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Issei Kitagawa

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A STOCHASTIC PROCESS ASSOCIATED WITH THE WEIGHTED WHITE NOISE DIFFERENTIATION

ISSEI KITAGAWA

Abstract. In this paper, we give a relationship between the weighted white noise differentiation and the Lévy Laplacian introducing an operator changing the white noise $\dot{B}(t)$ by $\dot{B}(t)^2$. In addition, we give an infinite dimensional stochastic process generated by the weighted white noise differentiation and a relationship between the stochastic process and the Lévy Laplacian.

1. Introduction

In [2], Kondratiev and Streit introduced a family $(\mathcal{E})^*_\beta$, $0 \le \beta < 1$, of generalized functions and a family $(\mathcal{E})_\beta$, $0 \le \beta < 1$ of test functions. Let $\mathcal{E} = \mathcal{S}(\mathbf{R})$ be the Schwartz space of rapidly decreasing functions on $\mathbf{R}$. Then, taking a complete orthonormal basis $\{\zeta_n\}_{n=0}^\infty \subset \mathcal{E}$ for $L^2(T)$ with a fixed finite interval $T \subset \mathbf{R}$, we define the generalized Lévy Laplacian $\Delta_L(h)\Phi$ of $\Phi \in (\mathcal{E})^*_\beta$ by

$$S[\Delta_L(h)\Phi](\xi) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} S[\Phi]''(\xi)(h\zeta_n, h\zeta_n)$$

for each ξ in the complexification $\mathcal{E}_c$ of $\mathcal{E}$ and $h \in \mathcal{E}$ with $\mathrm{supp}(h) \subset T$. Define an operator L on $(\mathcal{E})_\beta$ by

$$L\varphi = S^{-1}[S\varphi(\xi^2)], \quad \varphi \in (\mathcal{E})_\beta.$$

Then the operator L is a continuous linear operator from $(\mathcal{E})_\beta$ into $(\mathcal{E})^*_\beta$ for $\frac{1}{2} \le \beta < 1$. For $\frac{1}{2} \le \beta < 1$, we define an operator $\overline{\Delta_L(h)}$ on $\overline{L[(\mathcal{E})_\beta]}$ by $\overline{\Delta_L(h)}L\varphi = \sum_{n=0}^\infty \Delta_L(h)L\varphi_n$ for $\varphi = \sum_{n=0}^\infty \varphi_n \in (\mathcal{E})_\beta$, and denote $\overline{\Delta_L(h)}$ by the same notation $\Delta_L(h)$. Hence the operator $\Delta_L(h)$ is a continuous linear operator from $L[(\mathcal{E})_\beta]$ into $(\mathcal{E})^*_\beta$. Moreover the operator $\Delta_L(h)$ is a continuous linear operator from $L[(\mathcal{E})_\beta]$ into itself. Let $D_h = \int_T h(u)\frac{\partial}{\partial x(u)}du$, $x \in \mathcal{E}^*$, for $h \in \mathcal{E}$ with $\mathrm{supp}(h) \subset T$. The weighted white noise differentiation D_h is a continuous linear operator from $(\mathcal{E})_\beta$ into itself (see [4]). This operator on $(\mathcal{E})_\beta$ is same as the Gross differentiation on $(\mathcal{E})_\beta$. Then we can give a relationship between the operator D_h and the generalized Lévy Laplacian $\Delta_L(h)$ through the operator L as

$$\frac{1}{2}\Delta_L(h)L\varphi = \frac{1}{|T|}LD_{h^2}\varphi$$

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for any $\varphi \in (\mathcal{E})_\beta$. Hence we have also

$$e^{-\frac{t}{2}|T|\Delta_L(h)} L\varphi = L[e^{-tD_{h^2}}\varphi]$$

for $t \geq 0$ and $\varphi \in (\mathcal{E})_\beta$. Let $\{e_k\}_{k=0}^\infty$ be an orthonormal basis for $L^2(\mathbf{R})$. Define an infinite dimensional stochastic process $\mathbf{X}(t)$ by

$$\mathbf{X}(t) = -te^t \sum_{k=0}^\infty e^{iX_k(t)} \langle h^2, e_k \rangle e_k, \quad t \geq 0,$$

for $h \in \mathcal{E}$ with $\mathrm{supp}(h) \subset T$, where $\{X_k(t); t \geq 0\}$, $k = 0, 1, 2, \ldots$, is a sequence consisting of independent Cauchy processes with the characteristic functions given by

$$E\left[e^{izX_k(t)}\right] = e^{-t|z|}, \quad z \in \mathbf{R},\ k = 0, 1, 2, \ldots.$$

Then $\mathbf{X}(t)$ is an $\mathcal{E}_c$-valued stochastic process generated by $\widetilde{D_{h^2}}S \equiv SD_{h^2}$ on $S[(\mathcal{E})_\beta]$. For $y \in \mathcal{E}_c$ let T_y be a translation operator defined on $(\mathcal{E})_\beta$ by

$$S(T_y\varphi)(\xi) = S\varphi(\xi + y), \quad \varphi \in (\mathcal{E})_\beta.$$

Then we have

$$e^{-\frac{t}{2}|T|\Delta_L(h)} L\varphi = L[E[T_{\mathbf{X}(t)}\varphi]], \quad t \geq 0$$

for any $\varphi \in (\mathcal{E})_\beta$.

In this paper, we give a relationship between the weighted white noise differentation and the Lévy Laplacian [4, 6, 7, 8] acting on a space of white noise distributions introducing an operator changing the white noise $\dot{B}(t)$ by $\dot{B}(t)^2$. Moreover based on infinitely many Cauchy processes, we give a relationship between an infinite dimensional stochastic process generated by the weighted white noise differentiation and the Lévy Laplacian.

The paper is organized as follows. In Section 2 we summarize some basic definitions and results in the white noise analysis. In Section 3 we give a relationship between the weighted white noise differentiation and the Lévy Laplacian acting on a space of white noise distributions introducing an operator changing the white noise $\dot{B}(t)$ by $\dot{B}(t)^2$. In Section 4, based on infinitely many Cauchy processes, we give an infinite dimensional stochastic process generated by the weighted white noise differentiation and a relationship between the stochastic process and the Lévy Laplacian.

2. White Noise Background

Let $L^2(\mathbf{R})$ be a real separable Hilbert space with norm $|\cdot|_0$ and $\mathcal{E} = \mathcal{S}(\mathbf{R})$ be the Schwartz space of rapidly decreasing functions on $\mathbf{R}$. Let $A = -d^2/du^2 + u^2 + 1$. This is a densely defined self-adjoint operator on $L^2(\mathbf{R})$ and there exists an orthonormal basis $\{e_\nu; \nu \geq 0\}$ for $L^2(\mathbf{R})$ such that $Ae_\nu = (2\nu + 2)e_\nu$, $\nu = 0, 1, 2, \ldots$. We define the norm $|\cdot|_p$ by $|f|_p = |A^p f|_0$ for $f \in \mathcal{E}$ and $p \in \mathbf{R}$. Let $\mathcal{E}_p = \{f \in \mathcal{E}; |f|_p < \infty\}$. Then $\mathcal{E}_p$ is a real separable Hilbert space with the norm $|\cdot|_p$ and the dual space $\mathcal{E}_p'$ of $\mathcal{E}_p$ is the same as $\mathcal{E}_{-p}$. Let $\mathcal{E}$ be the projective limit space of $\{\mathcal{E}_p; p \geq 0\}$ and $\mathcal{E}'$ the dual space of $\mathcal{E}$. Then $\mathcal{E}$ becomes a nuclear space with the Gel'fand triple $\mathcal{E} \subset L^2(\mathbf{R}) \subset \mathcal{E}'$. We denote the complexifications of $L^2(\mathbf{R})$, $\mathcal{E}$ and $\mathcal{E}_p$ by $L_c^2(\mathbf{R})$, $\mathcal{E}_c$ and $\mathcal{E}_{c,p}$, respectively.

By the Bochner-Minlos theorem there is a unique probability measure μ on $\mathcal{E}'$ such that

$$\int_{\mathcal{E}'} \exp\{i\langle x,\xi\rangle\} d\mu(x) = \exp\left(-\frac{1}{2}|\xi|_0^2\right), \quad \xi \in \mathcal{E},$$

where $\langle\cdot,\cdot\rangle$ is the canonical bilinear form on $\mathcal{E}' \times \mathcal{E}$.

The space $(L^2) = L^2(\mathcal{E}',\mu)$ of complex-valued square-integrable functionals defined on $\mathcal{E}'$ admits the well-known Wiener-Itô decomposition :

$$(L^2) = \bigoplus_{n=0}^{\infty} H_n,$$

where H_n is the space of multiple Wiener integrals of order $n \in \mathbf{N}$ and $H_0 = c$. Let $L_c^2(\mathbf{R})^{\widehat{\otimes} n}$ denote the n-fold symmetric tensor product of $L_c^2(\mathbf{R})$. If $\varphi \in (L^2)$ is represented by $\varphi = \sum_{n=0}^{\infty} \mathbf{I}_n(f_n)$, $f_n \in L_c^2(\mathbf{R})^{\widehat{\otimes} n}$, then the (L^2)-norm $\|\varphi\|_0$ is given by

$$\|\varphi\|_0 = \left(\sum_{n=0}^{\infty} n!|f_n|_0^2\right)^{\frac{1}{2}},$$

where $|\cdot|_0$ means also the norm of $L_c^2(\mathbf{R})^{\widehat{\otimes} n}$.

For $p \in \mathbf{R}$, let $\|\varphi\|_p = \|\Gamma(A)^p\varphi\|_0$, where $\Gamma(A)$ is the second quantization operator of A. If $p \geq 0$, let $(\mathcal{E}_p)$ be the domain of $\Gamma(A)^p$. If $p < 0$, let $(\mathcal{E}_p)$ be the completion of (L^2) with respect to the norm $\|\cdot\|_p$. Then $(\mathcal{E}_p)$, $p \in \mathbf{R}$, is a Hilbert space with the norm $\|\cdot\|_p$. It is easy to see that for $p > 0$, the dual space $(\mathcal{E}_p)^*$ of $(\mathcal{E}_p)$ is given by $(\mathcal{E}_{-p})$. Moreover, for any $p \in \mathbf{R}$, we have the decomposition

$$(\mathcal{E}_p) = \bigoplus_{n=0}^{\infty} H_n^{(p)},$$

where $H_n^{(p)}$ is the completion of $\{\mathbf{I}_n(f); f \in \mathcal{E}_c^{\widehat{\otimes} n}\}$ with respect to $\|\cdot\|_p$. Here $\mathcal{E}_c^{\widehat{\otimes} n}$ is the n-fold symmetric tensor product of $\mathcal{E}_c$. We also have $H_n^{(p)} = \{\mathbf{I}_n(f); f \in \mathcal{E}_{c,p}^{\widehat{\otimes} n}\}$ for any $p \in \mathbf{R}$, where $\mathcal{E}_{c,p}^{\widehat{\otimes} n}$ is also the n-fold symmetric tensor product of $\mathcal{E}_{c,p}$. The norm $\|\varphi\|_p$ of $\varphi = \sum_{n=0}^{\infty} \mathbf{I}_n(f_n) \in (\mathcal{E}_p)$ is given by

$$\|\varphi\|_p = \left(\sum_{n=0}^{\infty} n!|f_n|_p^2\right)^{\frac{1}{2}}, \quad f_n \in \mathcal{E}_{c,p}^{\widehat{\otimes} n},$$

where the norm of $\mathcal{E}_{c,p}^{\widehat{\otimes} n}$ is denoted also by $|\cdot|_p$.

Let $0 \leq \beta < 1$ be a fixed number. We define the norm $\|\cdot\|_{p,\beta}$ by

$$\|\varphi\|_{p,\beta} = \left(\sum_{n=0}^{\infty} (n!)^{1+\beta}|f_n|_p^2\right)^{\frac{1}{2}}, \quad f_n \in \mathcal{E}_{c,p}^{\widehat{\otimes} n}.$$

Let $(\mathcal{E}_p)_\beta = \{\varphi \in (L^2); \|\varphi\|_{p,\beta} < \infty\}$. Then $(\mathcal{E}_p)_\beta$ is a separable Hilbert space with the norm $\|\cdot\|_{p,\beta}$ and the dual space $(\mathcal{E}_p)_\beta^*$ of $(\mathcal{E}_p)_\beta$ is the same as $(\mathcal{E}_{-p})_{-\beta}$. Let $(\mathcal{E})_\beta$ be the projective limit space of $\{(\mathcal{E}_p)_\beta; p \geq 0\}$ and $(\mathcal{E})_\beta^*$ the dual space of $(\mathcal{E})_\beta$.

Then $(\mathcal{E})_\beta$ becomes a nuclear space with the Gel'fand triple $(\mathcal{E})_\beta \subset (L^2) \subset (\mathcal{E})_\beta^*$. We denote by $\ll \cdot, \cdot \gg$ the canonical bilinear form on $(\mathcal{E})_\beta^* \times (\mathcal{E})_\beta$. Then we have

$$\ll \Phi, \varphi \gg = \sum_{n=0}^{\infty} n! \langle F_n, f_n \rangle$$

for any $\Phi = \sum_{n=0}^{\infty} \mathbf{I}_n(F_n) \in (\mathcal{E})_\beta^*$ and $\varphi = \sum_{n=0}^{\infty} \mathbf{I}_n(f_n) \in (\mathcal{E})_\beta$, where the canonical bilinear form on $(\mathcal{E}_c^{\otimes n})^* \times (\mathcal{E}_c^{\otimes n})$ is denoted also by $\langle \cdot, \cdot \rangle$.

Since $\exp\langle \cdot, \xi \rangle \in (\mathcal{E})_\beta$, the *S-transform* is defined on $(\mathcal{E})_\beta^*$ by

$$S[\Phi](\xi) = \exp\left(-\frac{1}{2}\langle \xi, \xi \rangle \right) \ll \Phi, \exp\langle \cdot, \xi \rangle \gg, \quad \xi \in \mathcal{E}_c.$$

We have the following characterization theorem of the S-transform.

Theorem 2.1 ([2, 4]). *A complex-valued function F on $\mathcal{E}_c$ is the S-transform of an element in $(\mathcal{E})_\beta^*$ if and only if F satisfies the conditions :*

1) *For any ξ and η in $\mathcal{E}_c$, the function $F(z\xi + \eta)$ is an entire function of $z \in \boldsymbol{C}$.*
2) *There exist nonnegative constants K, a, and p such that*

$$|F(\xi)| \le K \exp\left[a|\xi|_p^{\frac{2}{1-\beta}} \right], \quad \forall \xi \in \mathcal{E}_c.$$

Consider $F = S[\Phi]$ with $\Phi \in (\mathcal{E})_\beta^*$. By Theorem 2.1, for any $\xi, \eta \in \mathcal{E}_c$ the function $F(\xi + z\eta)$ admits the series expansion:

$$F(\xi + z\eta) = \sum_{n=0}^{\infty} \frac{z^n}{n!} F^{(n)}(\xi)(\eta, \ldots, \eta),$$

where $F^{(n)}(\xi) : \mathcal{E}_c \times \cdots \times \mathcal{E}_c \to \mathbf{C}$ is a continuous n-linear functional.

Fix a finite interval T in $\mathbf{R}$. Take an orthonormal basis $\{\zeta_n\}_{n=0}^{\infty} \subset \mathcal{E}$ for $L^2(T)$ satisfying the equally dense and uniform boundedness property. Let $\mathcal{D}_L$ denote the set of all $\Phi \in (\mathcal{E})_\beta^*$ such that the limit

$$\widetilde{\Delta}_L(h) S[\Phi](\xi) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} S[\Phi]''(\xi)(h\zeta_n, h\zeta_n)$$

exists for any $\xi \in \mathcal{E}_c$ and $h \in \mathcal{E}$ with $\mathrm{supp}(h) \subset T$. The generalized Lévy Laplacian $\Delta_L(h)$ is defined by

$$\Delta_L(h)\Phi = S^{-1} \widetilde{\Delta}_L(h) S\Phi, \quad \Phi \in \mathcal{D}_L.$$

3. Weighted White Noise Differentiation and the Lévy Laplacian

Define an operator L on $(\mathcal{E})_\beta$ by

$$L\varphi = S^{-1}[S\varphi(\xi^2)], \quad \varphi \in (\mathcal{E})_\beta.$$

Then we have the following theorem.

Theorem 3.1. *Let $\frac{1}{2} \le \beta < 1$. For all $\varphi \in (\mathcal{E})_\beta$, $L\varphi$ is in $(\mathcal{E})_\beta^*$, the operator L is a continuous linear operator from $(\mathcal{E})_\beta$ into $(\mathcal{E})_\beta^*$.*

Proof. For $\varphi = \sum_{n=0}^{\infty} \mathbf{I}_n(f_n) \in (\mathcal{E})_\beta$, we have

$$\begin{aligned}
L\varphi &= \sum_{n=0}^{\infty} \int_{\mathbf{R}^n} f_n(u_1, \dots, u_n) : x(u_1)^2 \cdots x(u_n)^2 : d\mathbf{u} \\
&= \sum_{n=0}^{\infty} \int_{\mathbf{R}^{2n}} \int_{\mathbf{R}^n} f_n(u_1, \dots, u_n) \delta_{u_1}^{\otimes 2} \widehat{\otimes} \cdots \widehat{\otimes} \delta_{u_n}^{\otimes 2}(v_1, \dots, v_{2n}) d\mathbf{u} \\
&\qquad : x(v_1) \cdots x(v_{2n}) : dv_1 \cdots dv_{2n}.
\end{aligned}$$

Denote $G_n(v_1, \dots, v_{2n}) = \int_{\mathbf{R}^n} f_n(u_1, \dots, u_n) \delta_{u_1}^{\otimes 2} \widehat{\otimes} \cdots \widehat{\otimes} \delta_{u_n}^{\otimes 2}(v_1, \dots, v_{2n}) d\mathbf{u}$ and $g_{k_1, \cdots, k_{2n}}(u_1, \cdots, u_n) = e_{k_1}(u_1) \cdots e_{k_{2n}}(u_n)$. Then we obtain

$$\begin{aligned}
&\langle G_n, e_{k_1} \otimes \cdots \otimes e_{k_{2n}} \rangle^2 \\
&= \left(\int_{\mathbf{R}^n} f_n(u_1, \dots, u_n) e_{k_1}(u_1) e_{k_2}(u_1) \cdots e_{k_{2n}}(u_n) d\mathbf{u} \right)^2 \\
&\leq |f_n|_q^2 |g_{k_1, \cdots, k_{2n}}|_{-q}^2 \\
&= |f_n|_q^2 \sum_{l_1, \cdots, l_{2n}=0}^{\infty} (2l_1 + 2)^{-2q} \cdots (2l_{2n} + 2)^{-2q} |\langle g_{k_1, \cdots, k_{2n}}, e_{l_1} \otimes \cdots \otimes e_{l_{2n}} \rangle^2| \\
&= |f_n|_q^2 (2k_1 + 2)^{-2q} \cdots (2k_{2n} + 2)^{-2q}
\end{aligned}$$

for some $q > 0$. Hence for any $p > 0$ and $\frac{1}{2} \leq \beta < 1$ there exists $q > 0$ such that

$$\begin{aligned}
\|L\varphi\|_{-p,-\beta}^2 &= \sum_{n=0}^{\infty} (2n)!^{1-\beta} |G_n|_{-p,-\beta}^2 \\
&= \sum_{n=0}^{\infty} (2n)!^{1-\beta} \sum_{k_1, \cdots, k_{2n}=0}^{\infty} (2k_1 + 2)^{-2p} \cdots (2k_{2n} + 2)^{-2p} \times \\
&\qquad |\langle G_n, e_{k_1} \otimes \cdots \otimes e_{k_{2n}} \rangle^2| \\
&\leq \sum_{n=0}^{\infty} (2n)!^{1-\beta} \sum_{k_1, \cdots, k_{2n}=0}^{\infty} (2k_1 + 2)^{-2(p+q)} \cdots (2k_{2n} + 2)^{-2(p+q)} |f_n|_q^2 \\
&= \sum_{n=0}^{\infty} (n!)^{1+\beta} \frac{(2n)!^{1-\beta}}{(n!)^{1+\beta}} \left(\sum_{k=0}^{\infty} (2k+2)^{-2(p+q)} \right)^{2n} |f_n|_q^2.
\end{aligned}$$

For $\beta \geq \frac{1}{2}$, there exists $N \in \mathbf{N}$ such that $(2n)!^{1-\beta} < (n!)^{1+\beta}$ for all $n \geq N$. Hence for $\beta \geq \frac{1}{2}$ we obtain

$$\begin{aligned}
\|L\varphi\|_{-p,-\beta}^2 \leq &\sum_{n=0}^{N-1} (n!)^{1+\beta} \frac{(2n)!^{1-\beta}}{(n!)^{1+\beta}} \left(\sum_{k=0}^{\infty} (2k+2)^{-2(p+q)} \right)^{2n} |f_n|_q^2 \\
&+ \sum_{n=N}^{\infty} (n!)^{1+\beta} \left(\sum_{k=0}^{\infty} (2k+2)^{-2(p+q)} \right)^{2n} |f_n|_q^2.
\end{aligned}$$

Define $M = \max\left\{\frac{(2n)!^{1-\beta}}{(n!)^{1+\beta}} \mid n = 0, 1, 2, \ldots, N-1\right\}$. Then for large $p > 0$ there exists $q > 0$ such that

$$\begin{aligned}\|L\varphi\|_{-p,-\beta}^2 &\le M \sum_{n=0}^{\infty} (n!)^{1+\beta} \left(\sum_{k=0}^{\infty} (2k+2)^{-2(p+q)}\right)^{2n} |f_n|_q^2 \\ &\le M \sum_{n=0}^{\infty} (n!)^{1+\beta} |f_n|_q^2 \\ &= M\|\varphi\|_{q,\beta}^2.\end{aligned}$$

The poof is completed. □

We define an operator $\overline{\Delta_L(h)}$ on $L[(\mathcal{E})_\beta]$ by

$$\overline{\Delta_L(h)} L\varphi = \sum_{n=0}^{\infty} \Delta_L(h) L\varphi_n \quad \text{for } \varphi = \sum_{n=0}^{\infty} \varphi_n \in (\mathcal{E})_\beta,$$

and denote $\overline{\Delta_L(h)}$ also by the same notation $\Delta_L(h)$.

Theorem 3.2. *Let $\frac{1}{2} \le \beta < 1$. The operator $\Delta_L(h)$ is a continuous linear operator from $L[(\mathcal{E})_\beta]$ into $(\mathcal{E})_\beta^*$.*

Proof. For $\varphi = \sum_{n=0}^{\infty} \mathbf{I}_n(f_n) \in (\mathcal{E})_\beta$, we have

$$\begin{aligned}&\sum_{n=0}^{\infty} \Delta_L(h) L[\mathbf{I}_n(f_n)] \\ &= \sum_{n=0}^{\infty} 2n \int_{\mathbf{R}^n} h(u_n)^2 f_n(u_1, \ldots, u_n) : x(u_1)^2 \cdots x(u_{n-1})^2 : d\mathbf{u} \\ &= \sum_{n=0}^{\infty} 2n \int_{\mathbf{R}^{2(n-1)}} \int_{\mathbf{R}^n} h(u_n)^2 f_n(u_1, \ldots, u_n) \delta_{u_1}^{\otimes 2} \widehat{\otimes} \cdots \widehat{\otimes} \delta_{u_{n-1}}^{\otimes 2}(v_1, \ldots, v_{2(n-1)}) d\mathbf{u} \\ &\qquad\qquad : x(v_1) \cdots x(v_{2(n-1)}) : dv_1 \cdots dv_{2(n-1)}.\end{aligned}$$

Let $H_n(v_1, \ldots, v_{2(n-1)}) = 2n \int_{\mathbf{R}^n} h(u_n)^2 f_n(\mathbf{u}) \delta_{u_1}^{\otimes 2} \widehat{\otimes} \cdots \widehat{\otimes} \delta_{u_{n-1}}^{\otimes 2}(v_1, \ldots, v_{2(n-1)}) d\mathbf{u}$. Then for $p > 0$ and $\frac{1}{2} \le \beta < 1$, it holds that

$$\begin{aligned}\sum_{n=0}^{\infty} (2n-2)!^{1-\beta} |H_n|_{-p,-\beta}^2 = \sum_{n=0}^{\infty} (2n-2)!^{1-\beta} \sum_{k_1, \cdots, k_{2(n-1)}=0}^{\infty} (2k_1+2)^{-2p} \cdots \\ (2k_{2(n-1)}+2)^{-2p} |\langle H_n, e_{k_1} \otimes \cdots \otimes e_{k_{2(n-1)}} \rangle^2|.\end{aligned}$$

Let $g_{k_1, \cdots, k_{2(n-1)}}(u_1, \cdots, u_{n-1}) = e_{k_1}(u_1) \cdots e_{k_{2(n-1)}}(u_{n-1})$. Then we obtain

$$\begin{aligned}&\langle H_n, e_{k_1} \otimes \cdots \otimes e_{k_{2(n-1)}} \rangle^2 \\ &= \left(2n \int_{\mathbf{R}^n} h(u_n)^2 f_n(u_1, \ldots, u_n) e_{k_1}(u_1) e_{k_2}(u_1) \cdots e_{k_{2(n-1)}}(u_{n-1}) d\mathbf{u}\right)^2 \\ &\le (2n)^2 |h|_\infty^4 |f_n|_q^2 |g_{k_1, \cdots, k_{2(n-1)}}|_{-q}^2\end{aligned}$$

for some $q > 0$. Since

$$\begin{aligned}
&|g_{k_1,\cdots,k_{2(n-1)}}|_{-q}^2 \\
&= \sum_{l_1,\cdots,l_{2(n-1)}=0}^{\infty} (2l_1+2)^{-2q}\cdots(2l_{2(n-1)}+2)^{-2q}\times \\
&\qquad\qquad |\langle g_{k_1,\cdots,k_{2(n-1)}}, e_{l_1}\otimes\cdots\otimes e_{l_{2(n-1)}}\rangle^2| \\
&= \sum_{l_1,\cdots,l_{2(n-1)}=0}^{\infty} (2l_1+2)^{-2q}\cdots(2l_{2(n-1)}+2)^{-2q}\delta_{k_1,l_1}\cdots\delta_{k_{2(n-1)},l_{2(n-1)}} \\
&= (2k_1+2)^{-2q}\cdots(2k_{2(n-1)}+2)^{-2q},
\end{aligned}$$

we see that for any $p > 0$ there exists $q > 0$ such that

$$\begin{aligned}
&\|\Delta_L(h)L\varphi\|_{-p,-\beta}^2 \\
&\le |h|_\infty^4 \sum_{n=0}^{\infty}(2n-2)!^{1-\beta}(2n)^2\left(\sum_{k=0}^{\infty}(2k+2)^{-2(p+q)}\right)^{2(n-1)} |f_n|_q^2 \\
&= |h|_\infty^4 \sum_{n=0}^{\infty}(n!)^{1+\beta}\frac{(2n-2)!^{1-\beta}(2n)^2}{(n!)^{1+\beta}}\left(\sum_{k=0}^{\infty}(2k+2)^{-2(p+q)}\right)^{2(n-1)} |f_n|_q^2.
\end{aligned}$$

For $\beta \ge \frac{1}{2}$, there exists $N \in \mathbf{N}$ such that $(2n-2)!^{1-\beta}(2n)^2 \le (n!)^{1+\beta}$ for all $n \ge N$. Hence for $\beta \ge \frac{1}{2}$ we obtain

$$\begin{aligned}
&\|\Delta_L(h)L\varphi\|_{-p,-\beta}^2 \\
&\le |h|_\infty^4 \sum_{n=0}^{N-1}(n!)^{1+\beta}\frac{(2n-2)!^{1-\beta}(2n)^2}{(n!)^{1+\beta}}\left(\sum_{k=0}^{\infty}(2k+2)^{-2(p+q)}\right)^{2(n-1)} |f_n|_q^2 \\
&\qquad + |h|_\infty^4 \sum_{n=N}^{\infty}(n!)^{1+\beta}\left(\sum_{k=0}^{\infty}(2k+2)^{-2(p+q)}\right)^{2(n-1)} |f_n|_q^2.
\end{aligned}$$

Define $K = |h|_\infty^4 \max\left\{\frac{(2n-2)!^{1-\beta}(2n)^2}{(n!)^{1+\beta}} \mid n = 0, 1, 2, \ldots, N-1\right\}$. Then for large $p > 0$ there exists $q > 0$ such that

$$\begin{aligned}
\|\Delta_L(h)L\varphi\|_{-p,-\beta}^2 &\le K\sum_{n=0}^{\infty}(n!)^{1+\beta}\left(\sum_{k=0}^{\infty}(2k+2)^{-2(p+q)}\right)^{2(n-1)} |f_n|_q^2 \\
&\le K\sum_{n=0}^{\infty}(n!)^{1+\beta}|f_n|_q^2 \\
&= K\|\varphi\|_{q,\beta}^2.
\end{aligned}$$

Thus by Theorem 3.1, the poof is completed. □

We can easily check that $\Delta_L(h)\Phi \in L[(\mathcal{E})_\beta]$ for each $\Phi \in L[(\mathcal{E})_\beta]$. Then we have the following corollary.

Corollary 3.3. *The operator $\Delta_L(h)$ is a continuous linear operator from $L[(\mathcal{E})_\beta]$ into itself.*

Let $D_h = \int_T h(u)\frac{\partial}{\partial x(u)}du$, $x \in \mathcal{E}^*$, for $h \in \mathcal{E}$ with $\operatorname{supp}(h) \subset T$. Then we have the next theorem.

Theorem 3.4 ([4])**.** *The weighted white noise differentiation D_h is a continuous linear operator from $(\mathcal{E})_\beta$ into itself.*

Example 3.5. For $\varphi = \,:\exp(c\int_{\mathbf{R}} g(u)x(u)du):$, the equality holds :

$$\Delta_L(h)\varphi = \sum_{n=0}^{\infty} \frac{c^n}{n!}\Delta_L(h) : \left(\int_{\mathbf{R}} g(u)x(u)du\right)^n : .$$

Theorem 3.6. *For any $\varphi \in (\mathcal{E})_\beta$, the equality holds :*

$$\frac{1}{2}\Delta_L(h)L\varphi = \frac{1}{|T|}LD_{h^2}\varphi. \tag{3.1}$$

Proof. For $\varphi = \,:\exp(c\int_{\mathbf{R}} g(u)x(u)du): \,\in (\mathcal{E})_\beta$, $g \in \mathcal{E}$, $c \in \mathbf{C}$, we have

$$LD_{h^2}\varphi = c\int_T g(u)h(u)^2du L\varphi.$$

On the other hand, by Example 3.5, we have

$$\frac{1}{2}\Delta_L(h)L\varphi = \frac{c}{|T|}\int_T g(u)h(u)^2du L\varphi.$$

Since the linear span of $:\exp(c\int_{\mathbf{R}} g(u)x(u)du):$, $g \in \mathcal{E}$, $c \in \mathbf{C}$ is dense in $(\mathcal{E})_\beta$, by the continuities of L, D_{h^2} and $\Delta_L(h)$ from Theorem 3.1, 3.4 and Corollary 3.3, we obtain (3.1). □

Corollary 3.7. *For any $t \geq 0$ and $\varphi \in (\mathcal{E})_\beta$, the equality holds :*

$$e^{-\frac{t}{2}|T|\Delta_L(h)}L\varphi = L[e^{-tD_{h^2}}\varphi]. \tag{3.2}$$

Proof. For $\varphi = \,:\exp(c\int_{\mathbf{R}} g(u)x(u)du): \,\in (\mathcal{E})_\beta$, $g \in \mathcal{E}$, $c \in \mathbf{C}$. By Theorem 3.1, 3.6, we have

$$\begin{aligned} e^{-\frac{t}{2}|T|\Delta_L(h)}L\varphi &= \sum_{n=0}^{\infty}\frac{(-\frac{t}{2}|T|)^n}{n!}\Delta_L^n(h)L\varphi \\ &= \sum_{n=0}^{\infty}\frac{(-t)^n c^n}{n!}\left(\int_T g(u)h(u)^2du\right)^n L\varphi \\ &= L\left[\sum_{n=0}^{\infty}\frac{(-t)^n c^n}{n!}\left(\int_T g(u)h(u)^2du\right)^n \varphi\right] \\ &= L[e^{-tD_{h^2}}\varphi]. \end{aligned}$$

Since the linear span of $:\exp(c\int_{\mathbf{R}} g(u)x(u)du):$, $g \in \mathcal{E}$, $c \in \mathbf{C}$ is dense in $(\mathcal{E})_\beta$, by the continuities of L, D_{h^2} and $\Delta_L(h)$ from Theorem 3.1, 3.4 and Corollary 3.3, we obtain (3.2). □

4. An Infinite Dimensional Stochastic Process

Let $\{X_k(t); t \geq 0\}$, $k = 0, 1, 2, \ldots$, be an infinite sequence consisting of independent Cauchy processes with the characteristic functions are given by

$$E\left[e^{izX_k(t)}\right] = e^{-t|z|}, \quad z \in \mathbf{R},\ k = 0, 1, 2, \ldots.$$

Let $\{\mathbf{X}(t); t \geq 0\}$ be an infinite dimensional stochastic process defined by

$$\mathbf{X}(t) = -te^t \sum_{k=0}^{\infty} e^{iX_k(t)} \langle h^2, e_k \rangle e_k, \quad t \geq 0,$$

for $h \in \mathcal{E}$ with $\operatorname{supp}(h) \subset T$.

Proposition 4.1. *For all $t \geq 0$ we have $\boldsymbol{X}(t) \in \mathcal{E}_c$ (a.e.).*

Proof. For all $p \in \mathbf{R}$, we have

$$\begin{aligned} E\left[|\mathbf{X}(t)|_p^2\right] &= E\left[\sum_{\nu=0}^{\infty} (2\nu+2)^{2p} |\langle \mathbf{X}(t), e_\nu \rangle|^2\right] \\ &= \sum_{\nu=0}^{\infty} (2\nu+2)^{2p} E\left[|\langle \mathbf{X}(t), e_\nu \rangle|^2\right]. \end{aligned}$$

Since

$$\begin{aligned} E\left[|\langle \mathbf{X}(t), e_\nu \rangle|^2\right] &= E\left[\left| te^t \left\langle \sum_{k=0}^{\infty} e^{iX_k(t)} \langle h^2, e_k \rangle e_k, e_\nu \right\rangle \right|^2\right] \\ &= E\left[t^2 e^{2t} \left| e^{iX_\nu(t)} \langle h^2, e_\nu \rangle \right|^2\right] \\ &= t^2 e^{2t} E\left[\left| e^{iX_\nu(t)} \right|^2 \left| \langle h^2, e_\nu \rangle \right|^2\right] \\ &= t^2 e^{2t} \left| \langle h^2, e_\nu \rangle \right|^2, \end{aligned}$$

we obtain

$$\begin{aligned} E\left[|\mathbf{X}(t)|_p^2\right] &= t^2 e^{2t} \sum_{\nu=0}^{\infty} (2\nu+2)^{2p} \left| \langle h^2, e_\nu \rangle \right|^2 \\ &= t^2 e^{2t} |h^2|_p^2. \end{aligned}$$

Hence for all $p \in \mathbf{R}$, $E\left[|\mathbf{X}(t)|_p^2\right] < \infty$ holds. Therefore we have $\mathbf{X}(t) \in \mathcal{E}_c$ (*a.e.*). □

For $y \in \mathcal{E}_c$ let T_y be a translation operator defined on $(\mathcal{E})_\beta$ by

$$S(T_y\varphi)(\xi) = S\varphi(\xi + y), \quad \varphi \in (\mathcal{E})_\beta.$$

Theorem 4.2 ([4])**.** *Let $y \in \mathcal{E}_c'$. For any $p \geq 0$, $q > 0$ with $|y|_{-p} < \infty$ and $2^{2q-1+\beta} \geq 1$, it holds that*

$$\|T_y\varphi\|_{p,\beta} \leq \|\varphi\|_{p+q,\beta} (1 - 2^{-2q+1-\beta})^{-\frac{1}{2}} \exp\left[(1+\beta) 4^{-\frac{q+\beta}{1+\beta}} |y|_{-p}^{\frac{2}{1+\beta}}\right], \quad \varphi \in (\mathcal{E})_\beta.$$

Lemma 4.3. *Let $\widetilde{\mathcal{E}}$ be a linear span of* $:\exp(c\int_{\mathbf{R}} g(u)x(u)du):$, $g \in \mathcal{E}$, $c \in \boldsymbol{C}$. *Then, for any $\varphi \in (\mathcal{E})_\beta$, there exists a sequence $(\varphi_l)_{l=1}^\infty \subset \widetilde{\mathcal{E}}$ such that*

$$E[ST_{\boldsymbol{X}(t)}\varphi(\xi)] = \lim_{l\to\infty} E[ST_{\boldsymbol{X}(t)}\varphi_l(\xi)].$$

Proof. Since $\mathbf{X}(t) \in \mathcal{E}_c$, by Theorem 4.2 there exists $K > 0$ such that

$$E\left[\|T_{\mathbf{X}(t)}\varphi\|_{p,\beta}^2\right] \leq K\|\varphi\|_{p+q,\beta}^2, \quad \varphi \in (\mathcal{E})_\beta,$$

for any $p \geq 0$ and $q > 0$. Since $\widetilde{\mathcal{E}}$ is dense in $(\mathcal{E})_\beta$ (see [1]), for any $\varphi \in (\mathcal{E})_\beta$ there exists a sequence $(\varphi_l)_{l=1}^\infty \subset \widetilde{\mathcal{E}}$ such that $\|\varphi_l - \varphi\|_{p,\beta} \to 0$ as $l \to \infty$, for all p. Hence we have

$$\begin{aligned}
E[|ST_{\mathbf{X}(t)}\varphi_l(\xi) - ST_{\mathbf{X}(t)}\varphi(\xi)|] &= E[|\langle T_{\mathbf{X}(t)}(\varphi_l - \varphi), \phi_\xi\rangle|] \\
&\leq E[\|T_{\mathbf{X}(t)}(\varphi_l - \varphi)\|_{p,\beta}]\|\phi_\xi\|_{-p,-\beta} \\
&\leq K\|\varphi_l - \varphi\|_{p+q,\beta}\|\phi_\xi\|_{-p,-\beta}.
\end{aligned}$$

Consequently we obtain

$$\begin{aligned}
E[|ST_{\mathbf{X}(t)}\varphi_l(\xi) - ST_{\mathbf{X}(t)}\varphi(\xi)|] &\leq K\|\varphi_l - \varphi\|_{p+q,\beta}\|\phi_\xi\|_{-p,-\beta} \\
&\to 0 \quad (l \to \infty).
\end{aligned}$$

□

Theorem 4.4. *The stochastic process $\boldsymbol{X}(t)$ is generated by $\widetilde{D_{h^2}S} \equiv SD_{h^2}$.*

Proof. For $\varphi = :\exp(c\int_{\mathbf{R}} g(u)x(u)du): \in (\mathcal{E})_\beta$, $g \in \mathcal{E}$, $c \in \mathbf{C}$, we have

$$\begin{aligned}
&E[S\varphi(\xi + \mathbf{X}(t))] \\
&= \sum_{n=0}^\infty \frac{c^n}{n!} E\left[\left\{\int_{\mathbf{R}} g(u)\xi(u)du - te^t \int_T g(u) \sum_{k=0}^\infty e^{iX_k(t)}\langle h^2, e_k\rangle e_k(u)du\right\}^n\right] \\
&= \sum_{n=0}^\infty \frac{c^n}{n!}\left(\int_{\mathbf{R}} g(u)\xi(u)du - t\int_T g(u)h(u)^2du\right)^n \\
&= \exp\left(-ct\int_T g(u)h(u)^2du\right) S\varphi(\xi) \\
&= e^{-t\widetilde{D_{h^2}}} S\varphi(\xi).
\end{aligned}$$

Since the linear span of $:\exp(c\int_{\mathbf{R}} g(u)x(u)du):$, $g \in \mathcal{E}$, $c \in \mathbf{C}$ is dense in $(\mathcal{E})_\beta$, and by the continuity of $e^{-t\widetilde{D_{h^2}}}$ and Lemma 4.3, the proof is completed. □

Theorem 4.5. *For any $\varphi \in (\mathcal{E})_\beta$, the following equality holds*

$$e^{-\frac{t}{2}|T|\Delta_L(h)}L\varphi = L[E[T_{\boldsymbol{X}(t)}\varphi]], \quad t \geq 0. \tag{4.1}$$

Proof. By Proposition 4.1 for $\varphi \in (\mathcal{E})_\beta$, we have

$$S(T_{\mathbf{X}(t)}\varphi)(\xi^2) = S\varphi(\xi^2 + \mathbf{X}(t)).$$

Hence for $\varphi = :\exp(c\int_{\mathbf{R}} g(u)x(u)du):\in(\mathcal{E})_\beta$, $g\in\mathcal{E}$, $c\in\mathbf{C}$, we see that

$$\begin{aligned}
&E[S(T_{\mathbf{X}(t)}\varphi)(\xi^2)] \\
&= \sum_{n=0}^\infty \frac{c^n}{n!} E\left[\left(\int_{\mathbf{R}} g(u)\xi(u)^2du - te^t\int_T g(u)\sum_{k=0}^\infty e^{iX_k(t)}\langle h^2, e_k\rangle e_k(u)du\right)^n\right] \\
&= \sum_{n=0}^\infty \frac{c^n}{n!}\left(\int_{\mathbf{R}} g(u)\xi(u)^2du - t\int_T g(u)h(u)^2du\right)^n \\
&= \exp\left(-ct\int_T g(u)h(u)^2du\right) S\varphi(\xi^2).
\end{aligned}$$

On the other hand, by Corollary 3.7 for $\varphi = :\exp(c\int_{\mathbf{R}} g(u)x(u)du):\in(\mathcal{E})_\beta$, $g\in\mathcal{E}$, $c\in\mathbf{C}$, we have

$$e^{-\frac{t}{2}|T|\widetilde{\Delta}_L(h)}SL\varphi(\xi) = \exp\left(-ct\int_T g(u)h(u)^2du\right)S\varphi(\xi^2).$$

Since the linear span of $:\exp(c\int_{\mathbf{R}} g(u)x(u)du):$, $g\in\mathcal{E}$, $c\in\mathbf{C}$ is dense in $(\mathcal{E})_\beta$, by Corollary 3.7 and Lemma 4.3, we obtain (4.1) for all $\varphi\in(\mathcal{E})_\beta$. □

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Issei Kitagawa: Department of Mathematics, Meijo University, Nagoya 468-8502, Japan

E-mail address: m0651501@ccmailg.meijo-u.ac.jp