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A GENERAL SUPERAPPROXIMATION RESULT

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Abstract. A general superapproximation result is derived in this paper which is useful for the local/interior error analysis of finite element methods.

Superapproximation results are useful tools for the local/interior error analysis of finite element methods [10]. Our goal is to derive a general superapproximation result that includes many of the results in the literature as special cases.

We will adopt the usual notation of function spaces and norms that can be found for example in [1, 6, 5]. The set-up is as follows.

- ω is a C^∞ function on $\mathbb{R}^N$ such that, for a positive number d,

$$|\omega|_{W^j_\infty(\mathbb{R}^N)} \leq C_\dagger d^{-j} \qquad j = 0, 1, 2, \dots. \tag{1}$$

- $(K, \mathcal{P}, \mathcal{N})$ is a finite element space (à la Ciarlet), where K is a connected compact subset of $\mathbb{R}^N$, $\mathcal{P}$ is a finite dimensional vector space of polynomials, $\mathcal{N}$ is a set of nodal variables (degrees of freedom), and the diameter h_K of K satisfies

$$h_K \leq d. \tag{2}$$

We will allow a slight abuse of notation to write the Sobolev spaces $H^\ell(\mathrm{int}(K))$ (resp., $W^\ell_\infty(\mathrm{int}(K))$) as $H^\ell(K)$ (resp., $W^\ell_\infty(K)$).

- ω_K is the mean of ω over K so that

$$|\omega_K| \leq C_\dagger, \tag{3}$$

$$|\omega - \omega_K|_{L_\infty(K)} \leq C_\dagger d^{-1} h_K. \tag{4}$$

- The linear operator $\Pi_K : C^\infty(K) \to \mathcal{P}$ is the nodal interpolation operator that satisfies

$$\zeta - \Pi_K \zeta = 0 \qquad \forall\, \zeta \in \mathcal{P}, \tag{5}$$

and there exists a positive integer k (depending on the maximum order of differentiation among the nodal variables in $\mathcal{N}$) such that

$$|\zeta - \Pi_K \zeta|_{W^\ell_\infty(K)} \leq C_\flat h_K^{k+1-\ell} |\zeta|_{W^{k+1}_\infty(K)} \qquad \forall\, \zeta \in C^\infty(K),\ 0 \leq \ell \leq k+1. \tag{6}$$

- The following inverse estimates are satisfied:

$$|\chi|_{W^q_s(K)} \leq C_\sharp h_K^{p-q} |\chi|_{W^p_s(K)} \qquad \forall\, \chi \in \mathcal{P},\ s = 2, \infty, \text{ and } 0 \leq p < q \leq k+1, \tag{7}$$

$$|\chi|_{W^p_\infty(K)} \leq C_\diamond h_K^{-N/2} |\chi|_{H^p(K)} \qquad \forall\, \chi \in \mathcal{P} \text{ and } 0 \leq p \leq k. \tag{8}$$

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Inverse estimates and interpolation error estimates for finite elements can be found in [6, 5]. Note that, by the Bramble-Hilbert lemma [3], the interpolation error estimates in (6) are valid provided the space $\mathcal{P}$ of shape functions contains all the polynomials of total degree $\leq k$.

The following theorem is the main result of this paper.

Theorem 1. *Given $0 \leq m \leq k$, $0 \leq \ell \leq m+1$ and $n \geq 1$, we have, under assumptions* (1)–(8),

$$|\omega^n \chi - \Pi_K(\omega^n \chi)|_{H^\ell(K)} \leq C h_K^{m+1-\ell} d^{-(m+1)} \sum_{j=0}^{m} d^j |\chi|_{H^j(K)} \qquad \forall\, \chi \in \mathcal{P}, \tag{9}$$

and, for $n \geq m+1$,

$$|\omega^n \chi - \Pi_K(\omega^n \chi)|_{H^\ell(K)} \leq C h_K^{m+1-\ell} d^{-(m+1)} \sum_{j=0}^{m} d^j |\omega^j \chi|_{H^j(K)} \qquad \forall\, \chi \in \mathcal{P}, \tag{10}$$

where the positive constant C in (9) *and* (10) *depends only on N, n, k, $C_\dagger$, $C_\flat$, $C_\sharp$ and $C_\diamond$.*

Remark 1. For an arbitrary smooth function χ, the interpolation error estimates in (9) and (10) will involve the $(m+1)$-order Sobolev norm $|\chi|_{H^{m+1}(K)}$. The fact that the right-hand sides of (9) and (10) only involve Sobolev norms up to order m can be exploited to show that the energy norm of a discrete harmonic (or biharmonic) function on a subdomain can be bounded by a lower order norm on a larger subdomain, which is the discrete version of a Caccioppoli estimate. Such estimates play a key role in the local/interior error analysis of finite element methods (cf. [10, Section 9.1] and [7, Section 3]).

Remark 2. The key observation is that, by the binomial theorem,

$$\omega^n \chi - \Pi_K(\omega^n \chi) = \sum_{q=1}^{n} \binom{n}{q} \Big[(\omega - \omega_K)^q \omega_K^{n-q} \chi - \Pi_K\big((\omega - \omega_K)^q \omega_K^{n-q} \chi \big) \Big] \qquad \forall\, \chi \in \mathcal{P}, \tag{11}$$

where the summation is over $q \geq 1$ because $\omega_K^n \chi - \Pi_K(\omega_K^n \chi) = 0$ by (5). This is why only derivatives up to order m appear on the right-hand side of (9) and (10) (cf. Lemma 1 below).

Example 1. For $n = 1$, $m = \ell = 1$ or $m = \ell = 0$, it follows from (9) that

$$|\omega \chi - \Pi_K(\omega \chi)|_{H^1(K)} \leq C h_K d^{-2} \big(\|\chi\|_{L_2(K)} + d |\chi|_{H^1(K)} \big),$$
$$\|\omega \chi - \Pi_K(\omega \chi)\|_{L_2(K)} \leq C h_K d^{-1} \|\chi\|_{L_2(K)},$$

which cover the results in [9, Section 2], [4, Section 6] and [2, Section 3].

Example 2. For $n = 2$ and $m = \ell = 1$, it follows from (10) that

$$|\omega^2 \chi - \Pi_K(\omega^2 \chi)|_{H^1(K)} \leq C h_K d^{-2} \big(\|\chi\|_{L_2(K)} + d |\omega \chi|_{H^1(K)} \big),$$

which is the result in [7, Section 2].

Example 3. For $n = 4$, $m = 2$ and $1 \leq \ell \leq 3$, it follows from (10) that

$$|\omega^4 - \Pi_K(\omega^4 \chi)|_{H^\ell(K)} \leq C h_K^{3-\ell} d^{-3} \big(\|\chi\|_{L_2(K)} + d |\omega \chi|_{H^1(K)} + d^2 |\omega^2 \chi|_{H^2(K)} \big),$$

which is the result in [8, Section 3.3].

We will use the notation $A \lesssim B$ to represent the inequality $A \leq (\text{constant})B$, where the hidden generic positive constant only depends on N, n, k, $C_\dagger$, $C_\flat$, $C_\sharp$ and $C_\diamond$.

The proof of Theorem 1 requires two lemmas that are based on the product rule (Leibniz formula) and the binomial theorem. The first lemma allows the reduction of the order of the Sobolev seminorm.

Lemma 1. *For $1 \leq q \leq n$, $0 \leq p \leq k+1$ and $s = 2, \infty$, we have*

$$d^p |(\omega - \omega_K)^q \chi|_{W^p_s(K)} \lesssim \sum_{j=0}^{(p-q)\vee 0} d^j |\chi|_{W^j_s(K)} \qquad \forall\, \chi \in \mathcal{P}, \tag{12}$$

where $a \vee b = \max(a, b)$.

Proof. Note that (1), (2), (4) and the product rule imply

$$|(\omega - \omega_K)^q|_{W^r_\infty(K)} \lesssim \begin{cases} d^{-r} & \text{if } r > q \\ d^{-q} h_K^{q-r} & \text{if } r \leq q \end{cases}. \tag{13}$$

Let $\chi \in \mathcal{P}$ be arbitrary. It follows from (13) and the product rule again that

$$\begin{aligned} d^p |(\omega - \omega_K)^q \chi|_{W^p_s(K)} &\lesssim d^p \sum_{r=0}^{p} |(\omega - \omega_K)^q|_{W^r_\infty(K)} |\chi|_{W^{p-r}_s(K)} \\ &\lesssim \sum_{r=0}^{p\wedge q} d^{p-q} h_K^{q-r} |\chi|_{W^{p-r}_s(K)} + \sum_{r=(p\wedge q)+1}^{p} d^{p-r} |\chi|_{W^{p-r}_s(K)}, \end{aligned} \tag{14}$$

where $p \wedge q = \min(p, q)$ and we have adopted the convention that a sum is void if the lower index is strictly greater than the upper index. Note that the second sum on the right-hand side of (14) is void if and only if $p \leq q$.

In the case where $p \leq q$, the first sum on the right-hand side of (14) satisfies

$$\sum_{r=0}^{p\wedge q} d^{p-q} h_K^{q-r} |\chi|_{W^{p-r}_s(K)} = \sum_{r=0}^{p} (h_K/d)^{q-p} h_K^{p-r} |\chi|_{W^{p-r}_s(K)} \lesssim \|\chi\|_{L_s(K)}$$

because of (2) and (7). On the other hand, if $p > q$, then we have

$$\sum_{r=0}^{p\wedge q} d^{p-q} h_K^{q-r} |\chi|_{W^{p-r}_s(K)} = \sum_{r=0}^{q} d^{p-q} h_K^{q-r} |\chi|_{W^{p-r}_s(K)} \lesssim d^{p-q} |\chi|_{W^{p-q}_s(K)}$$

because of (7).

It follows that

$$\sum_{r=0}^{p\wedge q} d^{p-q} h_K^{q-r} |\chi|_{W^{p-r}_s(K)} \lesssim d^{p-(p\wedge q)} |\chi|_{W^{p-(p\wedge q)}_s(K)},$$

which together with (14) implies

$$d^p |(\omega - \omega_K)^q \chi|_{W^p_s(K)} \lesssim \sum_{r=p\wedge q}^{p} d^{p-r} |\chi|_{W^{p-r}_s(K)}.$$

The estimate (12) is then obtained by the change of index $j = p - r$. □

The second lemma allows the switching from $|\omega_K^j\chi|_{H^j(K)}$ to $|\omega^j\chi|_{H^j(K)}$.

Lemma 2. *For $0 \le j \le m$, we have*

$$d^j|\omega_K^j\chi|_{H^j(K)} \lesssim \sum_{p=0}^{j} d^p|\omega^p\chi|_{H^p(K)} \qquad \forall\, \chi \in \mathcal{P}. \tag{15}$$

Proof. According to the binomial theorem, we have

$$\omega_K^j = \omega^j - \sum_{\ell=1}^{j} \binom{j}{\ell} (\omega - \omega_K)^\ell \omega_K^{j-\ell}. \tag{16}$$

It follows from (3), Lemma 1 (with $s = 2$) and (16) that

$$\begin{aligned} d^j|\omega_K^j\chi|_{H^j(K)} &\lesssim d^j|\omega^j\chi|_{H^j(K)} + \sum_{\ell=1}^{j} |\omega_K|^{j-\ell} d^j |(\omega-\omega_K)^\ell\chi|_{H^j(K)} \\ &\lesssim d^j|\omega^j\chi|_{H^j(K)} + \sum_{\ell=1}^{j}\sum_{p=0}^{j-\ell} d^p|\omega_K^{j-\ell}\chi|_{H^p(K)} \\ &\lesssim d^j|\omega^j\chi|_{H^j(K)} + \sum_{\ell=1}^{j}\sum_{p=0}^{j-\ell} d^p|\omega_K^{p}\chi|_{H^p(K)} \\ &= d^j|\omega^j\chi|_{H^j(K)} + \sum_{p=0}^{j-1}(j-p) d^p|\omega_K^{p}\chi|_{H^p(K)} \\ &\lesssim d^j|\omega^j\chi|_{H^j(K)} + \sum_{p=0}^{j-1} d^p|\omega_K^{p}\chi|_{H^p(K)}, \end{aligned}$$

which implies (15) through a recursive argument. □

We are now ready to prove Theorem 1. From (6) and Lemma 1 (with $s = \infty$), we find, for $n \ge q \ge 1$,

$$\begin{aligned} &|(\omega-\omega_K)^q\omega_K^{n-q}\chi - \Pi_K\big((\omega-\omega_K)^q\omega_K^{n-q}\chi\big)|_{H^\ell(K)} \\ &\qquad\lesssim (h_K^{N/2}|\omega_K^{n-q}|)|(\omega-\omega_K)^q\chi - \Pi_K\big((\omega-\omega_K)^q\chi\big)|_{W_\infty^\ell(K)} \\ &\qquad\lesssim (h_K^{N/2}|\omega_K^{n-q}|)h_K^{k+1-\ell}|(\omega-\omega_K)^q\chi|_{W_\infty^{k+1}(K)} \\ &\qquad\lesssim (h_K^{N/2}|\omega_K^{n-q}|)\Big(h_K^{k+1-\ell}d^{-(k+1)}\sum_{j=0}^{(k+1-q)\vee 0} d^j|\chi|_{W_\infty^j(K)}\Big). \end{aligned} \tag{17}$$

The next step is to reduce the summation on the right-hand side of (17) from the range $0 \le j \le [(k+1-q)\vee 0]$ to the range $0 \le j \le [(m+1-q)\vee 0]$.

In the case where $k+1-q \le 0$, we also have $m+1-q \le 0$ (since $m \le k$ is one of the assumptions in Theorem 1) and hence, in view of (2),

$$h_K^{k+1-\ell} d^{-(k+1)} \sum_{j=0}^{(k+1-q)\vee 0} d^j |\chi|_{W_\infty^j(K)} = (h_K/d)^{k-m} h_K^{m+1-\ell} d^{-(m+1)} \|\chi\|_{L_\infty(K)}$$
$$\le h_K^{m+1-\ell} d^{-(m+1)} \sum_{j=0}^{(m+1-q)\vee 0} d^j |\chi|_{W_\infty^j(K)}. \tag{18}$$

Suppose $k+1-q > 0$ and let j be an index within the range of the summation on the right-hand side of (17), i.e.,

$$0 \le j \le k+1-q. \tag{19}$$

If j is strictly greater than $m+1-q$, we have, by (2), (7) and (19),

$$\begin{aligned} h_K^{k+1-\ell} d^{-(k+1)} \big(d^j |\chi|_{W_\infty^j(K)}\big) &\lesssim h_K^{k+1-\ell} d^{-(k+1)} \big(d^j h_K^{m+1-q-j} |\chi|_{W_\infty^{m+1-q}(K)}\big) \\ &= (h_K/d)^{k+1-q-j} h_K^{m+1-\ell} d^{-(m+1)} \big(d^{m+1-q} |\chi|_{W_\infty^{m+1-q}(K)}\big) \\ &\le h_K^{m+1-\ell} d^{-(m+1)} \big(d^{m+1-q} |\chi|_{W_\infty^{m+1-q}(K)}\big). \end{aligned} \tag{20}$$

If j is less than or equal to $m+1-q$, we have

$$\begin{aligned} h_K^{k+1-\ell} d^{-(k+1)} \big(d^j |\chi|_{W_\infty^j(K)}\big) &= (h_K/d)^{k-m} h_K^{m+1-\ell} d^{-(m+1)} \big(d^j |\chi|_{W_\infty^j(K)}\big) \\ &\le h_K^{m+1-\ell} d^{-(m+1)} \big(d^j |\chi|_{W_\infty^j(K)}\big), \end{aligned} \tag{21}$$

again because $m \le k$ is one of the assumptions in Theorem 1.

Since all the possibilities are covered by (18), (20) and (21), we conclude from (8) and (17) that

$$\begin{aligned} &|(\omega-\omega_K)^q \omega_K^{n-q} \chi - \Pi_K\big((\omega-\omega_K)^q \omega_K^{n-q} \chi\big)|_{H^\ell(K)} \\ &\quad\lesssim (h_K^{N/2} |\omega_K^{n-q}|) \Big(h_K^{m+1-\ell} d^{-(m+1)} \sum_{j=0}^{(m+1-q)\vee 0} d^j |\chi|_{W_\infty^j(K)}\Big) \\ &\quad\lesssim h_K^{m+1-\ell} d^{-(m+1)} \sum_{j=0}^{(m+1-q)\vee 0} d^j |\omega_K^{n-q} \chi|_{H^j(K)}. \end{aligned} \tag{22}$$

Note that $n-q \ge 0$ and hence, in view of (3), we can simply drop ω_K^{n-q} from the right-hand side of (22) to arrive at

$$|(\omega-\omega_K)^q \omega_K^{n-q} \chi - \Pi_K\big((\omega-\omega_K)^q \omega_K^{n-q} \chi\big)|_{H^\ell(K)} \lesssim h_K^{m+1-\ell} d^{-(m+1)} \sum_{j=0}^{(m+1-q)\vee 0} d^j |\chi|_{H^j(K)}, \tag{23}$$

and the estimate (9) follows immediately from (11) and (23).

In the case where $n \ge m+1$, we have

$$n-q \ge [(m+1-q)\vee 0]$$

and hence

$$n - q - j \geq 0$$

for any index j in the range of the summation on the right-hand side of (22). Therefore we can replace ω_K^{n-q} by ω_K^j inside the summation to obtain, through Lemma 2,

$$\begin{aligned} &|(\omega-\omega_K)^q\omega_K^{n-q}\chi - \Pi_K\big((\omega-\omega_K)^q\omega_K^{n-q}\chi\big)|_{H^\ell(K)} \\ &\qquad \lesssim h_K^{m+1-\ell} d^{-(m+1)} \sum_{j=0}^{(m+1-q)\vee 0} d^j |\omega_K^j\chi|_{H^j(K)} \\ &\qquad \lesssim h_K^{m+1-\ell} d^{-(m+1)} \sum_{j=0}^{(m+1-q)\vee 0} d^j |\omega^j\chi|_{H^j(K)}. \end{aligned} \tag{24}$$

The estimate (10) follows immediately from (11) and (24).

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