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TRANSFER OF REGULARITY FOR MARKOV SEMIGROUPS BY USING AN INTERPOLATION TECHNIQUE

VLAD BALLY AND LUCIA CARAMELLINO*

Dedicated to the memory of Professor Hiroshi Kunita

Abstract. We study the regularity of a Markov semigroup $(P_t)_{t>0}$, that is, when $P_t(x,dy) = p_t(x,y)dy$ for a suitable smooth function $p_t(x,y)$. This is done by transferring the regularity from an approximating Markov semigroup sequence $(P_t^n)_{t>0}$, $n \in \mathbb{N}$, whose associated densities $p_t^n(x,y)$ are smooth and can blow up as $n \to \infty$. We use an interpolation type result and we show that if there exists a good equilibrium between the blow-up and the speed of convergence, then $P_t(x,dy) = p_t(x,y)dy$ and p_t has some regularity properties.

1. Introduction

In this paper we study Markov semigroups, that is, positive semigroups $(P_t)_{t\geq 0}$, such that $P_t 1 = 1$. The link with Markov processes is given by a family $P_t(x,dy)$, $t \geq 0$, $x \in \mathbb{R}^d$, of transition probability measures in $\mathbb{R}^d$ such that

$$P_t f(x) = \int_{\mathbb{R}^d} f(y) P_t(x,dy), \quad t \geq 0.$$

We study here the regularity of P_t, which is the property $P_t(x,dy) = p_t(x,y)dy$, $t > 0$, for a suitable smooth function $p_t(x,y)$, by transferring the regularity from an approximating Markov semigroup sequence $(P_t^n)_{t\geq 0}$, $n \in \mathbb{N}$.

Hereafter we assume that the domain of the Markov semigroup $(P_t)_{t\geq 0}$ contains the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ of the $C^\infty(\mathbb{R}^d)$ functions all of whose derivatives are rapidly decreasing. We assume that the semigroup is strongly continuous in its domain and we call L its infinitesimal generator. We suppose also that the domain of L contains $\mathcal{S}(\mathbb{R}^d)$ and for every $f \in \mathcal{S}(\mathbb{R}^d)$, $P_t f \in \mathcal{S}(\mathbb{R}^d)$, $t \geq 0$, and $Lf \in \mathcal{S}(\mathbb{R}^d)$.

Let $(P_t^n)_{t\geq 0}$, $n \in \mathbb{N}$, be a sequence of Markov semigroups:

$$P_t^n f(x) = \int_{\mathbb{R}^d} f(y) P_t^n(x,dy), \quad t \geq 0, n \in \mathbb{N}.$$

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For every n, we assume that $(P_t^n)_{t\geq 0}$ satisfies the same properties as $(P_t)_{t\geq 0}$: $\mathcal{S}(\mathbb{R}^d)$ is included in the domain of P_t^n and if $f\in\mathcal{S}(\mathbb{R}^d)$ then $P_tf\in\mathcal{S}(\mathbb{R}^d)$, $t\geq 0$; $(P_t^n)_{t\geq 0}$ is strongly continuous in its domain; the domain of its infinitesimal operator L_n contains $\mathcal{S}(\mathbb{R}^d)$ and $L_nf\in\mathcal{S}(\mathbb{R}^d)$ if $f\in\mathcal{S}(\mathbb{R}^d)$.

Classical results (Trotter Kato theorem, see e.g. [14]) assert that, as $n\to\infty$, if $L_n\to L$ then $P_t^n\to P_t$. The problem that we address in this paper is the following. We suppose that P_t^n has the regularity (density) property $P_t^n(x,dy)=p_t^n(x,y)dy$ with $p_t^n\in C^\infty(\mathbb{R}^d\times\mathbb{R}^d)$ and we ask under which hypotheses this property is inherited by the limit semigroup $(P_t)_{t\geq 0}$. If we know that p_t^n converges to some p_t in a sufficiently strong sense, of course we obtain $P_t(x,dy)=p_t(x,y)dy$. But in our framework p_t^n does not converge: here, p_t^n can even "blow up" as $n\to\infty$. However, if we may find a good equilibrium between the blow-up and the speed of convergence, then we are able to conclude that $P_t(x,dy)=p_t(x,y)dy$ and p_t has some regularity properties. This is an interpolation type result.

Roughly speaking our main result is as follows. We assume that the speed of convergence is controlled in the following sense: there exists some $a\in\mathbb{N}$ such that for every $q\in\mathbb{N}$

$$\left\|(L-L_n)f\right\|_{q,\infty}\leq\varepsilon_n\left\|f\right\|_{q+a,\infty}. \tag{1.1}$$

Here $\left\|f\right\|_{q,\infty}$ is the norm in the standard Sobolev space $W^{q,\infty}$. In fact we will work with weighted Sobolev spaces, and this is an important point. And also, we will assume a similar hypothesis for the adjoint $(L-L_n)^*$ (see Assumption 2.1 for a precise statement).

Moreover we assume a "propagation of regularity" property: there exist $b\in\mathbb{N}$ and $\Lambda_n\geq 1$ such that for every $q\in\mathbb{N}$

$$\left\|P_t^nf\right\|_{q,\infty}\leq\Lambda_n\left\|f\right\|_{q+b,\infty}. \tag{1.2}$$

Here also we will work with weighted Sobolev norms. And a similar hypothesis is supposed to hold for the adjoint $P_t^{*,n}$ (see Assumption 2.2 for a precise statement).

Finally we assume the following regularity property: for every $t\in(0,1]$, $P_t^n(x,dy)=p_t^n(x,y)dy$ with $p_t^n\in C^\infty(\mathbb{R}^d\times\mathbb{R}^d)$ and for every $\kappa\geq 0$, $t\in(0,1]$,

$$\left|\partial_x^\alpha\partial_y^\beta p_t^n(x,y)\right|\leq\frac{C}{(\lambda_nt)^{\theta_0(|\alpha|+|\beta|+\theta_1)}}\times\frac{(1+|x|^2)^{\pi(\kappa)}}{(1+|x-y|^2)^\kappa}. \tag{1.3}$$

Here, α,β are multi-indexes and $\partial_x^\alpha,\partial_y^\beta$ are the corresponding differential operators. Moreover, $\pi(\kappa)$, θ_0 and θ_1 are suitable parameters and $\lambda_n\to 0$ as $n\to\infty$ (we refer to Assumption 2.3). In concrete examples (jump type stochastic differential equations) λ_n is related to the lower eigenvalue of the Malliavin covariance matrix - essentially this is of order $\lambda_n^{\theta_0}$. And in order to handle the derivatives ∂_x^α and ∂_y^β we need to make $|\alpha|$ respectively $|\beta|$ integrations by parts (which involve $\lambda_n^{\theta_0}$). See also Assumption 3.6.

By (1.1)–(1.3), the rate of convergence is controlled by $\varepsilon_n\to 0$ and the blow-up of p_t^n is controlled by $\lambda_n^{-\theta_0}\to\infty$. So the regularity property may be lost as $n\to\infty$. However, if there is a good equilibrium between $\varepsilon_n\to 0$ and $\lambda_n^{-\theta_0}\to\infty$

and $\Lambda_n \to \infty$ then the regularity is saved: we ask that for some $\delta > 0$

$$\limsup_{n\to\infty} \frac{\varepsilon_n \Lambda_n}{\lambda_n^{\theta_0(a+b+\delta)}} < \infty, \tag{1.4}$$

the parameters a, b and θ_0 being given in (1.1), (1.2) and (1.3) respectively. Then $P_t(x,dy) = p_t(x,y)dy$ with $p_t \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ and the following upper bound holds: for every $\varepsilon > 0$, $\kappa \in \mathbb{N}$ and $R > 0$, one may find some constant $C, \pi(\kappa) > 0$ such that for very $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d$ with $|x| < R$ and $t \in (0,1]$

$$\left|\partial_x^\alpha \partial_y^\beta p_t(x,y)\right| \leq \frac{C}{t^{\theta_0(1+\frac{a+b}{\delta})(|\alpha|+|\beta|+2d+\varepsilon)}} \times \frac{(1+|x|^2)^{\pi(\kappa)}}{(1+|x-y|^2)^\kappa}. \tag{1.5}$$

This is the "transfer of regularity" that we mention in the title and which is stated in Theorem 2.6. The proof is based on a criterion of regularity for probability measures given in [4], which is close to interpolation spaces techniques.

The regularity criterion presented in this paper is tailored in order to handle the following example (which will be treated in a forthcoming paper). We consider the integro-differential operator

$$Lf(x) = \langle b(x), \nabla f(x)\rangle + \int_E \big(f(x+c(z,x)) - f(x) - \langle c(z,x), \nabla f(x)\rangle\big) d\mu(z) \tag{1.6}$$

where μ is an infinite measure on the normed space $(E, |\cdot|_E)$ such that $\int_E 1 \wedge |c(z,x)|^2 \, d\mu(z) < \infty$. Moreover, for a sequence $\delta_n \downarrow 0$, we denote

$$A_n^{i,j}(x) = \int_{\{|z|_E \leq \delta_n\}} c^i(z,x) c^j(z,x) d\mu(z)$$

and we define

$$\begin{aligned} L_n f(x) = \quad & \langle b(x), \nabla f(x)\rangle \\ & + \textstyle\int_{\{|z|_E \geq \delta_n\}} (f(x+c(z,x)) - f(x) - \langle c(z,x), \nabla f(x)\rangle) d\mu(z) \\ & + \frac{1}{2}\mathrm{tr}(A_n(x)\nabla^2 f(x)). \end{aligned} \tag{1.7}$$

By Taylor's formula,

$$\|Lf - L_n f\|_\infty \leq \|f\|_{3,\infty}\, \varepsilon_n \quad \text{with} \quad \varepsilon_n = \sup_x \int_{\{|z|_E \leq \delta_n\}} |c(z,x)|^3 \, d\mu(z)$$

(recall that $\|\cdot\|_{3,\infty}$ is the norm in the standard Sobolev space $W^{3,\infty}$). Under the uniform ellipticity assumption $A_n(x) \geq \lambda_n$ for every $x \in \mathbb{R}^d$, the semigroup $(P_t^n)_{t\geq 0}$ associated to L_n has the regularity property (1.3) with θ_0 depending on the measure μ. The speed of convergence in (1.1), with $a = 3$, is controlled by $\varepsilon_n \downarrow 0$. So, if (1.4) holds, then we obtain the regularity of P_t and the short time estimates (1.5).

The semigroup $(P_t)_{t\geq 0}$ associated to L corresponds to stochastic equations driven by the Poisson point measure $N_\mu(dt,dz)$ with intensity measure μ, so the problem of the regularity of P_t has been extensively discussed in the probabilistic literature. A first approach initiated by Bismut [9], Léandre [20] and Bichteler, Gravereaux and Jacod [8] (see also the recent monograph of Bouleau and Denis [10] and the bibliography therein), is done under the hypothesis that $E = \mathbb{R}^m$ and $\mu(dz) = h(z)dz$ with $h \in C^\infty(\mathbb{R}^m)$. Then one constructs a Malliavin type calculus based on the amplitude of the jumps of the Poisson point measure N_μ and employs this calculus in order to study the regularity of P_t. A second approach initiated by Carlen and Pardoux [12] (see also Bally and Clément [6]) follows the ideas in Malliavin calculus based on the exponential density of the jump times in order to study the same problem. Finally a third approach is due to Picard [22, 23], but see also Ishikawa and Kunita [16], the contributions of Kunita [17, 18] and the recent monograph by Ishikawa [15] for many references and developments in this direction. Picard constructs a Malliavin type calculus based on finite differences (instead of standard Malliavin derivatives) and obtains the regularity of P_t for a general class of intensity measures μ including purely atomic measures (in contrast with $\mu(dz) = h(z)dz$). We stress that all the above approaches work under different non degeneracy hypotheses, each of them corresponding to the specific noise that is used in the calculus. So in some sense we have not a single problem but three different classes of problems. The common feature is that the strategy in order to solve the problem follows the ideas from Malliavin calculus based on some noise contained in N_μ. Our approach is completely different because, as described above, we use the regularization effect of $\mathrm{tr}(A_n(x)\nabla^2)$. This regularization effect may be exploited either by using the standard Malliavin calculus based on the Brownian motion or using some analytical arguments. The approach that we propose in [5] is probabilistic, so employs the standard Malliavin calculus. But anyway, as mentioned above, the regularization effect vanishes as $n \to \infty$ and a supplementary argument based on the equilibrium given in (1.4) is used. We precise that the non degeneracy condition $A_n(x) \geq \lambda_n > 0$ is of the same nature as the one employed by J. Picard so the problem we solve is in the same class.

The idea of replacing "small jumps" (the ones in $\{|z|_E \leq \varepsilon_n\}$ here) by a Brownian part (that is $\mathrm{tr}(A_n(x)\nabla^2)$ in L_n) is not new – it has been introduced by Asmussen and Rosinski in [2] and has been extensively employed in papers concerned with simulation problems: since there is a huge amount of small jumps, they are difficult to simulate and then one approximates them by the Brownian part corresponding to $\mathrm{tr}(A_n(x)\nabla^2)$. See for example [1, 7, 13] and many others. However, at our knowledge, this idea has not been yet used in order to study the regularity of P_t.

The paper is organized as follows. In Section 2 we give the notation and the main results mentioned above and in Section 4 we give the proof of these results. Section 3 is devoted to some preliminary results about regularity. Namely, in Section 3.1 we recall and develop some results concerning regularity of probability measures, based on interpolation type arguments, coming from [4]. These are the main instruments used in the paper. In Section 3.2 we prove a regularity result which is a key point in our approach. In fact, it allows to handle the multiple integrals coming from the application of a Lindeberg method for the decomposition of $P_t - P_t^n$. Finally, in Appendix A and B we prove some technical results used in the paper.

2. Notation and Main Results

2.1. Notation. For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_m) \in \{1, \ldots, d\}^m$ we denote $|\alpha| = m$ (the length of the multi-index) and ∂^α is the derivative corresponding to

α, that is $\partial^{\alpha_m}\cdots\partial^{\alpha_1}$, with $\partial^{\alpha_i}=\partial_{x_{\alpha_i}}$. For $f\in C^\infty(\mathbb{R}^d\times\mathbb{R}^d)$, $(x,y)\in\mathbb{R}^d\times\mathbb{R}^d$ and two multi-indexes α and β, we denote by ∂_x^α the derivative with respect to x and by ∂_y^β the derivative with respect to y. Moreover, for $f\in C^\infty(\mathbb{R}^d)$ and $q\in\mathbb{N}$ we denote

$$\left|f\right|_q(x)=\sum_{0\leq|\alpha|\leq q}\left|\partial^\alpha f(x)\right|. \tag{2.1}$$

If f is not a scalar function, that is, $f=(f^i)_{i=1,\ldots,d}$ or $f=(f^{i,j})_{i,j=1,\ldots,d}$, we denote $\left|f\right|_q=\sum_{i=1}^d\left|f^i\right|_q$ respectively $\left|f\right|_q=\sum_{i,j=1}^d\left|f^{i,j}\right|_q$.

We will work with the weights

$$\psi_\kappa(x)=(1+|x|^2)^\kappa,\quad \kappa\in\mathbb{Z}. \tag{2.2}$$

The following properties hold:

- for every $\kappa\geq\kappa'\geq 0$,
$$\psi_\kappa(x)\leq\psi_{\kappa'}(x); \tag{2.3}$$
- for every $\kappa\geq 0$, there exists $C_\kappa>0$ such that
$$\psi_\kappa(x)\leq C_\kappa\psi_\kappa(y)\psi_\kappa(x-y); \tag{2.4}$$
- for every $\kappa\geq 0$, there exists $C_\kappa>0$ such that for every $\phi\in C_b^\infty(\mathbb{R}^d)$,
$$\psi_\kappa(\phi(x))\leq C_\kappa\psi_\kappa(\phi(0))(1+\|\nabla\phi\|_\infty^2)^\kappa\psi_\kappa(x); \tag{2.5}$$
- for every $q\in\mathbb{N}$ there exist $\overline{C}_q,\underline{C}_q>0$ such that for every $\kappa\in\mathbb{R}$ and $f\in C^\infty(\mathbb{R}^d)$,
$$\underline{C}_q\psi_\kappa\left|f\right|_q(x)\leq\left|\psi_\kappa f\right|_q(x)\leq\overline{C}_q\psi_\kappa\left|f\right|_q(x). \tag{2.6}$$

Note that (2.3)–(2.5) are immediate, whereas (2.6) is proved in Appendix A (see Lemma A.1).

For $q\in\mathbb{N}$, $\kappa\in\mathbb{R}$ and $p\in(1,\infty]$ (we stress that we include the case $p=+\infty$), we set $\|\cdot\|_p$ the usual norm in $L^p(\mathbb{R}^d)$ and

$$\|f\|_{q,\kappa,p}=\left\||\psi_\kappa f|_q\right\|_p. \tag{2.7}$$

We denote $W^{q,\kappa,p}$ to be the closure of $C^\infty(\mathbb{R}^d)$ with respect to the above norm. If $\kappa=0$ we just denote $\|f\|_{q,p}=\|f\|_{q,0,p}$ and $W^{q,p}=W^{q,0,p}$ (which is the usual Sobolev space). So, we are working with weighted Sobolev spaces. The weighted Sobolev spaces $W^{q,\kappa,p}$ are the natural framework in the paper [4] where the "balance argument" is obtained. There (see Theorem A.2 in [4]) we have used a crucial result of Petrushev and Xu [21] concerning the construction of kernels with polynomial decay at infinity. Then the weights ψ_κ appear in a natural way in order to capture the behaviour of the kernel at infinity.

The following properties hold:

- for every $q\in\mathbb{N}$ there exists $\overline{C}_q\geq\underline{C}_q>0$ such that for every $\kappa\in\mathbb{R}$, $p>1$ and $f\in W^{q,\kappa,p}(\mathbb{R}^d)$,
$$\underline{C}_q\|\psi_\kappa|f|_q\|_p\leq\|f\|_{q,\kappa,p}\leq\overline{C}_q\|\psi_\kappa|f|_q\|_p; \tag{2.8}$$

- for every $q\in\mathbb{N}$ and $p>1$ there exists $C_{q,p}>0$ such that for every $\kappa\in\mathbb{R}$ and $f\in W^{q,k,p}(\mathbb{R}^d)$,

$$\left\Vert f\right\Vert_{q,\kappa,p}\leq C_{q,p}\left\Vert f\right\Vert_{q,\kappa+d,\infty} \tag{2.9}$$

and if $p>d$,

$$\left\Vert f\right\Vert_{q,\kappa,\infty}\leq C_{q,p}\left\Vert f\right\Vert_{q+1,\kappa,p}\,; \tag{2.10}$$

- for $\kappa,\kappa'\in\mathbb{R}$, $q,q'\in\mathbb{N}$, $p\in(1,\infty]$ and $U:C^\infty(\mathbb{R}^d)\to C^\infty(\mathbb{R}^d)$, the following two assertions are equivalent: there exists a constant $C_*\geq 1$ such that for every f,

$$\left\Vert Uf\right\Vert_{q,\kappa,\infty}\leq C_*\left\Vert f\right\Vert_{q',\kappa',p} \tag{2.11}$$

and there exists a constant $C^*\geq 1$ such that for every f,

$$\left\Vert \psi_\kappa U\Big(\frac{1}{\psi_{\kappa'}}f\Big)\right\Vert_{q,\infty}\leq C^*\left\Vert f\right\Vert_{q',p}. \tag{2.12}$$

Notice that (2.8) is a consequence of (2.6). The inequality (2.9) is an immediate consequence of (2.6) and of the fact that $\psi_{-d}\in L^p(\mathbb{R}^d)$ for every $p\geq 1$. And the inequality (2.10) is a consequence of Morrey's inequality (Corollary IX.13 in [11]), whose use gives $\left\Vert f\right\Vert_{0,0,\infty}\leq\left\Vert f\right\Vert_{1,0,p}$, and of (2.6). In order to prove the equivalence between (2.11) and (2.12), one takes $g=\psi_{\kappa'}f$ (respectively $g=\frac{1}{\psi_{\kappa'}}f$) and uses (2.6) as well.

2.2. Main results. We consider a Markov semigroup $(P_t)_{t\geq 0}$ with infinitesimal operator L and a sequence $(P^n_t)_{t\geq 0}$, $n\in\mathbb{N}$, of Markov semigroups with infinitesimal operator L_n. We suppose that $\mathcal{S}(\mathbb{R}^d)$ is included in the domain of $(P_t)_{t\geq 0}$, $(P^n_t)_{t\geq 0}$, L and of L_n and we suppose that for $f\in\mathcal{S}(\mathbb{R}^d)$ we have $P_tf, P^n_tf, Lf, L_nf\in\mathcal{S}(\mathbb{R}^d)$.

We denote $\Delta_n=L-L_n$. Moreover, we denote by $P^{*,n}_t$ the formal adjoint of P^n_t and by Δ^*_n the formal adjoint of Δ_n that is

$$\left\langle P^{*,n}_tf,g\right\rangle=\left\langle f,P^n_tg\right\rangle\quad\text{and}\quad\left\langle \Delta^*_nf,g\right\rangle=\left\langle f,\Delta_ng\right\rangle, \tag{2.13}$$

$\langle\cdot,\cdot\rangle$ being the scalar product in $L^2(\mathbb{R}^d,dx)$.

We present now our hypotheses. The first one concerns the speed of convergence of $L_n\to L$.

Assumption 2.1. Let $a\in\mathbb{N}$, and let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a decreasing sequence such that $\lim_{n\to\infty}\varepsilon_n=0$.We assume that for every $q\in\mathbb{N},\kappa\geq 0$ and $p>1$ there exists $C>0$ such that for every $n\in\mathbb{N}$ and $f\in\mathcal{S}(\mathbb{R}^d)$,

$$(A_1)\qquad\left\Vert \Delta_nf\right\Vert_{q,-\kappa,\infty}\leq C\varepsilon_n\left\Vert f\right\Vert_{q+a,-\kappa,\infty}, \tag{2.14}$$

$$(A_1^*)\qquad\left\Vert \Delta^*_nf\right\Vert_{q,\kappa,p}\leq C\varepsilon_n\left\Vert f\right\Vert_{q+a,\kappa,p}. \tag{2.15}$$

Our second hypothesis concerns the "propagation of regularity" for the semigroups $(P^n_t)_{t\geq 0}$.

Assumption 2.2. Let $\Lambda_n \geq 1, n \in \mathbb{N}$, be an increasing sequence such that $\Lambda_{n+1} \leq \gamma\Lambda_n$ for some $\gamma \geq 1$. For every $q \in \mathbb{N}$ and $\kappa \geq 0, p > 1$, there exist $C > 0$ and $b \in \mathbb{N}$, such that for every $n \in \mathbb{N}$ and $f \in \mathcal{S}(\mathbb{R}^d)$

$$(A_2) \qquad \sup_{s\leq t} \|P_s^n f\|_{q,-\kappa,\infty} \leq C\Lambda_n \|f\|_{q+b,-\kappa,\infty}, \tag{2.16}$$

$$(A_2^*) \qquad \sup_{s\leq t} \|P_s^{*,n} f\|_{q,\kappa,p} \leq C\Lambda_n \|f\|_{q+b,\kappa,p}. \tag{2.17}$$

The hypothesis (A_2^*) is rather difficult to verify so, in Appendix B, we give some sufficient conditions in order to check it (see Proposition B.4).

Our third hypothesis concerns the "regularization effect" of the semi-group $(P_t^n)_{t\geq 0}$.

Assumption 2.3. We assume that

$$P_t^n f(x) = \int_{\mathbb{R}^d} p_t^n(x,y) f(y) dy \tag{2.18}$$

with $p_t^n \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$. Moreover, we assume there exist $\theta_0 > 0$ and a sequence λ_n, $n \in \mathbb{N}$, with, as $n \to \infty$,

$$\lambda_n \downarrow 0, \quad \lambda_n \leq \gamma\lambda_{n+1}, \tag{2.19}$$

for some $\gamma \geq 1$, such that the following property holds: for every $\kappa \geq 0, q \in \mathbb{N}$ there exist $\pi(q,\kappa)$, increasing in q and in κ, a constant $\theta_1 \geq 0$, and a constant $C > 0$ such that for every $n \in \mathbb{N}$, $t \in (0,1]$, for every multi-indexes α and β with $|\alpha| + |\beta| \leq q$ and $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d$

$$(A_3) \quad \left|\partial_x^\alpha \partial_y^\beta p_t^n(x,y)\right| \leq \frac{C}{(\lambda_n t)^{\theta_0(q+\theta_1)}} \times \frac{\psi_{\pi(q,\kappa)}(x)}{\psi_\kappa(x-y)}. \tag{2.20}$$

Note that in (2.20) we are quantifying the possible blow-up of $|\partial_x^\alpha \partial_y^\beta p_t^n(x,y)|$ as $n \to \infty$.

We also assume the following statements hold for the semigroup $(P_t)_{t\geq 0}$.

Assumption 2.4. For every $\kappa \geq 0, k \in \mathbb{N}$ there exists $C \geq 1$ such that

$$(A_4) \quad \|P_t f\|_{k,-\kappa,\infty} \leq C \|f\|_{k,-\kappa,\infty}. \tag{2.21}$$

Assumption 2.5. For every $\kappa \geq 0, k \in \mathbb{N}$ there exists $C \geq 1, \overline{\kappa} \geq \kappa$ such that

$$(A_5) \quad P_t\psi_\kappa(x) \leq C\psi_{\overline{\kappa}}(x). \tag{2.22}$$

Our main result is the following:

Theorem 2.6. *Suppose that Assumption 2.1, 2.2 , 2.3, 2.4 and 2.5 hold. Suppose also that for some* $\delta > 0$

$$\limsup_{n\to\infty} \frac{\varepsilon_n \Lambda_n}{\lambda_n^{\theta_0(a+b+\delta)}} < \infty.$$

Then $P_t(x,y) = p_t(x,y)$ *with* $p_t \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$. *Moreover, for every* $\kappa \in \mathbb{N}, R \in \mathbb{N}, \varepsilon > 0$ *and every multi-indexes* α *and* β *there exists some constants*

$C = C(R, \kappa, \varepsilon, \alpha, \beta)$ such that for every $t \in (0,1]$, $x \in \mathbb{R}^d$ with $|x| < R$ and $y \in \mathbb{R}^d$,

$$\left|\partial_x^\alpha \partial_y^\beta p_t(x,y)\right| \leq C \times t^{-\theta_0(1+\frac{a+b}{\delta})(|\alpha|+|\beta|+2d+\varepsilon)} \times \frac{1}{\psi_\kappa(x-y)} \tag{2.23}$$

with θ_0 from (2.20).

3. Regularity Results

This section is devoted to some preliminary results allowing us to prove the statements resumed in Section 2.2: in Section 3.1 we give an abstract regularity criterion, while in Section 3.2 we prove a regularity result for iterated integrals, that will be useful to handle a Lindeberg type decomposition of $P_t - P_t^n$.

3.1. A regularity criterion based on interpolation. Let us first recall some results obtained in [4] concerning the regularity of a measure μ on $\mathbb{R}^d$ (with the Borel σ-field). For two signed finite measures μ, ν and for $k \in \mathbb{N}$ we define the distance

$$d_k(\mu,\nu) = \sup\Big\{\Big|\int f d\mu - \int f d\nu\Big| : \|f\|_{k,\infty} \leq 1\Big\}. \tag{3.1}$$

If μ and ν are probability measures, d_0 is the total variation distance and d_1 is the Fortet Mourier distance. In this paper we will work with an arbitrary $k \in \mathbb{N}$. Notice also that $d_k(\mu,\nu) = \|\mu - \nu\|_{W_*^{k,\infty}}$ where $W_*^{k,\infty}$ is the dual of $W^{k,\infty}$.

We fix now $k, q, h \in \mathbb{N}$, with $h \geq 1$, and $p > 1$. Hereafter, we denote by $p_* = p/(p-1)$ the conjugate of p. Then, for a signed finite measure μ and for a sequence of absolutely continuous signed finite measures $\mu_n(dx) = f_n(x)dx$ with $f_n \in C^{2h+q}(\mathbb{R}^d)$, we define

$$\pi_{k,q,h,p}(\mu,(\mu_n)_n) = \sum_{n=0}^{\infty} 2^{n(k+q+d/p_*)} d_k(\mu,\mu_n) + \sum_{n=0}^{\infty} \frac{1}{2^{2nh}} \|f_n\|_{2h+q,2h,p}. \tag{3.2}$$

The following result is the key point in our approach:

Lemma 3.1. *Let $k, q, h \in \mathbb{N}$ with $h \geq 1$, and $p > 1$ be given. There exists a constant C_* (depending on k, q, h and p only) such that the following holds. Let μ be a finite measure for which one may find a sequence $\mu_n(dx) = f_n(x)dx$, $n \in \mathbb{N}$ such that $\pi_{k,q,h,p}(\mu,(\mu_n)_n) < \infty$. Then $\mu(dx) = f(x)dx$ with $f \in W^{q,p}$ and moreover*

$$\|f\|_{q,p} \leq C_* \times \pi_{k,q,h,p}(\mu,(\mu_n)_n). \tag{3.3}$$

The proof of Lemma 3.1 is given in [4], being a particular case (take $\mathbf{e} = \mathbf{e}_p$) of Proposition A.1 in Appendix A. We give a first simple consequence:

Lemma 3.2. *Let $p_t \in C^\infty(\mathbb{R}^d), t > 0$, be a family of probability densities such that $\int \psi_\kappa(x) p_t(x) dx \leq m_\kappa < \infty$ for every $\kappa \in \mathbb{N}$. We assume that for some $\theta_0 > 0$ and $\theta_1 > 0$ the following holds. For every $q, \kappa \in \mathbb{N}$ and $p \geq 1$ there exists a constant $C = C(q,\kappa,p)$ such that*

$$\|\psi_\kappa p_t\|_{q,p} \leq C t^{-\theta_0(q+\theta_1)}. \tag{3.4}$$

Then, for every $\delta > 0$ there exists a constant $C = C(q, \kappa, p, \delta)$ such that

$$\|\psi_\kappa p_t\|_{q,p} \leq Ct^{-\theta_0(q+\frac{d}{p_*}+\delta)}. \tag{3.5}$$

So does not matter the value of θ_1, one may (morally) replace it by $\frac{d}{p_*}$.

Proof. We take $n_* \in \mathbb{N}$ and we define $f_n = 0$ for $n \leq n_*$ and $f_n = \psi_\kappa p_t$ for $n > n_*$. Notice that $d_0(\psi_\kappa p_t, 0) = m_\kappa$. Then (3.3) with $k = 0$ gives

$$\begin{aligned}
\|\psi_\kappa p_t\|_{q,p} &\leq C_*\Big(m_\kappa \sum_{n=0}^{n_*} 2^{n(q+\frac{d}{p_*})} + \|\psi_\kappa p_t\|_{2h+q,2h,p} \sum_{n=n_*+1}^{\infty} \frac{1}{2^{2nh}}\Big) \\
&\leq C_*\Big(m_\kappa 2^{n_*(q+\frac{d}{p_*})} + \|\psi_\kappa p_t\|_{2h+q,2h,p} \frac{1}{2^{2n_* h}}\Big).
\end{aligned}$$

We denote $\rho_h = (q + \frac{d}{p_*})/2h$. We optimize over n_* and we obtain

$$\begin{aligned}
\|\psi_\kappa p_t\|_{q,p} &\leq 2C_* \times m_\kappa^{\frac{1}{1+\rho_h}} \times \|\psi_\kappa p_t\|_{2h+q,2h,p}^{\frac{\rho_h}{1+\rho_h}} \\
&\leq 2C_* \times m_\kappa^{\frac{1}{1+\rho_h}} \times Ct^{-\theta_0(2h+q+\theta_1)\frac{\rho_h}{1+\rho_h}},
\end{aligned}$$

because $\|\psi_\kappa p_t\|_{2h+q,2h,p} = \|\psi_{2h+\kappa} p_t\|_{2h+q,p}$ and we can use (3.4). Since $(2h + q + \theta_1)\frac{\rho_h}{1+\rho_h} \downarrow q + \frac{d}{p_*}$ as $h \to \infty$, the proof is completed. □

We will also use the following consequence of Lemma 3.1 (the proof is given in [3] and we do not repeat it here):

Lemma 3.3. *Let $k, q, h \in \mathbb{N}$, with $h \geq 1$, and $p > 1$ be given and set*

$$\rho_h := \frac{k + q + d/p_*}{2h}. \tag{3.6}$$

We consider an increasing sequence $\theta(n) \geq 1, n \in \mathbb{N}$, such that $\lim_{n\to\infty} \theta(n) = \infty$ and $\theta(n+1) \leq \Theta \times \theta(n)$ for some constant $\Theta \geq 1$. Suppose that we may find a sequence of functions $f_n \in C^{2h+q}(\mathbb{R}^d), n \in \mathbb{N}$, such that

$$\|f_n\|_{2h+q,2h,p} \leq \theta(n) \tag{3.7}$$

and, with $\mu_n(dx) = f_n(x)dx$,

$$\limsup_{n\to\infty} d_k(\mu, \mu_n) \times \theta^{\rho_h+\varepsilon}(n) < \infty \tag{3.8}$$

for some $\varepsilon > 0$. Then $\mu(dx) = f(x)dx$ with $f \in W^{q,p}$.

Moreover, for $\delta, \varepsilon > 0$ and $n_* \in \mathbb{N}$, let

$$A(\delta) = |\mu|(\mathbb{R}^d) \times 2^{l(\delta)(1+\delta)(q+k+d/p_*)} \quad \text{with } l(\delta) = \min\{l : 2^{l\times\frac{\delta}{1+\delta}} \geq l\}, \tag{3.9}$$

$$B(\varepsilon) = \sum_{l=1}^{\infty} \frac{l^{2(q+k+d/p_*+\varepsilon)}}{2^{2\varepsilon l}}, \tag{3.10}$$

$$C_{h,n_*}(\varepsilon) = \sup_{n\geq n_*} d_k(\mu, \mu_n) \times \theta^{\rho_h+\varepsilon}(n). \tag{3.11}$$

Then, for every $\delta > 0$

$$\|f\|_{q,p} \leq C_*(\Theta + A(\delta)\theta(n_*)^{\rho_h(1+\delta)} + B(\varepsilon)C_{h,n_*}(\varepsilon)), \tag{3.12}$$

C_* being the constant in (3.3) and ρ_h being given in (3.6).

3.2. A regularity lemma. We give here a regularization result in the following abstract framework. We consider a sequence of operators $U_j : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$, $j \in \mathbb{N}$, and we denote by U_j^* the formal adjoint defined by $\langle U_j^* f, g\rangle = \langle f, U_j g\rangle$ with the scalar product in $L^2(\mathbb{R}^d)$.

Assumption 3.4. Let $a \in \mathbb{N}$ be fixed. We assume that for every $q \in \mathbb{N}, \kappa \geq 0$ and $p \in [1,\infty)$ there exist constants $C_{q,\kappa,p}(U)$ and $C_{q,\kappa,\infty}(U)$ such that for every j and f,

$$(H_1) \qquad \left\Vert U_j f\right\Vert_{q,-\kappa,\infty} \leq C_{q,\kappa,\infty}(U)\left\Vert f\right\Vert_{q+a,-\kappa,\infty}, \tag{3.13}$$

$$(H_1^*) \qquad \left\Vert U_j^* f\right\Vert_{q,\kappa,p} \leq C_{q,\kappa,p}(U)\left\Vert f\right\Vert_{q+a,\kappa,p}. \tag{3.14}$$

We assume that $C_{q,\kappa,p}(U)$, $p \in [1,\infty]$, is non decreasing with respect to q and κ.

We also consider a semigroup $(S_t)_{t\geq 0}$ of the form

$$S_t(x,dy) = s_t(x,y)dy \quad \text{with} \quad s_t \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d).$$

We define the formal adjoint operator

$$S_t^* f(y) = \int_{\mathbb{R}^d} s_t(x,y) f(x)dx, \quad t>0.$$

Assumption 3.5. If $f \in \mathcal{S}(\mathbb{R}^d)$ then $S_t f \in \mathcal{S}(\mathbb{R}^d)$. Moreover, there exists $b \in \mathbb{N}$ such that for every $q \in \mathbb{N}, \kappa \geq 0$ and $p \in [1,\infty)$ there exist constants $C_{q,\kappa,p}(S)$ such that for every $t > 0$,

$$(H_2) \qquad \left\Vert S_t f\right\Vert_{q,-\kappa,\infty} \leq C_{q,\kappa,\infty}(S)\left\Vert f\right\Vert_{q+b,-\kappa,\infty}, \tag{3.15}$$

$$(H_2^*) \qquad \left\Vert S_t^* f\right\Vert_{q,\kappa,p} \leq C_{q,\kappa,p}(S)\left\Vert f\right\Vert_{q+b,\kappa,p}. \tag{3.16}$$

We assume that $C_{q,\kappa,p}(S)$, $p \in [1,\infty]$, is non decreasing with respect to q and κ.

We denote

$$C_{q,\kappa,\infty}(U,S) = C_{q,\kappa,\infty}(U)C_{q,\kappa,\infty}(S), \quad C_{q,\kappa,p}(U,S) = C_{q,\kappa,p}(U)C_{q,\kappa,p}(S), \tag{3.17}$$

$$C_{q,\kappa,\infty,p}(U,S) = C_{q,\kappa,\infty}(U,S) \vee C_{q,\kappa,p}(U,S). \tag{3.18}$$

Under Assumptions 3.4 and 3.5, one immediately obtains

$$\left\Vert (S_t U_j) f\right\Vert_{q,-\kappa,\infty} \leq C_{q,\kappa,\infty}(U,S)\left\Vert f\right\Vert_{q+a+b,-\kappa,\infty}, \tag{3.19}$$

$$\left\Vert (S_t^* U_j^*) f\right\Vert_{q,\kappa,p} \leq C_{q,\kappa,p}(U,S)\left\Vert f\right\Vert_{q+a+b,\kappa,p}. \tag{3.20}$$

In fact these are the inequalities that we will employ in the following. We stress that the above constants $C_{q,\kappa,\infty}(U,S)$ and $C_{q,\kappa,p}(U,S)$ may depend on a, b and are increasing w.r.t. q and κ.

Finally we assume that the (possible) blow-up of $s_t \to \infty$ as $t \to 0$ is controlled in the following way.

Assumption 3.6. Let $\theta_0, \lambda > 0$ be fixed. We assume that for every $\kappa \geq 0$ and $q \in \mathbb{N}$ there exist $\pi(q,\kappa)$, $\theta_1 \geq 0$ and $C_{q,\kappa} > 0$ such that for every multi-indexes α and β with $|\alpha| + |\beta| \leq q$, $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d$ and $t \in (0,1]$ one has

$$(H_3) \qquad \left|\partial_x^\alpha \partial_y^\beta s_t(x,y)\right| \leq \frac{C_{q,\kappa}}{(\lambda t)^{\theta_0(q+\theta_1)}} \times \frac{\psi_{\pi(q,\kappa)}(x)}{\psi_\kappa(x-y)}. \tag{3.21}$$

We also assume that $\pi(q,k)$ and $C_{q,\kappa}$ are both increasing in q and κ.

This property will be used by means of the following lemma:

Lemma 3.7. *Suppose that Assumption 3.6 holds.*

A. *For every $\kappa \geq 0$, $q \in \mathbb{N}$ and $p > 1$ there exists $C > 0$ such that for every $t \in (0,1]$ and f one has*

$$\|S_t^* f\|_{q,\kappa,p} \leq \frac{C}{(\lambda t)^{\theta_0(q+\theta_1)}} \|f\|_{0,\nu,1} \tag{3.22}$$

where $\nu = \pi(q,\kappa+d)+\kappa+d$

B. *For every $\kappa \geq 0$, $q_1, q_2 \in \mathbb{N}$, there exists $C > 0$ such that for every $t \in (0,1]$, for every multi-index α with $|\alpha| \leq q_2$ and f one has*

$$\left\| \frac{1}{\psi_\eta} S_t(\psi_\kappa \partial^\alpha f) \right\|_{q_1,\infty} \leq \frac{C}{(\lambda t)^{\theta_0(q_1+q_2+\theta_1)}} \|f\|_\infty \tag{3.23}$$

where $\eta = \pi(q_1+q_2, \kappa+d+1)+\kappa$.

Proof. In the sequel, C will denote a positive constant which may vary from a line to another and which may depend only on κ and q for the proof of **A.** and only on κ, q_1 and q_2 for the proof of **B.**

A. Using (3.21) if $|\alpha| \leq q$,

$$|\partial^\alpha S_t^* f(x)| \leq \int |\partial_x^\alpha s_t(y,x)| \times |f(y)|\, dy \leq \frac{C}{(\lambda t)^{\theta_0(q+\theta_1)}} \int \frac{\psi_{\pi(q,\kappa+d)}(y)}{\psi_{\kappa+d}(x-y)} \times |f(y)|\, dy.$$

By (2.4) $\psi_{\kappa+d}(x)/\psi_{\kappa+d}(x-y) \leq C\psi_{\kappa+d}(y)$ so that

$$\begin{aligned}
\psi_{\kappa+d}(x)\, |\partial^\alpha S_t^* f(x)| &\leq \frac{C}{(\lambda t)^{\theta_0(q+\theta_1)}} \int \frac{\psi_{\kappa+d}(x)\psi_{\pi(q,\kappa+d)}(y)}{\psi_{\kappa+d}(x-y)} \times |f(y)|\, dy \\
&\leq \frac{C}{(\lambda t)^{\theta_0(q+\theta_1)}} \int \psi_{\pi(q,\kappa+d)+\kappa+d}(y) \times |f(y)|\, dy \\
&= \frac{C}{(\lambda t)^{\theta_0(q+\theta_1)}} \|f\|_{0,\nu,1}.
\end{aligned}$$

We conclude that

$$\|S_t^* f\|_{q,\kappa+d,\infty} \leq \frac{C}{(\lambda t)^{\theta_0(q+\theta_1)}} \|f\|_{0,\nu,1}.$$

By (2.9) $\|S_t^* f\|_{q,\kappa,p} \leq C \|S_t^* f\|_{q,\kappa+d,\infty}$ so the proof of (3.22) is completed.

B. Let γ with $|\gamma| \leq q_1$. Using integration by parts

$$\begin{aligned}
\partial^\gamma S_t(\psi_\kappa \partial^\alpha f)(x) &= \int_{\mathbb{R}^d} \partial_x^\gamma s_t(x,y)\psi_\kappa(y)\partial^\alpha f(y)dy \\
&= (-1)^{|\alpha|} \int_{\mathbb{R}^d} \partial_y^\alpha(\partial_x^\gamma s_t(x,y)\psi_\kappa(y)) \times f(y)dy.
\end{aligned}$$

Using (2.6), (3.21) and (2.4), it follows that

$$|\partial^\gamma S_t(\psi_\kappa \partial^\alpha f)(x)| \leq \int_{\mathbb{R}^d} \left|\partial_y^\alpha(\partial_x^\gamma s_t(x,y)\psi_\kappa(y))\right| \times |f(y)|\, dy$$

$$\leq \int_{\mathbb{R}^d} |s_t(x,y)\psi_\kappa(y)|_{q_1+q_2} \times |f(y)|\, dy$$
$$\leq C\int_{\mathbb{R}^d} |s_t(x,y)|_{q_1+q_2}\, \psi_\kappa(y) \times |f(y)|\, dy$$
$$\leq \frac{C}{(\lambda t)^{\theta_0(q_1+q_2+\theta_1)}} \|f\|_\infty \int_{\mathbb{R}^d} \frac{\psi_{\pi(q_1+q_2,\kappa+d+1)}(x)}{\psi_{\kappa+d+1}(x-y)} \times \psi_\kappa(y)dy$$
$$\leq \frac{C}{(\lambda t)^{\theta_0(q_1+q_2+\theta_1)}} \|f\|_\infty \int_{\mathbb{R}^d} \frac{\psi_{\pi(q_1+q_2,\kappa+d+1)+\kappa}(x)}{\psi_{d+1}(x-y)} dy$$
$$\leq \frac{C}{(\lambda t)^{\theta_0(q_1+q_2+\theta_1)}} \|f\|_\infty\, \psi_{\pi(q_1+q_2,\kappa+d+1)+\kappa}(x).$$

This implies (3.23). □

We are now able to give the "regularity lemma". This is the core of our approach.

Lemma 3.8. *Suppose that Assumption 3.4, 3.5 and 3.6 hold. We fix* $t \in (0,1]$, $m \geq 1$ *and* $\delta_i > 0$, $i = 1, \dots, m$ *such that* $\sum_{i=1}^m \delta_i = t$.

A. *There exists a function* $\tilde{p}_{\delta_1,\dots,\delta_m} \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ *such that*

$$\prod_{i=1}^{m-1} (S_{\delta_i} U_i) S_{\delta_m} f(x) = \int \tilde{p}_{\delta_1,\dots,\delta_m}(x,y) f(y) dy. \tag{3.24}$$

B. *We fix* $q_1, q_2 \in \mathbb{N}, \kappa \geq 0, p > 1$ *and we denote* $q = q_1 + q_2 + (a+b)(m-1)$. *One may find universal constants* $C, \chi, \bar{p} \geq 1$ *(depending on* κ, p *and* $q_1 + q_2$*) such that for every multi-index* β *with* $|\beta| \leq q_2$ *and every* $x \in \mathbb{R}^d$

$$\left\|\partial_x^\beta \tilde{p}_{\delta_1,\dots,\delta_m}(x,\cdot)\right\|_{q_1,\kappa,p} \leq C\Big(\frac{2m}{\lambda t}\Big)^{\theta_0(q_1+q_2+d+2\theta_1)} \Big(C_{q,\chi,\bar{p},\infty}(U,S)\Big(\frac{2m}{\lambda t}\Big)^{\theta_0(a+b)}\Big)^{m-1} \psi_\chi(x). \tag{3.25}$$

Proof. **A.** For $g = g(x,y)$, we denote $g^x(y) := g(x,y)$. By the very definition of U_i^* one has

$$S_t U_i f(x) = \int_{\mathbb{R}^d} U_i^* s_t^x(y) f(y) dy.$$

As a consequence, one gets the kernel in (3.24):

$$\tilde{p}_{\delta_1,\dots,\delta_m}(x,y) = \int_{\mathbb{R}^{d\times(m-1)}} U_1^* s_{\delta_1}^x(y_1) \Big(\prod_{j=2}^{m-1} U_j^* s_{\delta_j}^{y_{j-1}}(y_j)\Big) s_{\delta_m}(y_{m-1},y) dy_1 \cdots dy_{m-1},$$

and the regularity immediately follows.

B. We split the proof in several steps.

Step 1: decomposition. Since $\sum_{i=1}^m \delta_i = t$ we may find $j \in \{1,\dots,m\}$ such that $\delta_j \geq \frac{t}{m}$. We fix this j and we write

$$\prod_{i=1}^{m-1} (S_{\delta_i} U_i) S_{\delta_m} = Q_1 Q_2$$

with

$$Q_1 = \prod_{i=1}^{j-1} (S_{\delta_i} U_i) S_{\frac{1}{2}\delta_j} \quad \text{and} \quad Q_2 = S_{\frac{1}{2}\delta_j} U_j \prod_{i=j+1}^{m-1} (S_{\delta_i} U_i) S_{\delta_m} = S_{\frac{1}{2}\delta_j} \prod_{i=j}^{m-1} (U_i S_{\delta_{i+1}}).$$

Here we use the semi-group property $S_{\frac{1}{2}\delta_j} S_{\frac{1}{2}\delta_j} = S_{\delta_j}$.

We suppose that $j \leq m-1$. In the case $j = m$ the proof is analogous but simpler. We will use Lemma 3.7 in order to estimate the terms corresponding to each of these two operators. As already seen, both Q_1 and Q_2 are given by means of smooth kernels, that we call $p_1(x,y)$ and $p_2(x,y)$ respectively.

Step 2. We take β with $|\beta| \leq q_2$ and we denote $g^{\beta,x}(y) := \partial_x^\beta g(x,y)$. For $h \in L^1$ we write

$$\begin{aligned}\int_{\mathbb{R}^d} h(z) \partial_x^\beta \tilde{p}_{\delta_1,\ldots,\delta_m}(x,z) dz &= \int_{\mathbb{R}^d} h(z) \int_{\mathbb{R}^d} \partial_x^\beta p_1(x,y) p_2(y,z) dy dz \\ &= \int_{\mathbb{R}^d} \partial_x^\beta p_1(x,y) \int h(z) p_2(y,z) dz dy = \int_{\mathbb{R}^d} \partial_x^\beta p_1(x,y) Q_2 h(y) dy \\ &= \int_{\mathbb{R}^d} Q_2^* p_1^{\beta,x}(y) h(y) dy.\end{aligned}$$

It follows that

$$\partial_x^\beta \tilde{p}_{\delta_1,\ldots,\delta_m}(x,z) = Q_2^* p_1^{\beta,x}(z) = \prod_{i=1}^{m-j} (S^*_{\delta_{m-i+1}} U^*_{m-i}) S^*_{\frac{1}{2}\delta_j} p_1^{\beta,x}(z).$$

We will use (3.20) $m-j$ times first and (3.22) then. We denote

$$q_1' = q_1 + (m-j)(a+b)$$

and we write

$$\begin{aligned}\|\partial_x^\beta \tilde{p}_{\delta_1,\ldots,\delta_m}(x,\cdot)\|_{q_1,\kappa,p} &\leq C^{m-j}_{q_1',\kappa,p}(U,S) \|S^*_{\frac{1}{2}\delta_j} p_1^{\beta,x}\|_{q_1',\kappa,p} \\ &\leq C^{m-j}_{q_1',\kappa,p}(U,S)\, C\Big(\frac{2m}{\lambda t}\Big)^{\theta_0(q_1'+\theta_1)} \|p_1^{\beta,x}\|_{0,\nu,1}\end{aligned} \tag{3.26}$$

with

$$\nu = \pi(q_1', \kappa + d) + \kappa + d.$$

Step 3. We denote $g_z(u) = \prod_{l=1}^d 1_{(0,\infty)}(u_l - z_l)$, so that $\delta_0(u-z) = \partial_u^\rho g_z(u)$ with $\rho = (1,2,\ldots,d)$. We take $\mu = \nu + d + 1$ and we formally write

$$p_1(x,z) = \frac{1}{\psi_\mu(z)} Q_1(\psi_\mu \partial^\rho g_z)(x).$$

This formal equality can be rigorously written by using the regularization by convolution of the Dirac function.

We denote

$$q_2' = q_2 + (j-1)(a+b), \quad \eta = \pi(d + q_2', \mu + d + 1) + \mu$$

and we write

$$|p_1^{\beta,x}(z)| = |\partial_x^\beta p_1(x,z)| \leq \frac{\psi_\eta(x)}{\psi_\mu(z)} \Big\| \frac{1}{\psi_\eta} \partial^\beta Q_1(\psi_\mu \partial^\rho g_z) \Big\|_\infty.$$

Since $\mu=\nu+d+1$, $\int\psi_\nu\times\frac{1}{\psi_\mu}<\infty$, so using (2.6), we obtain (recall that $|\beta|\leq q_2$)

$$\begin{aligned}\|p_1^{\beta,x}\|_{0,\nu,1} &\leq C\psi_\eta(x)\sup_{z\in\mathbb{R}^d}\Big\|\frac{1}{\psi_\eta}\partial^\beta Q_1(\psi_\mu\partial^\rho g_z)\Big\|_\infty\\ &\leq C\psi_\eta(x)\sup_{z\in\mathbb{R}^d}\Big\|\frac{1}{\psi_\eta}Q_1(\psi_\mu\partial^\rho g_z)\Big\|_{q_2,\infty}\\ &\leq C\psi_{\eta'}(x)\sup_{z\in\mathbb{R}^d}\big\|Q_1(\psi_\mu\partial^\rho g_z)\big\|_{q_2,-\eta,\infty}.\end{aligned}$$

Using (3.19) $j-1$ times and (3.23) (with $\kappa=\mu$) we get

$$\begin{aligned}\|Q_1(\psi_\mu\partial^\rho g_z)\|_{q_2,-\eta,\infty} &\leq C^{j-1}_{q_2',\eta,\infty}(U,S)\|S_{\frac12\delta_j}(\psi_\mu\partial^\rho g_z)\|_{q_2',-\eta,\infty}\\ &\leq C^{j-1}_{q_2',\eta,\infty}(U,S)\,\|g_z\|_\infty\,C\Big(\frac{2m}{\lambda t}\Big)^{\theta_0(q_2'+d+\theta_1)}.\end{aligned}$$

Since $\|g_z\|_\infty=1$ we obtain

$$\|p_1^{\beta,x}\|_{0,\nu,1}\leq\psi_\eta(x)C^{j-1}_{q_2',\eta,\infty}(U,S)\,C\Big(\frac{2m}{\lambda t}\Big)^{\theta_0(q_2'+d+\theta_1)}.$$

By inserting in (3.26) we obtain (3.25), so the proof is completed. □

4. Proofs of the Main Results

In this section we prove Theorem 2.6. But before we give an intermediary result, Theorem 4.1 below, which is more precise concerning constants. Let us introduce some notation. For $\delta\geq 0$ we denote

$$\Phi_n(\delta)=\varepsilon_n\Lambda_n\times\lambda_n^{-\theta_0(a+b+\delta)}. \tag{4.1}$$

We recall that the constants ε_n, a, Λ_n, b and λ_n are defined in Assumption 2.1, Assumption 2.2 and Assumption 2.3. Under Assumption 2.3, $\lambda_n\leq\gamma\lambda_{n+1}$ for some $\gamma\geq 1$, so we have

$$\Phi_n(\delta)\leq\gamma^{1+\theta_0(a+b+\delta)}\Phi_{n+1}(\delta). \tag{4.2}$$

For $\kappa\geq 0,\eta\geq 0$ we set

$$\Psi_{\eta,\kappa}(x,y):=\frac{\psi_\kappa(y)}{\psi_\eta(x)},\quad (x,y)\in\mathbb{R}^d\times\mathbb{R}^d. \tag{4.3}$$

Our intermediary result concerning the regularity of the semigroup $(P_t)_{t\geq 0}$ is the following.

Theorem 4.1. *Suppose that Assumption 2.1, 2.2 , 2.3 and 2.4 hold. Moreover we suppose there exists* $\delta>0$ *such that*

$$\limsup_{n\to\infty}\Phi_n(\delta)<\infty, \tag{4.4}$$

$\Phi_n(\delta)$ *being given in (4.1). Then the following statements hold.*

A. $P_tf(x)=\int_{\mathbb{R}^d}p_t(x,y)dy$ *with* $p_t\in C^\infty(\mathbb{R}^d\times\mathbb{R}^d)$.

B. *Let* $n_*\in\mathbb{N}$ *and* $\delta_*>0$ *be such that*

$$\overline{\Phi}_*:=\sup_{n\geq n_*}\Phi_n(\delta_*)<\infty. \tag{4.5}$$

We fix $q \in \mathbb{N}$, $p > 1$, $\varepsilon_* > 0$, $\kappa \geq 0$ and we put $\mathfrak{m} = 1 + \frac{q+2d/p_*}{\delta_*}$ with p_* the conjugate of p. There exist $C \geq 1$ and $\eta > 1$ (depending on $q, p, \varepsilon_*, \delta_*, \kappa$ and γ) such that for every $t \in (0,1]$

$$\left\|\Psi_{\eta,\kappa} p_t\right\|_{q,p} \leq C \times Q_*(q,\mathfrak{m}) \times t^{-\theta_0((a+b)\mathfrak{m}+q+2d/p_*)(1+\varepsilon_*)} \quad \textit{with} \tag{4.6}$$

$$Q_*(q,\mathfrak{m}) = \Big(\frac{1}{\lambda_{n_*}^{\theta_0(a+b)\mathfrak{m}+q+2d/p_*}} + \overline{\Phi}_*^{\mathfrak{m}}\Big)^{1+\varepsilon_*}. \tag{4.7}$$

***C**. Let $p > 2d$. Set $\bar{\mathfrak{m}} = 1 + \frac{q+1+2d/p_*}{\delta_*}$. There exist $C \geq 1, \eta > 1$ (depending on $q, p, \varepsilon_*, \delta_*, \kappa$) such that for every $t \in (0,1]$, $x, y \in \mathbb{R}^d$ and for every multi-indexes α, β such that $|\alpha| + |\beta| \leq q$,*

$$\left|\partial_x^\alpha \partial_y^\beta p_t(x,y)\right| \leq C \times Q_*(q+1,\bar{\mathfrak{m}}) \times t^{-\theta_0((a+b)\bar{\mathfrak{m}}+q+1+2d/p_*)(1+\varepsilon_*)} \times \frac{\psi_{\eta+\kappa}(x)}{\psi_\kappa(x-y)} \tag{4.8}$$

Remark 4.2. We stress that in hypothesis (4.5) the order of derivation q does not appear. However the conclusions (4.6) and (4.8) hold for every q. The motivation of this is given by the following heuristics. The hypothesis (2.20) says that the semi-group P_t^n has a regularization effect controlled by $1/(\lambda_n t)^{\theta_0}$. If we want to decouple this effect m_0 times we write $P_t^n = P_{t/m_0}^n \cdots P_{t/m_0}^n$ and then each of the m_0 operators P_{t/m_0}^n acts with a regularization effect of order $(\lambda_n \times t/m_0)^{\theta_0}$. But this heuristics does not work directly: in order to use it, in the proof we have to develop a Taylor expansion coupled with the interpolation criterion studied in Section 3.

Proof. **Step 0: constants and parameters set-up.** We first choose some parameters which will be used in the following steps. To begin we stress that we work with measures on $\mathbb{R}^d \times \mathbb{R}^d$ so the dimension of the space is $2d$ (and not d). We recall that in our statement the quantities $q, d, p, \delta_*, \varepsilon_*, \kappa$ and n are given and fixed. In the following we will denote by C a constant depending on all these parameters and which may change from a line to another. We define

$$m_0 = 1 + \Big\lfloor \frac{q + 2d/p_*}{\delta_*} \Big\rfloor > 0 \tag{4.9}$$

and given $h \in \mathbb{N}$ we denote

$$\rho_h = \frac{(a+b)m_0 + q + 2d/p_*}{2h}. \tag{4.10}$$

Notice that this is equal to the constant ρ_h defined in (3.6) corresponding to $k = (a+b)m_0$ and q and to $2d$ (instead of d).

Step 1: a Lindeberg-type method to decompose $P_t - P_t^n$. We fix (once for all) $t \in (0,1]$ and we write

$$P_t f - P_t^n f = \int_0^t \partial_s (P_{t-s}^n P_s) f ds = \int_0^t P_{t-s}^n (L - L_n) P_s f ds = \int_0^t P_{t-s}^n \Delta_n P_s f ds.$$

We iterate this formula m_0 times (with m_0 chosen in (4.9)) and we obtain

$$P_t f(x) - P_t^n f(x) = \sum_{m=1}^{m_0-1} I_n^m f(x) + R_n^{m_0} f(x) \tag{4.11}$$

with (we put $t_0 = t$)

$$I_n^m f(x) = \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{m-1}} dt_m \prod_{i=0}^{m-1} (P^n_{t_i - t_{i+1}} \Delta_n) P^n_{t_m} f(x),$$

$1 \leq m \leq m_0 - 1$,

$$R_n^{m_0} f(x) = \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{m_0-1}} dt_{m_0} \prod_{i=0}^{m_0-1} (P^n_{t_i - t_{i+1}} \Delta_n) P_{t_{m_0}} f(x).$$

In order to analyze $I_n^m f$ we use Lemma 3.8 for the semigroup $S_t = P_t^n$ and for the operators $U_i = \Delta_n = L - L_n$ (the same for each i), with $\delta_i = t_i - t_{i+1}$, $i = 0, \ldots, m$ (with $t_{m+1} = 0$). So the hypotheses (3.13) and (3.14) in Assumption 3.4 coincide with the requests (2.14) and (2.15) in Assumption 2.1. And we have $C_{q,\kappa,\infty}(U) = C_{q,\kappa,p}(U) = C\varepsilon_n$. Moreover the hypotheses (3.15) and (3.16) in Assumption 3.5 coincide with the hypotheses (2.16) and (2.17) in Assumption 2.2. And we have $C_{q,\kappa,\infty}(P^n) = C_{q,\kappa,p}(P^n) = \Lambda_n$. Hence,

$$C_{q,\kappa,\infty,p}(\Delta_n, P^n) = C\,\varepsilon_n \times \Lambda_n. \tag{4.12}$$

Finally, the hypothesis (3.21) in Assumption 3.6 coincides with (2.20) in Assumption 2.3. So, we can apply Lemma 3.8: by using (3.24) we obtain

$$I_n^m f(x) = \int_0^t dt_1 \cdots \int_0^{t_{m-1}} dt_m \int p^{n,m}_{t-t_1, t_1-t_2, \ldots, t_m}(x,y) f(y) dy.$$

We denote

$$\phi_t^{n,m_0}(x,y) = p_t^n(x,y) + \sum_{m=1}^{m_0-1} \int_0^t dt_1 \cdots \int_0^{t_{m-1}} dt_m p^{n,m}_{t-t_1, t_2-t_1, \ldots, t_m}(x,y)$$

so that (4.11) reads

$$\int f(y) P_t(x,dy) = \int f(y) \phi_t^{n,m_0}(x,y) dy + R_n^{m_0} f(x).$$

We recall that $\Psi_{\eta,\kappa}$ is defined in (4.3) and we define the measures on $\mathbb{R}^d \times \mathbb{R}^d$ defined by

$$\mu^{\eta,\kappa}(dx,dy) = \Psi_{\eta,\kappa}(x,y) P_t(x,dy) dx$$

and

$$\mu_n^{\eta,\kappa,m_0}(dx,dy) = \Psi_{\eta,\kappa}(x,y) \phi_t^{n,m_0}(x,y) dx dy.$$

So, the proof consists in applying Lemma 3.3 to $\mu = \mu^{\eta,\kappa}$ and $\mu_n = \mu_n^{\eta,\kappa,m_0}$.

Step 2: analysis of the principal term. We study here the estimates for $f_n(x,y) = \Psi_{\eta,\kappa} \phi_t^{n,m_0}(x,y)$ which are required in (3.7).

We first use (3.25) in order to get estimates for $p^{n,m}_{t-t_1, t_2-t_1, \ldots, t_m}(x,y)$. We fix $q_1, q_2 \in \mathbb{N}, \kappa \geq 0, p > 1$ and we recall that in Lemma 3.8 we introduced $\overline{q} = q_1 + q_2 + (a+b)(m_0 - 1)$. Moreover in Lemma 3.8 one produces χ such that (3.25) holds true: for every multi-index β with $|\beta| \leq q_2$

$$\begin{aligned} &\left\| \psi_\kappa \partial_x^\beta p^{n,m}_{t-t_1, t_1-t_2, \ldots, t_m}(x, \cdot) \right\|_{q_1,p} \\ &\quad \leq C \Big(\frac{1}{\lambda_n t} \Big)^{\theta_0(q_1+q_2+d+2\theta_1)} \times \bigg(\varepsilon_n \Lambda_n \Big(\frac{1}{\lambda_n t} \Big)^{\theta_0(a+b)} \bigg)^m \psi_\chi(x). \end{aligned}$$

Denote

$$\xi_1(q) = q + d + 2\theta_1 + m_0(a+b), \qquad \omega_1(q) = q + d + 2\theta_1.$$

With this notation, if $|\beta| \leq q_2$ we have

$$\begin{aligned}
&\left\|\psi_\kappa \partial_x^\beta \phi_t^{n,m_0}(x,\cdot)\right\|_{q_1,p} \\
&\quad \leq C\Big(\frac{1}{\lambda_n t}\Big)^{\theta_0(q_1+q_2+d+2\theta_1)} \times \left(\varepsilon_n \Lambda_n \Big(\frac{1}{\lambda_n t}\Big)^{\theta_0(a+b)}\right)^{m_0} \psi_\chi(x) \qquad (4.13)\\
&\quad = C t^{-\theta_0 \xi_1(q_1+q_2)} \lambda_n^{-\theta_0 \omega_1(q_1+q_2)} \Phi_n^{m_0}(0) \psi_\chi(x), \qquad (4.14)
\end{aligned}$$

where $\Phi_n(\delta)$ is the constant defined in (4.1). We take $l = 2h + q, l' = 2h$ and we take $q(l) = l + (a+b)m_0$. Moreover we fix q_1 and q_2 (so $q = q_1 + q_2 \leq l$) and we take χ to be the one in (4.13). Moreover we take η sufficiently large in order to have $p\eta - 2h - p\chi \geq d + 1$. This guarantees that

$$\int_{\mathbb{R}^d} \frac{dx}{\psi_{p\eta - l' - p\chi}(x)} = C < \infty. \qquad (4.15)$$

By (2.6) and (4.13)

$$\begin{aligned}
&\|\Psi_{\eta,\kappa} \phi_t^{n,m_0}\|_{l,l',p}^p \\
&\quad \leq C \sum_{|\alpha|+|\beta| \leq l} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Psi_{\eta,\kappa}^p(x,y) \left|\partial_x^\alpha \partial_y^\beta \phi_t^{n,m_0}(x,y)\right|^p \psi_{l'}(x) \psi_{l'}(y) dy dx \\
&\quad = C \sum_{|\alpha|+|\beta| \leq l} \int_{\mathbb{R}^d} \frac{1}{\psi_{p\eta - l'}(x)} \int_{\mathbb{R}^d} \left|\psi_{\kappa + l'/p}(y) \partial_x^\alpha \partial_y^\beta \phi_t^{n,m_0}(x,y)\right|^p dy dx \\
&\quad \leq C \sum_{|\alpha|+|\beta| \leq l} \int_{\mathbb{R}^d} \frac{1}{\psi_{p\eta - l'}(x)} \left\|\psi_{\kappa + l'/p} \partial_x^\alpha \phi_t^{n,m_0}(x,\cdot)\right\|_{|\beta|,p}^p dx \\
&\quad \leq C (t^{-\theta_0 \xi_1(l)} \lambda_n^{-\theta_0 \omega_1(l)} \Phi_n(0))^{p m_0} \int_{\mathbb{R}^d} \frac{dx}{\psi_{p\eta - l' - p\chi}(x)}.
\end{aligned}$$

We conclude that

$$\|\Psi_{\eta,\kappa} \phi_t^{n,m_0}\|_{2h+q,2h,p} \leq C t^{-\theta_0 \xi_1(q+2h)} \times \lambda_n^{-\theta_0 \omega_1(q+2h)} \Phi_n^{m_0}(0) =: \theta(n). \qquad (4.16)$$

By (4.2) $\theta(n) \uparrow +\infty$ and $\Theta \theta(n) \geq \theta(n+1)$ with

$$\Theta = \gamma^{\theta_0((a+b)m_0 + q + 2h + d + 2\theta_1) + m_0} \geq 1.$$

In the following we will choose h sufficiently large, depending on δ_*, m_0, q, d and p. So Θ is a constant depending on $\delta_*, m_0, q, d, a, b, \gamma$ and p, as the constants considered in the statement of our theorem.

Step 3: analysis of the remainder. We study here

$$d_{m_0}(n) := d_{(a+b)m_0}(\mu^{\eta,\kappa}, \mu_n^{\eta,\kappa,m_0})$$

as required in (3.8): we prove that, if $\eta \geq \kappa + d + 1$, then

$$d_{m_0}(n) \leq C(\Lambda_n \varepsilon_n)^{m_0} \leq \lambda_n^{\theta_0(a+b+\delta_*)m_0} \Phi_n^{m_0}(\delta_*). \qquad (4.17)$$

Using first (A_1) and (A_2) (see (2.14) and (2.16)) and then (A_4) (see (2.21)) we obtain

$$\left\| \prod_{i=0}^{m_0-1} (P^n_{t_i-t_{i+1}}\Delta_n) P_{t_{m_0}} f \right\|_{0,-\kappa,\infty} \leq C \left\| f \right\|_{(a+b)m_0,-\kappa,\infty} (\Lambda_n \varepsilon_n)^{m_0}$$

which gives

$$\| R_n^{m_0} f \|_{0,-\kappa,\infty} \leq C \left\| f \right\|_{(a+b)m_0,-\kappa,\infty} (\Lambda_n \varepsilon_n)^{m_0}.$$

Using now the equivalence between (2.11) and (2.12) we obtain

$$\left\| \frac{1}{\psi_\kappa} R_n^{m_0}(\psi_\kappa f) \right\|_{\infty} \leq C \left\| f \right\|_{(a+b)m_0,\infty} (\Lambda_n \varepsilon_n)^{m_0}. \tag{4.18}$$

We take now $g \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$, we denote $g_x(y) = g(x, y)$, and we write

$$\begin{aligned} & \left| \int_{\mathbb{R}^d \times \mathbb{R}^d} g(x, y)(\mu^{\eta,\kappa} - \mu_n^{,\kappa,m_0})(dx, dy) \right| \\ & \leq \int_{\mathbb{R}^d} \frac{dx}{\psi_\eta(x)} \left| \int_{\mathbb{R}^d} g_x(y)\psi_\kappa(y)(P_t(x, dy) - \phi_t^{n,m_0}(x, y))dy \right| \\ & \leq \int_{\mathbb{R}^d} \frac{dx}{\psi_{\eta-\kappa}(x)} \left| \frac{1}{\psi_\kappa(x)} R_n^{m_0}(\psi_\kappa g_x)(x) \right| dx \\ & \leq C \sup_{x\in\mathbb{R}^d} \| g_x \|_{(a+b)m_0,\infty} (\Lambda_n \varepsilon_n)^{m_0} \end{aligned}$$

the last inequality being a consequence of (4.18) and of $\eta - \kappa \geq d + 1$. Now (4.17) is proved because $\sup_{x\in\mathbb{R}^d} \| g_x \|_{(a+b)m_0,\infty} \leq \| g \|_{(a+b)m_0,\infty}$.

Step 4: use of Lemma 3.3 and proof of A. and B. We recall that ρ_h is defined in (4.10) and we estimate

$$d_{m_0}(n) \times \theta(n)^{\rho_h} \leq C t^{-\theta_0 \xi_2(h)} \lambda_n^{\theta_0 \omega_2(h)} \Phi_n^{m_0(1+\rho_h)}(\delta_*)$$

with

$$\xi_2(h) = \rho_h \xi_1(q + 2h) = \rho_h(q + 2h + d + 2\theta_1 + m_0(a + b))$$

and

$$\begin{aligned} \omega_2(h) &= (a + b + \delta_*)m_0 - \rho_h(q + 2h + d + 2\theta_1) \\ &= \delta_* m_0 - \frac{(a + b)m_0 + q + 2d/p_*}{2h}(q + d + 2\theta_1) - (q + 2d/p_*). \end{aligned}$$

By our choice of m_0 we have

$$\delta_* m_0 > q + 2d/p_*$$

so, taking h sufficiently large we get $\omega_2(h) > 0$. And we also have $\xi_2(h) \leq \xi_3 := (a + b)m_0 + q + \frac{2d}{p_*} + \varepsilon_*$ and $\rho_h \leq \varepsilon_*$. So we finally get

$$d_{m_0}(n) \times \theta(n)^{\rho_h} \leq C t^{-\theta_0 \xi_3} \Phi_n^{m_0(1+\varepsilon_*)}(\delta_*). \tag{4.19}$$

The above inequality guarantees that (3.8) holds so that we may use Lemma 3.3. We take $\eta > \kappa + d$ and, using (A_4) (see (2.21)) we obtain

$$|\mu^{\eta,\kappa}| = \int_{\mathbb{R}^2} \frac{\psi_\eta(x)}{\psi_\kappa(y)} P_t(x, dy)dx \leq C \int_{\mathbb{R}} \frac{dx}{\psi_{\kappa-\eta}(x)} < \infty.$$

Then, $A(\delta) < C$ (see (3.9)). One also has $B(\varepsilon) < \infty$ (see (3.10)) and finally (see (3.11))

$$C_{h,n_*}(\varepsilon) \leq Ct^{-\theta_0 \xi_3} \Phi_n^{m_0(1+\varepsilon_*)}(\delta_*).$$

We have used here (4.19). For large h we also have

$$\theta(n)^{\rho_h} \leq C(\lambda_n t)^{-\theta_0((a+b)m_0+q+\frac{2d}{p_*})(1+\varepsilon_*)} \Phi_n^{\varepsilon_*}(0).$$

Now (3.12) gives (4.6). So **A** and **B** are proved.

Step 5: proof of C. We apply **B.** with q replaced by $\bar{q} = q+1$, so $\Psi_{\eta,\kappa} p_t \in W^{\bar{q},p}(\mathbb{R}^d \times \mathbb{R}^d) = W^{\bar{q},p}(\mathbb{R}^{2d})$. Since $\bar{q} > 2d/p$ (here the dimension is $2d$), we can use the Morrey's inequality: for every α, β with $|\alpha| + |\beta| \leq \lfloor \bar{q} - 2d/p \rfloor = q$, then $|\partial_x^\alpha \partial_y^\beta (\Psi_{\eta,\kappa} p_t)(x,y)| \leq C \|\Psi_{\eta,\kappa} p_t\|_{\bar{q},p}$. By (4.6), one has

$$\left|\partial_x^\alpha \partial_y^\beta (\Psi_{\eta,\kappa} p_t)(x,y)\right| \leq CQ_*(\bar{q}, \bar{\mathfrak{m}}) t^{-\theta_0((a+b)\bar{\mathfrak{m}}+\bar{q}+2d/p_*)(1+\varepsilon_*)}$$

i.e. (using (2.6)),

$$\left|\partial_x^\alpha \partial_y^\beta p_t(x,y)\right| \leq CQ_*(\bar{q}, \bar{\mathfrak{m}}) t^{-\theta_0((a+b)\bar{\mathfrak{m}}+\bar{q}+2d/p_*)(1+\varepsilon_*)} \times \frac{1}{\Psi_{\eta,\kappa}(x,y)}.$$

Now, by a standard calculus, $\Psi_{\eta,\kappa}(x,y) \geq C_\kappa \frac{\psi_\kappa(x-y)}{\psi_{\eta+\kappa}(x)}$ (use that $\psi_\kappa(x-y) \leq C_\kappa \psi_\kappa(x)\psi_\kappa(-y) = C_\kappa \psi_\kappa(x)\psi_\kappa(y)$), so (4.8) follows. □

We are finally ready for the

Proof of Theorem 2.6. Our assumptions guarantees that $P_t(x,dy) = p_t(x,y)dy$ and p_t satisfies (4.8). We take a cut-off function $F_R \in C^\infty(\mathbb{R}^d)$ such that $1_{B_R(0)} \leq F_R \leq 1_{B_{R+1}(0)}$ ($B_r(0)$ denoting the open ball centered at 0 with radius r) and we denote $p_t^R(x,y) = F_R(x)p_t(x,y)$. By (4.8) we know that, for every $\kappa \in \mathbb{N}, \varepsilon > 0$ and every $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d$ one has

$$\left|\partial_x^\alpha \partial_y^\beta p_t^R(x,y)\right| \leq Ct^{-\theta_*(|\alpha|+|\beta|+\theta_1)} \times \frac{\psi_{\eta+\kappa}(x)}{\psi_\kappa(x-y)}$$

where $\theta_* = \theta_0(1+\frac{a+b}{\delta_*})(1+\varepsilon)$, θ_1 is computed from (4.8) (the precise value is not important here) and C and η both depend on $\kappa, \varepsilon, \delta_*, |\alpha|$ and $|\beta|$. Since the above left hand side is identically null when $|x| > R+1$, we can write

$$\left|\partial_x^\alpha \partial_y^\beta p_t^R(x,y)\right| \leq Ct^{-\theta_*(|\alpha|+|\beta|+\theta_1)} \times \psi_{-\kappa}(x,y)$$

where C is a new constant depending on R as well (we also stress that here $\psi_{-\kappa}(x,y) = (1+|x|^2+|y|^2)^{-\kappa}$, so the underlying dimension is $2d$). This allows one to apply Lemma 3.2: for every $p \geq 1$, $\kappa \in \mathbb{N}$ and $\delta > 0$,

$$\left\|\psi_\kappa p_t^R\right\|_{q,p} \leq Ct^{-\theta_*(q+\frac{2d}{p_*}+\delta)}.$$

Then by Morrey's Lemma, for every $p > 2d$

$$\left\|\psi_\kappa p_t^R\right\|_{q,\infty} \leq \left\|\psi_\kappa p_t^R\right\|_{q+1,p} \leq Ct^{-\theta_*(q+1+\frac{2d}{p_*}+\delta)} \leq Ct^{-\theta_*(q+2d+\varepsilon)}$$

the last inequality being true if we take p close to $2d$ and $\delta < \varepsilon$. And this gives (2.23). □

Appendix A. Weights

For $k \in \mathbb{Z}$ and $x \in \mathbb{R}^d$, we denote

$$\psi_k(x) = (1+|x|^2)^k. \tag{A.1}$$

Lemma A.1. *For every multi-index α there exists a constant C_α such that*

$$\Big|\partial^\alpha\Big(\frac{1}{\psi_k}\Big)\Big| \leq \frac{C_\alpha}{\psi_k}. \tag{A.2}$$

Moreover, for every q there is a constant $C_q \geq 1$ such that for every $f \in C_b^\infty(\mathbb{R}^d)$

$$\frac{1}{C_q}\sum_{0\leq|\alpha|\leq q}\Big|\partial^\alpha\Big(\frac{f}{\psi_k}\Big)\Big| \leq \sum_{0\leq|\alpha|\leq q}\frac{1}{\psi_k}\left|\partial^\alpha f\right| \leq C_q\sum_{0\leq|\alpha|\leq q}\Big|\partial^\alpha\Big(\frac{f}{\psi_k}\Big)\Big|. \tag{A.3}$$

Proof. One checks by recurrence that

$$\partial^\alpha\Big(\frac{1}{\psi_k}\Big) = \sum_{q=1}^{|\alpha|}\frac{P_{\alpha,q}}{\psi_{k+q}}$$

where $P_{\alpha,q}$ is a polynomial of order q. And since

$$\frac{(1+|x|)^q}{(1+|x|^2)^{q+k}} \leq \frac{C}{(1+|x|^2)^k}$$

the proof (A.2) is completed. In order to prove (A.3) we write

$$\partial^\alpha\Big(\frac{f}{\psi_k}\Big) = \frac{1}{\psi_k}\partial^\alpha f + \sum_{\substack{(\beta,\gamma)=\alpha\\ |\beta|\geq 1}} c(\beta,\gamma)\partial^\beta\Big(\frac{1}{\psi_k}\Big)\partial^\gamma f.$$

This, together with (A.2) implies

$$\Big|\partial^\alpha\Big(\frac{f}{\psi_k}\Big)\Big| \leq C\sum_{0\leq|\gamma|\leq|\alpha|}\frac{1}{\psi_k}\left|\partial^\gamma f\right|$$

so the first inequality in (A.3) is proved. In order to prove the second inequality we proceed by recurrence on q. The inequality is true for $q=0$. Suppose that it is true for $q-1$. Then we write

$$\frac{1}{\psi_k}\partial^\alpha f = \partial^\alpha\Big(\frac{f}{\psi_k}\Big) - \sum_{\substack{(\beta,\gamma)=\alpha\\ |\beta|\geq 1}} c(\beta,\gamma)\partial^\beta\Big(\frac{1}{\psi_k}\Big)\partial^\gamma f$$

and we use again (A.2) in order to obtain

$$\frac{1}{\psi_k}\left|\partial^\alpha f\right| \leq \Big|\partial^\alpha\Big(\frac{f}{\psi_k}\Big)\Big| + C\sum_{|\gamma|<|\alpha|}\frac{1}{\psi_k}\left|\partial^\gamma f\right| \leq C\sum_{0\leq|\beta|\leq q}\Big|\partial^\beta\Big(\frac{f}{\psi_k}\Big)\Big|$$

the second inequality being a consequence of the recurrence hypothesis. □

Remark A.2. The assertion is false if we define $\psi_k(x) = (1+|x|)^k$ because $\partial_i\partial_j\left|x\right| = \frac{\delta_{i,j}}{|x|} - \frac{x_ix_j}{|x|^2}$ blows up in zero.

We look now to ψ_k itself.

Lemma A.3. *For every multi-index α there exists a constant C_α such that*

$$|\partial^\alpha \psi_k| \le C_\alpha \psi_k. \tag{A.4}$$

Moreover, for every q there is a constant $C_q \ge 1$ such that for every $f \in C_b^\infty(\mathbb{R}^d)$

$$\frac{1}{C_q} \sum_{0\le|\alpha|\le q} |\partial^\alpha(\psi_k f)| \le \sum_{0\le|\alpha|\le q} \psi_k \, |\partial^\alpha f| \le C_q \sum_{0\le|\alpha|\le q} |\partial^\alpha(\psi_k f)| \,. \tag{A.5}$$

Proof. One proves by recurrence that, if $|\alpha| \ge 1$ then $\partial^\alpha \psi_k = \sum_{q=1}^{|\alpha|} \psi_{k-q} P_q$ with P_q a polynomial of order q. Since $1+|x| \le 2(1+|x|^2)$ it follows that $|P_q| \le C\psi_q$ and (A.4) follows. Now we write

$$\psi_k \partial^\alpha f = \partial^\alpha(\psi_k f) - \sum_{\substack{(\beta,\gamma)=\alpha \\ |\beta|\ge 1}} c(\beta,\gamma) \partial^\beta \psi_k \partial^\gamma f$$

and the same arguments as in the proof of (A.3) give (A.5). □

Appendix B. Semigroup Estimates

We consider a semigroup $(P_t)_{t\ge 0}$ on $C^\infty(\mathbb{R}^d)$ such that $P_t f(x) = \int f(y) P_t(x,dy)$ where $P_t(x,dy)$ is a probability transition kernel and we denote by P_t^* its formal adjoint.

Assumption B.1. There exists $Q \ge 1$ such that for every $t \le T$ and every $f \in C^\infty(\mathbb{R}^d)$

$$\|P_t f\|_1 \le Q \, \|f\|_1 \,. \tag{B.1}$$

Moreover, for every $k \in \mathbb{N}$ there exists $K_k \ge 1$ such that for every $x \in \mathbb{R}^d$ and $t \le T$

$$|P_t(\psi_k)(x)| \le K_k \psi_k(x). \tag{B.2}$$

Lemma B.2. *Under Assumption B.1, for every $t \le T$ one has*

$$\|\psi_k P_t^*(f/\psi_k)\|_p \le K_{kp}^{1/p} Q^{1/p_*} \, \|f\|_p \,. \tag{B.3}$$

Proof. Using Hölder's inequality, the identity $\psi_k^p = \psi_{kp}$, and (B.2)

$$|P_t(\psi_k g)(x)| \le |P_t(\psi_k^p)(x)|^{1/p} \, |P_t(|g|^{p_*})(x)|^{1/p_*} \le K_{kp}^{1/p} \psi_k(x) \, |P_t(|g|^{p_*})(x)|^{1/p_*} \,.$$

Then, using (B.1)

$$\begin{aligned} \left\| \frac{1}{\psi_k} P_t(\psi_k g) \right\|_{p_*} &\le K_{kp}^{1/p} \left\| |P_t(|g|^{p_*})|^{1/p_*} \right\|_{p_*} = K_{kp}^{1/p} (\|P_t(|g|^{p_*})\|_1)^{1/p_*} \\ &\le K_{kp}^{1/p} Q^{1/p_*} (\||g|^{p_*}\|_1)^{1/p_*} = K_{kp}^{1/p} Q^{1/p_*} \, \|g\|_{p_*} \,. \end{aligned}$$

Using Hölder's inequality first and the above inequality we obtain

$$\begin{aligned} |\langle g, \psi_k P_t^*(f/\psi_k)\rangle| &= \left| \left\langle \frac{1}{\psi_k} P_t(g\psi_k), f \right\rangle \right| \le \|f\|_p \left\| \frac{1}{\psi_k} P_t(g\psi_k) \right\|_{p_*} \\ &\le K_{kp}^{1/p} Q^{1/p_*} \, \|g\|_{p_*} \, \|f\|_p \,. \end{aligned}$$

□

We consider also the following hypothesis.

Assumption B.3. There exists $\rho > 1$ such that for every $q \in \mathbb{N}$ there exists $D^*_{(q)}(\rho) \geq 1$ such that for every $x \in \mathbb{R}^d$ and $t \leq T$

$$\sum_{|\alpha|\leq q} |\partial^\alpha P^*_t f(x)| \leq D^*_{(q)}(\rho) \sum_{|\alpha|\leq q} (P^*_t(|\partial^\alpha f|^\rho)(x))^{1/\rho}. \tag{B.4}$$

Proposition B.4. *Suppose that Assumption B.1 and B.3 hold. Then for every $k, q \in \mathbb{N}$ and $p > \rho$ there exists a universal constant C (depending on k and q only) such that for every $t \leq T$*

$$\|\psi_k P^*_t(f/\psi_k)\|_{q,p} \leq C K^{1/p}_{kp} Q^{(p-\rho)/\rho p} D^*_{(q)}(\rho) \|f\|_{q,p}. \tag{B.5}$$

Proof. We will prove (B.5). Let α with $|\alpha| \leq q$. By (B.4)

$$\begin{aligned}
|\partial^\alpha(\psi_k P^*_t(f/\psi_k)(x))| &\leq C\psi_k(x) \sum_{|\gamma|\leq q} |\partial^\gamma(P^*_t(f/\psi_k)(x))| \\
&\leq CD^*_{(q)}(\rho)\psi_k(x) \sum_{|\beta|\leq q} (P^*_t(\left|\partial^\beta(f/\psi_k)\right|^\rho)(x))^{1/\rho} \\
&= CD^*_{(q)}(\rho) \sum_{|\beta|\leq q} (\psi_{\rho k}(x) P^*_t(\left|\partial^\beta(f/\psi_k)\right|^\rho)(x))^{1/\rho} \\
&= CD^*_{(q)}(\rho) \sum_{|\beta|\leq q} (\psi_{\rho k}(x) P^*_t(g_\beta/\psi_{\rho k})(x))^{1/\rho}
\end{aligned}$$

with

$$g_\beta(x) = \psi_{\rho k}(x) \left|\partial^\beta(f/\psi_k)(x)\right|^\rho = \left|\psi_k(x)\partial^\beta(f/\psi_k)(x)\right|^\rho.$$

Taking $p > \rho$ and using (B.3)

$$\left\|(\psi_{\rho k} P^*_t(g_\beta/\psi_{\rho k}))^{1/\rho}\right\|_p = \|\psi_{\rho k} P^*_t(g_\beta/\psi_{\rho k})\|^{1/\rho}_{p/\rho} \leq K^{1/p}_{kp} Q^{(p-\rho)/\rho p} \|g_\beta\|^{1/\rho}_{p/\rho}.$$

We also have

$$\begin{aligned}
\|g_\beta\|^{1/\rho}_{p/\rho} &= \left(\int \left|\psi_k(x)\partial^\beta(f/\psi_k)(x)\right|^p dx\right)^{1/p} \\
&\leq C \sum_{|\gamma|\leq q} (\int |\partial^\gamma f(x)|^p dx)^{1/p} = C \|f\|_{q,p}.
\end{aligned}$$

We conclude that

$$\|\psi_k P^*_t(f/\psi_k)\|_{q,p} \leq C K^{1/p}_{kp} Q^{(p-\rho)/\rho p} D^*_{(q)}(\rho) \|f\|_{q,p}.$$

□

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