section carries over from the classical literature, especially from [4] or [5], with only minor modifications. The following definition of separability is modelled after the one given in [4] for stochastic processes.

Definition 2.1. A real valued random field ϕ on $(\Omega, \mathcal{A}, P)$ indexed by a metric space (M, d) is called *separable*, if there exists an at most countable subset S of M which is dense in (M, d), so that for all closed intervals C in $\mathbb{R}$, and all open subsets O of M,

$$\big\{\phi(x) \in C,\, x \in O\big\} = \big\{\phi(x) \in C,\, x \in O \cap S\big\}$$

holds. Then S is called a *separating set* for ϕ.

As in [5], separability of ϕ can be expressed equivalently in various ways:

Lemma 2.2. *A real valued random field ϕ on $(\Omega, \mathcal{A}, P)$ indexed by (M, d) is separable with separating set S, if and only if one of the following equivalent statements holds:*

(S_1) *For every open subset O in M,*

$$\inf_{y \in O \cap S} \phi(y) = \inf_{x \in O} \phi(x), \ \text{and}$$
$$\sup_{y \in O \cap S} \phi(y) = \sup_{x \in O} \phi(x);$$

(S_2) *For every open subset O in M,*

$$\inf_{y\in O\cap S}\phi(y)\le\inf_{x\in O}\phi(x),\ \textit{and}$$
$$\sup_{y\in O\cap S}\phi(y)\ge\sup_{x\in O}\phi(x);$$

(S_3) *For every open subset O in M and every $x\in O$,*

$$\inf_{y\in O\cap S}\phi(y)\le\phi(x)\le\sup_{y\in O\cap S}\phi(y);$$

(S_1') *For every $x\in M$,*

$$\liminf_{y\to x,\,y\in S}\phi(y)=\liminf_{y\to x}\phi(y),\ \textit{and}$$
$$\limsup_{y\to x,\,y\in S}\phi(y)=\limsup_{y\to x}\phi(y);$$

(S_2') *For every $x\in M$,*

$$\liminf_{y\to x,\,y\in S}\phi(y)\le\liminf_{y\to x}\phi(y),\ \textit{and}$$
$$\limsup_{y\to x,\,y\in S}\phi(y)\ge\limsup_{y\to x}\phi(y);$$

(S_3') *For every $x\in M$,*

$$\liminf_{y\to x,\,y\in S}\phi(y)\le\phi(x)\le\limsup_{y\to x,\,y\in S}\phi(y).$$

Proof. The equivalence of statements (S_1), (S_2), (S_3) is obvious. Also the equivalence of (S_1'), (S_2'), (S_3') is clear. Assume that (S_2) holds. Let $x\in M$, and choose the open set O in (S_2) as the ball $B_{1/n}(x)$ of radius $1/n$, $n\in\mathbb{N}$, with center x. Taking the limit $n\to+\infty$, we obtain (S_2'). On the other hand, (S_3') implies (S_3): Let O be open in M, $x\in O$, and choose n large enough, so that $B_{1/n}(x)\subset O$. Then

$$\begin{aligned}\inf_{y\in O\cap S}\phi(y)&\le\inf_{y\in B_{1/n}\cap S}\phi(y)\\&\le\sup_n\inf_{y\in B_{1/n}\cap S}\phi(y),\end{aligned}$$

and

$$\begin{aligned}\sup_{y\in O\cap S}\phi(y)&\ge\sup_{y\in B_{1/n}\cap S}\phi(y)\\&\ge\inf_n\sup_{y\in B_{1/n}\cap S}\phi(y).\end{aligned}$$

From (S_3') we have

$$\sup_n\inf_{y\in B_{1/n}\cap S}\phi(y)\le\phi(x)\le\inf_n\sup_{y\in B_{1/n}\cap S}\phi(y),$$

and therefore (S_3) holds. Thus, the equivalence of all statements (S_i), (S_i'), $i=1,2,3$, has been proven.

Finally we show that the statements (S_i), (S_i'), $i = 1, 2, 3$, are equivalent to the separability of ϕ with separating set S. To this end assume first that ϕ is separable with separating set S. Let $\omega \in \Omega$, and suppose that O is open in M. Define

$$a(\omega) := \inf_{y \in O \cap S} \phi(y, \omega)$$
$$b(\omega) := \sup_{y \in O \cap S} \phi(y, \omega),$$

where we also allow $a(\omega) = -\infty$ or $b(\omega) = +\infty$. We set $C(\omega) := [a(\omega), b(\omega)]$ if $a(\omega)$ and $b(\omega)$ are finite, and define the closed interval $C(\omega)$ in the obvious way in the case that one of them or both are infinite. Then given ω is such that for all $y \in O \cap S$, we have $\phi(y, \omega) \in C(\omega)$. Then for $\omega \in \Omega$ we have that for all $x \in O \cap S$, $\phi(x, \omega) \in C(\omega)$. Because $C(\omega)$ is closed, we have

$$\inf_{x \in O} \phi(x, \omega) \in C(\omega), \text{ and } \sup_{x \in O} \phi(x, \omega) \in C(\omega).$$

Consequently, (S_2) holds. Now suppose that (S_1) is true. Given an open set O and a closed interval $C = [a, b]$, let $\omega \in \Omega$ be such that for all $y \in O \cap S$, $\phi(y, \omega) \in C$. Then

$$\inf_{x \in O} \phi(x, \omega) = \inf_{y \in O \cap S} \phi(y, \omega)$$
$$\geq a.$$

Similarly, we derive $\sup_{x \in O} \phi(x, \omega) \leq b$. Therefore we must have $\phi(x, \omega) \in C$ for all $x \in O$, and therefore ϕ is separable with separating set S. ☐

Lemma 2.3. *Let H be a non-empty set, and let ψ be a real valued random field on $(\Omega, \mathcal{A}, P)$ indexed by H. Then there exists a non-empty, at most countable subset S of H, so that for all $x \in H$, and all $B \in \mathcal{B}(\mathbb{R})$,*

$$P\Big(\big\{\psi(y) \in B,\, y \in S\big\} \cap \big\{\psi(x) \notin B\big\}\Big) = 0.$$

Corollary 2.4. *Let H and ψ be as above, and suppose that $(C_k,\, k \in \mathbb{N})$ is a sequence in $\mathcal{B}(\mathbb{R})$. Let $\mathcal{C} \subset \mathcal{B}(\mathbb{R})$ be the family of all countable intersections of the family $(C_k,\, k \in \mathbb{N})$. Then there exists a non-empty, at most countable subset S of H, and for every $x \in H$ there is a P-null set $N(x)$ so that for every $B \in \mathcal{C}$,*

$$\big\{\psi(y) \in B,\, y \in S\big\} \cap \big\{\psi(x) \notin B\big\} \subset N(x).$$

Lemma 2.3 and Corollary 2.4 are proved in [4] (cf. also [5]) for the case that H is a subset of $\mathbb{R}$. But it has been remarked in [4], that they hold for a general set H. In fact, the arguments in [4] can be taken over word by word, and therefore the proofs are omitted here.

Lemma 2.5. *Let ϕ be a real valued random field on $(\Omega, \mathcal{A}, P)$ which is indexed by M. Then there exists an at most countable set $S \subset M$, which is dense in (M, d), and for every $x \in M$ there is a P-null set $N(x)$ so that for every open subset O of M, which contains x, and every closed subset C of $\mathbb{R}$,*

$$\big\{\phi(y) \in C,\, y \in O \cap S\big\} \cap \big\{\phi(x) \notin C\big\} \subset N(x).$$

Proof. Recall that by hypothesis (M,d) is separable. Let M_0 be an at most countable dense subset of M. We may choose as a countable base of the topology of (M,d) the open balls $B_r(z)$ with radius $r>0$, $r\in\mathbb{Q}$, and centers $z\in M_0$. We apply Corollary 2.4 to the following situation: We choose as the family $(C_k,\, k\in\mathbb{N})$ of Borel sets in $\mathbb{R}$ the family of all (bounded or unbounded) closed intervals with rational endpoints. Then the family $\mathcal{C}$ is the family of all closed subsets of $\mathbb{R}$. Furthermore, we choose $H=B_r(z)$, $r>0$, $r\in\mathbb{Q}$, $z\in M_0$, $\psi=\phi$. As a result we obtain a non-empty, at most countable subset $S_{r,z}$ of $B_r(z)$, so that for every $x\in B_r(z)$ there is a P-null set $N_{r,z}(x)$, and the inclusion

$$\bigl\{\phi(y)\in C,\, y\in S_{r,z}\bigr\}\cap\bigl\{\phi(x)\notin C\bigr\}\subset N_{r,z}(x)$$

holds for every $C\in\mathcal{C}$. Now set

$$S:=\bigcup_{r>0,\, r\in\mathbb{Q},\, z\in M_0} S_{r,z},$$

and for $x\in M$,

$$N(x):=\bigcup_{r>0,\, r\in\mathbb{Q},\, z\in M_0} N_{r,z}(x).$$

Then S is at most countable, and we have $S\cap B_r(z)\neq\emptyset$ for all $r>0$, $z\in M_0$. Hence S is dense in (M,d). Furthermore, for every $x\in M$, $P\bigl(N(x)\bigr)=0$.

Next let $C\in\mathcal{C}$, $x\in M$, and let $O\subset M$ be open with $x\in O$. Then there are $r>0$, $r\in\mathbb{Q}$, and $z\in M_0$ with $x\in B_r(z)\subset O$. Therefore we get

$$\begin{aligned}\bigl\{\phi(y)\in C,\, y\in O\cap S\bigr\}&\cap\bigl\{\phi(x)\notin C\bigr\}\\ &\subset\bigl\{\phi(y)\in C,\, y\in B_r(z)\cap S\bigr\}\cap\bigl\{\phi(x)\notin C\bigr\}\\ &=\bigl\{\phi(y)\in C,\, y\in S_{r,z}\bigr\}\cap\bigl\{\phi(x)\notin C\bigr\}\\ &\subset N_{r,z}(x)\\ &\subset N(x),\end{aligned}$$

and the proof is finished. □

Theorem 2.6. *Let (M,d) be a separable metric space, and let ϕ be a real valued random field indexed by M. Then ϕ has a separable modification.*

Proof. Let $x\in M$, and let S and $N(x)$ be as in the statement of Lemma 2.5. Let $\omega\in\complement N(x)$. For $r>0$, $r\in\mathbb{Q}$, and $z\in M_0$, so that $x\in B_r(z)$, we set

$$\begin{aligned}C_{r,z}(\omega)&:=\overline{\bigl\{\phi(y,\omega),\, y\in B_r(z)\cap S\bigr\}}\\ &=\overline{\bigl\{\phi(y,\omega),\, y\in S_{r,z}\bigr\}},\end{aligned}$$

where $\overline{A}$ indicates the closure of the set A in $\mathbb{R}$, and $S_{r,z}:=B_r(z)\cap S$. By construction $C_{r,z}(\omega)$ is closed, and because $S_{r,z}$ is non-empty, we have that $C_{r,z}(\omega)$ is also non-empty. Moreover, since $\omega\in\complement N(x)$ is such that for all $y\in S_{r,z}$ the values $\phi(y,\omega)$ belong to $C_{r,z}(\omega)$, Lemma 2.5 entails that $\phi(x,\omega)\in C_{r,z}(\omega)$. Therefore

$$C(x,\omega):=\bigcap_{r>0,\, r\in\mathbb{Q},\, z\in M_0,\, x\in B_r(z)} C_{r,z}(\omega)$$

is closed and $\phi(x,\omega) \in C(x,\omega)$. For $x \in S$, $\omega \in \Omega$ or $x \notin S$, $\omega \notin N(x)$ set

$$\phi^*(x,\omega) := \phi(x,\omega),$$

and for $x \notin S$, $\omega \in N(x)$ define

$$\phi^*(x,\omega) := \liminf_{y \to x,\, y \in S} \phi(y,\omega).$$

It is clear that ϕ^* is a modification of ϕ. Moreover, by construction we have for all $\omega \in \Omega$, $x \in M$ that $\phi'(x,\omega) \in C(x,\omega)$. We use this to show that ϕ^* is separable with separating set S: Let C be a closed interval, and suppose that $O \subset M$ is open. We have to prove that if $\omega \in \Omega$ is such that $\phi^*(y,\omega) \in C$ for all $y \in O \cap S$, then $\phi^*(x,\omega) \in C$ for all $x \in O$. First let $O = B_r(z)$, $r > 0$, $r \in \mathbb{Q}$, $z \in M_0$, and let $\omega \in \Omega$ be such that $\phi^*(y,\omega) \in C$ for all $y \in B_r(z) \cap S = S_{r,z}$. The definition of ϕ^* implies that $\phi(y,\omega) \in C$ for all $y \in B_r(z) \cap S = S_{r,z}$. Then by the construction of $C(x,\omega)$ we have that $C(x,\omega) \subset C$ for all $x \in B_r(z)$. Since for all $(x',\omega') \in M \times \Omega$, $\phi^*(x',\omega') \in C(x',\omega')$ holds, we find $\phi^*(x,\omega) \in C$. We have shown

$$\bigl\{\phi^*(y) \in C,\, y \in B_r(z) \cap S\bigr\} = \bigl\{\phi^*(x) \in C,\, x \in B_r(z)\bigr\}.$$

Now let O be a general open set. Then O can be written as a (countable) union of balls of the type $B_r(z)$. Therefore, it suffices to take the corresponding intersection on both sides of the last equality to finish the proof. □

Definition 2.7. A real valued random field ϕ on $(\Omega, \mathcal{A}, P)$ which is indexed by M is called *a.s. separable*, if it is a.s. equal to a separable random field ϕ^*. If S is then a separating set for ϕ^*, it is called an *a.s. separating set* for ϕ.

The following two results can be proved as in [4] or [5] without any modification, and therefore the proofs are omitted here.

Lemma 2.8. *Assume that ϕ is a.s. separable with a.s. separating set S. Let M_0 be any at most countable dense subset of M, and suppose that for every $x \in M$, there exists a P-null set $N(x)$, so that one of the properties (S_1'), (S_2'), or (S_3') holds outside of $N(x)$. Then M_0 is a.s. separating for ϕ.*

Theorem 2.9. *Let ϕ be a real valued random field indexed by M, which is continuous in probability and is a.s. separable. Then any at most countable dense subset of M is a.s. separating for ϕ.*

Corollary 2.10. *Let ϕ be a real valued random field indexed by M, which is continuous in probability. Then for any at most countable dense subset M_0 in M, ϕ has a modification which is continuous in probability and separable with separating set M_0.*

Proof. According to Theorem 2.6, we can choose a modification ϕ^* of ϕ which is separable for some at most countable dense subset S of M. As a modification of ϕ, ϕ^* has the same finite dimensional distributions as ϕ, and therefore also ϕ^* is continuous in probability. By Theorem 2.9, for any at most countable dense subset M_0 of M ϕ^* is a.s. separable. Let N_{M_0} be the exceptional set, and define ϕ^{**} as identically zero on N_{M_0} and as equal to ϕ^* on its complement. Then it is obvious that ϕ^{**} is a separable modification of ϕ which is continuous in probability. □

3. Measurability

Throughout this section we assume that (M, d) is a separable, locally compact metric space. We equip M with its Borel σ-algebra denoted by $\mathcal{B}(M)$, and suppose that a σ-finite measure μ is given on $(M, \mathcal{B}(M))$.

Definition 3.1. Let ϕ be a real valued random field on $(\Omega, \mathcal{A}, P)$ indexed by M.

(a) ϕ is called *measurable*, if the mapping
$$\phi : M \times \Omega \ \to \mathbb{R}$$
is $(\mathcal{B}(M) \otimes \mathcal{A})$-$\mathcal{B}(\mathbb{R})$-measurable.

(b) ϕ is called *a.e. measurable (with respect to $\mu \otimes P$)*, if there is a $\mu \otimes P$-null set, so that on its complement ϕ coincides with a measurable random field.

We investigate in this section the question under which conditions a given real valued random field ϕ indexed by M has a measurable modification. To this end, we combine the arguments given in [4] with the existence of an appropriate partition of unity (cf., e.g., [3]).

We begin with a lemma which will later on allow us to assume without loss of generality that μ is finite.

Lemma 3.2. *There exists a finite measure on $(M, \mathcal{B}(M))$ which is equivalent to μ.*

Proof. By hypothesis there exists a sequence $(B_n,\, n \in \mathbb{N})$ in $\mathcal{B}(M)$ so that $M = \bigcup_n B_n$, and for every $n \in \mathbb{N}$ we have $\mu(B_n) < +\infty$. For $A \in \mathcal{B}(M)$ set
$$\widehat{\mu}(A) := \sum_{n=1}^{\infty} 2^{-n} \frac{\mu(A \cap B_n)}{1 + \mu(B_n)}.$$
It is straightforward to check that $\widehat{\mu}$ is a finite measure on $(M, \mathcal{B}(M))$. Also, it is obvious that $\widehat{\mu}$ is absolutely continuous with respect to μ. On the other hand, suppose that $A \in \mathcal{B}(M)$ is such that $\widehat{\mu}(A) = 0$. Then it follows that $\mu(A \cap B_n) = 0$ for every $n \in \mathbb{N}$. Since $M = \bigcup_n B_n$, we find that $\mu(A) = 0$, and therefore μ is absolutely continuous with respect to $\widehat{\mu}$. □

Given the random field ϕ as above, we construct a sequence $(\phi_n,\, n \in \mathbb{N})$ of real valued random fields indexed by M as follows.

By hypothesis there exists an at most countable subset M_0 of M which is dense in (M, d). We choose as a base $\mathcal{B}$ of the topology of (M, d) the family of open balls with rational radii and centers in the set M_0. Fix $n \in \mathbb{N}$. Let $\mathcal{C}_n^0$ denote the family of open balls of radius $1/n$ with centers in M_0. Then $\mathcal{C}_n^0$ is an open covering of M. According to [3, No. 12.6.1], there exists an at most countable finer covering $\mathcal{C}_n$ of M by sets in $\mathcal{B}$, which is locally finite: There exists a sequence $(x_{n,m},\, m \in \mathbb{N})$ in M_0, and a sequence $(r_{n,m},\, m \in \mathbb{N})$, $r_{n,m} > 0$, $r_{n,m} \in \mathbb{Q}$, so that
$$\mathcal{C}_n = (B_{n,m},\, m \in \mathbb{N}),$$
where $B_{n,m}$ is the ball of radius $r_{n,m}$ with center $x_{n,m}$. $\mathcal{C}_n$ is finer than $\mathcal{C}_n^0$ in the sense that for every $m \in N$ there exists a set $C \in \mathcal{C}_n^0$ so that $B_{n,m} \subset C$. Consequently, $r_{n,m} \leq 1/n$ for all $m \in \mathbb{N}$. Moreover, for every $x \in M$ there is a

neighborhood U of x, so that $U \cap B_{n,m} = \emptyset$ for almost all $m \in \mathbb{N}$. In particular, every $x \in M$ belongs only to finitely many balls in $\mathcal{C}_n$. In [3], 12.6.3, it is stated that there exists a continuous partition of unity $(f_{n,m},\, m \in \mathbb{N})$ subordinate to $\mathcal{C}_n$: For every $m \in \mathbb{N}$, $f_{n,m}$ is a continuous function from M to $\mathbb{R}$, such that for all $x \in M$, $0 \le f_{n,m}(x) \le 1$,

$$\sum_{m=1}^{\infty} f_{n,m}(x) = 1,$$

and $\operatorname{supp} f_{n,m} \subset B_{n,m}$. We define

$$\phi_n(x) := \sum_{m=1}^{\infty} \phi(x_{n,m})\, f_{n,m}(x), \qquad x \in M. \tag{3.1}$$

It is an easy exercise to show that the random fields $(x,\omega) \mapsto \phi(x_{n,m},\omega)\, f_{n,m}(x)$ are measurable, and therefore so is ϕ_n for every $n \in \mathbb{N}$.

Furthermore, if for every $x \in M$ we have $\phi(x) \in \mathcal{L}^1(P)$, then for every $n \in \mathbb{N}$ and every $x \in M$, $\phi_n(x) \in \mathcal{L}^1(P)$: In view of equation (3.1) this follows from the fact that for every $n \in \mathbb{N}$ and every $x \in M$ there are only finitely many $m \in \mathbb{N}$ so that $f_{n,m}(x) \neq 0$, and that we have $|f_{n,m}(x)| \le 1$.

Lemma 3.3. *Suppose that for every $x \in M$, $\phi(x)$ belongs to $\mathcal{L}^1(P)$ and that*

$$\begin{aligned} \phi\ :\ & M \to \mathcal{L}^1(P) \\ & x \mapsto \phi(x) \end{aligned}$$

is continuous. Then for every $x \in M$, the sequence $(\phi_n(x),\, n \in \mathbb{N})$ converges in $\mathcal{L}^1(P)$ to $\phi(x)$.

Proof. We shall write $\|\cdot\|_1$ for the pseudo-norm of $\mathcal{L}^1(P)$. Let $x \in M$, $\varepsilon > 0$. Choose $\delta > 0$ so that for all $y \in M$, $d(x,y) < \delta$ implies $\|\phi(x) - \phi(y)\|_1 < \varepsilon$. Choose $n_0 \in \mathbb{N}$ with $1/n_0 < \delta$. Let $n \in \mathbb{N}$ be such that $n \ge n_0$. Note that for $m \in \mathbb{N}$, we have that $f_{n,m}(x) > 0$ implies $x \in B_{n,m}$, i.e., $d(x, x_{n,m}) < r_{n,m} \le 1/n < \delta$. Thus we can estimate as follows

$$\begin{aligned} \|\phi(x) - \phi_n(x)\|_1 &= \Big\| \sum_{m=1}^{\infty} \big(\phi(x) - \phi(x_{n,m})\big)\ f_{n,m}(x) \Big\|_1 \\ &\le \sum_{m=1}^{\infty} \|\phi(x) - \phi(x_{n,m})\|_1\ f_{n,m}(x) \\ &< \sum_{m=1}^{\infty} \varepsilon\, f_{n,m}(x) \\ &= \varepsilon, \end{aligned}$$

and the proof is finished. □

Theorem 3.4. *Assume that ϕ is a real valued random field indexed by M which is continuous in probability. Then ϕ has an a.e. measurable modification. Furthermore, if in addition ϕ is separable with separating set $S \subset M$, then the a.e. measurable modification can be chosen in such way that it is separable with separating set S.*

Remark 3.5. If we assume in addition that (M, d) is complete with respect to d, then it becomes a Borel space, and in this case (even without the assumption of local compactness) the statement of the theorem follows directly from Doob's classical results [4, p.60 ff].

Proof. Throughout this proof we use the notation already employed above. Without loss of generality we may assume that ϕ is uniformly bounded. (Otherwise, we consider instead of ϕ the random field $\arctan\circ\phi$, construct its modification, and undo the transformation by arctan at the end of the proof.) Also, by Corollary 2.10, we may assume that ϕ is separable with separating set S, where S is any at most countable dense subset of M, and we choose $M_0 = S$ in the above construction of the sequence $(\phi_n\, n \in \mathbb{N})$.

First observe that for every $x \in M$ we have $\{x\} \in \mathcal{B}(M)$, because $\mathcal{B}(M)$ contains all closed sets. This entails that the separating set S belongs to $\mathcal{B}(M)$, and hence the restriction ϕ_S of ϕ to $S \times \Omega$ is measurable: If $B \in \mathcal{B}(\mathbb{R})$, then

$$\begin{aligned}\phi_S^{-1}(B) &= \phi^{-1}(B) \cap (S \times \Omega)\\ &= \bigcup_{x\in S} \phi^{-1}(B) \cap (\{x\} \times \Omega)\\ &= \bigcup_{x\in S} \{x\} \times \phi(x)^{-1}(B),\end{aligned}$$

and the sets $\{x\} \times \phi(x)^{-1}(B)$, $x \in S$, belong to $\mathcal{B}(M) \otimes \mathcal{A}$. Since S is at most countable, it follows that also their union over $x \in S$ is in $\mathcal{B}(M) \otimes \mathcal{A}$. Therefore we may leave ϕ on $S \times \Omega$ unchanged, and it remains to construct the desired modification on $\complement S \times \Omega$.

Since ϕ is uniformly bounded, the family $(\phi(x), x \in M)$ is trivially uniformly integrable. Thus the assumption of continuity in probability implies that $x \mapsto \phi(x)$ is continuous from M into $\mathcal{L}^1(P)$. Consider now the sequence $(\phi_n, n \in \mathbb{N})$ as in equation 3.1, with $M_0 = S$. By construction, for every $n \in \mathbb{N}$, ϕ_n is measurable, and by Lemma 3.3 we know that for every $x \in M$, $(\phi_n(x), n \in \mathbb{N})$ converges in $\mathcal{L}^1(P)$ to $\phi(x)$. Because ϕ is uniformly bounded, we see from equation 3.1 that so is the sequence $(\phi_n, n \in \mathbb{N})$. Moreover, the measure μ is bounded, so that the dominated convergence theorem gives us that

$$\int_{\complement S} \|\phi(x) - \phi_n(x)\|_1 \, d\mu(x) \to 0, \qquad n \to +\infty.$$

By an application of Fubini's theorem, we therefore find that the sequence $(\phi_n, n \in \mathbb{N})$ is Cauchy in $\mathcal{L}^1(\complement S \times \Omega, \mathcal{B}(\complement S) \otimes \mathcal{A}, \mu \otimes P)$, where $\mathcal{B}(\complement S)$ is the trace of $\mathcal{B}(M)$ on $\complement S$. We abbreviate the latter $\mathcal{L}^1$-space with $\mathcal{L}^1(\complement S \times \Omega)$ in the sequel. The Riesz-Fischer-theorem implies that there exists ψ in $\mathcal{L}^1(\complement S \times \Omega)$ so that $(\phi_n, n \in \mathbb{N})$ converges in $\mathcal{L}^1(\complement S \times \Omega)$ to ψ. In particular, ψ is measurable from $\complement S \times \Omega$ into $\mathbb{R}$. Moreover, by selection of a subsequence, we may suppose that there is a $\mu \otimes P$-null set $N \in \mathcal{B}(\complement S) \otimes \mathcal{A}$, so that on $\complement S \times \Omega \setminus N$ the sequence $(\phi_n, n \in \mathbb{N})$ converges pointwise to ψ.

We use again Fubini's theorem and observe that

$$\int_{\complement S} \|\phi_n(x) - \psi(x)\|_1 \, d\mu(x) \to 0, \qquad n \to +\infty.$$

By selection of another subsequence, if necessary, we therefore obtain that there is a μ-null set S_0 in $\mathcal{B}(\complement S)$, so that for all x in its complement we have $\phi_n(x) \to \psi(x)$, $n \to +\infty$, in $\mathcal{L}^1(P)$. Since this subsequence converges also to $\phi(x)$ we have for all $x \in \complement S_0$, $P(\phi(x) = \psi(x)) = 1$.

We now define the modification ϕ^* of ϕ as follows:

$$\phi^*(x,\omega) := \begin{cases} \phi(x,\omega), & (x,\omega) \in \big((S \cup S_0) \times \Omega\big) \cup N \\ \psi(x,\omega), & \text{otherwise.} \end{cases}$$

We have already shown above that for all $x \in M$, $P\big(\phi^*(x) = \phi(x)\big) = 1$, i.e., ϕ^* is indeed a modification of ϕ. Furthermore, ϕ^* is measurable when restricted to $S \times \Omega$ or to $\complement S_0 \times \Omega$. Since S_0 is a μ-null set, $S_0 \times \Omega$ is a $\mu \otimes P$-null set, and consequently ϕ^* is a.e. measurable.

Finally we show that ϕ^* is separable with separating set S. Let O be open in M, let C be a closed interval, and assume $\omega \in \Omega$ is such that for all $y \in O \cap S$ we have $\phi^*(y,\omega) \in C$. By construction, ϕ^* and ϕ coincide on $S \times \Omega$, so that we obtain $\phi(y,\omega) \in C$ for all $y \in O \cap S$. Let $x \in O$. We have to show that $\phi^*(x,\omega) \in C$. This is trivial for $x \in S$. For $(x,\omega) \in M \times \Omega$ so that $x \in S_0$ or $(x,\omega) \in N$ this follows from the fact that ϕ is separable with separating set S. It remains to consider the case where $(x,\omega) \in M \times \Omega$ is such that $x \in \complement S_0$ and $(x,\omega) \in \complement S \times \Omega \setminus N$. Let $r > 0$ be such that $B_r(x) \subset O$. Choose $n_0 \in \mathbb{N}$ so that $1/n_0 \le r$, and consider $n \in \mathbb{N}$ with $n \ge n_0$. Then by construction of $\phi_n(x)$ in equation (3.1), we have that those $m \in \mathbb{N}$, for which $f_{n,m}(x) > 0$, are such that $d(x, x_{n,m}) < 1/n \le r$. Thus $x_{n,m} \in B_r(x)$ for those terms which contribute to (3.1), and the corresponding values of $\phi(x_{n,m},\omega)$ are by assumption in C. $\phi_n(x,\omega)$ is a convex combination of these values, and therefore $\phi_n(x,\omega) \in C$, for all $n \in \mathbb{N}$, $n \ge n_0$. Now $\phi^*(x,\omega)$ is by construction the limit of a subsequence of $(\phi_n(x,\omega), n \in \mathbb{N}, n \ge n_0)$, and C is closed. Hence we get $\phi^*(x,\omega) \in C$, and the proof is finished. □

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