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A One Dimensional Elliptic Distributed Optimal Control Problem with Pointwise Derivative Constraints

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A ONE DIMENSIONAL ELLIPTIC DISTRIBUTED OPTIMAL CONTROL PROBLEM WITH POINTWISE DERIVATIVE CONSTRAINTS

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Abstract. We consider a one dimensional elliptic distributed optimal control problem with pointwise constraints on the derivative of the state. By exploiting the variational inequality satisfied by the derivative of the optimal state, we obtain higher regularity for the optimal state under appropriate assumptions on the data. We also solve the optimal control problem as a fourth order variational inequality by a C^1 finite element method, and present the error analysis together with numerical results.

1. Introduction

Let I be the interval $(-1,1)$ and the function $J : L_2(I) \times L_2(I) \longrightarrow \mathbb{R}$ be defined by

$$J(y,u) = \frac{1}{2}\big(\|y - y_d\|^2_{L_2(I)} + \beta\|u\|^2_{L_2(I)}\big), \tag{1.1}$$

where $y_d \in L_2(I)$ and β is a positive constant.

The optimal control problem is to

$$\text{find} \quad (\bar{y},\bar{u}) = \operatorname*{argmin}_{(y,u)\in\mathbb{K}} J(y,u), \tag{1.2}$$

where $(y,u) \in H^1_0(I) \times L_2(I)$ belongs to $\mathbb{K}$ if and only if

$$\int_I y'z'dx = \int_I (u+f)z\,dx \qquad \forall\, z \in H^1_0(I), \tag{1.3}$$

$$y' \le \psi \qquad \text{a.e. on } I. \tag{1.4}$$

We assume that

$$f \in H^1(I),\ \psi \in H^2(I) \tag{1.5}$$

and

$$\int_I \psi\,dx > 0. \tag{1.6}$$

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Remark 1.1. The optimal control problem defined by (1.1)–(1.4) is a one dimensional analog of the optimal control problems considered in [10, 11, 13, 21, 24]. It was solved by a C^1 finite element method in [9] under the assumptions that

$$f \in H^{\frac{1}{2}-\epsilon}(I) \quad \text{and} \quad \psi \in H^{\frac{3}{2}-\epsilon}(I). \tag{1.7}$$

Since the constraint (1.3) implies $y \in H^2(I)$ by elliptic regularity, we can reformulate the optimization problem (1.1)–(1.4) as follows:

$$\text{Find} \quad \bar{y} = \operatorname*{argmin}_{y \in K} \frac{1}{2}\big(\|y - y_d\|^2_{L_2(I)} + \beta\|y'' + f\|^2_{L_2(I)}\big), \tag{1.8}$$

where

$$K = \{y \in H^2(I) \cap H^1_0(I) : y' \le \psi \text{ on } I\}. \tag{1.9}$$

According to the standard theory [14, 18], the minimization problem defined by (1.8)-(1.9) has a unique solution characterized by the fourth order variational inequality

$$\beta \int_I (\bar{y}'' + f)(y'' - \bar{y}'')dx + \int_I (\bar{y} - y_d)(y - \bar{y})dx \ge 0 \qquad \forall\, y \in K,$$

which can also be written as

$$a(\bar{y}, y - \bar{y}) \ge \int_I y_d(y - \bar{y})dx - \beta \int_I f(y'' - \bar{y}'')dx \qquad \forall\, y \in K, \tag{1.10}$$

where

$$a(y, z) = \beta \int_I y'z'dx + \int_I yz\, dx. \tag{1.11}$$

Remark 1.2. The reformulation of state constraint optimal control problems as fourth order variational inequalities was discussed in [22], and a nonconforming finite element based on this idea was introduced in [19]. Other finite element methods can be found in [16, 7, 1, 4, 3, 2, 8].

Remark 1.3. Note that (1.4) implies

$$\int_I \psi\, dx \ge \int_I y'\, dx = 0 \qquad \forall\, y \in K$$

and hence $\int_I \psi\, dx \ge 0$ is a necessary condition for K to be nonempty. It is also a sufficient condition because the function y defined by

$$y(x) = \int_{-1}^{x} (\psi(t) - \bar{\psi})dt$$

belongs to K, where $\bar{\psi}$ is the mean of ψ over I. Furthermore,

$$0 = \int_I \psi\, dx = \int_I (\psi - y')dx$$

together with (1.4) implies $\psi = y'$ identically on I and hence $K = \{\psi\}$ is a singleton. Therefore we impose the condition (1.6) to ensure that the optimization problem defined by (1.8)–(1.9) is nontrivial.

Our goal is to show that $\bar{y} \in H^3(I)$ under the assumptions in (1.5) and consequently (1.8)/(1.10) can be solved by a C^1 finite element method with $O(h)$ convergence in the energy norm. Note that previously $\bar{y} \in H^{\frac{5}{2}-\epsilon}$ was the best regularity result in the literature for Dirichlet elliptic distributed optimal control problems on smooth/convex domains with pointwise constraints on the gradient of the state.

The rest of the paper is organized as follows. The H^3 regularity of $\bar{y}$ is obtained in Section 2 through a variational inequality for $\bar{y}'$ that can be interpreted as a Neumann obstacle problem for the Laplace operator. The C^1 finite element method for (1.8)/(1.10) is analyzed in Section 3, followed by numerical results in Section 4. We end with some remarks on the extension to higher dimensions in Section 5.

2. A Variational Inequality for $\bar{y}'$

Observe that the set $\{y' : y \in K\}$ is the subset $\mathscr{K}$ of $H^1(I)$ given by

$$\mathscr{K} = \{v \in H^1(I) : \int_I v\,dx = 0 \quad \text{and} \quad v \le \psi \text{ on } I\}, \tag{2.1}$$

and the variational inequality (1.10) is equivalent to

$$\begin{aligned} &\int_I (\Phi - f')(q-p)dx + \int_I p'(q'-p')dx \\ &\qquad + [f(1)(q(1)-p(1)) - f(-1)(q(-1)-p(-1))] \ge 0 \qquad \forall\, q \in \mathscr{K}, \end{aligned} \tag{2.2}$$

where $p = \bar{y}'$, $q = y'$, and $\Phi \in H^1(I)$ is determined by

$$\beta\Phi' = y_d - \bar{y} \tag{2.3}$$

and

$$\int_I \Phi\,dx = 0. \tag{2.4}$$

Moreover (2.2) is the variational inequality that characterizes the solution of the following minimization problem:

$$\text{Find} \quad p = \operatorname*{argmin}_{q\in\mathscr{K}} \Big[\frac{1}{2}\int_I (q')^2 dx + \int_I (\Phi - f')q\,dx + f(1)q(1) - f(-1)q(-1)\Big]. \tag{2.5}$$

2.1. **A Neumann Obstacle Problem.** The minimization problem (2.5), which is a Neumann obstacle problem, can be written more conveniently as

$$p = \operatorname*{argmin}_{q\in\mathscr{K}} \Big[\frac{1}{2} b(q,q) + (\phi, q) + \tau q(1) - \sigma q(-1)\Big], \tag{2.6}$$

where $\sigma = f(-1)$, $\tau = f(1)$,

$$b(q,r) = \int_I q' r'\,dx, \quad (\phi, q) = \int_I \phi\, q\,dx \quad \text{and} \quad \phi = \Phi - f'. \tag{2.7}$$

Note that we have a compatibility condition

$$\int_I \phi\,dx + \tau - \sigma = 0 \tag{2.8}$$

that follows from (1.5), (2.4) and the Fundamental Theorem of Calculus for absolutely continuous functions.

Since $b(\cdot,\cdot)$ is coercive on $H^1(I)/\mathbb{R}$, the obstacle problem defined by (2.1) and (2.6) has a unique solution p characterized by the variational inequality

$$b(p, q-p) + (\phi, q-p) + \tau(q(1) - p(1)) - \sigma(q(-1) - p(-1)) \geq 0 \qquad \forall\, q \in \mathscr{K}. \tag{2.9}$$

Theorem 2.1. *The solution $p = \bar{y}' \in \mathscr{K}$ of (2.6)/(2.9) belongs to $H^2(I)$.*

Proof. We begin by observing that

$$b(p, q-p) + (\phi, q-p) + \tau(q(1) - p(1)) - \sigma(q(-1) - p(-1)) \geq 0 \qquad \forall\, q \in \tilde{K}, \tag{2.10}$$

where

$$\tilde{K} = \{q \in H^1(I) :\ q \leq \psi \ \text{ in } I \ \text{ and } \int_I q\,dx \geq 0\}. \tag{2.11}$$

Indeed, $q \in \tilde{K}$ implies $q - \bar{q} \in K$, where $\bar{q}$ is the mean of q over I, and hence, in view of (2.8), the definition of $b(\cdot,\cdot)$ in (2.7) and (2.9),

$$\begin{aligned} &b(p, q-p) + (\phi, q-p) + \tau(q(1) - p(1)) - \sigma((q(-1) - p(-1)) \\ &\quad = b(p, q-\bar{q}-p) + (\phi, q-\bar{q}-p) + \tau(q(1) - \bar{q} - p(1)) - \sigma(q(-1) - \bar{q} - p(-1)) \\ &\quad \geq 0 \end{aligned}$$

for all $q \in \tilde{K}$.

Let $\mathfrak{K} \subset H^1(I)$ be defined by

$$\mathfrak{K} = \{q \in H^1(I) :\ q \leq\ \psi \ \text{ in } I\}, \tag{2.12}$$

and $G : H^1(I) \longrightarrow [0,\infty)$ be defined by

$$G(q) = \int_I q\,dx. \tag{2.13}$$

Then the function ψ belongs to $\mathfrak{K}$ and

$$G(\psi) > 0 \tag{2.14}$$

by (1.6).

It follows from the Slater condition (2.14) and the theory of Lagrange multipliers [17, Chapter 1, Theorem 1.6] that there exists a nonnegative number λ such that

$$b(p,q-p)+(\phi,q-p)+\tau((q(1)-p(1))-\sigma(q(-1)-p(-1))-\lambda\int_I(q-p)dx\geq 0 \tag{2.15}$$

for all $q\in\mathfrak{K}$.

Finally we observe that

$$\tilde{b}(p,q-p)+(F,q-p)+\tau(q(1)-p(1))-\sigma(q(-1)-p(-1))\geq 0 \qquad \forall\, q\in\mathfrak{K}, \tag{2.16}$$

where

$$\tilde{b}(q,r)=\int_I q'r'dx+\int_I qr\,dx \tag{2.17}$$

and

$$F=\phi-\lambda-p. \tag{2.18}$$

The variational inequality defined by (2.12), (2.16) and (2.17) characterizes the solution of a coercive Neumann obstacle problem on $H^1(I)$. Since $F\in L_2(I)$ and $\psi\in H^2(I)$, we can apply the result in [23, Chapter 5, Theorem 3.4] to conclude that $p\in H^2(I)$. $\square$

We can deduce the regularity of $(\bar{y},\bar{u})$ from the relations $p=\bar{y}'$ and $\bar{u}=-(\bar{y}''+f)$.

Corollary 2.2. *Under the assumption* (1.5) *on the data, the solution* $(\bar{y},\bar{u})$ *of the optimal control problem* (1.1)–(1.6) *belongs to* $H^3(I)\times H^1(I)$.

Remark 2.3. The result in [23], which is for dimensions ≥ 2, requires a compatibility condition between $\partial\psi/\partial n$ and the Neumann boundary condition so that the boundary trace of the normal derivative of the solution of the obstacle problem belongs to the correct Sobolev space. This is not needed in one dimension since the boundary values of the normal derivative are just numbers.

2.2. **The Karush-Kuhn-Tucker Conditions.** It follows from (2.7), Theorem 2.1 and integration by parts that

$$b(p,q)+(\phi,q)+\tau q(1)-\sigma q(-1)-\lambda\int_I q\,dx+\int_I q\,d\nu=0 \qquad \forall\, q\in H^1(I), \tag{2.19}$$

where the regular Borel measure ν is given by

$$d\nu=(p''-\phi+\lambda)dx+[p'(-1)+\sigma]d\delta_{-1}-[p'(1)+\tau]d\delta_1, \tag{2.20}$$

and δ_{-1} (resp., δ_1) is the Dirac point measure at -1 (resp., 1).

Let $\mathscr{A}$ be the active set of the derivative constraint (1.4), i.e.,

$$\mathscr{A}=\{x\in[-1,1]:\,\bar{y}'(x)=\psi(x)\}=\{x\in[-1,1]:\,p(x)=\psi(x)\}. \tag{2.21}$$

By a standard argument, p satisfies (2.15) if and only if

$$\nu \text{ is nonnegative and supported on } \mathscr{A}. \tag{2.22}$$

We can translate (2.19)–(2.22) into Karush-Kuhn-Tucker (KKT) conditions for the solution $\bar{y}' = p \in \mathscr{K}$ of (2.5)/(2.9), which is summarized in the following theorem.

Theorem 2.4. *There exists a nonnegative number λ such that*

$$\begin{aligned}(2.23)\quad &\int_I p'q'dx + \int_I (\Phi - f')q\,dx + f(1)q(1) - f(-1)q(-1) + \int_I q\,d\nu \\ &\qquad = \lambda \int_I q\,dx \qquad \forall\, q \in H^1(I),\end{aligned}$$

$$(2.24)\quad \int_{[-1,1]} (p - \psi)d\nu = 0,$$

$$(2.25)\quad d\nu = \rho\,dx + \gamma d\delta_{-1} + \zeta d\delta_1,$$

where

$$(2.26)\quad \rho = p'' + f' - \Phi + \lambda \in L_2(I) \text{ is nonnegative a.e.},$$

$$(2.27)\quad \gamma = p'(-1) + f(-1) \text{ and } \zeta = -[p'(1) + f(1)] \text{ are nonnegative numbers},$$

and $\Phi \in H^1(I)$ satisfies (2.3)–(2.4).

Remark 2.5. The KKT conditions (2.23)–(2.27) are also sufficient conditions for (2.15). Indeed, they imply, for any $q \in \mathscr{K}$,

$$\begin{aligned}&\int_I p'(q' - p')dx + \int_I (\Phi - f')q\,dx + f(1)\big(q(1) - p(1)\big) - f(-1)\big(q(-1) - p(-1)\big) \\ &\qquad = \lambda \int_I (q - p)dx - \int_I (q - p)d\nu \\ &\qquad = -\int_I (q - \psi)d\nu \geq 0,\end{aligned}$$

which then also implies $\bar{y}(x) = \int_{-1}^{x} p(t)dt$ is the solution of (1.8).

Finally we observe that Theorem 2.4 implies

$$(2.28)\quad \beta \int_I (\bar{y}'' + f)z''dx + \int_I (\bar{y} - y_d)z\,dx + \int_{[-1,1]} z'd\mu = 0 \qquad \forall\, z \in H^2(I) \cap H_0^1(I),$$

where

$$(2.29)\quad \mu = \beta\nu \text{ is a nonnegative Borel measure},$$

and

$$(2.30)\quad \int_{[-1,1]} (\bar{y}' - \psi)d\mu = 0.$$

3. The Discrete Problem

Let $V_h \subset H^2(I) \cap H_0^1(I)$ be the cubic Hermite finite element space (cf. [12, 5]) associated with a triangulation/partition $\mathcal{T}_h$ of I with mesh size h. The discrete problem is to find

$$\bar{y}_h = \operatorname*{argmin}_{y_h \in K_h} \frac{1}{2}\big(\|y_h - y_d\|_{L_2(I)}^2 + \beta\|y_h'' + f\|_{L_2(I)}^2\big), \tag{3.1}$$

where

$$K_h = \{y_h \in V_h : \, P_h y_h' \le P_h \psi\}, \tag{3.2}$$

and P_h is the interpolation operator associated with the P_1 finite element space associated with $\mathcal{T}_h$, i.e., the constraint (1.3) is only enforced at the grid points.

The nodal interpolation operator from $C^1([-1,1])$ onto V_h is denoted by Π_h.

We will use the following standard estimates for P_h and Π_h (cf. [12, 5]) in the error analysis:

$$\|\zeta - P_h\zeta\|_{L_2(I)} \le Ch|\zeta|_{H^1(I)} \qquad \forall\, \zeta \in H^1(I), \tag{3.3}$$

$$\|\zeta - P_h\zeta\|_{L_2(I)} \le Ch^2|\zeta|_{H^2(I)} \qquad \forall\, \zeta \in H^2(I), \tag{3.4}$$

$$|\zeta - \Pi_h\zeta|_{H^1(I)} + h|\zeta - \Pi_h\zeta|_{H^2(I)} \le Ch^2|\zeta|_{H^3(I)} \qquad \forall\, \zeta \in H^3(I). \tag{3.5}$$

Here and below we use C to denote a generic positive constant that is independent of the mesh size h.

The unique solution $\bar{y}_h \in K_h$ of the minimization problem defined by (3.1) and (3.2) is characterized by the discrete variational inequality

$$\beta \int_I (\bar{y}_h'' + f)(y_h'' - \bar{y}_h'')dx + \int_I (\bar{y}_h - y_d)(y_h - \bar{y}_h)dx \ge 0 \qquad \forall\, y_h \in K_h,$$

which can also be written as

$$a(\bar{y}_h, y_h - \bar{y}_h) \ge \int_I y_d(y_h - \bar{y}_h)dx - \beta \int_I f(y_h'' - \bar{y}_h'')dx \qquad \forall\, y_h \in K_h, \tag{3.6}$$

where the bilinear form $a(\cdot,\cdot)$ is defined in (1.11).

The error analysis of the finite element method is based on the approach in [6] for state constrained optimal control problems that was extended to one dimensional problems with constraints on the derivative of the state in [9].

We will use the energy norm $\|\cdot\|_a$ defined by

$$\|v\|_a^2 = a(v,v) = \|v\|_{L_2(I)}^2 + \beta|v|_{H^2(I)}^2. \tag{3.7}$$

Note that

$$\|v\|_a \approx \|v\|_{H^2(I)} \qquad \forall\, v \in H^2(I) \cap H_0^1(I) \tag{3.8}$$

by a Poincaré-Friedrichs inequality [20].

3.1. **An Abstract Error Estimate.** In view of (3.6), (3.7) and the Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
\|\bar{y}-\bar{y}_h\|_a^2 &= a(\bar{y}-\bar{y}_h,\bar{y}-y_h)+a(\bar{y}-\bar{y}_h,y_h-\bar{y}_h)\\
&\leq \frac{1}{2}\|\bar{y}-\bar{y}_h\|_a^2+\frac{1}{2}\|\bar{y}-y_h\|_a^2+a(\bar{y},y_h-\bar{y}_h)\\
&\qquad -\int_I y_d(y_h-\bar{y}_h)dx+\beta\int_I f(y_h''-\bar{y}_h'')dx \qquad \forall\, y_h\in K_h.
\end{aligned}
\tag{3.9}
$$

It follows from (2.28), (2.30) and (3.2) that

$$
\begin{aligned}
&a(\bar{y},y_h-\bar{y}_h)-\int_I y_d(y_h-\bar{y}_h)dx+\beta\int_I f(y_h''-\bar{y}_h'')dx\\
&\quad=\int_{[-1,1]}(\bar{y}_h'-y_h')d\mu\\
&\quad=\int_{[-1,1]}(\bar{y}_h'-P_h\bar{y}_h')d\mu+\int_{[-1,1]}(P_h\bar{y}_h'-P_h\psi)d\mu+\int_{[-1,1]}(P_h\psi-\psi)d\mu\\
&\qquad\quad+\int_{[-1,1]}(\psi-\bar{y}')d\mu+\int_{[-1,1]}(\bar{y}'-y_h')d\mu\\
&\quad\leq\int_{[-1,1]}(\bar{y}_h'-P_h\bar{y}_h')d\mu+\int_{[-1,1]}(P_h\psi-\psi)d\mu+\int_{[-1,1]}(\bar{y}'-y_h')d\mu
\end{aligned}
\tag{3.10}
$$

for all $y_h\in K_h$.

Putting (3.9) and (3.10) together, we arrive at the abstract error estimate

$$
\begin{aligned}
\|\bar{y}-\bar{y}_h\|_a^2 &\leq 2\Big(\int_{[-1,1]}(\bar{y}_h'-P_h\bar{y}_h')d\mu+\int_{[-1,1]}(P_h\psi-\psi)d\mu\Big)\\
&\qquad+\inf_{y_h\in K_h}\Big(\|\bar{y}-y_h\|_a^2+2\int_{[-1,1]}(\bar{y}'-y_h')d\mu\Big).
\end{aligned}
\tag{3.11}
$$

3.2. **Concrete Error Estimates.** The three terms on the right-hand side of (3.11) can be estimated as follows.

First of all, we have

$$
\begin{aligned}
\int_{[-1,1]}(\bar{y}_h'-P_h\bar{y}_h')d\mu &= \int_{[-1,1]}\big[(\bar{y}_h'-\bar{y}')-P_h(\bar{y}_h'-\bar{y}')\big]d\mu+\int_{[-1,1]}(\bar{y}'-P_h\bar{y}')d\mu\\
&=\beta\Big(\int_I\big[(\bar{y}_h'-\bar{y}')-P_h(\bar{y}_h'-\bar{y}')\big]\rho\,dx+\int_I(\bar{y}'-P_h\bar{y}')\rho\,dx\Big)\\
&\leq C\big(h\|\bar{y}-\bar{y}_h\|_a+h^2|y|_{H^3(I)}\big),
\end{aligned}
\tag{3.12}
$$

by Corollary 2.2, (2.25), (2.29), (3.3), (3.4), (3.8) and the fact that $\zeta-P_h\zeta$ vanishes at the points ±1 for any $\zeta\in H^1(I)$.

Similarly we can derive

$$\int_{[-1,1]} (P_h\psi - \psi)d\mu = \beta \int_I (P_h\psi - \psi)\rho\, dx \le Ch^2|\psi|_{H^2(I)} \tag{3.13}$$

by (1.5) and (3.4).

Finally we have

$$\begin{aligned} &\inf_{y_h \in K_h} \Big(\|\bar{y} - y_h\|_a^2 + 2\int_{[-1,1]} (\bar{y}' - y_h')d\mu \Big) \\ &\qquad \le \|\bar{y} - \Pi_h\bar{y}\|_a^2 + 2\int_{[-1,1]} \big[\bar{y}' - (\Pi_h\bar{y})'\big] d\mu \\ &\qquad = \|\bar{y} - \Pi_h\bar{y}\|_a^2 + 2\beta\int_I \big[\bar{y}' - (\Pi_h\bar{y})'\big]\rho\, dx \le Ch^2\big[|\bar{y}|_{H^3(I)}^2 + |\bar{y}|_{H^3(I)}\big], \end{aligned} \tag{3.14}$$

by Corollary 2.2, (2.25), (2.29), (3.5), (3.8) and the fact that $\bar{y}' - (\Pi_h\bar{y})'$ vanishes at ± 1.

It follows from (3.11)–(3.14) and Young's inequality that

$$\|\bar{y} - \bar{y}_h\|_a \le Ch, \tag{3.15}$$

which immediately implies the following result, where $\bar{u}_h = -(\bar{y}_h'' + f)$ is the approximation for $\bar{u} = -(\bar{y} + f)$.

Theorem 3.1. *Under the assumptions on the data in* (1.5)*, we have*

$$|\bar{y} - \bar{y}_h|_{H^1(I)} + \|\bar{u} - \bar{u}_h\|_{L_2(I)} \le Ch. \tag{3.16}$$

Remark 3.2. Numerical results in Section 4 indicate that the estimate for $\|\bar{u} - \bar{u}_h\|_{L_2(I)}$ in Theorem 3.1 is sharp.

Remark 3.3. For comparison, the error estimate

$$|\bar{y} - \bar{y}_h|_{H^1(I)} + \|\bar{u} - \bar{u}_h\|_{L_2(I)} \le C_\epsilon h^{\frac{1}{2}-\epsilon}$$

was obtained in [9] under the assumptions in (1.7).

4. A Numerical Experiment

We begin by constructing an example for the problem (1.8)/(1.10) with a known exact solution.

4.1. **An Example.** Let $\beta = 1$,

$$\psi(x) = \begin{cases} 1 - \frac{9}{2}x^2 & -1 \le x \le 0 \\ 1 & 0 \le x \le 1 \end{cases}, \tag{4.1}$$

and

$$\bar{y}(x) = \int_{-1}^{x} p(t)dt, \tag{4.2}$$

where

$$p(x) = \begin{cases} 1 - \frac{81}{32}(x - \frac{1}{3})^2 & -1 \le x \le \frac{1}{3} \\ 1 & \frac{1}{3} \le x \le 1 \end{cases}. \tag{4.3}$$

We have $\psi \in H^2(I)$,

$$\int_I \psi \, dx = \frac{1}{2}, \tag{4.4}$$

$p \in H^2(I)$,

$$p''(x) = \begin{cases} -81/16 & -1 < x < \frac{1}{3} \\ 0 & \frac{1}{3} < x < 1 \end{cases}, \tag{4.5}$$

$p'(1) = 0$, $p'(-1) = 27/4$,

$$\int_I p \, dx = 0, \quad p \le \psi \quad \text{and} \quad \mathscr{A} = \{-1\} \cup [1/3, 1]. \tag{4.6}$$

Let $f \in H^1(I)$ be defined by

$$f(x) = \begin{cases} \dfrac{2}{9\pi} \sin(\pi(3x - 1)) & -1 < x \le \frac{1}{3} \\ -(x - \frac{1}{3})^2 & \frac{1}{3} \le x < 1 \end{cases}. \tag{4.7}$$

We have $f(-1) = 0$, $f(1/3) = 0$, $f'_-(1/3) = 2/3$, $f'_+(1/3) = 0$ and $f(1) = -4/9$. Therefore the function

$$\Phi(x) = \begin{cases} f'(x) & -1 < x < \frac{1}{3} \\ f'(x) + \frac{2}{3} & \frac{1}{3} < x < 1 \end{cases} \tag{4.8}$$

belongs to $H^1(I)$ and

$$\int_I \Phi \, dx = \int_I f'(x) + \int_{\frac{1}{3}}^1 \frac{2}{3} \, dx = f(1) - f(-1) + \frac{4}{9} = 0. \tag{4.9}$$

Finally we take $\lambda = 81/16$ and $y_d = \bar{y} + \Phi'$. Then the KKT conditions (2.23)–(2.27) are satisfied with

$$\begin{aligned} d\nu &= [p'' + f' - \Phi + \lambda]dx + [p'(-1) + f(-1)]d\delta_{-1} - [p'(1) + f(1)]d\delta_1 \\ &= (211/48)\chi_{[1/3,1]}dx + (27/4)d\delta_{-1} + (4/9)d\delta_1, \end{aligned}$$

where $\chi_{[1/3,1]}$ is the characteristic function of the interval $[1/3, 1]$.

4.2. **Numerical Results.** We solved the problem in Section 4.1 by the finite element method in Section 3 on uniform meshes. The results are displayed in Table 4.1.

We observe $O(h)$ convergence in the H^2 norm which agrees with Theorem 3.1. On the other hand the convergence in the H^1 norm is $O(h^2)$, better than the $O(h)$ convergence predicted by Theorem 3.1. The convergence in L_2 and L_∞ is also $O(h^2)$.

$2/h$	$\|\bar{y}-\bar{y}_h\|_{L_2(I)}$	$\|\bar{y}-\bar{y}_h\|_{L_\infty(I)}$	$\lvert\bar{y}-\bar{y}_h\rvert_{H^1(I)}$	$\lvert\bar{y}-\bar{y}_h\rvert_{H^2(I)}$
$1+2^0$	1.430334e-01	1.625937e-01	2.581989e-01	8.660252e-01
$1+2^1$	1.216070e-01	1.385037e-01	2.199480e-01	7.486796e-01
$1+2^2$	4.306657e-02	4.679253e-02	8.061916e-02	4.485156e-01
$1+2^3$	1.613494e-02	1.850729e-02	2.919318e-02	2.573315e-01
$1+2^4$	3.439341e-03	3.849954e-03	6.315816e-03	1.266029e-01
$1+2^5$	9.590453e-04	1.087740e-03	1.741244e-03	6.470514e-02
$1+2^6$	2.256478e-04	2.542346e-04	4.125212e-04	3.223430e-02
$1+2^7$	5.874304e-05	6.639870e-05	1.067193e-04	1.618687e-02
$1+2^8$	1.425640e-05	1.608790e-05	2.549283e-05	8.086258e-03
$1+2^9$	3.657433e-06	4.124680e-06	6.430499e-06	4.047165e-03

TABLE 4.1. Numerical results for the example in Section 4.1

5. Concluding Remarks

We have shown that higher regularity for the solutions of one dimensional Dirichlet elliptic distributed optimal control problems with pointwise constraints on the derivative of the state can be obtained through a variational inequality satisfied by the derivative of the optimal state. A similar result for one dimensional optimal control problems with mixed boundary conditions was obtained earlier in [9]. A natural question is: Can these results be extended to higher dimensions?

For analogs of (1.1)–(1.4) on a smooth/convex domain $\Omega \in \mathbb{R}^d$ $(d=2,3)$, where $f \in H^1(\Omega)$ and $\boldsymbol{\Psi} \in [H^2(\Omega)]^d$, one can also derive a variational inequality for the gradient of the optimal state. Observe that the space $\boldsymbol{G}$ of the gradients of the states is characterized by (cf. [15, Chapter I, Section 2.3])

$$\begin{aligned}\boldsymbol{G} &= \{\nabla y :\ y \in H^2(\Omega) \cap H_0^1(\Omega)\}\\ &= \{\boldsymbol{q} \in [H^1(\Omega)]^d :\ \mathrm{curl}\,\boldsymbol{q} = 0 \text{ on } \Omega \text{ and } \boldsymbol{n} \times \boldsymbol{q} = 0 \text{ on } \partial\Omega\},\end{aligned}$$

where $\boldsymbol{n}$ is the unit outward normal along $\partial\Omega$.

Let $\boldsymbol{K}$ be the subset of $\boldsymbol{G}$ defined by

$$\boldsymbol{K} = \{\boldsymbol{q} \in \boldsymbol{G} :\ \boldsymbol{q} \le \boldsymbol{\Psi} \text{ a.e. in } \Omega\}.$$

We assume that $\boldsymbol{K}$ is nonempty, which is the case for example if $\boldsymbol{\Psi} \ge \mathbf{0}$.

The analog of (2.2) is given by

$$\int_\Omega (\boldsymbol{\Phi} - \nabla f)\cdot(\boldsymbol{q}-\boldsymbol{p})dx + \int_\Omega \mathrm{div}\,\boldsymbol{p}\,\mathrm{div}\,(\boldsymbol{q}-\boldsymbol{p})\,dx + \int_{\partial\Omega} f(\boldsymbol{q}-\boldsymbol{p})\cdot\boldsymbol{n}\,dS \ge 0 \tag{5.1}$$

for all $\boldsymbol{q} \in \boldsymbol{K}$, where $\boldsymbol{p} = \nabla\bar{y} \in \boldsymbol{K}$, and $\boldsymbol{\Phi} \in \boldsymbol{G}$ is defined by $\beta \mathrm{div}\,\boldsymbol{\Phi} = y_d - \bar{y}$, which is an analog of (2.3). The variational inequality (5.1) is uniquely solvable because (cf. [15, Chapter I, Sections 3.2 and 3.4])

$$\int_\Omega (\operatorname{div}\boldsymbol{q})^2 dx \geq C_\Omega |\boldsymbol{q}|^2_{H^1(\Omega)} \qquad \forall\, \boldsymbol{q} \in \boldsymbol{G}.$$

We can also write (5.1) as

$$\begin{aligned}\int_\Omega (\boldsymbol{\Phi} - \nabla f)\cdot(\boldsymbol{q}-\boldsymbol{p})dx + \int_\Omega \big[\operatorname{div}\boldsymbol{p}\operatorname{div}(\boldsymbol{q}-\boldsymbol{p}) + \operatorname{curl}\boldsymbol{p}\cdot\operatorname{curl}(\boldsymbol{q}-\boldsymbol{p})\big]\,dx \\ + \int_{\partial\Omega} f(\boldsymbol{q}-\boldsymbol{p})\cdot\boldsymbol{n}\,dS \geq 0 \qquad \forall\,\boldsymbol{q}\in\boldsymbol{K},\end{aligned} \tag{5.2}$$

which can be interpreted as an obstacle problem for the vector Laplacian operator with natural boundary conditions.

In order to obtain higher regularity for the optimal state $\bar{y}$, one will need regularity results for (5.1)/(5.2), which unfortunately are not available. Therefore the problem of extending the results in this paper to higher dimensions remains open.

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