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GENERALIZED GIRSANOV TRANSFORM OF PROCESSES AND ZAKAI EQUATION WITH JUMPS

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Dedicated to the memory of Professor Hiroshi Kunita

Abstract. It is well known that the Girsanov transform (or, Girsanov's theorem) plays an important role in the stochastic analysis and this transform is closely related with the uniform integrability of local martingales. The first aim of this article is to give concrete, necessary and sufficient conditions of uniform integrability of positive local martingales with jumps. Then we shall apply Girsanov transform to Zakai equation (Zakai SDE) arisen from the filtering problem of stochastic processes with jumps. Using Girsanov transform for Lévy processes, Malliavin calculus could be applied to show the existence of smooth density of the filtering measure. The second aim of this article is to show the uniqueness of solutions of Zakai equation. This is worthwhile from the fact that the solution of Zakai equation can be obtained from the filtering measure by using Girsanov transform.

1. Introduction

Absolutely continuous transform of laws of stochastic processes which are given by SDE (stochastic differential equation) is called the Girsanov transform. This is common and effective means in the stochastic analysis. In our contexts, we assume that these SDE can be formulated as martingale problems for integro-differential operators, and that transforms are made by positive local martingales with jumps. Since the transform is closely related with the uniform integrability of positive local martingales, the first aim of this article is to obtain concrete, necessary and sufficient conditions of uniform integrability of positive local martingales with jumps.Then using Girsanov transform, we shall discuss Zakai SDE.

The Zakai SDE ([14]) is a measure-valued linear SDE, and it is originally arisen from the nonlinear filtering problem with respect to stochastic signal-observation systems. It is shown in [2] that filtering measures satisfy filtering equations for systems of continuous processes, and in [6] for systems of discontinuous semi-martingales. However, it is not easy to analyze filtering equations directly, for they are nonlinear SDEs. On the other hand, it can be proved that a certain unnormalized filtering measure becomes a solution to the Zakai SDE and that the

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relation between the unnormalized filtering measure and the original one is given by the Bayes formula ([3], [13]). From the Bayes formula, it is possible to apply the Malliavin calculus to show the existence of smooth densities of filtering measures ([3], [4], [11], see also [12]).

It is worthwhile to discuss the uniqueness of solutions of Zakai SDE. Indeed, if we can prove the uniqueness of solutions of the Cauchy problem for Zakai SDE, then this implies the uniqueness of solutions of the filtering equation and the existence of smooth densities of the solution of the Zakai SDE. However, even in the continuous diffusion systems, it is not easy to show the uniqueness of solutions of Zakai SDE with unbounded coefficients ([7],[8] and [13]). On the other hand, by using a simple replacement of solutions it is shown that Zakai SDE can be transformed into a parabolic integro-differential equation in the cases where noises of the signal and those of the observation are independent ([3]). Then this enables us to apply the usual functional analysis methods to the proof of the uniqueness of solutions of Zakai SDE with linear growth coefficients.

2. Martingale Problem for SDE and Girsanov Transform

Let $W = D([0,1] \to \boldsymbol{R}^n)$ be the space of càd - làg functions. Set $Y_t(w) = w(t)$ for $w = (w(t)) \in W$, and $\mathcal{W}_t = \bigcap_{\varepsilon>0} \sigma(Y_s, s < t+\varepsilon)$. Consider a triplet $(G_t(w), H_t(w), K_t(w,d\eta))$ of $(\mathcal{W}_t)$−adapted process. For fixed (t,w), $G_t(w)$ is a non-negative $n \times n$−matrix, $H_t(w) \in \boldsymbol{R}^n$ and $K_t(w,d\eta)$ is a measure on $\boldsymbol{R}^n \setminus \{0\}$. Hereafter, $G_t(w)$ (resp. $H_t(w)$ and $K_t(w,d\eta)$) is simply denoted by G_t (resp. H_t and $K_t(d\eta)$). Let Q be a probability on $(W, \mathcal{W}_1)$ such that

$$Q\left[\int_0^1 \{ \text{ trace } G_s + |H_s| + \int (|\eta|^2 \wedge 1)\ K_s(d\eta)\ \}\ ds\ <\ \infty\right] = 1.$$

Using the Itô formula (cf. [1]) for semi-martingales with jumps, it is a routine work to prove the lemma (cf. [9]).

Lemma 2.1. *The following three conditions are equivalent.*

(1) *For any $z \in \boldsymbol{R}^n$, the process*

$$\exp\left[iz\cdot Y_t - \int_0^t \left\{-\frac{1}{2}z\cdot G_s z + iz\cdot H_s + \int \left(e^{iz\cdot\eta} - 1 - iz\cdot[\eta]_1\right) K_s(d\eta)\right\} ds\right]$$

is a $(\mathcal{W}_t, Q)$−martingale, where $[\eta]_1 = \eta\ I(|\eta| \le 1)$.

(2) *Let $\boldsymbol{A}_t \equiv \boldsymbol{A}_t(w)$ be the integro-differential operator on $C_0^\infty(\boldsymbol{R}^n)$:*

$$\begin{aligned}\boldsymbol{A}_s f(y) \quad &= \quad \frac{1}{2}\text{ tr }[G_s\ ({\partial_y}^2 f(y))] + H_s\cdot\partial_y f(y) \\ &\quad + \int \{f(y+\eta) - f(y) - [\eta]_1\cdot\partial_y f\}\ K_s(d\eta),\end{aligned}$$

where $\partial_y = (\partial/\partial y_i)_{1\le i\le n}$ and ${\partial_y}^2 = (\partial^2/\partial y_i\partial y_j)_{1\le i,j\le n}$.
For any $f = (f(y)) \in C_0^\infty(\boldsymbol{R}^n)$, the process

$$M_t^f \ := \ f(Y_t)\ -\ f(Y_0)\ -\ \int_0^t (\boldsymbol{A}_s f)(Y_s)\ ds$$

is a local $(\mathcal{W}_t, Q)$− martingale.

(3) *The process (Y_t, Q) is decomposed into the form*

$$Y_t \; = \; \beta_Y(t) + \int_0^t H_s \; ds + \int_0^t\!\!\int_{|\eta|\le 1} \eta \; \widetilde{J}_Y(dsd\eta) + \int_0^t\!\!\int_{|\eta|>1} \eta \; J_Y(dsd\eta),$$

where $\beta_Y = (\beta_Y(t))$ is a $\boldsymbol{R}^n-$valued continuous $(\mathcal{W}_t, Q)-$local martingale with

$$d\langle z\cdot\beta_Y\rangle_s \equiv d\langle z\cdot\beta_Y(s),\; z\cdot\beta_Y(s)\rangle = (z\cdot G_s z)\; ds,$$

and $J_Y(dsd\eta) := \#\{\; \tau\in ds \;\mid\; 0\neq \Delta Y_\tau \in d\eta \;\}$ is a measure with the property that the signed random measure $\widetilde{J}_Y(dsd\eta) := J_Y(dsd\eta) - K_s(d\eta)\; ds$ generates $(\mathcal{W}_t, Q)-$local martingales.

If conditions in the above Lemma are satisfied, let the process (Y_t, Q) be called " a solution to the martingale problem for the triplet $(G_t, H_t, K_t(d\eta))$", or simply be called " a $(G_t, H_t, K_t)-$process ".
We shall introduce an assumption.

[A1] The existence and the uniqueness of solutions to the martingale problem for the triplet $(G_t, H_t, K_t(d\eta))$ hold.

There are many cases where **[A1]** is satisfied (cf. [10]). The most simple but important example is the case where (Y_t, Q) is a Lévy process.

We shall consider the representation of local martingales. Let $\mathcal{H}^2_{loc}(\beta_Y)$ (resp. $\mathcal{H}^2(\beta_Y)$) denote the space of $\boldsymbol{R}^n-$valued, $(\mathcal{W}_t)-$adapted processes $\varGamma = (\varGamma_t(w))$ satisfying

$$\int_0^1 \varGamma_s\cdot G_s\varGamma_s \; ds < \;\infty \;\; \text{a.e.}(Q) \quad \left(\text{resp. } E_Q\left[\int_0^1 \varGamma_s\cdot G_s\varGamma_s \; ds\right] < \;\infty\right).$$

For $\varGamma = (\varGamma_t(w)) \in \mathcal{H}^2_{loc}(\beta_Y)$, the stochastic integral

$$(\varGamma\cdot\beta_Y)_t \;:=\; \int_0^t \varGamma_s \; d\beta_Y(s)$$

is well defined, and the process $((\varGamma\cdot\beta_Y)_t)$ is a continuous $(\mathcal{W}_t, Q)-$local martingale.
Let $\mathcal{L}_{loc}(J_Y)$ (resp. $\mathcal{L}^p(J_Y)\;\; p=1,2$) be the class of function valued $(\mathcal{W}_t)-$adapted processes $\varLambda = (\varLambda_{t,\eta}) = (\varLambda_{t,\eta}(w))$ satisfying

$$\int_0^1\!\!\int |\varLambda_{s,\eta}| \;\; J_Y(dsd\eta) < \infty \;\; \text{a.e.}(Q) \quad \left(\text{resp. } E_Q\left[\int_0^1\!\!\int |\varLambda_{s,\eta}|^p \;\; K_s(d\eta)ds\right] < \infty\right).$$

Moreover, let $\mathcal{L}^p_{loc}(J_Y)$ be the class of function valued, $(\mathcal{W}_t)-$adapted processes $\varLambda = (\varLambda_{t,\eta}(w))$ such that there exists a sequence of $(\mathcal{W}_t)-$stopping times $T_n \uparrow 1$ satisfying $\;(I(t\le T_n)\times\varLambda_{t,\eta}) \in \mathcal{L}^p(J_Y)\;$ for each n.
For any $\varLambda = (\varLambda_{t,\eta}(w)) \in \mathcal{L}^1_{loc}(J_Y)\bigcup\mathcal{L}^2_{loc}(J_Y)$, the stochastic integral

$$(\varLambda\cdot\widetilde{J}_Y)_t \;:=\; \int_0^t\!\!\int \varLambda_{s,\eta}\;\widetilde{J}_Y(dsd\eta)$$

is well defined so that $((\varLambda\cdot\widetilde{J}_Y)_t)$ is a $(\mathcal{W}_t, Q)-$local martingale.

Lemma 2.2. *Assume that $(G_t, H_t, K_t(d\eta))$ satisfies* **[A1]**, *and let (Y_t, Q) be a $(G_t, H_t, K_t)-$process. Let (M_t) be a given locally bounded $(\mathcal{W}_t, Q)-$martingale. Then there exist processes $(\Gamma_t) \in \mathcal{H}^2_{loc}(\beta_Y)$ and $(\Lambda_{t,\eta}) \in \mathcal{L}_{loc}(J_Y)$ having properties that $(\Lambda_{t,\eta} \times I(|\Lambda_{t,\eta}| > 1)) \in \mathcal{L}^1_{loc}(J_Y)$ and $(\Lambda_{t,\eta} \times I(|\Lambda_{t,\eta}| \leq 1)) \in \mathcal{L}^2_{loc}(J_Y)$, and*

$$M_t \;=\; M_0 + \int_0^t \Gamma_s \; d\beta_Y(s) + \int_0^t \!\! \int \Lambda_{s,\eta} \; \widetilde{J}_Y(dsd\eta).$$

The above lemma is necessary to prove Theorem 2.4. We shall give the proof of the lemma in Appendix, for it needs a special technique in the martigale theory. Consider the class of functions

$$\mathcal{V} := \{ \; v(\cdot) \in C^2(\boldsymbol{R}_+) \mid v(0) = 1, \; 0 < v'(s) < 1, \; -1 < v''(s) \leq 0, \; v(+\infty) = +\infty \; \}.$$

Lemma 2.3. *Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{H}$, a subset of $L^1(\Omega, \mathcal{F}, P)$. $\mathcal{H}$ is uniformly integrable if and only if there exists a positive function $v(\cdot) \in \mathcal{V}$ such that*

$$\text{(U)} \qquad \sup_{\phi \in \mathcal{H}} \; E_P[\; |\phi| \; v(\log^+ |\phi|) \;] < \infty.$$

Proof. Suppose that $\mathcal{H}$ is uniformly integrable. Since

$$\sup_{\phi \in \mathcal{H}} \; E_P[\; |\phi| \;] \; < \infty, \qquad \lim_{\ell \to \infty} \; \sup_{\phi \in \mathcal{H}} \; E_P[\; |\phi| \; I(|\phi| \geq \ell) \;] \; = \; 0,$$

it is possible to choose constants $c_0 = 0, \; 1 < c_1 < c_2 < c_3 < \cdots$ so that

$$\sup_{\phi \in \mathcal{H}} \; E_P[\; |\phi| \; I(\log^+ |\phi| \geq c_n) \;] \; \leq \; 2^{-n}, \qquad c_{n+1} - c_n \; > \; c_n - c_{n-1} \qquad (n \geq 1).$$

Let us consider a decreasing step function $u(t)$ given by

$$u(t) \; := \; \frac{1}{c_1} \; I(t < 0) \; + \; \sum_{n=0}^{\infty} \; \frac{1}{c_{n+1} - c_n} \; I(c_n \leq t < c_{n+1}),$$

and define functions $\wp(\cdot) \in C([-1, \infty))$ and $v(\cdot)$ by

$$\wp(\tau) \; := \; \int_0^\tau u(t) \; dt, \quad v(t) \; := \; 1 + \int_{-1}^{+1} (1 - |s|) \; \wp(t - s) \; ds.$$

Since $u(\cdot)$ is a decreasing function with $0 < u(t) < 1$, from equalities

$$v'(t) \; = \; \int_{-1}^{+1} (1 - |s|) \; u(t - s) \; ds, \quad v''(t) \; = \; - \int_0^1 (u(t - s) - u(t + s)) \; ds,$$

we see that $v(\cdot) \in C^2(\boldsymbol{R}_+)$ with $0 < v'(t) < 1$ and $-1 < v''(t) \leq 0$. From $\wp(c_n) = n$ and $\wp(t) < v(t) < \wp(t) + 1$, we see that $v(t)$ is an increasing function tending to infinity with t. Set $a_n = E_P[|\phi| I(\log^+ |\phi| \geq c_n)]$. Then $a_0 = E_P[|\phi|]$ and

$$\begin{aligned}
&E_P[|\phi| v(\log^+ |\phi|)] \; \leq \; \sum_{n=1}^{\infty} v(c_n) E_P[|\phi| I(c_n > \log^+ |\phi| \geq c_{n-1})] \\
&\leq \; \sum_{n=1}^{\infty} (n+1)(a_{n-1} - a_n) \; = \; a_0 + \sum_{n=1}^{\infty} \; (\sum_{k=1}^{n} 1) \; (a_{n-1} - a_n) \\
&= \; a_0 + \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} (a_{n-1} - a_n) \; = \; 2a_0 + \sum_{k=2}^{\infty} a_{k-1} \; \leq \; 2a_0 + \sum_{n=1}^{\infty} 2^{-n}.
\end{aligned}$$

Then the uniform integrability of $\mathcal{H}$ implies the property (U). The converse implication is proved from that $v(\log \ell) \to \infty$ as $\ell \to \infty$ and the simple inequality: $E_P[|\phi| I(|\phi| \geq \ell)] \ \leq \ E_P[|\phi| v(\log^+ |\phi|)] \ / \ v(\log \ell)$. □

The aim of this section is to generalize the Girsanov theorem for diffusions into that for a family of processes including solutions to the martigale problem for the triplet $(G_t, H_t, K_t(d\eta))$.

Assume **[A1]**, and let (Y_t, Q) be the $(G_t, H_t, K_t)-$process. The process (Y_t) is represented as follows:

$$Y_t \ = \ \beta_Y(t) \ + \ \int_0^t H_s \ ds \ + \int_0^t\!\!\int \eta \ \{J_Y(dsd\eta) \ - \ I(|\eta| \leq 1) \ K_s(d\eta)ds\}.$$

We say that $(h(t,w), k(t,w,\eta))$ belongs to the class $\mathcal{A}[\mathcal{W}]$ iff $(t,w) \longrightarrow h(t,w) \in \boldsymbol{R}^n$ and $(t,w) \longrightarrow k(t,w,\eta) > 0$ are $(\mathcal{W}_t)-$adapted process, and the function $\eta \longrightarrow k(t,w,\eta)$ is Borel measurable. For $(h,k) \in \mathcal{A}[\mathcal{W}]$, set

$$\begin{aligned} \Phi_t[h,k;Q] \ &:= \ \exp\Bigg[\int_0^t h(s,w)\cdot d\beta_Y(s) - \frac{1}{2}\int_0^t h(s,w)\cdot G_s \ h(s,w) \ ds \\ &\quad + \int_0^t\!\!\int \{\log k(s,w,\eta) \ J(dsd\eta) - (k(s,w,\eta)-1) \ K_s(d\eta)ds\}\Bigg], \end{aligned}$$

and define $H_t^{h,k}$, $K_t^k(d\eta)$ by

$$\text{(R)} \qquad \begin{cases} H_t^{h,k} \ &:= \ H_t \ + \ G_t \ h(t,w) \ + \ \displaystyle\int_{|\eta|\leq 1} \eta \ (k(t,w,\eta)-1) \ K_t(d\eta) \\ K_t^k(d\eta) \ &;= \ k(t,w,\eta) \ K_t(d\eta). \end{cases}$$

Theorem 2.4. *Given a $(G_t, H_t, K_t)-$process (Y_t, Q) and a $(\widetilde{G}_t, \widetilde{H}_t, \widetilde{K}_t)-$process $(Y_t, \widetilde{Q})$. Assume that probabilities Q and $\widetilde{Q}$ are mutually absolutely continuous. Then there are $(h,k) \in \mathcal{A}[\mathcal{W}]$ and $v(\cdot) \in \mathcal{V}$ such that*

$$\begin{aligned} &\int_0^1 \{(h\cdot G_s \ h) + \int ((\sqrt{k}-1)^2 v(|\log k|)) \ K_s(d\eta)\} \ ds < \infty \ \text{ a.e. } (Q), \\ &E_Q[\ d\widetilde{Q}/dQ \mid \mathcal{W}_t] \ = \ \Phi_t[h,k;Q]. \end{aligned}$$

Then we have $(\widetilde{G}_t, \widetilde{H}_t, \widetilde{K}_t(d\eta)) \ = \ (G_t, H_t^{h,k}, K_t^k(d\eta))$.

Proof. (Step 1) Set $\varphi_t := E_Q[\ d\widetilde{Q}/dQ \mid \mathcal{W}_t]$. Then (φ_t) is a $(\mathcal{W}_t, Q)-$martingale. From the Itô formula for semi-martingales (cf. [1]) and the representation lemma for local martingales (Lemma 2.2), it is a routine work to show that there exists $(h,k) \in \mathcal{A}[\mathcal{W}]$ satisfying

$$\int_0^1 \{ \ h(s,w)\cdot G_s \ h(s,w) \ + \ \int (\sqrt{k(s,w,\eta)}-1)^2 \ K_s(d\eta) \ \} \ ds < \infty \ \text{ a.e.}$$

and $\varphi_t = \Phi_t[h,k;Q]$. Set $r(t,w,\eta) = \log k(t,w,\eta)$ and note that

$$\int_{|r|\leq 1} r^2 \ K_s(d\eta) + \int_{|r|>1} |e^r - 1| \ K_s(d\eta) \ \leq \ 5 \ \int (\sqrt{k}-1)^2 \ K_s(d\eta).$$

The relation $(\widetilde{G}_t, \widetilde{H}_t, \widetilde{K}_t(d\eta)) = (G_t, H_t^{h,k}, K_t^k(d\eta))$ is a consequence of the uniqueness of Meyer's decomposition.

(Step 2) The set $\{\varphi_T \mid T : \text{stopping time}\}$ is uniformly integrable with respect to Q. From Lemma 2.3, there is a positive increasing function $v \in C^2(\boldsymbol{R})$ such that

$$(v(s) \times I(s \geq 0)) \in \mathcal{V}, \qquad \sup_T \; E_Q[\varphi_T \; v(\log \varphi_T)\;] < \infty.$$

Set $L_t = \log \varphi_t$ and $v_1(s) = v(s) + v'(s)$. From the Itô formula, we have

$$\begin{aligned}
&\varphi_t \; v(L_t) - 1 \\
&= \left[\int_0^t \varphi_s \; v_1(L_s) \; h \cdot d\beta_Y(s) + \frac{1}{2}\int_0^t \varphi_s \; v_1'(L_s) \; h \cdot G_s h \; ds\right] \\
&\quad + \left[\int_0^t\int_{|r|\leq 1} \varphi_{s-} \; (e^r v(L_{s-} + r) - v(L_{s-})) \; \widetilde{J}_Y(dsd\eta)\right. \\
&\qquad \left. + \int_0^t\int_{|r|\leq 1} \varphi_s \; \{e^r(v(L_s + r) - v(L_s)) - (e^r - 1) \; v'(L_s)\} \; K_s(d\eta)ds\right] \\
&\quad + \left[\left(\int_0^t\int_{r>1} + \int_0^t\int_{r<-1}\right) \varphi_{s-} \; (e^r v(L_{s-} + r) - v(L_{s-})) \; J_Y(dsd\eta)\right. \\
&\qquad \left. - \int_0^t\int_{|r|>1} \varphi_s \; (e^r - 1) \; v_1(L_s) \; K_s(d\eta)ds\right] \\
&=: \left[\hat{A}_t\right] + \left[\hat{B}_t\right] + \left[(C_t^+ + C_t^-) - \hat{C}_t\right].
\end{aligned}$$

Define a sequence (T_ν) of stopping times by

$$T_\nu = \inf \{\; t \in [0,1] \mid |L_t| > \nu \text{ or } \int_0^t (\; h \cdot G_s h + \int (e^{r/2} - 1)^2 \; K_s \;) \; ds > \nu \;\}.$$

Since $\hat{A}_{T_\nu}$, $\hat{B}_{T_\nu}$, $C^-_{T_\nu}$ and $\hat{C}_{T_\nu}$ are integrable with respect to Q, we have

$$E_Q\left[\int_0^{T_\nu} \varphi_s \int_{r>1} (e^r v(L_s + r) - v(L_s)) \; K_s(d\eta) \; ds\right] = E_Q\left[\; C^+_{T_\nu} \;\right] < \infty.$$

Under restraints $s < T_\nu$ and $r = r(s, w, \eta) > 1$, inequalities

$$\begin{aligned}
&v(r) - v(L_s + r) < v(-L_s) \; I(L_s + r < 0) + v(|L_s|) \; I(L_s + r \geq 0) \leq v(\nu), \\
&e^r v(r) \;=\; [e^r v(L_s + r) - v(L_s)] + v(L_s) + e^r[v(r) - v(L_s + r)] \\
&\qquad\;\; \leq\; [e^r v(L_s + r) - v(L_s)] + v(\nu) + e^r v(\nu)
\end{aligned}$$

hold. Then we have

$$E_Q\left[\int_0^{T_\nu} \varphi_s \int_{r>1} e^r v(r) \; K_s(d\eta) \; ds\right] < \infty.$$

Since $Q[\; T_\nu < 1 \;] \longrightarrow 0$, we see that

$$\int_0^1\int_{r>1} e^r v(|r|) \; K_s(d\eta)ds \; < \; \infty \quad \text{a.e.}$$

(Step 3) Note that ${\varphi_t}^{-1} = \Phi_t[h,k;Q]^{-1} = \Phi_t[-h,k^{-1};\widetilde{Q}]$. From the same reason as in Step 2, the uniform integrability of the $(\mathcal{W}_t,\widetilde{Q})-$martingale $({\varphi_t}^{-1})$ implies the existence of a function $\tilde{v}\in\mathcal{V}$ such that

$$\int_0^1\int_{-r>1} e^{-r}\ \tilde{v}(-r)\ \tilde{K}_s(d\eta)ds \ \equiv\ \int_0^1\int_{r<-1} \tilde{v}(|r|)\ K_s(d\eta)ds\ <\ \infty \quad \text{a.e.}$$

There exists $\hat{v}\in\mathcal{V}$ such that $\hat{v}(s)\le 2\ (v(s)\wedge\tilde{v}(s))$. Then we have

$$\begin{aligned}\int_0^1\int & (\sqrt{k(s,w,\eta)}-1)^2\ \hat{v}(|\log k(s,w,\eta)|)\ K_s(d\eta)ds\\ &\le \int_0^1\int (I(r>1)\ e^r + I(r<-1))\ \hat{v}(|r|)\ K_s(d\eta)ds\\ &\quad + \hat{v}(1)\int_0^1\int_{|r|\le 1}(e^{r/2}-1)^2\ K_s(d\eta)ds\ <\ \infty \quad \text{a.e.}\end{aligned}$$

□

Given a $(G_t,H_t,K_t)-$process (Y_t,Q) and a pair $(h,k)\in\mathcal{A}[\mathcal{W}]$. For $v\in\mathcal{V}$, set

$$\begin{aligned}\Psi_t[h,k,v] \quad &= \int_0^t (h\cdot G_s h)(s,w)\ ds\\ &\quad + \int_0^t\int ((\sqrt{k}-1)^2 v(|\log k|))(s,w,\eta)\ K_s(d\eta)\ ds. \qquad (2.1)\end{aligned}$$

Theorem 2.5. *If there exists $v\in\mathcal{V}$ such that $\Psi_1[h,k,v]\equiv\Psi_t[h,k,v]\ |_{t=1}$ is bounded, then the process $(\ \Phi_t[h,k;Q]\ )_{0\le t\le 1}$ is a $(\mathcal{W}_t,Q)-$martingale.*

Proof. Set $\varphi_t = \Phi_t[h,k;Q]$, $L_t = \log\varphi_t$. Since (φ_t) is a local martingale, it suffices to prove the uniform integrability of the local martingale. Choose an even function $u\in C^2(\boldsymbol{R})$ such that $u(r)=u(-r)\ge v(|r|)$, and

$$u(r)\equiv v(r)\ \ (\text{for}\ \ r\ge 1);\qquad 0\le u'(r)\le 1,\ |u''(r)|\le 1\ \ (\text{for}\ \ 0<r<1).$$

Set $r(t,w,\eta):=\log k(t,w,\eta)$ and

$$\hat{h}(t,w):=h(t,w)\ \frac{u(L_t)+u'(L_t)}{u(L_t)},\quad \hat{k}(t,w,\eta):=k(t,w,\eta)\ \frac{u(L_t+r(t,w,\eta))}{u(L_t)}.$$

Using the Itô formula, we have

$$\begin{aligned}&\varphi_t\ u(L_t)\ /\ \Phi_t[\hat{h},\hat{k};Q]\\ &=\exp\Bigg[\ \frac{1}{2}\int_0^t \frac{u'(L_s)+u''(L_s)}{u(L_s)}\ h\cdot G_s h\ ds\\ &\qquad + \int_0^t\int\left(e^r\ \frac{u(L_s+r)-u(L_s)}{u(L_s)} - (e^r-1)\ \frac{u'(L_s)}{u(L_s)}\right)K_s(d\eta)ds\ \Bigg].\end{aligned}$$

Since $u(L_s+r)-u(L_s)\le v(|r|+1)-v(0)\le v(|r|)$, we have

$$e^r(u(L_s+r)-u(L_s))-(e^r-1)u'(L_s)\ \le\ \begin{cases} e\ r^2 & (|r|\le 1)\\ e^r\ (v(|r|)+1) & (r>1)\\ v(|r|)+1 & (r<-1)\end{cases}.$$

Then there is a constant c_1 such that

$$\int_0^t\int \left(e^r\ \frac{u(L_s+r)-u(L_s)}{u(L_s)}\ -\ (e^r-1)\ \frac{u'(L_s)}{u(L_s)}\right) K_s(d\eta)ds$$
$$\leq\ c_1\ \int_0^1\int\ (e^{r/2}-1)^2\ v(|r|)\ K_s(d\eta)ds.$$

On the other hand

$$\frac{1}{2}\int_0^t\ \frac{u'(L_s)+u''(L_s)}{u(L_s)}\ h\cdot G_s h\ ds\ \leq\ \int_0^1\ h\cdot G_s h\ ds.$$

Then there is a constant c_0 such that $\varphi_t\ u(L_t)\ \leq\ c_0\ \Phi_t[\hat{h},\hat{k};Q]$. Since $(\Phi_t[\hat{h},\hat{k};Q])$ is a positive local martingale, $E_Q[\ \Phi_t[\hat{h},\hat{k};Q]\ ]\leq 1$. Then

$$\sup_t\ E_Q[\varphi_t\ v(|\log\varphi_t|)\ ]\ \leq\ \sup_t\ E_Q[\varphi_t\ u(L_t)]\ \leq\ c_0,$$

which implies that (φ_t) is uniformly integrable. □

Corollary 2.6. *Assume* [**A1**] *for the triplet* $(G_t,H_t,K_t(d\eta))$ *and let* $H_t^{h,k}$, K_t^k *be defined by* (R) *just before Theorem 2.4. If the random variable* $\Psi_1[h,k,v]$ *is bounded, then there exists a* $(G_t,H_t^{h,k},K_t^k)-$*process* (Y_t,P)*. And the probability* P *is uniquely determined.*

Proof. Let $\Phi_1[h,k;Q]\ :=\ \Phi_t[h,k;Q]|_{t=1}$ and set $P(dw)\ =\ \Phi_1[h,k;Q]\ Q(dw)$, which becomes a probability by Theorem 2.5. From the Itô formula, it is immediate to show that (Y_t,P) is a $(G_t,H_t^{h,k},K_t^k)-$process. Similarly if (Y_t,P') is any $(G_t,H_t^{h,k},K_t^k)-$process, and $Q'(dw):=\Phi_1[-h,k^{-1};P']\ P'(dw)$. Then Q' is a probability and (Y_t,Q') is a $(G_t,H_t,K_t)-$process. From [**A1**], we have $Q=Q'$. Since $\Phi_1[-h,k^{-1};P']=\Phi_1[h,k;Q]^{-1}$, we see $P=P'$. □

Remark 2.7. The existence of $(G_t,H_t,K_t)-$process is premised in [**A1**]. The usual method to construct $(G_t,H_t,K_t)-$process is the Skorokhod method under the condition that $(G_t(w),H_t(w),K_t(w,d\eta))$ are continuous in $w\in W$. But in the above corollary, to be continuous is not necessary for functions $h(t,w)$ and $k(t,w,\eta)$ in order that the existence and the uniqueness of solutions to the martingale problem for the triplet $(G_t,H_t^{h,k},K_t^k(d\eta))$ holds.

Example 2.8. We shall give an example of Girsanov transform used in the Malliavin calcurus for SDE with jumps (cf. [12]). Consider the Lévy process $\{\Omega,(\mathcal{F}_t),P;\ X=(X_t)\}$ of the form

$$X_t\ =\ X_0+\beta_X(t)+\int_0^t\int\theta\ J_X(dsd\theta),$$

where $\beta_X(t)=(\beta_X^1(t),\cdots,\beta_X^m(t))$ is an $m-$dimensional Brownian motion, $J_X\equiv J_X(dtd\theta)$ is a Poisson random measure with $E_P[J_X(dtd\theta)]\ =\ \pi(d\theta)dt$, where $\pi(d\theta)=|\theta|^{-m-\alpha}d\theta$ ($\theta\in\boldsymbol{R}^m$, $0<\alpha<2$). Let $\xi_t=\phi_t^s(X;x^{\Delta})$ be the path-wise solution to the $d-$dimensional SDE:

$$d\xi_t\ =\ a_0(\xi_t)\ dt+\sum_{k=1}^m a_k(\xi_t)\circ d\beta_X^k(t)+\int b(\xi_t,\theta)J_X(dtd\theta),\quad \xi_s=x^{\Delta},$$

where $a_0(x), \cdots, a_m(x)$ and $b(x,\theta)$ are $\boldsymbol{R}^d-$valued smooth functions of $(x,\theta) \in \boldsymbol{R}^d \times \boldsymbol{R}^m$ and $b(x,0) = 0$. The system $(\phi_t^s(X;\cdot))$ gives a stochastic flow on $\boldsymbol{R}^d$ (cf. [5]). Set $\mathcal{F}_t^X = \sigma(X_s; s \le t+0)$. Let $l_1(t,X), \cdots, l_m(t,X)$ be $\boldsymbol{R}^d-$valued bounded processes adapted to $(\mathcal{F}_t^X)$ and $h_\theta \equiv h_\theta(t,X)$ be $C^\infty(\boldsymbol{R}^m \to \boldsymbol{R}^d)-$valued bounded process adapted to $(\mathcal{F}_t^X)$ with $h_0(t,X) = 0$. For $z \in \boldsymbol{R}^d$, Set $\Gamma^z(t,X) = (l_1(t,X)\cdot z, \cdots, l_m(t,X)\cdot z)^*$. (For a vector v, Transpose$[v]$ is denoted by v^*.) Consider the perturbed process $X^z = (X_t^z)$ defined by

$$X_t^z \ := \ X_0 + \beta_X(t) + \int_0^t \Gamma^z(s,X)\ ds + \int_0^t\!\!\int \exp[h_\theta(s,X)\cdot z]\ \theta\ J_X(dsd\theta).$$

Let $P^z = P \circ (X^z)^{-1}$ be the law of the perturbed process X^z. Then P^0 and P^z are mutually absolutely continuous on $\mathcal{F}_t^X$, and

$$\begin{aligned} M_t^z \ &:= \ E_P\Big[dP^z/dP^0 \mid \mathcal{F}_t\Big] \ = \ E_P\Big[dP^z/dP^0 \mid \mathcal{F}_t^X\Big] \\ &= \ \exp\Big[-\sum_k \int_0^t \Big\{(l_k\cdot z)\ d\beta_X^k(s) + (l_k\cdot z)^2/2\ ds\Big\} \\ &\qquad + \int_0^t\!\!\int \Big\{\log\ k_\theta^z\ J_X(dsd\theta) - (k_\theta^z - 1)\ \pi(d\theta)ds\Big\}\Big], \\ l_k \ &= l_k(s,X), \quad k_\theta^z \ = \ \exp[-\alpha\ h_\theta(s,X)\cdot z]\ \det(I + \theta\ \partial_\theta(h_\theta(s,X)\cdot z)). \end{aligned}$$

Set $\xi_t^z := \phi_t^0(X^z; x^\Delta)$, $\tilde{a}_0(x) \ := \ a_0(x) \ + \ (1/2)\ \sum_{k=1}^m \ a_k'(x)a_k(x)$. We see that

$$d\xi_t^z = \tilde{a}_0(\xi_t^z)\ dt + \sum_k a_k(\xi_t^z)(d\beta_X^k(t) + l_k\cdot z\ dt) + \int b(\xi_t^z, \exp[h_\theta\cdot z]\ \theta)\ J_X.$$

For any bounded process $V_t(X)$ and $f(x) \in C_0^\infty(\boldsymbol{R}^d)$, we have

$$\partial_z\ E_P[\ f(\xi_t^z)\ M_t^z\ V_t(X^z)\ ] \ = \ \partial_z\ E_P[\ f(\xi_t^0)\ V_t(X)\ ] \ = 0.$$

These facts lead to the integration by parts formula in the Malliavin calculus.

3. Girsanov Transform for SDE With Linear Growth Coefficients

Notations for $(G_t, H_t, K_t(d\eta))$, $\mathcal{A}[\mathcal{W}]$, $\mathcal{V}$, $H_t^{h,k}$, K_t^k and $\Psi_t[h,k,v]$ are the same as in Section 2. Namely $H_t^{h,k}$, K_t^k are defined by (R), and $\Psi_t[h,k,v]$, by (2.1). We shall introduce conditions **[B1]** and **[B2]** on these elements.

[B1] There exist a function $v \in \mathcal{V}$, a sequence of stopping time T_ν and probability measure P_ν on $(W, \mathcal{W}_{T_\nu})$ such that

(a) $\Psi_1[h,k,v] \ < \ \infty$ a.e. (Q),

(b) $T_1 \le T_2 \le T_3 \le \cdots$ and $\lim_{\nu\to\infty} Q[T_\nu < 1] = 0$,

(c) (Y_t, P_ν) is a $(G_t, H_t^{h,k}, K_t^k)-$process with restricted time interval $[0, T_\nu]$ and

$$\lim_{\ell\to\infty}\ \sup_\nu\ P_\nu[\ \Psi_{T_\nu}[h,k,v] > \ell\ ] \ = \ 0.$$

Note that **[B1]** does not require the existence or the uniqueness of solutions to the martingale problem for the triplet $(G_t, H_t^{h,k}, K_t^k(d\eta))$ with time interval $[0,1]$.

[B2] Let $\|Y\|_t := \sup_{s\le t}|Y_s|$, and each of $c.$'s denote absolute constant.

(a) There are functions $v\in\mathcal{V}$ and $\rho\in C(\boldsymbol{R}_+)$ such that $\Psi_t[h,k,v] \le \rho(\|Y\|_t)$.

(b) Elements $\{G_t,\ H_t,\ K_t,\ H_t^{h,k},\ K_t^k\}$ admit following estimates:

$$\begin{aligned}
&\text{trace } G_t + \int_{|\eta|\le 1} |\eta|^2\ (K_t(d\eta)+K_t^k(d\eta)) \ \le\ c.\ (1+\|Y\|_t)^2,\\
&Y_t\cdot H_t \ \le\ c.\ (1+\|Y\|_t)^2, \qquad Y_t\cdot H_t^{h,k} \ \le\ c.\ (1+\|Y\|_t)^2,\\
&\int_{|\eta|>1}(K_t(d\eta)+K_t^k(d\eta)) \ \le\ c.
\end{aligned}$$

Lemma 3.1. *If* **[A1]** *and* **[B2]** *are satisfied, then* **[B1]** *is also satisfied.*

Proof. Let (Y_t,Q) be a $(G_t,H_t,K_t)-$process. For $N>1$, set

$$\begin{aligned}
T^N &:= \inf\{\ t\le 1 \mid |\Delta Y_t|>N\}, \quad H_t^{(N)} \ :=\ H_t + \int_{1<|\eta|\le N}\eta\ K_t(d\eta),\\
Y_t^{(N)} &:= \beta_Y(t) + \int_0^t H_s^{(N)}\ ds + \int_0^t\!\!\int_{|\eta|\le N}\eta\ \widetilde{J}(dsd\eta).
\end{aligned}$$

We see that $Y_t^{(N)} = Y_t\ \ (t<T^N)$ and

$$\left(\int_{|\eta|\le 1} + \int_{1<|\eta|\le N}\right)|\eta|^2\ K_t(d\eta)\ \le\ c.\ (1+\|Y\|_t)^2 + c.\ N^2.$$

Using the Itô formula we have, for $t\le T^N$,

$$\begin{aligned}
|Y_t^{(N)}|^2 \ &=\ \int_0^t\left[2\ Y_s\cdot H_s^{(N)} + \text{trace } G_s + \int_{|\eta|\le N}|\eta|^2\ K_s(d\eta)\right]\ ds\\
&\quad + \int_0^t 2\ Y_s\cdot d\beta_Y(s) + \int_0^t\!\!\int_{|\eta|\le N}(2Y_s^{(N)}\cdot\eta + |\eta|^2)\ \widetilde{J}_Y(dsd\eta)\\
&\le\ \int_0^t c.\ (N+\|Y\|_s)^2\ ds\\
&\quad + \sup_{\tau\le t}\left|\int_0^\tau 2\ Y_s\cdot d\beta_Y(s) + \int_0^\tau\!\!\int(2Y_s^{(N)}\cdot\eta + |\eta|^2)\ \widetilde{J}_Y(dsd\eta)\right|.
\end{aligned}$$

From the martingale inequality, for $S := t\wedge T^N$, we have

$$\begin{aligned}
E_Q[(1+\|Y^{(N)}\|_{t\wedge T^N})^4] \ &\le\ c. + c.\ E_Q\left[\left(\int_0^S (1+\|Y\|_s)^2\ ds\ \right)^2\right]\\
&\quad + c.\ E_Q\left[\int_0^S\left(4\ Y_s\cdot G_sY_s + \int(2Y_s\cdot\eta + |\eta|^2)^2\ K_s(d\eta)\ \right)ds\right]\\
&\le\ c.\int_0^t E_Q[(1+\|Y^{(N)}\|_{s\wedge T^N})^4]\ ds.
\end{aligned}$$

This inequality implies that $E_Q[(1+\|Y^{(N)}\|_{T^N})^4] < \infty$. Then we have

$$E_Q[\sup_{t<T^N} |Y_t|^4\ ] \ \le\ E_Q[\sup_{t\le T^N} |Y_t^{(N)}|^4\ ]\ < c.$$

From **[B2]**(b) we have $Q[\ T^N < 1\ ] \to 0$ as $N \to \infty$. These facts and **[B2]**(a) imply that **[B1]**(a) holds good. Define

$$T_\nu = \inf\{t \in [0,1] \mid \ \Psi_t[h,k,v] > \nu\ \}, \quad P_\nu(dw) = \Phi_{T_\nu}[h,k;Q]\ Q(dw).$$

Since $Q[T_\nu < 1] \le Q[\Psi_{T_\nu}[h,k,v] \ge \nu] \le Q[\rho(\|Y\|_1) \ge \nu]$, **[B1]**(b) holds good. From Corollary 2.6, each (Y_t, P_ν) is a $(G_t, H_t^{h,k}, K_t^k)-$process with restricted time interval $[0, T_\nu]$, and $P_{\nu+1} = P_\nu$ on $\mathcal{W}_{T_\nu}$. By a similar consideration to the above, it can be proved that

$$\sup_\nu E_{P_\nu}[\sup_{t<T_\nu\wedge T^N} |Y_t|^4\ ] \le c, \quad \sup_\nu P_\nu[\ T^N < T_\nu\ ] \longrightarrow 0 \ \ (\text{as } N\to\infty).$$

From **[B2]**(a), we see that **[B1]**(c) holds good. □

Theorem 3.2. *Assume that $(h,k) \in \mathcal{A}[\mathcal{W}]$ satisfies* **[B1]** *or* **[B2]**. *Then there exists uniquely a $(G_t, H_t^{h,k}, K_t^k)-$process $\{W, \mathcal{W}_t, P; Y_t\}$. Probabilities P and Q are mutually absolutely continuous and*

$$[dP/dQ](w) \ = \ \Phi_1[h,k;Q] \ = \ \Phi_1[-h,k^{-1};P]^{-1}\ .$$

Proof. (Step 1) We shall denote $H_t^{h,k}$ and K_t^k by $\widetilde{H}_t$ and $\widetilde{K}_t$. Let (T_ν) and (Y_t, P_ν) are the same ones in **[B1]**. Since $Q[\ T_\nu < 1\ ] \to 0$ as $\nu \to \infty$, we have $\Phi_{T_\nu}[h,k;Q] \longrightarrow \Phi_1[h,k;Q]$ a.e. Q. If the set $\{\Phi_{T_\nu}[h,k;Q]\}_\nu$ is uniformly integrable with respect to Q, then $P(dw) := \Phi_1[h,k;Q]\ Q(dw)$ is a probability and (Y_t, P) is a $(G_t, \widetilde{H}_t, \widetilde{K}_t)-$process.
Set $d\beta_Y^\nu(s) = d\beta_Y(s) - G_s h\ ds$, $\ \widetilde{J}_Y^\nu(dsd\eta) = J_Y(dsd\eta) - \widetilde{K}_s(d\eta)\ ds$. We have

$$\begin{aligned}
&\int I(\Phi_{T_\nu}[h,k;Q] > e^{4\ell})\ \Phi_{T_\nu}[h,k;Q]\ Q(dw) \\
&\quad = \ P_\nu[\ \log\ \Phi_{T_\nu}[h,k;Q] > 4\ell\ ] \\
&\quad \le \ P_\nu\left[\int_0^{T_\nu} h\cdot d\beta_Y^\nu(s) \ + \ \frac{1}{2}\int_0^{T_\nu} h\cdot G_s h\ ds\ > \ell\right] \\
&\qquad + P_\nu\left[\int_0^{T_\nu}\int_{|r|\le 1} r\ \widetilde{J}_Y^\nu \ + \ \int_0^{T_\nu}\int_{|r|\le 1} (e^{-r}-1+r)\ \widetilde{K}_s\ ds\ > \ell\right] \\
&\qquad + P_\nu\left[\int_0^{T_\nu}\int_{r<-1} r\ J_Y \ + \ \int_0^{T_\nu}\int_{r<-1} (e^{-r}-1)\ \widetilde{K}_s\ ds\ > \ell\right] \\
&\qquad + P_\nu\left[\int_0^{T_\nu}\int_{r>1} r\ J_Y \ + \ \int_0^{T_\nu}\int_{r>1} (e^{-r}-1)\ \widetilde{K}_s\ ds\ > \ell\right] \\
&\quad =: \ I_1^{(\nu)} + I_2^{(\nu)} + I_3^{(\nu)} + I_4^{(\nu)},
\end{aligned}$$

where $r = \log k(s,w,\eta)$, $\widetilde{J}_Y^\nu = \widetilde{J}_Y^\nu(dsd\eta)$, $J_Y = J_Y(dsd\eta)$ and $\widetilde{K}_s = \widetilde{K}_s(d\eta)$. We see that

$$\begin{aligned}
I_1^{(\nu)} &\le P_\nu\left[\int_0^{T_\nu} h\cdot G_s h\ ds\ > \ell\right] \\
&\qquad + P_\nu\left[\int_0^{T_\nu} h\cdot d\beta_Y^\nu(s)\ > \frac{\ell}{2},\ \int_0^{T_\nu} h\cdot G_s h\ ds\ \le \ell\right] \\
&\le P_\nu\left[\int_0^{T_\nu} h\cdot G_s h\ ds\ > \ell\right] + \frac{16}{\ell}, \\
I_2^{(\nu)} &\le P_\nu\left[\int_0^{T_\nu}\int_{|r|\le 1} r\ \widetilde{J}_Y^\nu\ +\ \int_0^{T_\nu}\int_{|r|\le 1} r^2\ \widetilde{K}_s\ ds\ > \ell\right] \\
&\le P_\nu\left[\int_0^{T_\nu}\int_{|r|\le 1} r^2\ \widetilde{K}_s\ ds\ > \frac{\ell}{2}\right] \\
&\qquad + P_\nu\left[\int_0^{T_\nu}\int_{|r|\le 1} r\ \widetilde{J}_Y^\nu\ > \frac{\ell}{2},\ \int_0^{T_\nu}\int_{|r|\le 1} r^2\ \widetilde{K}_s\ ds\ \le \frac{\ell}{2}\right] \\
&\le P_\nu\left[\int_0^{T_\nu}\int_{|r|\le 1} r^2\ K_s\ ds\ > \frac{\ell}{2e}\right] + \exp(-\ell/30), \\
I_3^{(\nu)} &\le P_\nu\left[\int_0^{T_\nu}\int_{r<-1}(e^{-r}-1)\widetilde{K}_s ds > \ell\right]\ \le\ P_\nu\left[\int_0^{T_\nu}\int_{r<-1} K_s ds\ > \ell\right].
\end{aligned}$$

For a moment, set

$$Z_t := \int_0^t\int_{r>1} r\ J_Y(dsd\eta).$$

Since

$$\log v(Z_t)\ =\ \int_0^t\int_{r>1}(\log v(Z_{s-}+r) - \log v(Z_{s-}))\ J_Y\ \le\ \int_0^t\int_{r>1}\log v(r)\ J_Y,$$

$$\begin{aligned}
I_4^{(\nu)} &\le P_\nu[\ Z_{T_\nu} > \ell\ ]\ =\ P_\nu[\ \log v(Z_{T_\nu}) > \log v(\ell)\ ] \\
&\le P_\nu\left[\int_0^{T_\nu}\int_{r>1}\log v(r)\ J_Y > \log v(\ell)\right] \\
&\le P_\nu\left[\int_0^{T_\nu}\int_{r>1}(v(r)-1)\ \widetilde{K}_s\ ds > \log\sqrt{v(\ell)}\right] \\
&\qquad + P_\nu\left[\int_0^{T_\nu}\int_{r>1}\{\log v(r)\ J_Y - (v(r)-1)\ \widetilde{K}_s\ ds\}\ >\ \log\sqrt{v(\ell)}\right] \\
&\le P_\nu\left[\int_0^{T_\nu}\int_{r>1} v(r)e^r\ K_s\ ds > \log\sqrt{v(\ell)}\right] + \frac{1}{\sqrt{v(\ell)}}\ .
\end{aligned}$$

From **[B1]** (c), we have

$$\lim_{\ell\to\infty} \sup_{\nu} \int I(\Phi_{T_\nu}[h,k;Q] > e^{4\ell})\ \Phi_{T_\nu}[h,k;Q]\ Q(dw) \ = \ 0.$$

Therefore $\{\Phi_{T_\nu}[h,k;Q]\}_\nu$ is uniformly integrable with respect to Q.

(Step 2) Let (Y_t, P') be any $(G_t, \widetilde{H}_t, \widetilde{K}_t)-$process. Since

$$\int_0^{T_\nu} \left\{ (-h)\cdot G_s(-h) + \int ((\sqrt{k^{-1}}-1)^2 v(|\log k^{-1}|)\ \widetilde{K}_s \right\}\ ds \ = \ \Phi_{T_\nu}[h,k,v] \ \leq \ n,$$

from Theorem 2.5, we see that $(\Phi_{t\wedge T_\nu}[-h,k^{-1},P'])$ is a $(\mathcal{W}_t, P')-$martingale. From Corollary 2.6, we see that $\Phi_{T_\nu}[-h,k^{-1},P']\ P'(dw) = Q(dw)$ on $\mathcal{W}_{T_\nu}$. Since

$$\Phi_{T_\nu}[-h,k^{-1},P'] \ = \ \Phi_{T_\nu}[h,k,Q]^{-1},$$

we have $P' = P_\nu$ on $\mathcal{W}_{T_\nu}$. **[B1]** (c) also implies that $P'[\ T_\nu < 1\ ] \longrightarrow 0$ as $\nu \to \infty$. Therefore $P' = P$. Then the proof is complete. □

Example 3.3. Let $(W, (\mathcal{W}_t), Q; Y_t)$ be a 1-dimensional Brownian motion with $Y_0 = 0$, that is, (Y_t, Q) is $(1,0,0)-$process. Set

$$M_t^{(\lambda)} \ := \ \exp\left[\ \lambda\int_0^t Y_s\ dY_s \ - \ \frac{\lambda^2}{2}\int_0^t {Y_s}^2\ ds\ \right] \qquad (t \leq 1).$$

From Theorem 3.2, for any $\lambda \in \boldsymbol{R}$, the local martingale $(M_t^{(\lambda)}, Q)$ is uniformly integrable, and the probability measure $Q^{(\lambda)} = M_1^{(\lambda)} Q$ on $(W, \mathcal{W}_1)$ gives a solution to the martingale problem for the triplet $(1, \lambda Y_t, 0)$. For a moment, set

$$\varphi(\lambda) := E_Q\left[\ \exp\left(\frac{\lambda^2}{2}\int_0^1 Y_s^2\ ds\right)\ \right].$$

Using the complex analysis, we shall prove the formula:

$$\varphi(\lambda) \ = \ 1/\sqrt{\cos(\lambda)} \qquad (0 < \lambda < \pi/2).$$

Let $D = \{u + \sqrt{-1}v \mid u^2 < v^2 + (\pi/2)^2\}$ and

$$\log[f(\zeta,t,x)] = \frac{\zeta}{2}\tan(\zeta(1-t))\ x^2 - \frac{1}{2}\log[\cos(\zeta(1-t))] \qquad (\zeta \in D).$$

Set $T_\nu = \inf\{t \leq 1 \mid |Y_t| > \nu\}$. Since $[\partial_t + (1/2)\partial_x^2 + (1/2)(\zeta x)^2] f(\zeta,t,x) = 0$, using the Itô formula, we have

$$E_Q\left[f(\zeta, T_\nu, Y_{T_\nu})\ \exp\left(\frac{\zeta^2}{2}\int_0^{T_\nu} Y_s^2\ ds\right)\right] \ = \ f(\zeta,0,0).$$

From this equality and that $f(\zeta,1,x) = 1$, using the Fatou lemma, the inequality

$$E_Q\left[\exp\left(\frac{\lambda^2}{2}\int_0^1 Y_s^2\ ds\right)\right] \ \leq \ f(\lambda,0,0) = \frac{1}{\sqrt{\cos(\lambda)}} < \infty \qquad (0 < \lambda < \pi/2)$$

is obtained by letting $\nu \to \infty$. Then the following function is analytic.

$$D \ni \zeta \quad \longrightarrow \quad \varphi(\zeta) := E_Q\left[\exp\left(\frac{\zeta^2}{2}\int_0^1 Y_s^2\ ds\right)\ \right]$$

For any $v \in \boldsymbol{R}$, the function

$$f(\sqrt{-1}v, t, x) \;=\; \frac{1}{\sqrt{\cosh(v(1-t))}}\ \exp\Big(-\frac{v}{2}\tanh(v(1-t))\ x^2\Big)$$

is bounded and smooth in (t, x). Applying the Itô formula, we have

$$\begin{aligned}&\varphi(\sqrt{-1}v) \;-\; f(\sqrt{-1}v, 0, 0)\\ &= E_Q\left[\int_0^1 (\partial_x f)(\sqrt{-1}v, t, Y_t)\ \exp\left(-\frac{v^2}{2}\int_0^t Y_s^2\ ds\right)\ dY_t\right] \;=\; 0.\end{aligned}$$

Then $\varphi(\zeta_0) = f(\zeta_0, 0, 0) = 1/\sqrt{\cos(\zeta_0)}$ for pure imaginary numbers $\zeta_0 = \sqrt{-1}v$. Since the function $D \ni \zeta \longrightarrow 1/\sqrt{\cos(\zeta)}$ is also analytic, from theorem of identity, we see that $\varphi(\zeta) = 1/\sqrt{\cos(\zeta)},\ (\zeta \in D)$.

Remark that, for $\lambda \geq \pi/2$,

$$\varphi(\lambda) \;\geq\; \lim_{u\uparrow\pi/2}\varphi(u) \;=\; \lim_{u\uparrow\pi/2}\frac{1}{\sqrt{\cos(u)}} \;=\; \infty.$$

Then the Novikov condition about the uniform integrability of positive local martingales is not satisfied for local martingales $(M_t^{(\lambda)}),\ \lambda \geq \pi/2$. This shows that the Novikov condition is far from critical conditions.

4. Zakai SDE and Nonlinear Filtering

In this section, we shall study linear functional jump-SDE's arising from nonlinear filtering equations for filtering systems of jump-processes. Notations for $(W, \mathcal{W}_t),\ (G_t, H_t, K_t),\ (Y_t, \beta_Y, J_Y, \widetilde{J_Y})$ and Q are the same as in Section 2.

Let $a_j(t, Y, \cdot) \in C^1(\boldsymbol{R}^d \to \boldsymbol{R}^d)$ and $b(t, Y, \cdot, \cdot) \in C(\boldsymbol{R}^d \times \boldsymbol{R}^m \to \boldsymbol{R}^d)$ be processes adapted to $(\mathcal{W}_t)$. We shall define several operators on $C_0^\infty(\boldsymbol{R}^d)$.

$$\begin{aligned}A_{t,j}^Y \;&=\; a_j(t, Y, x)\cdot\partial_x, \qquad A_t^Y \;=\; (A_{t,j}^Y)_{1\leq j\leq m},\\ B_{t,\theta}^Y f(x) \;&=\; f(x + b(t, Y, x, \theta)) \quad (f \in C_0^\infty(\boldsymbol{R}^d)),\\ L_t^Y &= A_{t,0}^Y + \frac{1}{2}\ A_t^Y \cdot A_t^Y + \int (B_{t,\theta}^Y - I)\frac{d\theta}{|\theta|^{m+\alpha}}.\end{aligned}$$

For given function valued process $h_t^Y(\cdot) \in C(\boldsymbol{R}^d \to \boldsymbol{R}^n)$ and $k_t^Y(\cdot,\cdot) \in C(\boldsymbol{R}^d \times \boldsymbol{R}^n \to \boldsymbol{R}_+)$ adapted to $(\mathcal{W}_t)$, let us consider the linear functional jump-SDE (which shall be called "the Zakai SDE") for measure valued processes $\mu_t(dx)$:

$$\begin{aligned}d(\mu_t[f]) \;&=\; \mu_t[L_t^Y f]\ dt + \mu_t[h_t^Y f]\cdot d\beta_Y(t)\\ &\quad + \int \mu_t[(k_t^Y(\cdot,\eta) - 1)f]\ \widetilde{J}_Y(dtd\eta)\end{aligned} \tag{4.1}$$

for any $f \in C_0^\infty(\boldsymbol{R}^d)$, where $\mu_t[\ \cdot\ ]$ denotes the integral operator by $\mu_t(dx)$:

$$\mu_t[g] \;:=\; \int g(x)\ \mu_t(dx)\ \left(\equiv \int_{\boldsymbol{R}^d} g(x)\ \mu_t(dx)\right).$$

A measure valued process $\mu_t(\cdot) \equiv \mu_t(dx)$ is said to be a solution to Eq. (4.1) if $\mu_t[\ 1\ ]$ is a locally bounded process (i.e. there exists a sequence $\{T_\nu\}$ of $(\mathcal{W}_t)-$stopping times T_ν such that $Q[T_\nu < 1] \downarrow 0$ as $\nu \to \infty$ and $\mu_{t\wedge T_\nu}[\ 1\ ]$ is bounded for each ν), and the functional $\mu_t[\ \cdot\ ]$ satisfies Eq. (4.1).

Consider the space $\Omega = D([0,1] \to \boldsymbol{R}^{m+n})$ of càd-làg functions $\omega = ((X_t, Y_t))$ with $X_t(\omega) \in \boldsymbol{R}^m$, $X_0(\omega) = 0$, and $Y_t(\omega) \in \boldsymbol{R}^n$, $Y_0(\omega) = 0$. Define filtrations $\mathcal{F}_t^X = \sigma(X_s; s \le t+0)$, $\mathcal{F}_t^Y = \sigma(Y_s; s \le t+0)$ and $\mathcal{F}_t = \mathcal{F}_t^X \vee \mathcal{F}_t^Y$. Let $\widetilde{Q}$ be a probability on $(\Omega, \mathcal{F}_1)$ defined by following properties (a),(b) and (c).

(a) processes $(X_t, \widetilde{Q})$ and $(Y_t, \widetilde{Q})$ are independent.

(b) the process $\{W, \mathcal{W}_t, Q; Y_t\}$ is equivalent to the process $\{\Omega, \mathcal{F}_t^Y, \widetilde{Q}; Y_t\}$.

(c) $(X_t, \widetilde{Q})$ is the Lévy process with

$$E_{\widetilde{Q}}[\ \exp(i\xi \cdot X_t)\ ] \ = \ \exp(-\frac{1}{2}|\xi|^2 - c_\alpha |\xi|^\alpha)$$

where $0 < \alpha < 2$ and c_α is a certain positive constant.

We see that $X_t = \beta_X(t) + \alpha_X(t)$ with the Wiener process $(\beta_X(t), \widetilde{Q})$ and a symmetric $\alpha-$stable process $(\alpha_X(t), \widetilde{Q})$. We may suppose that the counting measure of jumps $J_X(dsd\theta) := \#\{\tau \in ds \mid 0 \ne \Delta X_\tau \in d\theta\}$ is a Poisson radom measure with $E_{\widetilde{Q}}[J_X(dsd\theta)] \ = \ |\theta|^{-m-\alpha}\ d\theta ds$, and that the signed measure

$$\widetilde{J}_X(dsd\theta) \ := \ J_X(dsd\theta) - |\theta|^{-m-\alpha}\ d\theta ds$$

generates discontinuous $(\mathcal{F}_t^X \vee \mathcal{F}_1^Y, \widetilde{Q})-$local martingales. Let $\xi_t = \xi_t(\omega)$ be the pathwise solution to the $d-$dimensional stochastic integro-differential equation:

$$\begin{aligned} d\xi_t \ &= \ a_0(t, Y, \xi_t)\ dt + \sum_{j=1}^m a_j(t, Y, \xi_t) \circ d\beta_X^j(t) \\ &\quad + \int b(t, Y, \xi_t, \theta)\ J_X(dtd\theta). \end{aligned} \tag{4.2}$$

We see from the Itô formula that processes

$$M_t^f \ := \ f(\xi_t) \ - \ f(\xi_0) \ - \int_0^t [L_s^Y f](\xi_s)\ ds \quad (f \in C_0^\infty)$$

are $(\mathcal{F}_t, \widetilde{Q})-$martingales. Fix $v \in \mathcal{V}$ and set

$$\begin{aligned} \rho(r) \ &:= \ (\sqrt{r} - 1)^2\ v(|\log r|), \\ \Psi[h,k] \ &:= \ \int_0^1 \{\ h_s^Y(\xi_s) \cdot G_s h_s^Y(\xi_s) + \int \rho(k_s^Y(\xi_s, \eta))\ K_s(d\eta)\}\ ds. \end{aligned}$$

Let $\widetilde{Q}[\ \Psi[h,k] < \infty\ ] = 1$ and introduce a positive $(\mathcal{F}_t, \widetilde{Q})-$ local martingale

$$\begin{aligned} \varphi_t[h,k] \ = \ \exp\Bigg[&\int_0^t h_s^Y(\xi_s) \cdot d\beta_Y(s) \ - \ \frac{1}{2}\int_0^t h_s^Y(\xi_s) \cdot G_s h_s^Y(\xi_s)\ ds \\ &+ \int_0^t\!\!\int \{\ \log k_s^Y(\xi_s, \eta)\ J_Y(dsd\eta) \ - \ (k_s^Y(\xi_s, \eta) - 1)\ K_s(d\eta)\ ds\ \} \Bigg]. \end{aligned}$$

Then we have the following lemma (see [3]).

Lemma 4.1. *Assume that* $\widetilde{Q}[\ \Psi[h,k] < \infty\ ] = 1$. *Then the measure valued process* $\mu_t(dx) := E_{\widetilde{Q}}[\ I(\xi_t \in dx)\ \varphi_t[h,k]\ \ |\ \mathcal{F}_t^Y]$ *gives a solution to the Zakai SDE* (4.1).

We shall give the martingale formulation for the filtering system of stochastic processes. Let $\beta_Y(t)$, $J_Y(dtd\eta)$ and $\xi_t(\omega)$ be the same objects as above, and define new objects $\beta_Y^*(t)$ and $J_Y^*(dsd\eta)$ by

$$\begin{aligned} \beta_Y^*(t) &:= \beta_Y(t) - \int_0^t G_s h_s^Y(\xi_s)\ ds, \\ J_Y^*(dsd\eta) &:= J_Y(dsd\eta) - k_s^Y(\xi_s, \eta) K_s(d\eta) ds. \end{aligned}$$

Let P be a probability measure on $(\Omega, \mathcal{F}_1)$ such that

(1) $P = \widetilde{Q}$ on $\mathcal{F}_1^X$,

(2) $(\beta_Y^*(t))$ is a continuous $(\mathcal{F}_t, P)-$local martingale,

(3) $J_Y^*(\cdot)$ generates discontinuous $(\mathcal{F}_t, P)-$local martingales.

In the filtering theory, the process (ξ_t, P) is called signal process, and the process (Y_t, P), observation process (cf. [13]). Set

$$[H_s^Y]^*(x) := H_s + G_s h_s^Y(x) + \int_{|\eta|\le 1} \eta\ (k_s^Y(x,\eta) - 1)\ K_s(d\eta).$$

Then we have

$$Y_t = \beta_Y^*(t) + \int_0^t [H_s^Y]^*(\xi_s)\ ds\ + \int_0^t\int_{|\eta|\le 1} \eta\ J_Y^* + \int_0^t\int_{|\eta|>1} \eta\ J_Y, \tag{4.3}$$

where $J_Y^* = J_Y^*(dsd\eta)$ and $J_Y = J_Y(dsd\eta)$. The measure valued process

$$\pi_t(dx) := P[\xi_t \in dx \mid \mathcal{F}_t^Y]$$

is called the filtering process.

Let $\Psi[h,k]$ be the same random variable as before, and introduce assumption.

$$\textbf{[A2]} \qquad \widetilde{Q}[\Psi[h,k] < \infty] = 1, \qquad P[\Psi[h,k] < \infty] = 1.$$

It can be proved by a similar way to the proof of Theorem 3.2 that if **[A1]** and **[A2]** are satisfied, then the $(\mathcal{F}_t, P)-$local martingale $((\varphi_t[h,k])^{-1})$ is uniformly integrable and $P(d\omega) = \varphi_1[h,k](\omega)\ \widetilde{Q}(d\omega)$. Moreover we have the following lemma (see [3]).

Lemma 4.2. *Assuming* **[A1]** *and* **[A2]**, *we have*

$$\pi_t[f] = \frac{E_{\widetilde{Q}}[\ f(\xi_t)\ \varphi_t[h,k] \mid \mathcal{F}_t^Y]}{E_{\widetilde{Q}}[\ \varphi_t[h,k] \mid \mathcal{F}_t^Y]} = \frac{E_{\widetilde{Q}}[\ f(\xi_t)\ \varphi_t[h,k] \mid \mathcal{F}_1^Y]}{E_{\widetilde{Q}}[\ \varphi_t[h,k] \mid \mathcal{F}_1^Y]}.$$

The above equality is called the Bayes formula (see [13]). This formula has been obtained in [7] or [8] in terms of different context from ours.

Let us consider the nonlinear functional SDE for measure valued processes $\pi_t(dx)$ adapted to $(\mathcal{W}_t)$ with $\pi_t[1] = 1$:

$$\begin{aligned} d\pi_t[f] \;&=\; \pi_t[L_t^Y f]\ dt \;+\; \big(\ \pi_t[f\ h_t^Y] - \pi_t[f]\ \pi_t[h_t^Y]\ \big)\ d\overline{\beta_Y}(t) \\ &\quad + \int \big(\ \pi_t[f\ k_t^Y(\cdot,\eta)]\ /\ \pi_t[k_t^Y(\cdot,\eta)] \;-\; \pi_t[f]\ \big)\ \overline{J_Y}(dtd\eta) \qquad (4.4) \\ d\overline{\beta_Y}(t) \;&:=\; d\beta_Y(t) - G_t\pi_t[h_t^Y]\ dt, \\ \overline{J_Y}(dtd\eta) \;&:=\; J_Y(dtd\eta) - \pi_t[k_t^Y(\cdot,\eta)]K_t(d\eta)\ dt. \end{aligned}$$

The objects $d\overline{\beta_Y}(t)$ and $\overline{J_Y}(dtd\eta)$ are called sometimes as the innovations of $d\beta_Y^*(t)$ and $J_Y^*(dtd\eta)$. We shall call Eq. (4.4) " the filtering equation", but its solutions $\pi_t(dx)$ are not necessarily filtering measures.

From the following theorem, we see that the Zakai SDE and the filtering equation are nearly equivalent.

Theorem 4.3. (1) *Let $\mu_t(dx)$ be a solution to Eq.(4.1) with $\mu_t[1] > 0$. Set*

$$\wp_t := \mu_t[1], \qquad \pi_t(dx) := {\wp_t}^{-1}\ \mu_t(dx).$$

Then the measure valued process $\pi_t(dx)$ is a solution to Eq.(4.4).

(2) *Let $\pi_t(dx)$ be a solution to Eq.(4.4). Set*

$$\overline{h}_t := \pi_t[h_t^Y], \quad \overline{k}_t(\eta) := \pi_t[k_t^Y(\cdot,\eta)], \qquad \wp_t := \Phi_t[\overline{h},\overline{k};Q] = \varphi_t[\overline{h},\overline{k}].$$

Then the measure valued process $\mu_t(dx) := \wp_t\ \pi_t(dx)$ is a solution to Eq.(4.1).

Proof. (1) Let $\pi_t(dx) := \mu_t(dx)/\mu_t[1] = {\wp_t}^{-1}\ \mu_t(dx)$ and

$$\overline{h}_t := \pi_t[h_t^Y], \qquad \overline{k}_t(\eta) := \pi_t[k_t^Y(\cdot,\eta)].$$

From (4.1) we have

$$d\wp_t \;=\; \wp_t\ \left(\overline{h}_t \cdot d\beta_Y(t) + \int (\overline{k}_t(\eta) - 1)\ \widetilde{J}_Y(dtd\eta)\right).$$

By the Itô formula, it is shown that

$$d{\wp_t}^{-1} \;=\; {\wp_t}^{-1}\left[-\overline{h}_t \cdot d\overline{\beta_Y}(t) + \int \left({\overline{k}_t(\eta)}^{-1} - 1\right)\ \overline{J_Y}(dtd\eta)\right],$$

where $d\overline{\beta_Y}(t) \;:=\; d\beta_Y(t) - G_t\overline{h}_t\ dt,\ \ \overline{J_Y}(dtd\eta) \;:=\; J_Y(dtd\eta) - \overline{k}_t(\eta)K_t(d\eta)\ dt.$ From (4.1), for any $f \in C_0^\infty(\boldsymbol{R}^d)$, we have

$$\begin{aligned} d\mu_t[f] \;&=\; \mu_t[L_t^Y f]\ dt + \mu_t[f\ h_t^Y] \cdot d\beta_Y(t) + \int \mu_t[f\ (k_t^Y - 1)]\ \widetilde{J}_Y(dtd\eta) \\ &=\; \wp_t\ \Bigg[\pi_t[L_t^Y f]\ dt + \pi_t[f\ h_t^Y] \cdot d\overline{\beta_Y}(t) + \int \pi_t[f\ (k_t^Y - 1)]\ \overline{J_Y}(dtd\eta) \\ &\qquad + \pi_t[f\ h_t^Y] \cdot G_t\overline{h}_t\ dt + \int \pi_t[f\ (k_t^Y - 1)]\ (\overline{k}_t - 1)K_t(d\eta)\ dt\Bigg]. \end{aligned}$$

By the Itô formula, we have

$$\begin{aligned} d\pi_t[f] &= \mu_t[f]\ d\wp_t{}^{-1} \ +\ \wp_t{}^{-1}\ d\mu_t[f] \ +\ [\ d\wp_t{}^{-1},\ d\mu_t[f]\ ] \\ &= \pi_t[f]\left[-\overline{h}_t \cdot d\overline{\beta_Y}(t) + \int \left(\overline{k}_t{}^{-1} - 1\right)\ \overline{J_Y}(dtd\eta)\right] \\ &\quad + \Big[\pi_t[L_t^Y f]\ dt + \pi_t[f\ h_t^Y] \cdot d\overline{\beta_Y}(t) + \int \pi_t[f\ (k_t^Y - 1)]\ \overline{J_Y}(dtd\eta) \\ &\qquad + \pi_t[f\ h_t^Y] \cdot G_t\overline{h}_t\ dt + \int \pi_t[f\ (k_t^Y - 1)]\ (\overline{k}_t - 1)K_t(d\eta)\ dt\Big] \\ &\quad - \pi_t[f\ h_t^Y] \cdot G_t\overline{h}_t\ dt + \int \pi_t[f\ (k_t^Y - 1)] \left(\overline{k}_t{}^{-1} - 1\right) J_Y(dtd\eta) \\ &= \pi_t[L_t^Y f]\ dt \ +\ \left(\ \pi_t[f\ h_t^Y] - \pi_t[f]\ \pi_t[h_t^Y]\ \right)\ d\overline{\beta_Y}(t) \\ &\quad + \int \left(\ \pi_t[f\ k_t^Y(\cdot,\eta)]\ /\ \pi_t[k_t^Y(\cdot,\eta)]\ -\ \pi_t[f]\ \right)\ \overline{J_Y}(dtd\eta). \end{aligned}$$

Then $\pi_t(dx)$ is a solution to the filtering equation (4.4).

(2) Using the Itô formula, it is a routine work to verify the equality

$$\begin{aligned} d\mu_t[f] &= \pi_t[f]\ d\wp_t \ +\ \wp_t\ d\pi_t[f] \ +\ [\ d\wp_t,\ d\pi_t[f]\ ] \\ &= \mu_t[L_t^Y f]\ dt + \mu_t[f\ h_t^Y] \cdot d\beta_Y(t) + \int \mu_t[f\ (k_t^Y(\cdot,\eta) - 1)]\ \widetilde{J}_Y(dtd\eta). \end{aligned}$$

Hence $\mu_t(dx)$ is a solution to Zakai SDE (4.1). □

Remark 4.4. (1) From Theorem 4.3, we notice that if the uniqueness of solutions to the Cauchy problem for Zakai SDE (4.1) is valid, the same holds for the filtering equation (4.4). (If $\pi_t(dx)$ and $\pi'_t(dx)$ are solutions to Eq.(4.4) with $\pi_0(dx) = \pi'_0(dx)$, then $\mu_t(dx) := \wp_t\pi_t(dx)$ and $\mu'_t(dx) := \wp'_t\pi'_t(dx)$ are solutions to Eq.(4.1). The uniqueness of solutions to Eq.(4.1) implies the equality $\wp_t\pi_t(dx) = \wp'_t\pi'_t(dx)$. Since $\wp_t = \mu_t[1] = \mu'_t[1] = \wp'_t$, we have $\pi_t(dx) = \pi'_t(dx)$.)

(2) The Bayes formula (Lemma 4.2) is useful to proceed the Malliavin calculus for processes (ξ_t, $\widetilde{Q}[\ \cdot \mid \mathcal{F}_1^Y]$). We shall give the illustration of the Malliavin calculus. In order to simplify the illustration, let us consider the case :

$$a_j(t,Y,x) = a_j(x) \in C^{\infty,b}(\boldsymbol{R}^d), \quad b(t,Y,x,\theta) = b(x,\theta) \in C^{\infty,b}(\boldsymbol{R}^d \times \boldsymbol{R}^m),$$

and assume that $\|\partial b(x,\theta)\| < c. < 1$. Let $(\phi_t^s(x))_{s<t}$ be the process of random mappings characterized by the property that $\xi_t = \phi_t^s(x)$ satisfies SDE (4.2) given the initial condition $\xi_s = x$. We see that $u_t := (\partial_x\phi_t^s(x))^{-1} \in \boldsymbol{R}^{d\times d}$ with $\xi_t = \phi_t^0(x)$ satisfies the equation

$$\begin{aligned} d_t\ u_t \ &= \ -\ u_t\ \{\ \partial a_0(\xi_t)\ dt + \Sigma_j\ \partial a_j(\xi_t) \circ d\beta_X^j(t) \\ &\qquad + \int (I - [I + \partial b(\xi_t,\theta)]^{-1}) J_X(dtd\theta)\ \}. \end{aligned}$$

Let $\hat{b}(x,\theta) := [I + \partial b(x,\theta)]^{-1}(\theta \cdot \partial_\theta)b(x,\theta)$ and $\varrho_t > 0$ be a certain normalizing factor. Then the Malliavin covariance V_t for the process (ξ_t) is given by

$$\begin{aligned} V_t \;\; &:= \;\; [u_t]^{-1} \int_0^t \varrho_s \; \{ \; \Sigma_j \; (u_s a_j(\xi_s))(u_s a_j(\xi_s))^* \; ds \\ &\qquad + \int \; (u_s \hat{b}(\xi_s,\theta))(u_s \hat{b}(\xi_s,\theta))^* J_X(dsd\theta) \; \}. \end{aligned}$$

The strong integrability condition : $E_{\widetilde{Q}}[\; |\det V_t|^{-p}] < \infty \quad (^\forall p > 1)$ is satisfied if "the generalized Hörmander condition" on $\{A^Y_{t,0},\; A^Y_{t,1},\; \cdots,\; A^Y_{t,m}\}$ and $\{\; B^Y_\theta \mid \theta \in \boldsymbol{R}^m\}$ is satisfied (cf. [12]). The integration by parts formula holds under the above integrability condition. Then, if h^Y_t and k^Y_t satisfy the condition:

[B3] For any $\nu \in \boldsymbol{Z}^d_+$, for a.a. ω (Q),

$$\begin{aligned} &\int_0^1 \{ \; \|\sqrt{G_t}\partial^\nu_x h^Y_t\|^2_\infty \; + \; \int_{|\eta|\le 1} \|\partial^\nu_x \log k^Y_t(\cdot,\eta)\|^2_\infty \; K_t(d\eta) \\ &\qquad + \int_{|\eta|>1} \|\partial^\nu_x k^Y_t(\cdot,\eta)\|_\infty \; K_t(d\eta) \; \} \; dt \; < \; \infty, \end{aligned}$$

filtered measures $\pi_t(dx)$ have smooth densities (cf. [11]).

Since processes $\{\Omega, \mathcal{F}^Y_t, \widetilde{Q}; Y_t\}$ and $\{W, \mathcal{W}_t, Q; Y_t\}$ are equivalent, hereafter we shall identify these processes. We shall show that the Zakai SDE can be changed into a parabolic equation using an absolutely continuous transform of measures.

Let $\mu_t(dx)$ be a solution to the Zakai SDE (4.1), and define processes $\Lambda^Y_t(x)$, $\psi_t(x)$, the operator $\widetilde{L}_t \equiv \widetilde{L}^Y_t$ and the mesure valued process $\nu_t(dx)$ by

$$\begin{aligned} \Lambda^Y_t(x) \; &:= \; \frac{1}{2}\, h^Y_t(x) \cdot G_t h^Y_t(x) \; + \; \int \left(k^Y_t(x,\eta) - 1 - \log k^Y_t(x,\eta)\right) \; K_t(d\eta), \\ \psi_t(x) &:= \int_0^t h^Y_s(x) \cdot d\beta_Y(s) \; + \; \int_0^t \int \; \log k^Y_s(x,\eta) \; \widetilde{J}_Y(dsd\eta), \\ \widetilde{L}_t f(x) \; &:= \; e^{\psi_t(x)} L^Y_t \left(e^{-\psi_t(x)} f(x)\right) - \Lambda^Y_t(x) f(x). \\ \nu_t(dx) \; &:= \; e^{-\psi_t(x)} \; \mu_t(dx). \end{aligned}$$

Lemma 4.5. *Let $f^Y_t(x) \equiv f(t,x,Y)$ be a function such that*

a) *the support of $f(\cdot,\cdot,Y)$ is compact in $[0,1] \times \boldsymbol{R}^m$,*

b) *functions $\partial^\gamma_x f(t,x,Y)$ $(\gamma \in \boldsymbol{Z}^d_+$ with $|\gamma| \le 2\,)$ and $(\partial/\partial t) f(t,x,Y)$ are bounded in (t,x) and continuous in x.*

Under some regularity conditions on functions h^Y_t and k^Y_t, for each Y, the process $\nu_t(dx)$ satisfies the following formula.

$$(d/dt)\; \nu_t[\; f^Y_t \;] \; = \; \nu_t[\; (\partial/\partial t + \widetilde{L}_t) f^Y_t \;] \tag{4.5}$$

Proof. Let $\rho(x) \in \mathcal{C}_0^\infty(\boldsymbol{R}^d)$, $\displaystyle\int \rho(x)dx = 1$, and set $\rho_\delta^z(x) := \delta^{-d}\rho((x-z)/\delta)$ where $\delta > 0$ and $z \in \boldsymbol{R}^d$. Then we have

$$d\ \mu_t[\rho_\delta^z] = \mu_t[L_t^Y \rho_\delta^z]dt + \mu_t[h_t^Y \rho_\delta^z] \cdot d\beta_Y(t) + \int \mu_t[(k_t^Y(\cdot,\eta)-1)\rho_\delta^z]\widetilde{J}_Y(dtd\eta),$$

and for $F_t(z) := f_t^Y(z)\ e^{-\psi_t(z)}$,

$$\begin{aligned} dF_t(z) &= (\partial/\partial t)f_t^Y(z)\ e^{-\psi_t(z)}\ dt + F_t(z)\{ -h_t^Y(z)\cdot(d\beta_Y(t) - G_t h_t^Y(z)dt) \\ &\quad + \int (k_t^Y(z,\eta)^{-1} - 1)\ (J_Y(dtd\eta) - k_t^Y(z,\eta)K_t(d\eta)dt) - \varLambda_t^Y(z)dt\ \}. \end{aligned}$$

Taking integration by dz for the differential form $d\,(\mu_t[\rho_\delta^z] \times F_t(z))$, we have

$$\begin{aligned} d\ (\mu_t[\rho_\delta^0 * F_t]) &= \int \{d\,(\mu_t[\rho_\delta^z] \times F_t(z))\}\ dz \\ &= \left\{\mu_t\left[\rho_\delta^0 * ((\partial/\partial t)f_t^Y \cdot e^{-\psi_t})\right] + \mu_t\left[L_t^Y(\rho_\delta^0 * F_t) - \rho_\delta^0 * (\varLambda_t^Y F_t)\right]\right\}\ dt \\ &\quad + \mu_t\left[h_t^Y\ \rho_\delta^0 * F_t - \rho_\delta^0 * (F_t h_t^Y)\right] \cdot d\beta_Y(t) \\ &\quad - \mu_t\left[h_t^Y \cdot \rho_\delta^0 * (F_t G_t h_t^Y) - \rho_\delta^0 * (F_t h_t^Y \cdot G_t h_t^Y)\right]\ dt \\ &\quad + \int \mu_t\left[(k_t^Y - 1)\ \rho_\delta^0 * (F_t(k_t^Y)^{-1}) + \rho_\delta^0 * (F_t((k_t^Y)^{-1} - 1))\right] J_Y(dtd\eta) \\ &\quad - \int \mu_t\left[(k_t^Y - 1)\ \rho_\delta^0 * F_t - \rho_\delta^0 * (F_t(k_t^Y - 1))\right] K_t(d\eta)dt, \end{aligned}$$

where $*$ denotes the convolution. This converges, as $\delta \to +0$, to the formula

$$\begin{aligned} d\ (\mu_t[F_t]) &= \left\{\mu_t[(\partial/\partial t)f_t^Y \cdot e^{-\psi_t}] + \mu_t[(L_t^Y - \varLambda_t^Y)F_t]\right\}\ dt \\ &= \nu_t[\ \left(\partial/\partial t + \widetilde{L}_t\right) f_t^Y\ ]\ dt. \end{aligned}$$

□

We shall introduce an assumption for $\{a_0,\ a_k,\ b(\cdot,\theta),\ h_t^Y,\ k_t^Y(\cdot,\theta)\}$.

[A4] Let (ζ_t^*) denote certain positive locally bounded processes.

(***a***) $\displaystyle\|\partial_x a_0\|_\infty + \sum_{k=1}^m(\|a_k\|_\infty + \|\partial_x a_k\|_\infty) \le \zeta_t^*$, $\|\partial_x\psi_t\|_\infty + \|\partial_x\partial_x^*\psi_t\|_\infty \le \zeta_t^*$.

(***b***) $|b(t,Y,x,\theta)| \le \zeta_t^*$, $x \cdot b(t,Y,x,\theta) \ge -\ \zeta_t^*$; and the measure

$$N_{t,x}^Y(du) := \int_{b(t,Y,x,\theta)\in du} |\theta|^{-m-\alpha}d\theta$$

satisfies

$$\left|\int u\ N_{t,x}^Y(du)\right|^2 + \int (|u|^2 \wedge 1)\ N_{t,x}^Y(du) \le \zeta_t^*.$$

(***c***) $\varLambda_t^Y(x) \ge (|x|^2 - 1)/\zeta_t^*$.

(The bounded-ness of $a_0(t,Y,\cdot)$, $h_t^Y(\cdot)$ and $k_t^Y(\cdot,\theta)$ is not required.)

Lemma 4.6. *Assume* **[A4]**. *If* $\mu_0[1] = \nu_0[1] < \infty$, *then* $\nu_t[1]$ *is locally bounded.*

Proof. Let $(\zeta_t), (\zeta'_t)$ and (ζ^*_t) denote certain positive locally bounded processes. From **[A4]** $(\boldsymbol{a})$ and the equality

$$\nu_t[\, e^{-\varepsilon|x|^2}] \;=\; \mu_t[\, e^{\psi_t(x)-\varepsilon|x|^2}] \qquad (\varepsilon > 0),$$

we see that the process $\nu_t[\, e^{-\varepsilon|x|^2}]$ is locally bounded. From **[A4]** $(\boldsymbol{a})$ and $(\boldsymbol{b})$, we have the estimate

$$e^{\varepsilon|x|^2}\widetilde{L}_t e^{-\varepsilon|x|^2} \;+\; \Lambda^Y_t(x) \le \zeta_t\ (\varepsilon|x|^2 + |x| + 1)$$

(see [4]). Combining this estimate and **[A4]** $(\boldsymbol{c})$, we have

$$e^{\varepsilon|x|^2}\tilde{L}_t e^{-\varepsilon|x|^2} \;\le\; -\,|x|^2/\zeta'_t \;+\; \zeta_t(|x|+1) \;\le\; \zeta^*_t$$

for sufficiently small ε. From (4.5), we see that

$$(d/dt)\ \nu_t[\, e^{-\varepsilon|x|^2}] \;\le\; \zeta^*_t\ \nu_t[\, e^{-\varepsilon|x|^2}].$$

Then $\nu_t[1] \;=\; \lim_{\varepsilon\to 0} \nu_t[\, e^{-\varepsilon|x|^2}] \;\le\; \mu_0[1]\ \zeta^*_t.$ □

We shall introduce an assumption. Set

$$\begin{aligned}
\mathcal{C}^{\ell,b}_* \;&=\; \{\ f\ \mid \text{for } |\gamma| \le \ell,\ \partial^\gamma_x f(t,x) \text{ are continuous in } x \text{ and bounded in } (t,x)\},\\
\mathcal{E} \;&=\; \{\ f \in \mathcal{C}^{2,b}_* \mid\ (\partial/\partial t) f(t,x) \in \mathcal{C}^{0,b}_*,\ f(t,x) \to 0 \ \text{ as } t \nearrow 1 \ \text{ or } |x| \to \infty\ \}.
\end{aligned}$$

[A5] Let $\lambda_0 := \sup_{t,x}\ [\widetilde{L}_t\ 1](t,x) < \infty$. For any $f \in \mathcal{C}^{2,b}_*$, the equation:

$$(\lambda - (\partial/\partial s) - \widetilde{L}_s)\ g(s,x) = f(s,x) \qquad (\lambda > \lambda_0)$$

has a solution $g \in \mathcal{E}$ such that $\|g\|_\infty \le (\lambda - \lambda_0)^{-1}\|f\|_\infty$.

For example, this assumption is satisfied in the case where the coefficients are sufficiently smooth in x and they satisfy the linear growth condition.

Theorem 4.7. *Assume* **[A4]** *and* **[A5]**. *Let* $\bar{\mu}(dx)$ *be a given finite measure. Then there exists at most a solution* (μ_t) *of Eq.* (4.1) *with the condition* $\mu_0 = \bar{\mu}$.

Proof. Set $\mathcal{A} := \partial/\partial s + \widetilde{L}_s - \lambda_0$ and

$$\begin{aligned}
&J_\lambda := \lambda(\lambda - \mathcal{A})^{-1}, \qquad \mathcal{A}_\lambda := A J_\lambda\ (\equiv \lambda J_\lambda - \lambda),\\
&\mathcal{C}^2_0 := \{f \in \mathcal{C}^{2,b}_* \mid f(t,x) \to 0 \text{ as } t \uparrow 1 \text{ or } |x| \to \infty\}, \qquad \mathcal{D}(\mathcal{A}) := J_\lambda(\mathcal{C}^2_0).
\end{aligned}$$

Then it can be proved that $\mathcal{D}(\mathcal{A})$ is dense in $\mathcal{C}^2_0$. Define $\tilde{\nu}_t$ and $\tilde{\nu}^\lambda_t$ by formulae

$$\tilde{\nu}_t[f] = e^{\lambda_0 t}\nu_t[f], \qquad \tilde{\nu}^\lambda_t[f] = \mu_0[e^{t\mathcal{A}_\lambda} f].$$

Then it is shown that $J_\lambda(\mathcal{C}^2_0) = \mathcal{D}(\mathcal{A})$, domain of $\mathcal{A}$. It is easy to see that for any $f \in \mathcal{D}(\mathcal{A})$, they satisfy the following equations:

$$\tilde{\nu}_t[f] = \mu_0[f] + \int_0^t \tilde{\nu}_s[\mathcal{A} f]ds, \qquad \tilde{\nu}^\lambda_t[f] = \mu_0[f] + \int_0^t \tilde{\nu}^\lambda_s[\mathcal{A}_\lambda f]ds.$$

Taking the differences of these equations, we have

$$(\tilde{\nu}_t^\lambda - \tilde{\nu}_t)[f] = \int_0^t (\tilde{\nu}_s^\lambda - \tilde{\nu}_s)[\mathcal{A}_\lambda f]\ ds + \int_0^t \tilde{\nu}_s[(\mathcal{A}_\lambda - \mathcal{A})f]\ ds.$$

Since $\tilde{\nu}_s[1]$ is bounded and $\|\mathcal{A}_\lambda\| \le 2\lambda$, by the iterative method, we have

$$(\tilde{\nu}_t^\lambda - \tilde{\nu}_t)[f] = \int_0^t \tilde{\nu}_s[(\mathcal{A}_\lambda - \mathcal{A})e^{(t-s)\mathcal{A}_\lambda} f]\ ds.$$

For any $g \in \mathcal{D}(\mathcal{A})$, $\|J_\lambda g - g\| = \|J_\lambda(\mathcal{A}f)\|/\lambda \to 0$ as $\lambda \to \infty$. Since $\mathcal{D}(\mathcal{A})$ is dence in $\mathcal{C}_0^2$, for $f \in \mathcal{D}(\mathcal{A})$, we have $\|J_\lambda(\mathcal{A}f) - (\mathcal{A}f)\| \to 0$ as $\lambda \to \infty$. Therefore

$$\|e^{(t-s)\mathcal{A}_\lambda}(\mathcal{A}_\lambda - \mathcal{A})f\| \longrightarrow 0 \quad (\lambda \to \infty).$$

Since $\|e^{(t-s)\mathcal{A}_\lambda}(\mathcal{A}_\lambda - \mathcal{A})f\| \le 2\|\mathcal{A}f\|$, we have

$$\int_0^t \tilde{\nu}_s \left[(\mathcal{A}_\lambda - \mathcal{A})e^{(t-s)\mathcal{A}_\lambda} f\right]\ ds \longrightarrow 0 \quad (\lambda \to \infty).$$

Then $\tilde{\nu}_t[f] = \lim_{\lambda\to\infty} \tilde{\nu}_t^\lambda[f]$, which means the uniquenss of $\nu_t[f]$ and $\mu_t[f]$. □

Example 4.8. Let the measure $K(d\eta) = K_t(d\eta)$ be independent of $\{t,\ Y\}$, and $\Gamma_1,\ \Gamma_2$ be disjoint sets with $K(\Gamma_1) = K(\Gamma_2) = (\gamma/\varepsilon)^2 > 0$. Let $\ell(x)$ be a smooth odd function on $\boldsymbol{R}$ such that $\ell(x) = x$ $(|x| \le 1)$ and $\ell(x) = 2$ $(x \ge 3)$. We shall consider the Zakai SDE in the case where

$$\begin{aligned} L &= \frac{1}{2}\frac{\partial^2}{\partial x^2}, \qquad h(t,x) = \lambda x, \\ k(t,x,\eta) &= e^{\ell(\varepsilon x)}\ I(\eta \in \Gamma_1) + e^{-\ell(\varepsilon x)}\ I(\eta \in \Gamma_2) + I(\eta \notin \Gamma_1 \cup \Gamma_2). \end{aligned}$$

Set $J_t^1 := J_Y([0,t] \times \Gamma_1)$, $J_t^2 := J_Y([0,t] \times \Gamma_2)$ and

$$\psi_t(x) := \lambda x\ \beta_Y(t) + \ell(\varepsilon x)\ (J_t^1 - J_t^2).$$

Let $\tilde{p}(t,Y,x)$ denote the formal density of the solution $\mu_t(dx)$ to the Zakai SDE, and set $\tilde{q}(t,Y,x) := \tilde{p}(t,Y,x)\ e^{-\psi_t(x)}$. Then the Zakai SDE for $\tilde{p}$ is equivalent to the 1-st parabolic equation for $\tilde{q} = \tilde{q}(t,Y,x)$:

$$\begin{aligned} \frac{\partial}{\partial t}\tilde{q} &= \frac{1}{2}\frac{\partial^2}{\partial x^2}\tilde{q} + \partial_x\psi_t(x)\frac{\partial}{\partial x}\tilde{q} + \frac{1}{2}\left[\partial_x^2\psi_t(x) + (\partial_x\psi_t(x))^2\right]\tilde{q} \\ &\quad - \left[\frac{1}{2}\lambda^2 x^2 + \frac{\gamma^2}{\varepsilon^2}\left(\cosh(\ell(\varepsilon x)) - 1\right)\right]\tilde{q}. \end{aligned}$$

For small ε, the above parabolic equation for $\tilde{q}$ is close to the 2-nd parabolic equation for the function $u = u(t,x)$ on the domain $\{(t,x) \mid\ t > 0,\ |x| < \varepsilon^{-1/2}\ \}$:

$$\begin{aligned} \frac{\partial}{\partial t}u &= \frac{1}{2}\frac{\partial^2}{\partial x^2}u + [\lambda\beta_Y(t) + \varepsilon(J_t^1 - J_t^2)]\frac{\partial}{\partial x}u \\ &\quad - \frac{1}{2}\left[(\lambda^2 + \gamma^2)\ x^2 - (\lambda\beta_Y(t) + \varepsilon(J_t^1 - J_t^2))^2\right]u\ . \end{aligned}$$

Assume that the function $u = U(t,x)$ of the form

$$U(t,x) = \exp[\ D_0(t) + D_1(t)\ x - (1/2)D_2(t)\ x^2\ ], \quad D_2(+0) = \infty$$

satisfies the 2-nd parabolic equation. Then

$$\frac{d}{dt}D_2 = \sigma^2 - {D_2}^2, \qquad \frac{d}{dt}D_1 + D_2 D_1 = -[\lambda\beta_Y(t) + \varepsilon(J_t^1 - J_t^2)]\ D_2\ ,$$

where $\sigma = \sqrt{\lambda^2+\gamma^2}$. We have $D_2(t) = \sigma/\tanh(\sigma t)$ and

$$\frac{\sinh(\sigma t)}{\sigma}\ D_1(t)\ =\ c_0 - \int_0^t\ [\lambda\beta_Y(s) + \varepsilon(J_s^1 - J_s^2)]\ \cosh(\sigma s)\ ds$$

with a certain constant c_0. Then we see that

$$U(t,x)\ e^{\psi_t(x)}\ =\ D_3(t)\ \exp\left[-\ \frac{\sigma}{2\tanh(\sigma t)}\ \left(x - \frac{M_t^\varepsilon}{\cosh(\sigma t)}\right)^2\right],$$

where $D_3(t)$ is a certain function of t, and

$$M_t^\varepsilon\ :=\ c_0 + \int_0^t\ \frac{\sinh(\sigma s)}{\sigma}\ [\lambda\ d\beta_Y(s) + \varepsilon\ (dJ_s^1 - dJ_s^2)]\ .$$

Consider the filtering system of signal and observation $\{(\xi_t, Y_t), P_\varepsilon\}$ associated with the above Zakai SDE with $P_\varepsilon[\xi_0 = c_0] = 1$. Then we see that, for very small ε, the density of filtered measure $P_\varepsilon[\ \xi_t \in dx \mid \mathcal{F}_t^Y\ ]$ is close to the function:

$$p_\varepsilon(t,Y,x)\ =\ \sqrt{\frac{\sigma}{2\pi\tanh(\sigma t)}}\ \exp\left[-\ \frac{\sigma}{2\tanh(\sigma t)}\ \left(x - \frac{M_t^\varepsilon}{\cosh(\sigma t)}\right)^2\right].$$

5. Appendix

We shall prove Lemma 2.2. The following lemma is the key for the proof.

Lemma 5.1. *Assume* **[A1]** *for the triplet* $(G_t, H_t, K_t(d\eta))$*, and let* (Y_t, Q) *be the* $(G_t, H_t, K_t)-$ *process. Let* (M_t) *be a uniformly bounded* $(\mathcal{W}_t, Q)-$*martingale with* $M_0 = 0$*. Suppose that, for a.a.* w (Q)*, equalities for signed measures :*

$$d\langle\beta_Y(t), M_t\rangle \equiv 0, \qquad |\Delta M_t| \times J_Y(dtd\eta) \equiv 0$$

hold. Then we have $Q[\ M_t = 0\ (0 \le t \le 1)\ ] = 1$.

Proof. Choose a constant $c > 0$ so that $2c\,|M_1| \le 1$. Define a probability measure $\widetilde{Q}$ on $(W, \mathcal{W}_1)$ by $\widetilde{Q}(dw)\ =\ (1 + c\ M_1)\ Q(dw)$ and set $Z_t := 1 + c\ M_t$.
Let $T_\nu := \inf\{t \mid |\beta_Y(t)| \ge \nu\}$ and $B_t^\nu := \beta_Y(t \wedge T_\nu)$. Since

$$Z_t B_t^\nu\ =\ \int_0^t B_s^\nu\ dZ_s + \int_0^t Z_s\ dB_s^\nu,$$

we have $E_{\widetilde{Q}}[B_T^\nu]\ =\ E_Q[Z_T B_T^\nu]\ =\ 0$ for any stopping time T. This implies that the process (B_t^ν) is $(\mathcal{W}_t, \widetilde{Q})-$martingale. Since Q and $\widetilde{Q}$ are mutually absolutely continuous, $\widetilde{Q}[T_\nu < 1]\ \downarrow 0$ as $\nu \uparrow \infty$ and $\widetilde{Q}[B_t^\nu$ is continuous in $t] = 1$. Then

$$(z \cdot \beta_Y(t)), \qquad \left((z \cdot \beta_Y(t))^2 - \int_0^t z \cdot G_s z\ ds\right) \qquad (z \in \boldsymbol{R}^n)$$

are continuous $(\mathcal{W}_t, \widetilde{Q})-$local martingales.
Define afresh an increasing sequence $\{T_\nu\}$ of stopping times by

$$T_\nu \ := \ \inf \left\{ t \mid \int_0^t \int (|\eta|^2 \wedge 1) \ K_s(d\eta) ds \geq \nu \right\}.$$

Let $\lambda(\eta)$ be a function such that $\lambda(\eta)^2 \leq \text{const.}(|\eta|^2 \wedge 1)$ and set

$$N_t^\nu \ := \ \int_0^{t \wedge T_\nu} \int \lambda(\eta) \ \widetilde{J}_Y(dsd\eta).$$

Since $\lambda(\eta) \Delta Z_s J_Y(dsd\eta) \equiv 0$, we have

$$Z_t N_t^\nu \ = \ \int_0^t Z_{s-} dN_s^\nu + \int_0^t N_{s-} dZ_s.$$

Therefore $E_{\widetilde{Q}}[N_T^\nu] = E_Q[Z_T N_T^\nu] = 0$ for any stopping time T. This implies

$$E_{\widetilde{Q}} \left[\int_0^{T \wedge T_\nu} \int \lambda(\eta) \ J_Y(dsd\eta) \right] \ = \ E_{\widetilde{Q}} \left[\int_0^{T \wedge T_\nu} \int \lambda(\eta) \ K_s(d\eta) ds \right].$$

Letting $\nu \to \infty$, we have

$$E_{\widetilde{Q}} \left[\int_0^T \int \lambda(\eta) \ J_Y(dsd\eta) \right] \ = \ E_{\widetilde{Q}} \left[\int_0^T \int \lambda(\eta) \ K_s(d\eta) ds \right].$$

It is a routine work to show that the above equality holds also for any non-negative function $\lambda(\eta)$. We see that $\widetilde{J}_Y(dsd\eta) = J_Y(dsd\eta) - K_s(d\eta)\ ds$ generates $(\mathcal{W}_t, \widetilde{Q})-$local martingales. Then $(Y_t, \widetilde{Q})$ is a solution to the martingale problem for the triplet $(G_t, H_t, K_t(d\eta))$. From **[A1]**, we have $Q \equiv \widetilde{Q}$, which implies $Q[\ M_t = 0\ (0 \leq t \leq 1)\ ] = 1$. □

[Proof of Lemma 2.2] Let (Y_t, Q) be a solution to the martingale problem for the triplet $(G_t, H_t, K_t(d\eta))$, and assume that $(G_t, H_t, K_t(d\eta))$ satisfies **[A1]**. Let (M_t) be a given locally bounded $(\mathcal{W}_t, Q)-$martingale.

(Step 1) The case where (M_t) is a bounded continuous martingale. Let $\mathcal{M} = \mathcal{M}(\mathcal{W}_t, Q)$ be the space of all square integrable $(\mathcal{W}_t, Q)-$martingales. $\mathcal{M}$ is a complete linear space endowed with "inner product" $\langle \cdot , \cdot \rangle$. Set

$$\mathcal{M}_0 \ := \ \left\{ \left(\int_0^t \varGamma_s \cdot d\beta_Y(s) \right) \mid (\varGamma_t) \in \mathcal{H}^2(\beta_Y) \right\}.$$

Since $\mathcal{M}_0$ is a closed subspace of $\mathcal{M}$, the space $\mathcal{M}$ is decomposed into the space $\mathcal{M}_0$ and its orthogonal $\mathcal{M}_1 := {\mathcal{M}_0}^\perp$. Then the martingale $(M_t - M_0)$ can be expressed in the form

$$M_t - M_0 \ = \ M_t^{(0)} + M_t^{(1)} \ ; \qquad (M_t^{(0)}) \in \mathcal{M}_0, \quad (M_t^{(1)}) \in \mathcal{M}_1.$$

Since $(M_t^{(1)}) \in {\mathcal{M}_0}^\perp$, we have $d\langle M_t^{(1)}, \beta_Y(t) \rangle = 0$. From the above lemma, we see $M_t^{(1)} \equiv 0$. This implies that the martingale $(M_t - M_0)$ can be expressed by a stochastic integral along the continuous local martingale $(\beta_Y(t))$. The expression is also possible for any continuous local martingale.

(Step 2) The case where (M_t) is a bounded purely discontinuous martingale. From the fact that the space

$$\mathcal{M}_2 \ := \ \left\{ \left(\int_0^t \int \Lambda_{s,\eta} \ \widetilde{J}_Y(dsd\eta) \right) \mid (\Lambda_{t,\eta}) \in \mathcal{L}^2(J_Y) \right\}.$$

is a closed subspace of $\mathcal{M}$, the space $\mathcal{M}$ is decomposed into the space $\mathcal{M}_2$ and its orthogonal $\mathcal{M}_3 := {\mathcal{M}_2}^{\perp}$. Then $(M_t - M_0)$ can be expressed in the form

$$M_t - M_0 \ = \ M_t^{(2)} + M_t^{(3)} \ ; \qquad (M_t^{(2)}) \in \mathcal{M}_2, \quad (M_t^{(3)}) \in \mathcal{M}_3.$$

Since $(M_t^{(3)}) \in {\mathcal{M}_2}^{\perp}$, we have $|\Delta M_t^{(3)}| \times J_Y(dtd\eta) \equiv 0$. From the above lemma, we see $M_t^{(3)} \equiv 0$. This implies that the martingale $(M_t - M_0)$ can be expressed by a stochastic integral by the martingale random measure $(\widetilde{J}_Y(dtd\eta))$. The expression is also possible for any purely discontinuous local martingale.

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