

September 2021

## Ergodicity of Burgers' System

Szymon Peszat

*Jagiellonian University, Lojasiewicza 6, 30-348 Kraków, Poland, napeszat@cyf-kr.edu.pl*

Krystyna Twardowska

*Warsaw University of Life Sciences - SGGW, ul. Nowoursynowska 159, 02-776 Warsaw, Poland, krystyna\_twardowska@sggw.pl*

Jerzy Zabczyk

*Institute of Mathematics, Polish Academy of Sciences, ul. Śniadeckich 8, 00-956 Warsaw, Poland, zabczyk@impan.pl*

Follow this and additional works at: <https://digitalcommons.lsu.edu/josa>



Part of the [Analysis Commons](#), and the [Other Mathematics Commons](#)

---

### Recommended Citation

Peszat, Szymon; Twardowska, Krystyna; and Zabczyk, Jerzy (2021) "Ergodicity of Burgers' System," *Journal of Stochastic Analysis*: Vol. 2 : No. 3 , Article 10.

DOI: 10.31390/josa.2.3.10

Available at: <https://digitalcommons.lsu.edu/josa/vol2/iss3/10>

## ERGODICITY OF BURGERS' SYSTEM

SZYMON PESZAT\*, KRYSZYNA TWARDOWSKA†, AND JERZY ZABCZYK

*Dedicated to the memory of Professor Hiroshi Kunita*

ABSTRACT. We consider a stochastic version of a system of coupled two equations formulated by Burgers [2] with the aim to describe the laminar and turbulent motions of a fluid in a channel. The existence and uniqueness of the solution as well as the irreducibility property of such system were given by Twardowska and Zabczyk [18, 19, 20]. In the paper the existence of a unique invariant measure is investigated. The paper generalizes the results of Da Prato, Debussche and Temam [4], and Da Prato and Gatarek [5], dealing with one equation describing the turbulent motion only.

### 1. Introduction

Let  $U = U(t)$  denote the *primary* velocity of the fluid, parallel to the walls of the channel, and let  $v = v(t, x)$  denote the *secondary* velocity of the turbulent motion. According to [2], they satisfy the following system of equations

$$\frac{dU}{dt}(t) = P - \nu U(t) - \int_0^1 v^2(t, y) dy, \quad (1.1)$$

$$\frac{\partial v}{\partial t}(t, x) = \nu \frac{\partial^2 v}{\partial x^2}(t, x) + U(t)v(t, x) - \frac{\partial}{\partial x}(v^2(t, x)) \quad (1.2)$$

for  $t > 0$  and  $x \in (0, 1)$ . The system is considered with the initial and boundary conditions

$$\begin{aligned} U(0) &= U_0, \quad v(0, x) = v_0(x), \quad x \in (0, 1) \\ v(t, 0) &= v(t, 1) = 0, \quad t > 0. \end{aligned} \quad (1.3)$$

In (1.1) and (1.2),  $P$  is a constant representing an exterior force, analogous to the mean pressure gradient in the hydrodynamic case, and  $\nu = \frac{\mu}{\rho} > 0$ , where  $\rho$  is the density and  $\mu$  is the viscosity of the fluid. We assume that  $\rho$  and  $\mu$  are constant. The system is derived from the theory of turbulent fluid motion and has similar properties to the Navier–Stokes equation, but is much simpler to study.

The existence and uniqueness of the global solution was established by Dłotko [8], using the Galerkin method. The necessary and sufficient conditions on  $P$  and

---

Received 2020-6-18; Accepted 2021-7-12; Communicated by S. Aida, D. Applebaum, Y. Ishikawa, A. Kohatsu-Higa, and N. Privault.

2010 *Mathematics Subject Classification.* 35Q72, 60H15.

*Key words and phrases.* Stochastic Burgers' system, the existence and uniqueness of an invariant measure, strong Feller and irreducibility.

\* Corresponding author.

† Passed away in 2011.

$\nu$  such that the solutions to (1.1)–(1.3) satisfy  $U(t) \rightarrow 0$  and  $v(t) \rightarrow 0$  as  $t \rightarrow +\infty$  are given in [3] and [8].

In order to obtain nontrivial limiting behaviour of solutions as  $t \rightarrow +\infty$ , the following stochastic perturbation was proposed

$$dU(t) = \left( P - \nu U(t) - \int_0^1 v^2(t, y) dy \right) dt + g_0(U(t), v(t)) dW_0(t), \quad (1.4)$$

$$dv(t, x) = \left( \nu \frac{\partial^2 v}{\partial x^2}(t, x) + U(t)v(t, x) - \frac{\partial}{\partial x}(v^2(t, x)) \right) dt + g_1(U(t), v(t, \cdot))(x) dW_1(t, x) \quad (1.5)$$

with the initial and boundary conditions (1.3). Above  $W_0$  is a real-valued Wiener process,  $W_1$  is an independent of  $W_0$  cylindrical Wiener process in  $L^2 := L^2(0, 1)$ , see the next section,  $g_0: \mathbb{R} \times L^2 \mapsto \mathbb{R}$  and  $g_1: \mathbb{R} \times L^2 \mapsto L^2$  are Lipschitz continuous functions.

The existence and uniqueness of the global solution to (1.3)–(1.5) in the space  $H := \mathbb{R} \times L^2$  was established by Twardowska and Zabczyk [18, 19, 20]. The existence and uniqueness of the solution for the classical one dimensional stochastic Burgers equation driven by cylindrical Wiener process, as well as the existence of an invariant measure was established by Da Prato, Debussche and Temam [4].

As the main result, we prove the existence of an invariant measure to system (1.3)–(1.5). In the proof we use the Krylov–Bogolyubov theorem adapting the method of Da Prato and Gatarek [5], see also [7]. The main difficulty is caused by the fact that since the noise is not an  $\mathbb{R} \times L^2$ -valued process we cannot use the Itô formula. Due to existence of nonlinear diffusion terms and different structure of our system we cannot apply the Hopf–Cole transformation as for example in [10].

A similar problem of an hydrodynamic equation coupled with heat equation was studied by Ferrario [11]. In the considered there Bénard problem: a viscous fluid, in a rectangular container, is heated from below and the top surface is taken at constant temperature. Heating the fluid, its density changes and the gradient of density gives rise to a motion of particles from the bottom to the top of the container.

## 2. Preliminaries and Formulation of the Main Result

Let  $L^2 := L^2(0, 1)$  be the Hilbert space equipped with the scalar product  $\langle \varphi, \psi \rangle = \int_0^1 \varphi(x)\psi(x)dx$  and the corresponding norm  $\|\cdot\|$ .

Let  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  be a filtered probability space. We assume that  $(\Omega, \mathcal{F}, \mathbb{P})$  is complete, the filtration is right-continuous and each  $\mathcal{F}_t$  contains all  $\mathbb{P}$ -null sets in  $\mathcal{F}$ . We consider the one-dimensional Wiener process  $W_0(t)$ ,  $t \geq 0$ , and a cylindrical Wiener process  $W_1(t, \cdot)$ ,  $t \geq 0$ , in  $L^2$ . It can be defined as the sum

$$W_1(t, x) = \sum_{k=1}^{\infty} W^k(t) e_k(x),$$

where  $(e_k)$  is any orthonormal basis of  $L^2$  and  $(W^k)$  are independent real-valued Wiener processes. The sum converges however not in  $L^2$  but in any Hilbert space  $V$  such that the imbedding  $L^2 \hookrightarrow V$  is Hilbert–Schmidt. We assume that for all

$t > s > 0$ ,  $k \in \mathbb{N}$ , the increments  $W_0(t) - W_0(s)$ ,  $W^k(t) - W^k(s)$  are independent of the  $\sigma$ -field  $\mathcal{F}_s$ .

From now on, we assume that  $(e_k)$  is the orthonormal basis of the eigenvectors of the Laplace operator on  $L^2$  with Dirichlet boundary conditions, that is

$$e_k(x) = \sqrt{\frac{2}{\pi}} \sin k\pi x, \quad x \in (0, 1), \quad k = 1, 2, \dots \quad (2.1)$$

**Definition 2.1.** A pair of processes  $(U, v)$  is a *weak solution* to problem (1.3)–(1.5) if and only if they are adapted with continuous trajectories in  $\mathbb{R}$  and  $L^2$ , respectively, and such that for arbitrary  $t \geq 0$ ,

$$\begin{aligned} U(t) &= U_0 + tP - \nu \int_0^t U(s) ds - \int_0^t \|v(s)\|^2 ds \\ &\quad + \int_0^t g_0(U(s), v(s)) dW_0(s), \quad \mathbb{P}\text{-a.s.}, \end{aligned}$$

and for arbitrary  $\varphi \in C_0^\infty(0, 1)$ ,

$$\begin{aligned} \langle v(t), \varphi \rangle &= \langle v_0, \varphi \rangle + \int_0^t U(s) \langle v(s), \varphi \rangle ds \\ &\quad + \int_0^t \left\langle v(s), \nu \frac{d^2 \varphi}{dx^2} \right\rangle ds + \int_0^t \left\langle v^2(s), \frac{d\varphi}{dx} \right\rangle ds \\ &\quad + \int_0^t \langle \varphi, g_1(U(s), v(s, \cdot)) dW_1(s) \rangle, \quad \mathbb{P}\text{-a.s.} \end{aligned}$$

We introduce now an equivalent concept of the *integral or mild solution*, see e.g. [6, 7]. Namely, let  $(e_k)$  be the orthonormal basis of  $L^2$  given by (2.1), and let  $S(t)$ ,  $t \geq 0$ , be the classical heat semigroup on  $L^2$ . Then, for any  $\psi \in L^2$ ,

$$S(t)\psi = \sum_{k=1}^{\infty} e^{-\frac{\pi^2}{\nu} k^2 t} \langle \psi, e_k \rangle e_k$$

with the convergence of the series in  $L^2$ . It is well known that the generator  $A$  of the semigroup  $S(t)$ ,  $t \geq 0$ , is identical with the second derivative operator  $\frac{d^2}{dx^2}$  on the domain  $D(A)$  consisting of functions  $\psi$  such that  $\psi$  and  $\frac{d\psi}{dx}$  are absolutely continuous,  $\psi$ ,  $\frac{d\psi}{dx}$ ,  $\frac{d^2\psi}{dx^2} \in L^2$ , and  $\psi(0) = \psi(1) = 0$ .

A proof of the following lemma can be find e.g. in [4, 18, 19, 20].

**Lemma 2.2.** *The operators  $S(t)$ ,  $t > 0$ , can be extended linearly to the space of all distributions of the form  $\frac{d\psi}{dx}$ ,  $\psi \in L^1(0, 1)$ , in such a way that they take values in  $L^2$  and for all  $\psi \in L^1$  and  $t \geq 0$ ,*

$$\left\| S(t) \frac{d\psi}{dx} \right\| \leq \|\psi\|_{L^1(0,1)} \left( \sum_{k=1}^{\infty} \frac{2\pi}{\sqrt{\nu}} k^2 e^{-\frac{2\pi^2}{\nu} k^2 t} \right)^{1/2} \leq \frac{C}{\sqrt{t}} \|\psi\|_{L^1(0,1)}.$$

**Definition 2.3.** A pair of continuous adapted processes  $(U, v)$  with values in  $\mathbb{R}$  and  $L^2$ , respectively, is said to be *an integral or mild solution to problem (1.3)–(1.5)* if

$$U(t) = e^{-\nu t} U_0 + \int_0^t e^{-\nu(t-s)} (P - \|v(s)\|^2) ds \\ + \int_0^t e^{-\nu(t-s)} g_0(U(s), v(s)) dW_0(s),$$

and

$$v(t) = S(t)v_0 + \int_0^t S(t-s) \left( U(s)v(s) - \frac{dv^2}{dx}(s) \right) ds \\ + \int_0^t S(t-s) g_1(U(s), v(s)) dW_1(s).$$

In the integral

$$\int_0^t S(t-s) \frac{dv^2}{dx}(s) ds$$

we use the extension of the operator  $S(t-s)$  described in Lemma 2.2.

Let  $Z_T^p$ ,  $p > 1$ , denote the space of all continuous adapted processes  $X = (U, v)$  on  $[0, T]$  with values on  $\mathbb{R} \times L^2$  such that

$$\|(U, v)\|_{p,T} = \|U\|_{1,p,T} + \|v\|_{2,p,T} < +\infty,$$

where

$$\|U\|_{1,p,T} = \left( \mathbb{E} \sup_{t \in [0,T]} |U(t)|^p \right)^{1/p}, \\ \|v\|_{2,p,T} = \left( \mathbb{E} \sup_{t \in [0,T]} \|v(t)\|^p \right)^{1/p}.$$

The first part of the theorem below concerning the existence of the solution was proven in [18, 19, 20] and it is stated here only for the sake of completeness. The second and the third parts concerning existence and uniqueness of the invariant measure are the main result of the present paper. Their proofs are presented in Sections 4 and 5, respectively.

**Theorem 2.4.** *Assume that the mappings  $g_0: \mathbb{R} \times L^2 \mapsto \mathbb{R}$  and  $g_1: \mathbb{R} \times L^2 \mapsto L^2$  are bounded and Lipschitz continuous. Then the following assertions are true:*

- (i) *For any initial values  $U_0 \in \mathbb{R}$  and  $v_0 \in L^2$ , system (1.3)–(1.5) has a unique weak solution in any  $Z_T^p$ -space,  $T > 0$ ,  $p \geq 1$ . Moreover, (1.3)–(1.5) defines Feller Markov process on the Hilbert space  $H := \mathbb{R} \times L^2$ .*
- (ii) *There is an invariant measure  $\mu$  for the Markov family defined by system (1.3)–(1.5).*
- (iii) *If  $g_0$  and  $g_1$  are separated from zero; i.e. there is a constant  $C > 0$  such that*

$$\left| \frac{1}{g_0(r, \psi)} \right| \leq C, \quad \left\| \frac{1}{g_1(r, \psi)} \right\| \leq C \quad \text{for all } r \in \mathbb{R} \text{ and } \psi \in L^2,$$

then the corresponding transition semigroup  $(P_t)$  is strong Feller and irreducible, and the invariant measure  $\mu$  is unique. Finally, for any initial value  $(U_0, v_0)$  the law  $\mathcal{L}(U(t), v(t))$  of the solution at time  $t$  converges in the total variation norm to  $\mu$  as  $t \rightarrow +\infty$ .

### 3. Auxiliary Results

To prove the existence of an invariant measure we shall apply the classical Krylov–Bogolyubov criterion in the form of the following proposition, see [7].

**Proposition 3.1. (Krylov–Bogolyubov)** *Assume that  $X$  is a Feller Markov process on a Hilbert space  $H$  and  $H_0 \subset H$  is a Hilbert space contained in  $H$  with compact embedding. If for some  $X(0)$  and for each  $\varepsilon > 0$  there exists a constant  $M$  such that for all  $t > 0$ ,*

$$\frac{1}{t} \int_0^t P(\|X(s)\|_{H_0} \geq M) ds \leq \varepsilon,$$

then there exists an invariant measure for  $X$  in  $H$ .

Let us recall that  $\nu$  is the viscosity parameter. The eigenvalues of  $A$  are  $-\lambda_m = \frac{\pi^2 m^2}{\nu}$ ,  $m = 1, \dots$ . The operator  $-A$  is self-adjoint and positive definite and for arbitrary  $\alpha > 0$  the fractional power  $(-A)^\alpha$  is given by the formulae,

$$(-A)^\alpha \psi = \sum_{m=1}^{\infty} \lambda_m^\alpha \psi_m e_m \quad \text{for} \quad \psi = \sum_{m=1}^{\infty} \psi_m e_m \in D(-A)^\alpha,$$

where  $\psi \in D(-A)^\alpha$  if and only if

$$\|(-A)^\alpha \psi\| = \left( \sum_{m=1}^{\infty} \lambda_m^{2\alpha} \psi_m^2 \right)^{1/2} < +\infty.$$

We set  $H^{2\alpha} = D(-A)^\alpha$ .

In our proof of the existence of an invariant measure to (1.3)–(1.5) we will show that the assumptions of the proposition above are satisfied for  $H = \mathbb{R} \times L^2$  and  $H_0 = \mathbb{R} \times H^{\frac{1}{4}}$ .

Let  $X(t) = (U(t), v(t))$  be the solution to problem (1.3)–(1.5). We assume that  $U(0) = 0$  and  $v_0 = 0$ . Given  $L > 0$  set

$$\begin{aligned} Z_L(t) &:= \int_0^t e^{-L(t-s)} S(t-s) g_1(U(s), v(s)) dW_1(s), & t > 0, \\ Y_L(t) &:= \int_0^t e^{-(L+\nu)(t-s)} g_0(U(s), v(s)) dW_0(s), & t > 0. \end{aligned} \tag{3.1}$$

**Lemma 3.2.** *Let  $p \geq 2$  and  $\varepsilon > 0$ . There exists  $L_0 \geq 0$  such that for  $L \geq L_0$  we have*

$$\sup_{0 \leq t < +\infty} \mathbb{E} \left( \|Z_L(t)\|_{H^{\frac{1}{4}}}^p + |Y_L(t)|^p \right) \leq \varepsilon.$$

*Proof.* Let  $\|B\|_{(HS)}$  denote the Hilbert–Schmidt norm of a linear operator  $B: L^2 \mapsto L^2$ . In (3.1) we identify an element  $g_1(U(s), v(s))$  of  $L^2$  with the multiplication operator

$$g_1(U(s), v(s)) : L^2 \ni \psi \mapsto g_1(U(s), v(s)) \psi \in L^2.$$

Clearly, its operator norm

$$\|g_1(U(s), v(s))\|_{L(L^2, L^2)} \leq \sup_{r \in \mathbb{R}, \psi \in L^2} \|g_1(r, \psi)\| := K.$$

Hence using the Burkholder inequality and denoting by  $S_L(t) = e^{-Lt}S(t)$ , we obtain

$$\begin{aligned} \mathbb{E} \|Z_L(t)\|_{H^{\frac{1}{4}}}^p &\leq c_p \mathbb{E} \left( \int_0^t \left\| (-A)^{\frac{1}{8}} S_L(t-s) g_1(U(s), v(s)) \right\|_{(HS)}^2 ds \right)^{\frac{p}{2}} \\ &\leq c_p K^p \left( \int_0^{+\infty} \left\| (-A)^{\frac{1}{8}} S_L(s) \right\|_{(HS)}^2 ds \right)^{\frac{p}{2}} \\ &\leq c_p K^p \left( \int_0^{+\infty} e^{-2Ls} \left\| (-A)^{\frac{1}{8}} S(s/2) S(s/2) \right\|_{(HS)}^2 ds \right)^{\frac{p}{2}}. \end{aligned}$$

Since

$$\begin{aligned} \left\| (-A)^{\frac{1}{8}} S(s/2) S(s/2) \right\|_{(HS)} &\leq \left\| (-A)^{\frac{1}{8}} S(s/2) \right\|_{L(L^2, L^2)} \|S(s/2)\|_{(HS)} \\ &\leq C s^{-\frac{1}{8}} \|S(s/2)\|_{(HS)}, \end{aligned}$$

we have

$$\mathbb{E} \|Z_L(t)\|_{H^{\frac{1}{4}}}^p \leq C' \left( \int_0^{+\infty} e^{-2Ls} s^{-\frac{1}{4}} \sum_{k=1}^{\infty} e^{-\frac{\pi^2}{\nu} k^2 s} ds \right)^{\frac{p}{2}}.$$

Therefore the desired conclusion follows from the Lebesgue dominated convergence theorem and fact that

$$\int_0^{+\infty} s^{-\frac{1}{4}} \sum_{k=1}^{\infty} e^{-\frac{\pi^2}{\nu} k^2 s} ds = \int_0^{+\infty} s^{-\frac{1}{4}} e^{-s} ds \sum_{k=1}^{\infty} \left( \frac{\pi^2}{\nu} k^2 \right)^{-1+\frac{1}{4}} < +\infty.$$

Clearly, in the same way one can obtain the desired estimate for  $Y_L$ .  $\square$

**Corollary 3.3.** *Taking into account the Sobolev embedding  $H^{\frac{1}{4}} \hookrightarrow L^4$ , see Theorem 3, p. 8 in [15], we obtain*

$$\lim_{L \rightarrow +\infty} \sup_{t \geq 0} \mathbb{E} \|Z_L(t)\|_{L^4}^p = 0.$$

Let

$$\begin{aligned} V_L(t) &:= v(t) - Z_L(t), \\ U_L(t) &:= U(t) - Y_L(t). \end{aligned}$$

Note that  $V_L$  and  $U_L$  are solutions to the following system of equations with random coefficients:

$$\begin{aligned} \frac{dU_L}{dt}(t) &= P - \nu U_L(t) - \|V_L(t) + Z_L(t)\|^2 + LY_L(t), \\ U_L(0) &= U_0 \end{aligned}$$

and

$$\begin{aligned} \frac{\partial V_L}{\partial t}(t, x) &= \nu \frac{\partial^2 V_L}{\partial x^2}(t, x) + (U_L(t) + Y_L(t))(V_L(t, x) + Z_L(t, x)) \\ &\quad - \frac{\partial}{\partial x} (V_L(t, x) + Z_L(t, x))^2 + LZ_L(t, x), \\ V_L(0, x) &= v_0(x), \\ V_L(t, 0) &= 0 = V_L(t, 1), \quad t > 0. \end{aligned}$$

Recall, see e.g. [15], that  $H_0^1$  is the Sobolev space consisting of absolutely continuous functions  $\psi$  such that  $\frac{d\psi}{dx} \in L^2$ , and  $\psi(0) = \psi(1) = 0$ . We can now state the main result of this section.

**Lemma 3.4.** *Given  $L$  write*

$$y_L(t) := \|V_L(t)\|^2 + U_L^2(t), \quad t \geq 0.$$

*Then there are independent of  $L$  constants  $\beta > 0$  and  $C > 0$  such that*

$$\frac{d}{dt} y_L(t) + \beta \left[ y_L(t) + \|V_L(t)\|_{H_0^1} \right] \leq C y_L(t) \xi_L(t) + C \eta_L(t),$$

where

$$\begin{aligned} \xi_L(t) &:= |Y_L(t)| + \|Z_L(t)\|^2 + \|Z_L(t)\| + \|Z_L(t)\|_{L^4}^{\frac{8}{3}}, \\ \eta_L(t) &:= (1 + L^2) (Y_L^4(t) + \|Z_L(t)\|_{L^4}^4) + 1. \end{aligned}$$

*Proof.* We have

$$\begin{aligned} \left\langle \frac{\partial V_L}{\partial t}, V_L \right\rangle &= \nu \left\langle \frac{\partial^2 V_L}{\partial x^2}, V_L \right\rangle + U \langle V_L + Z_L, V_L \rangle \\ &\quad - \left\langle \frac{\partial}{\partial x} (V_L + Z_L)^2, V_L \right\rangle + L \langle Z_L, V_L \rangle. \end{aligned}$$

Therefore

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|V_L\|^2 &= -\nu \left\langle \frac{\partial V_L}{\partial x}, \frac{\partial V_L}{\partial x} \right\rangle + U \langle V_L, V_L \rangle + U \langle Z_L, V_L \rangle \\ &\quad + \left\langle V_L^2, \frac{\partial V_L}{\partial x} \right\rangle + 2 \left\langle V_L Z_L, \frac{\partial V_L}{\partial x} \right\rangle + \left\langle Z_L^2, \frac{\partial V_L}{\partial x} \right\rangle + L \langle Z_L, V_L \rangle \\ &= -\nu \left\langle \frac{\partial V_L}{\partial x}, \frac{\partial V_L}{\partial x} \right\rangle + U \langle V_L, V_L \rangle + U \langle Z_L, V_L \rangle \\ &\quad + 2 \left\langle V_L Z_L, \frac{\partial V_L}{\partial x} \right\rangle + \left\langle Z_L^2, \frac{\partial V_L}{\partial x} \right\rangle + L \langle Z_L, V_L \rangle, \end{aligned}$$

since  $\langle V_L^2, \frac{\partial V_L}{\partial x} \rangle = 0$ . In other words we have

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \|V_L\|^2 + \nu \|V_L\|_{H_0^1}^2 \\ &= U \|V_L\|^2 + U \langle V_L, Z_L \rangle + 2 \int_0^1 V_L Z_L \frac{\partial V_L}{\partial x} dx + \int_0^1 Z_L^2 \frac{\partial V_L}{\partial x} dx + L \langle Z_L, V_L \rangle. \end{aligned}$$



Further we have

$$\begin{aligned} \left| \int_0^1 Z_L^2 \frac{\partial V_L}{\partial x} dx \right| &\leq \|Z_L^2\|^{\frac{1}{2}} \left\| \frac{\partial V_L}{\partial x} \right\| = \|Z_L\|_{L^4}^2 \|V_L\|_{H_0^1} \\ &\leq \frac{\nu}{4} \|V_L\|_{H_0^1}^2 + \frac{1}{\nu} \|Z_L\|_{L^4}^4. \end{aligned}$$

Using an interpolation estimate, see e.g. [13], we obtain

$$\begin{aligned} 2 \left| \int_0^1 V_L Z_L \frac{\partial V_L}{\partial x} dx \right| &\leq 2C_1 \|V_L\|^{\frac{3}{4}} \|V_L\|_{H_0^1}^{\frac{5}{4}} \|Z_L\|_{L^4} \\ &\leq \frac{\nu}{4} \|V_L\|_{H_0^1}^2 + C_2 \|V_L\|^2 \|Z_L\|_{L^4}^{\frac{8}{3}} \end{aligned}$$

Hence

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|V_L\|^2 + \frac{\nu}{2} \|V_L\|_{H_0^1}^2 &\leq U \|V_L\|^2 + U \langle V_L, Z_L \rangle + L \|Z_L\| \|V_L\| \\ &\quad + C_2 \|V_L\|^2 \|Z_L\|_{L^4}^{\frac{8}{3}} + \frac{1}{\nu} \|Z_L\|_{L^4}^4. \end{aligned}$$

Recall that  $U = U_L + Y_L$  and

$$\begin{aligned} \frac{1}{2} \frac{dU_L^2}{dt} + \nu U_L^2 &= U_L (P - \|V_L + Z_L\|^2 + LY_L) \\ &= U_L (P - \|V_L\|^2 - 2\langle V_L, Z_L \rangle - \|Z_L\|^2) + LU_L Y_L \\ &\leq \frac{\nu}{2} U_L^2 + \frac{1}{2\nu} P^2 - U_L \|V_L\|^2 - 2U_L \langle V_L, Z_L \rangle - U_L \|Z_L\|^2 \\ &\quad + LU_L Y_L. \end{aligned}$$

Therefore adding the above inequality we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} [\|V_L\|^2 + U_L^2] + \frac{\nu}{2} [\|V_L\|_{H_0^1}^2 + U_L^2] \\ \leq Y_L \|V_L\|^2 + (Y_L - U_L) \langle V_L, Z_L \rangle + L \|Z_L\| \|V_L\| \\ + C_2 \|V_L\|^2 \|Z_L\|_{L^4}^{\frac{8}{3}} + \frac{1}{\nu} \|Z_L\|_{L^4}^4 + \frac{1}{2\nu} P^2 - U_L \|Z_L\|^2 + LU_L Y_L. \end{aligned}$$

Thus, using the Young inequality  $|ab| \leq a^p/p + b^q/q$ , we obtain

$$\begin{aligned} \frac{d}{dt} [\|V_L\|^2 + U_L^2] + \frac{\nu}{2} [\|V_L\|_{H_0^1}^2 + U_L^2] \\ \leq 2|Y_L| \|V_L\|^2 + 2(Y_L - U_L) \langle V_L, Z_L \rangle + C_3 L^2 \|Z_L\|^2 \\ + 2C_2 \|V_L\|^2 \|Z_L\|_{L^4}^{\frac{8}{3}} + \frac{2}{\nu} \|Z_L\|_{L^4}^4 + \frac{1}{\nu} P + C_3 \|Z_L\|^4 + C_3 L^2 Y_L^2 \\ \leq 2|Y_L| \|V_L\|^2 + 2|Y_L| \|Z_L\| \|V_L\| + 2|U_L| \|V_L\| \|Z_L\| + C_3 L^2 \|Z_L\|^2 \\ + 2C_2 \|V_L\|^2 \|Z_L\|_{L^4}^{\frac{8}{3}} + \frac{2}{\nu} \|Z_L\|_{L^4}^4 + \frac{1}{\nu} P + C_3 \|Z_L\|^4 + C_3 L^2 Y_L^2 \\ \leq 2|Y_L| \|V_L\|^2 + |Y_L|^2 + \|Z_L\|^2 \|V_L\|^2 + U_L^2 \|Z_L\| + \|V_L\|^2 \|Z_L\| \\ + C_3 L^2 \|Z_L\|^2 + 2C_2 \|V_L\|^2 \|Z_L\|_{L^4}^{\frac{8}{3}} + \frac{2}{\nu} \|Z_L\|_{L^4}^4 \end{aligned}$$

$$\begin{aligned}
 &\leq +\frac{1}{\nu}P + C_3\|Z_L\|^4 + C_3L^2Y_L^2 \\
 &\leq C_4[\|V_L\|^2 + U_L^2] \left( Y_L(t) + \|Z_L\|^2 + \|Z_L\| + \|Z_L\|_{L^4}^{\frac{8}{4}} \right) \\
 &\quad + C_4((1+L^2)|Y_L|^2 + (1+L^2)\|Z_L\|^2 + \|Z_L\|_{L^4}^4 + P)
 \end{aligned}$$

and the desired estimate follows from the Poincaré inequality

$$\pi^2\|V_L\|^2 \leq \|V_L\|_{H_0^1}^2$$

and from the fact that  $\|Z_L\| + \|Z_L\|^2 \leq 2\|Z_L\|_{L^4}^4 + 2$ .  $\square$

**Corollary 3.5.** *Let  $X = (U, v)$  be the solution to problem (1.3)–(1.5) with initial conditions  $U(0) = U_0 \in \mathbb{R}$  and  $v(0) = v_0 \in L^2$ . Then*

$$\lim_{M \rightarrow +\infty} \limsup_{T \rightarrow +\infty} \frac{1}{T} \int_0^T \mathbb{P}(\|U(t)\| + \|v(t)\| \geq M) dt = 0.$$

*Proof.* We use the idea of Da Prato and Gatarek [5], see also Da Prato and Zabczyk [7]. Namely, let us fix  $M \geq 2$ . Recall, see Lemma 3.4, that  $y_L(t) := \|V_L(t)\|^2 + U_L^2(t)$ . Given  $L > 0$  set

$$\zeta_L(t) := \log \left( y_L(t) \vee \frac{M}{2} \right).$$

Then

$$\frac{d}{dt} \zeta_L(t) = \chi_{\{y_L(t) \geq M/2\}} \frac{1}{y_L(t)} \frac{d}{dt} y_L(t).$$

Recall, see Lemma 3.4 that

$$\frac{d}{dt} y_L(t) + \beta y_L(t) \leq C y_L(t) \xi_L(t) + C \eta_L(t).$$

Multiplying both sides of the inequality above, by

$$\chi_{\{y_L(t) \geq M/2\}} \frac{1}{y_L(t)} \leq \frac{2}{M},$$

we obtain

$$\frac{d}{dt} \zeta_L(t) + \beta \chi_{\{y_L(t) \geq M/2\}} \leq C \xi_L(t) + \frac{2C}{M} \eta_L(t).$$

Hence, after integrating over  $t$  and taking expectation, we get

$$\begin{aligned}
 &\mathbb{E}(\zeta_L(T) - \zeta_L(0)) + \beta \int_0^T \mathbb{P} \left( y_L(t) \geq \frac{M}{2} \right) dt \\
 &\leq C \int_0^T \left( \mathbb{E} \xi_L(t) + \frac{2\mathbb{E} \eta_L}{M} \right) dt.
 \end{aligned}$$

Since  $\zeta_L(T) - \zeta_L(0) \geq 0$  because  $M \geq 2$ , we have

$$\frac{1}{T} \int_0^T \mathbb{P} \left( y_L(t) \geq \frac{M}{2} \right) dt \leq \frac{C}{\beta T} \int_0^T \left( \mathbb{E} \xi_L(t) + \frac{2\mathbb{E} \eta_L}{M} \right) dt.$$

Let us fix an  $\varepsilon > 0$ . Then, by Lemma 3.2 there is a constant  $L_0$  such that for any  $L \geq L_0$ ,

$$\sup_{t \geq 0} \mathbb{E} \xi_L(t) \leq \varepsilon \quad \text{and} \quad \sup_{t \geq 0} \mathbb{E} \eta_L(t) \leq (1 + L^2)\varepsilon + 1.$$

Therefore, for  $L \geq L_0$ ,

$$\frac{1}{T} \int_0^T \mathbb{P} \left( y_L(t) \geq \frac{M}{2} \right) dt \leq \frac{C}{\beta} \left[ \varepsilon + \frac{(1+L^2)\varepsilon + 1}{M} \right].$$

Since

$$U^2(t) + \|v(t)\|^2 = y_L(t) + Y_L^2(t) + \|Z_L(t)\|^2,$$

we have by the Chebyshev inequality

$$\begin{aligned} & \frac{1}{T} \int_0^T \mathbb{P} (U^2(t) + \|v(t)\|^2 \geq M) dt \\ & \leq \frac{1}{T} \int_0^T \left[ \mathbb{P} \left( y_L(t) \geq \frac{M}{2} \right) + \mathbb{P} \left( Y_L^2(t) + \|Z_L(t)\|^2 \geq \frac{M}{2} \right) \right] dt \\ & \leq \frac{C}{\beta} \left[ \varepsilon + \frac{(1+L^2)\varepsilon + 1}{M} \right] + \frac{2}{M} \sup_{t \geq 0} \mathbb{E} [Y_L^2(t) + \|Z_L(t)\|^2], \end{aligned}$$

and the desired conclusion follows since by Lemma 3.2,

$$\lim_{L \rightarrow +\infty} \sup_{t \geq 0} \mathbb{E} [Y_L^2(t) + \|Z_L(t)\|^2] = 0.$$

To be more precise, given  $\varepsilon > 0$  we first find  $L$  such that

$$\sup_{t \geq 0} \mathbb{E} [Y_L^2(t) + \|Z_L(t)\|^2] \leq \frac{\varepsilon}{2},$$

and then  $M > 2$  such that

$$\frac{C}{\beta} \left[ \varepsilon + \frac{(1+L^2)\varepsilon + 1}{M} \right] \leq \frac{\varepsilon}{2}.$$

□

#### 4. Proof of the Existence of an Invariant Measure

Taking into account the Krylov–Bogolyubov criterion (see Proposition 3.1) and Corollary 3.5 it is enough to show that the family of laws  $(\mathcal{L}(v(t)), t > 0)$  is tight in the space  $H^{\frac{1}{4}}$ . Recall that  $v(t) = V_L(t) + Z_L(t)$ . On the other hand by Lemma 3.4 we have the inequality

$$\beta \|V_L(t)\|_{H_0^1}^2 \leq C y_L(t) \xi_L(t) + C \eta_L(t) \leq C [y_L^2(t) + \xi_L^2(t) + \eta_L(t)].$$

Since

$$y_L(t) = \|V_L(t)\|^2 + U_L^2(t) \leq 2 (\|v(t)\|^2 + U^2(t)) + 2 (\|Z_L(t)\|^2 + Y_L^2(t)),$$

and  $\|\cdot\|_{H^{\frac{1}{4}}} \leq C_1 \|\cdot\|_{H_0^1}$  we have

$$\begin{aligned} \|v(t)\|_{H^{\frac{1}{4}}} & \leq \|Z_L(t)\|_{H^{\frac{1}{4}}} + C_2 (\|v(t)\|^2 + U^2(t)) \\ & \quad + C_2 (\|Z_L(t)\|^2 + Y_L^2(t)) + C_2 [\xi_L^2(t) + \eta_L(t)]^{\frac{1}{2}} \\ & \leq C_2 (\|v(t)\|^2 + U^2(t)) + R_L(t), \end{aligned}$$

where, by Lemma 3.2,

$$R_L(t) := \|Z_L(t)\|_{H^{\frac{1}{4}}} + C_2 (\|Z_L(t)\|^2 + Y_L^2(t)) + C_2 [\xi_L^2(t) + \eta_L(t)]^{\frac{1}{2}}$$

satisfies

$$\sup_{t>0} \mathbb{E} R_L(t) < +\infty.$$

Hence

$$\mathbb{P}\left(\|v(t)\|_{H^{\frac{1}{4}}} \geq M\right) \leq \mathbb{P}\left(\|v(t)\|^2 + U^2(t) \geq \frac{M}{2C_2}\right) + \mathbb{P}\left(R_L(t) \geq \frac{M}{2}\right).$$

By Corollary 3.5,

$$\lim_{M \rightarrow +\infty} \limsup_{T \rightarrow +\infty} \frac{1}{T} \int_0^T \mathbb{P}\left(\|v(t)\|^2 + U^2(t) \geq \frac{M}{2C_2}\right) dt = 0.$$

By Chebyshev's inequality

$$\begin{aligned} & \lim_{M \rightarrow +\infty} \limsup_{T \rightarrow +\infty} \frac{1}{T} \int_0^T \mathbb{P}\left(R_L(t) \geq \frac{M}{2}\right) dt \\ & \leq \lim_{M \rightarrow +\infty} \frac{2 \sup_{t>0} \mathbb{E} R_L(t)}{M} = 0. \end{aligned}$$

□

## 5. Strong Feller and Irreducibility

In this section we will show that if  $g_0$  and  $g_1$  are separated from 0, then the semigroup defined by problem (1.3)–(1.5) is strong Feller and irreducible.

Let us recall first some basic definitions and theorems. Assume that  $P_t(z, \Gamma)$ ,  $t \geq 0$ ,  $z \in E$ ,  $\Gamma \in \mathcal{B}(E)$  is a transition probability on a Polish space  $E$ . Let  $(P_t)$  be the corresponding transition semigroup.

**Definition 5.1.** We call a transition probability  $P$  *regular* if the measures  $P_t(u, \cdot)$  and  $P_t(z, \cdot)$ ,  $u, z \in E$  and  $t, s > 0$  are mutually absolutely continuous.

We call  $P$  *strong Feller* if the semigroup  $P_t: B_b(E) \mapsto C_b(E)$  for  $t > 0$ . We call it *irreducible* if  $P(t, z, \mathcal{O}) > 0$  for all  $t > 0$ ,  $z \in E$  and non-empty open set  $\mathcal{O} \subset E$ .

The first part of the theorem below is due to Khas'minskii [12]. The second part is due to Doob [9]. For the proofs of these two parts we refer the reader also to [7]. The third part contains results of Seidler [16] and Stettner [17].

**Theorem 5.2.** (i) *If a transition probability is strong Feller and irreducible then it is regular.* (ii) *If  $P$  is stochastically continuous and regular transition probability and  $\mu$  is an invariant measure with respect to  $P$ , then  $\mu$  is unique, equivalent to all measures  $P_t(z, \cdot)$  and for all  $z \in E$  and  $\Gamma \in \mathcal{B}(E)$ ,*

$$\lim_{t \rightarrow +\infty} P_t(z, \Gamma) = \mu(\Gamma),$$

(iii) *Assume that the Markov process  $X$  with a transition function  $P$  is right-continuous, strong Markov and regular. Assume that  $\mu$  is an invariant-measure for  $X$ . Then for any probability measure  $\tilde{\mu}$  on  $E$  we have*

$$\lim_{t \rightarrow +\infty} \|P_t^* \mu - \mu\|_{TV} = 0.$$

Recall that the *total variation norm* of two probability measures  $\mu$  and  $\tilde{\mu}$  on  $E$  is given as follows

$$\|\mu - \tilde{\mu}\|_{\text{TV}} = \sup_{\Gamma \in \mathcal{B}(E)} |\mu(\Gamma) - \tilde{\mu}(\Gamma)|.$$

**5.1. Proof of strong Feller property.** In this part we are going to apply some standard techniques. Therefore we present here only a sketch of the proof. First we need to approximate  $g_0$  and  $g_1$  by sequence of  $C^\infty$  mappings  $g_0^n$  and  $g_1^n$  such that the corresponding Lipschitz constants converge; i.e.

$$\|g_0^n\|_{\text{Lip}} \rightarrow \|g_0\|_{\text{Lip}} \quad \text{and} \quad \|g_1^n\|_{\text{Lip}} \rightarrow \|g_1\|_{\text{Lip}}.$$

We require also point convergence and the upper and lower bounds:

$$|g_0^n(r, \psi) - g_0(r, \psi)| \rightarrow 0 \quad \text{and} \quad \|g_1^n(r, \psi) - g_1(r, \psi)\| \rightarrow 0, \quad \forall r \in \mathbb{R}, \psi \in L^2,$$

and there is a constant  $C \in (0, +\infty)$  such that for all  $r \in \mathbb{R}$  and  $\psi \in L^2$  all the following numbers

$$|g_0^n(r, \psi)|, \quad \frac{1}{|g_0^n(r, \psi)|}, \quad \|g_1^n(r, \psi)\|, \quad \left\| \frac{1}{g_1(r, \psi)} \right\|$$

are less or equal to  $C$ . In [14] one can find a construction of such sequences.

We need also to regularize (in fact localize) the mappings  $\mathcal{N}(\psi) = \|\psi\|^2$  and  $B(\psi) = -\frac{\partial}{\partial x}\psi^2$ . We set

$$\mathcal{N}_n(\psi) = \mathcal{N}(\psi)\phi_n(\|\psi\|^2), \quad (5.1)$$

where  $\phi_n \in C^\infty(\mathbb{R})$  is such that  $0 \leq \phi_n(r) \leq 1$ ,  $|\phi_n'(r)| \leq 2$  for  $r \in \mathbb{R}$ ,  $\phi_n(r) = 1$  for  $r \leq n+1$  and  $\phi_n(r) = 0$  for  $r \geq n$ . In the same fashion we localize  $B$ ,

$$B_n(\psi) = B(\psi)\phi_n(\|\psi\|^2). \quad (5.2)$$

Let  $X_{n,m}^z(t) = (U_{n,m}^z(t), v_{n,m}^z(t))$  be the value at time  $t$  of the solution to the problem obtained from (1.3)–(1.5) by replacing  $g_i$  by  $g_i^m$ , and  $\mathcal{N}$  and  $B$  by  $\mathcal{N}_n$  and  $B_n$ . We denote by  $z = (U(0), v(0))$  the initial data. We take for granted the existence and uniqueness of the solution to the modify problem on the state space  $H = \mathbb{R} \times L^2$ .

Let

$$P_t^{n,m} f(z) = \mathbb{E} f(X_{n,m}^z(t)), \quad z = (r, \psi) \in \mathbb{R} \times L^2,$$

be the transition semigroup to the modify problem. As in [14] one can derive the Bismut–Elworthy–Li formula

$$\begin{aligned} & D_r P_t^{n,m} f((r, \psi))a + D_\psi P_t^{n,m} f((r, \psi))[\varphi] \\ &= \mathbb{E} f(X_{n,m}^z(t)) Y_{n,m}(r, \psi, t, a, \varphi), \end{aligned}$$

where:  $a \in \mathbb{R}$  and  $\varphi \in L^2$  are directions,  $D_r$  and  $D_\psi$  denote the derivatives in  $r$  and  $\psi$ ,

$$\begin{aligned} Y_{n,m}(r, \psi, t, a, \varphi) &:= Y_{n,m}^1(r, \psi, t)a + Y_{n,m}^2(r, \psi, t, \varphi), \\ Y_{n,m}^1(r, \psi, t) &:= \frac{1}{t} \int_0^t \frac{1}{g_0(X_{n,m}^{(r,\psi)}(s))} \frac{\partial}{\partial r} U_{n,m}^{(r,\psi)}(s) dW_0(s), \end{aligned}$$

$$Y_{n,m}^2(r, \psi, t, \varphi) := \frac{1}{t} \int_0^t \left\langle \frac{1}{g_1(X_{n,m}^{(r,\psi)}(s))} D_\psi U_{n,m}^{(r,\psi)}(s)[\varphi], dW_0(s) \right\rangle.$$

We left to the reader the verification of the fact that there is an independent of  $m, r, \psi, a$ , and  $\varphi$  constant  $C(n, t)$  such that

$$\mathbb{E} |Y_{n,m}(r, \psi, t, a, \varphi)| \leq C(n, t) [|a|^2 + \|\varphi\|^2]^{\frac{1}{2}}.$$

This ensures not only the strong Feller property of  $(P_t^{n,m})$  but also gives the estimate

$$\|DP_t^{n,m} f(z)\|_{L(H,H)} \leq C(n, t) \sup_{z \in H} |f(z)|.$$

Therefore  $P_t f$  is Lipschitz continuous function on  $H$ , and

$$\|P_t^{n,m} f\|_{\text{Lip}} \leq C(n, t).$$

Since  $C(n, t)$  does not depend on  $m$  it enables us to pass with  $n \rightarrow \infty$  and prove the strong Feller property of the transition semigroup  $(P_t^n)$  for the problem, say problem  $\Pi_n$ , obtained from (1.3)–(1.5) by replacing only  $\mathcal{N}$  and  $B$  by  $\mathcal{N}_n$  and  $B_n$ . In fact,  $P^n f$  is Lipschitz continuous and for any bounded measurable function  $f$  we have

$$\|P_t^n f\|_{\text{Lip}} \leq C(n, t).$$

We can now conclude by using a standard localization procedure. Namely let  $X_n^z$  be the solution to problem  $\Pi_n$  and let  $X^z$  be the solution to problem (1.3)–(1.5). Then, from the uniqueness of the solutions and since for  $\psi: \|\psi\| \leq n$  we have  $\mathcal{N}_n(\psi) = \mathcal{N}(\psi)$  and  $B_n(\psi) = B(\psi)$ , we infer that

$$X_n^z(s) = X^z(s) \quad \text{for } s < \tau_z^n, \quad (5.3)$$

where

$$\tau_z^n := \inf\{s: \|X^z(s)\| \geq n\}.$$

We can now repeat arguments from [5, 7]. Namely, let  $f$  be a bounded measurable function on  $H = \mathbb{R} \times L^2$ . We have

$$\begin{aligned} |P_t^n f(z) - P_t f(z)| &= |\mathbb{E}[f(X_n^z(t)) - f(X^z(t))] \chi_{\{t \geq \tau_z^n\}}| \\ &\leq 2 \sup_{z \in H} |f(z)| \mathbb{P}(t \geq \tau_z^n). \end{aligned}$$

Since functions  $P_t^n f$  are continuous, it is enough to show that for all  $M > 0$  and  $t > 0$  we have

$$\lim_{n \rightarrow +\infty} \sup_{\|z\|_H \leq M} \mathbb{P}(t \geq \tau_z^n) = 0.$$

This can be written equivalently as

$$\lim_{n \rightarrow +\infty} \sup_{\|z\|_H \leq M} \mathbb{P}\left(\sup_{s \leq t} \|X^z(s)\|_H \geq n\right) = 0,$$

which can be easily deduced from Lemma 3.4, see [5] for more details, or from the existence of solution in the spaces  $Z_T^p$ ,  $p > 1$ , established in [18, 19, 20].

**5.2. Proof of irreducibility.** We will follow [5, 14]. Namely, from (5.3) we infer that it is enough to prove the irreducibility for the problem  $\Pi_n$  obtained from the original problem (1.3)–(1.5) by replacing  $\mathcal{N}$  and  $B$  by  $\mathcal{N}_n$  and  $B_n$ , see formulae (5.1) and (5.2) from Section 5.1. The idea is to use the Girsanov theorem. Let  $\mathcal{O} \neq \emptyset$  be an open set in  $H$  and let  $T > 0$ . Let  $z = (U_0, v_0)$  be a fixed initial datum. We are going to show that  $P_T^n(z, \mathcal{O}) > 0$ . Let  $\bar{z} = (\bar{U}, \bar{v}) \in \mathcal{O}$ . Since the domain  $D(A)$  is dense in  $L^2$  we can assume that  $\bar{v} \in D(A)$ . We will show that for any  $\varepsilon > 0$ ,  $P_T^n(z, B(\bar{z}, \varepsilon)) > 0$ , where  $B(\bar{z}, \varepsilon)$  is the ball in  $H$  of radius  $\varepsilon$  and center at  $\bar{z}$ . To explain the idea it will be convenient for us to write our problem in the following form

$$dX_n = [\mathcal{A}X_n + F_n(X_n)] dt + \sigma(X_n)dW,$$

with with properly chosen  $\mathcal{A}$ ,  $F_n$  and  $\sigma$ . In particular  $\mathcal{A} = (A, -\nu)$ . Given  $0 < h < T$  define

$$f^h(s) = \begin{cases} 0 & \text{for } s \leq T - h, \\ h^{-1}e^{\mathcal{A}(s-(T-h))}(\bar{z} - X_n(T-h)) - \mathcal{A}\bar{z} & \text{for } s > T - h. \end{cases}$$

Note that the mapping  $f^h$  is such that

$$X_n(T-h) + \int_{T-h}^T e^{\mathcal{A}(T-s)} f^{h,m}(s) ds = \bar{z}. \quad (5.4)$$

Let

$$\zeta_{h,m}(s) := \sigma(X_n(s))^{-1} [f^h(s)] \chi_{\{\|X_n(T-h)\|_H \leq m\}}.$$

Then under the probability measure

$$d\mathbb{P}^* = \exp \left\{ - \int_0^T \zeta_{h,m}(s) dW(s) - \frac{1}{2} \int_0^T \|\zeta_{h,m}(s)\|^2 ds \right\} d\mathbb{P},$$

$$W^*(t) = W(t) + \int_0^t \zeta_{h,m}(s) ds, \quad t \leq T,$$

is a cylindrical Wiener process on  $H$ . Therefore the law of  $X_n$  is equivalent to the law of the solution  $\tilde{X}_n$  to the following equation

$$d\tilde{X}_n = [\mathcal{A}\tilde{X}_n + F_n(\tilde{X}_n) + f^h \chi_{\{\|\tilde{X}_n\|_H \leq m\}}] dt + \sigma(\tilde{X}_n)dW.$$

Taking into account, (5.4), we have

$$\tilde{X}_n(T) = \bar{z} + R_h(T) \quad \text{if } \|X_n(T-h)\|_H \leq m,$$

where

$$R_h(T) := \int_{T-h}^T e^{\mathcal{A}(T-s)} F_n(\tilde{X}_n(s)) ds + \int_{T-h}^T e^{\mathcal{A}(T-s)} \sigma(\tilde{X}_n(s)) dW(s).$$

Obviously we have

$$\lim_{m \rightarrow +\infty} \mathbb{P}(\|X_n(T-h)\|_H \leq m) = 1.$$

Thus the proof will be completed as soon as we can show that for any  $\varepsilon > 0$  we have

$$\lim_{h \downarrow 0} \mathbb{P}(\|R_h(T)\|_H < \varepsilon) > 0.$$

Since it can be done in the same way as in Da Prato and Gatarek [5], we leave it to the reader.

**Acknowledgment.** The work of Szymon Peszat was supported by Polish National Science Center grant 2017/25/B/ST1/02584.

### References

1. Adams, R. A.: *Sobolev Spaces*, Academic Press, New York, 1975.
2. Burgers, J.M.: *Mathematical examples illustrating relations occurring in the theory of turbulent fluid motion*, Verh. Nederl. Akad. Wetensch. Afd. Natuurk. **17** (1939), 1–53.
3. Cholewa, J.W. and Dłotko, T.: *Global Attractors in Abstract Parabolic Problems*, Cambridge Univ. Press, Cambridge, 2000.
4. Da Prato, G., Debussche, A. and Temam, R.: *Stochastic Burgers' equation*, NoDEA Nonlinear Differential Equations Appl. **1** (1994), 389–402.
5. Da Prato, G. and Gatarek, D.: *Stochastic Burgers equation with correlated noise*, Stochastics Stochastics Rep. **52** (1995), 29–41.
6. Da Prato, G. and Zabczyk, J.: *Stochastic Equations in Infinite Dimensions*, Cambridge, 1992.
7. Da Prato, G. and Zabczyk, J.: *Ergodicity for Infinite Dimensional Systems*, Cambridge Univ. Press, Cambridge, 1996.
8. Dłotko, T.: *The one-dimensional Burgers' equation; existence, uniqueness and stability*, Zeszyty Naukowe Uniw. Jagielloń skiegoPrace Mat. **23** (1982), 157–172.
9. Doob, J. L.: *Asymptotic properties of Markoff transition probabilities*, Trans. Amer. Math. Soc. **63** (1948), 394–421.
10. E, W., Khanin, K., Mazel, A., and Sinai, Ya.: *Invariant measures for Burgers equation with stochastic forcing*, Ann. of Math. (2) **151** (2000), 877–960.
11. Ferrario, B.: *The Bénard problem with random perturbations: dissipativity and invariant measures*, NoDEA Nonlinear Differential Equations Appl. **4** (1997), 101–121.
12. Khas'minskii, R. Z.: *Ergodic properties of recurrent diffusion processes and stabilization of the solution to the Cauchy problem for parabolic equations*, Theor. Probability Appl. **5** (1960), 179–196.
13. Lions, J. L. and Magenes, E.: *Non-Homogeneous Boundary Value Problems and Applications Vol. I*, Springer 1972.
14. Peszat, S. and Zabczyk, J.: *Strong Feller property and irreducibility for diffusions on Hilbert spaces*, Ann. Probab. **23** (1995), 157–172.
15. Runst, T. and Sickel, W.: *Sobolev Spaces of Fractional Orders, Nemytskij Operators, and Nonlinear Partial Differential Equations*, de Gruyter, 1996.
16. Seidler, J.: *Ergodic behaviour of stochastic parabolic equations*, Czechoslovak Math. J. **47** (1997), 277–316.
17. Stettner, L.: *Remarks on ergodic conditions for Markov processes on Polish spaces*, Bull. Polish Acad. Sci. Math. **42** (1994), 103–114.
18. Twardowska, K. and Zabczyk, J.: *A note on stochastic Burgers' system of equations*, Preprint **646**, (2003) 1–32, Polish Academy of Sciences, Inst. of Math., Warsaw.
19. Twardowska, K. and Zabczyk, J.: *A note on stochastic Burgers' system of equations*, Stochastic Anal. Appl. **22** (2004), 1641–1670.
20. Twardowska, K. and Zabczyk, J.: *Qualitative properties of solutions to stochastic Burgers' system of equations*, in: *Stochastic Partial Differential Equations and Appl. VII, Ed. G. Da Prato and L. Tubaro, Proceedings SPDE's and Applications, Levico, Italy, 2004*; Lect. Notes Pure Appl. Math. **245** (2006) 311–322, Chapman & Hall, Boca Raton.



SZYMON PESZAT: INSTITUTE OF MATHEMATICS, JAGIELLONIAN UNIVERSITY, ŁOJASIEWICZA 6,  
30-348 KRAKÓW, POLAND

*E-mail address:* `napeszat@cyf-kr.edu.pl`

KRZYSTYNA TWARDOWSKA: FACULTY OF APPLIED INFORMATICS AND MATHEMATICS, WARSAW  
UNIVERSITY OF LIFE SCIENCES-SGGW, UL. NOWOURSYNOWSKA 159, 02-776 WARSAW, POLAND

JERZY ZABCZYK: INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES, UL. ŚNIADE-  
CKICH 8, 00-956 WARSAW, POLAND

*E-mail address:* `zabczyk@impan.pl`