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STABILITY AND UNCONDITIONAL UNIQUENESS OF SOLUTIONS FOR ENERGY CRITICAL WAVE EQUATIONS IN HIGH DIMENSIONS

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ABSTRACT. In this paper we establish a complete local theory for the energy-critical nonlinear wave equation (NLW) in high dimensions $\mathbb{R}\times\mathbb{R}^d$ with $d\geq 6$. We prove the stability of solutions under the weak condition that the perturbation of the linear flow is small in certain space-time norms. As a by-product of our stability analysis, we also prove local well-posedness of solutions for which we only assume the smallness of the linear evolution. These results provide essential technical tools that can be applied towards obtaining the extension to high dimensions of the analysis of Kenig and Merle [17] of the dynamics of the focusing (NLW) below the energy threshold. By employing refined paraproduct estimates we also prove unconditional uniqueness of solutions for $d\geq 5$ in the natural energy class. This extends an earlier result by Planchon [26].

1. INTRODUCTION

We consider the Cauchy problem for the energy critical nonlinear wave equation

$$\text{(NLW)}\qquad \begin{cases} u_{tt}-\Delta u = F(u),\\ u(0,x)=u_0(x),\\ \partial_t u(0,x)=u_1(x),\end{cases}$$

where $u(t,x)$ is a real valued function defined on $\mathbb{R}\times\mathbb{R}^d$ for $d\geq 6$, and $u_0\in\dot{H}^1(\mathbb{R}^d)$, $u_1\in L^2(\mathbb{R}^d)$. Moreover, the nonlinearity is of a power type given by

$$F(u)=\mu|u|^{\frac{4}{d-2}}u,$$

and $\mu\in\{-1,1\}$. We note that $\mu=-1$ corresponds to the defocusing problem, while $\mu=1$ corresponds to the focusing problem.

The energy for the (NLW) is given by

$$E\left(u(t),\partial_t u(t)\right)=\frac{1}{2}\|\partial_t u(t,\cdot)\|_{L^2}^2+\frac{1}{2}\|\nabla u(t,\cdot)\|_{L^2}^2-\mu\frac{d-2}{2d}\|u(t,\cdot)\|_{L^{\frac{2d}{d-2}}}^{\frac{2d}{d-2}},\qquad(1.1)$$

and it is conserved in time. Also we remark that if $u(t,x)$ is a solution to (NLW), then $u_\lambda(t,x)$ defined via

$$u_\lambda(t,x)=\frac{1}{\lambda^{\frac{d-2}{2}}}\,u(\frac{t}{\lambda},\frac{x}{\lambda})$$

is also a solution to (NLW). Since the above scaling leaves the energy invariant, the (NLW) problem is referred to as "energy critical".

Local well-posedness for the Cauchy problem (NLW) has been studied in many papers (see, e.g. [25, 9, 22, 28, 29, 30, 13, 17]). Here we recall a version of the local well-posedness result as presented in [17] (see also [25, 9, 28]) which states that for

$d = 3, 4, 5$ and initial data $(u_0, u_1) \in \dot{H}^1 \times L^2$, $\|(u_0, u_1)\|_{\dot{H}^1 \times L^2} \le A$, $0 \in I$, there exists $\delta = \delta(A)$ such that if

$$\|K(t)\,(\,u_0, u_1\,)\,\|_{L_{t,x}^{\frac{2(d+1)}{d-2}}(I \times \mathbb{R}^d)} < \delta,$$

there exists a unique solution to (NLW) in $I \times \mathbb{R}^d$ such that $(u, \partial_t u) \in C(I; \dot{H}^1 \times L^2)$ and $\|u\|_{L_{t,x}^{\frac{2(d+1)}{d-2}}(I \times \mathbb{R}^d)} \le 2\delta$. Here K denotes the associated linear operator i.e.,

$$K(t)(u_0, u_1) = \cos\left(t\sqrt{-\Delta}\right)\, u_0 + (-\Delta)^{-\frac{1}{2}} \sin\left(t\sqrt{-\Delta}\right)\, u_1.$$

The proof of this local well-posedness in dimensions $3 \le d \le 5$ is based on the use of the standard Strichartz estimates. However this proof does not carry directly to high dimensions $d > 6$. The main reason for this is that for $d > 6$ the derivative of the nonlinearity is no longer Lipschitz continuous in the standard Strichartz space.

A natural question related to the local well-posedness theory is the stability of solutions. Roughly speaking this amounts to showing the closeness of the solution and an approximate solution, which solves a perturbed equation, if the perturbations of the equation and of the initial data are small in a certain sense. More precisely, let $\tilde{u} :\ I \times \mathbb{R}^d \to \mathbb{R}$ be an approximate solution which solves the perturbed NLW:

$$\begin{cases} \tilde{u}_{tt} - \Delta \tilde{u} = F(\tilde{u}) + e, \\ \tilde{u}(t_0, x) = \tilde{u}_0(x), \\ \partial_t \tilde{u}(t_0, x) = \tilde{u}_1(x). \end{cases}$$

Assume the perturbation e is small in a certain norm and the difference of linear flow measured in terms of scattering size

$$\|K(t)(u_0 - \tilde{u}_0, u_1 - \tilde{u}_1)\|_{L_{t,x}^{\frac{2(d+1)}{d-2}}(I \times \mathbb{R}^d)} \tag{1.2}$$

is small, then the goal of a typical stability result is to show that there exists a unique solution u to (NLW) with initial data (u_0, u_1) such that u and $\tilde{u}$ stay close on the whole time interval I. Such a stability result for the (NLW) in $3 \le d \le 5$ was obtained in the work of Kenig and Merle [17]. However the proof does not carry directly to higher dimensions because the nonlinearity is no longer Lipschitz in the standard Strichartz space. This problem was first overcome in the context of the energy-critical nonlinear Schrödinger equation (NLS) in [32] in $d > 6$ by using certain "exotic Strichartz" spaces which have same scaling with standard Strichartz space but lower derivative.[1] The proof was later simplified in [18] (see Section 3 therein) where stability is established in Sobolev Strichartz spaces by using fractional chain rule. In the case of the energy-critical Klein-Gordon equation in high dimension stability was proved by Nakanishi in [24]. The main technical difficulty in the context of NLW, besides choosing the appropriate exotic Strichartz space, is that in order to show that nonlinearity is Lipschitz continuous in these spaces, one encounters a problem in establishing Hölder continuity of the nonlinearity in the standard Strichartz space. This is quite different from the NLS

[1] Actually for smallness condition of type (1.2), exotic Strichartz spaces are also employed to establish stability theory even in dimensions $3 \le d \le 5$, see, e.g. [15]. However if instead of (1.2) one assumes a stronger condition that $\|(u_0 - \tilde{u}_0, u_1 - \tilde{u}_1)\|_{\dot{H}^1 \times L^2}$ is small, then the proof of stability theory can be again carried out by standard Strichartz estimates in dimension $3 \le d \le 5$.

case since in the latter case one works with the local operator ∇, while in NLW one has to work with fractional derivatives which are nonlocal.

In the defocusing case the global well-posedness theory was worked out in seminal papers [31, 11, 12, 27]. In particular, Struwe [31] obtained global well-posedness for the (NLW) in the radial case when $d = 3$. Grillakis [11] removed the radial assumption in $d = 3$. The global well-posedness and persistence of regularity was shown for $3 \leq d \leq 5$ by Grillakis [12], Shatah-Struwe [27, 28, 29] and Kapitanski [13]. On the other hand, in the focusing case, Levine [19] proved that if the initial data $(u_0, u_1) \in \dot{H}^1 \times L^2$ are such that $E(u_0, u_1) < 0$, then the solution must blowup in finite time. Hence, in the focusing case, the global well-posedness does not hold in general. In particular, Kenig and Merle in [17] presented a detailed study of the focusing case for $3 \leq d \leq 5$ and showed that depending on the size of the initial data with respect to the size of the ground state, global well-posedness or blowup occurs. More precisely, in [17], Kenig and Merle employed sophisticated "concentrated compactness + rigidity method", introduced in their work [16] on the NLS, to obtain the following dichotomy-type result under the assumption that $E(u_0, u_1) < E(W, 0)$:

(i) If $\|u_0\|_{\dot{H}^1} < \|W\|_{\dot{H}^1}$, then the global well-posedness holds.
(ii) If $\|u_0\|_{\dot{H}^1} > \|W\|_{\dot{H}^1}$, then a finite time blowup occurs.

Here W denotes the solution to the stationary problem i.e., W satisfies the elliptic equation

$$\Delta W + |W|^{\frac{4}{d-2}} W = 0.$$

Many parts of the proof of this dichotomy argument carry out in high dimensions (e.g. the rigidity theorem is among them). However the local well-posedness as well as a certain stability result require revisiting in higher dimensions, since as noted above, one has to prove the Lipschitz continuity of the nonlinearity in the exotic Strichartz spaces and also the Hölder continuity in the standard Strichartz spaces.

The purpose of this paper is to establish a complete local theory for (NLW) in high dimensions $d \geq 6$ by providing a stability result for the (NLW) in $d \geq 6$ as well as an unconditional uniqueness result in $\mathbb{R} \times \mathbb{R}^d$ for $d \geq 5$. More precisely:

(1) We prove a stability result for the (NLW) for $d \geq 6$ via introducing appropriate exotic Strichartz spaces (in particular, see the definition of the space X in Section 2) and via working in Strichartz spaces of Besov type (see the definition of the space $\dot{S}^1$ in Section 2). In order to prove Lipschitz continuity of nonlinearity in the exotic Strichartz spaces, one usually proves the Hölder continuity of the nonlinearity in the standard Strichartz space of Sobolev type. As mentioned above this leads to a technical difficulty which is different from the NLS case. In the NLS case, the Hölder continuity can be easily established due to the fact that ∇ is a local operator. On the other hand, in the NLW case, the standard Strichartz space involves the fractional derivative which is nonlocal and this causes the technical difficulty to prove Hölder continuity in the Strichartz space of Sobolev type. We shall circumvent this difficulty by choosing the working space as Strichartz space of Besov type, space $\dot{S}^1$, and then transferring the corresponding result to the Sobolev setting (see Remark 2.2, Lemma 2.10 and Section 5

for more details). Hence we can prove the main stability result stated in Theorem 3.6 in the pure Sobolev setting [2].

We remark that a direct side-product of our stability result is continuous dependance of the data that follows from Theorem 3.6 by taking $e = 0$.

Also using the nonlinear estimates that we employ in the stability analysis, we obtain a local in time existence of solutions to (NLW) and a standard blow-up criterion, see Theorem 3.3 and Lemma 3.5 for the precise statements of these results.

(2) By using paraproduct estimates we prove unconditional uniqueness of strong solutions to the (NLW) as stated in Theorem 3.4. By unconditional uniqueness, we mean that for given initial data (u_0, u_1), there exists at most one solution of (NLW) in the class $C_t\dot{H}^1_x(I \times \mathbb{R}^d)$. In the context of $\dot{H}^s$ critical NLS, the unconditional uniqueness was first established by Furioli and Terraneo [7] using para-product analysis. In the context of energy critical NLW, this problem was first addressed by Planchon [26], where the unconditional uniqueness was established in dimensions $d = 4, 5$ (a review of the unconditional uniqueness for both the NLS and NLW can be found in the paper by Furioli, Planchon and Terraneo [8]). As a matter of fact, the proof presented in [26] can also cover the 6-dimensional case with quadratic nonlinearity u^2. The main technical barrier when extending the analysis to high dimensions is that the nonlinearity fails to be C^2. Therefore one cannot do Taylor expansion on the nonlinearity to second order as in the low dimensional case, see [26] for more details. The analysis used in this paper is reminiscent of the one in [26]; on the other hand, to remove the restriction on the dimension, we need more refined estimates on the nonlinearity.

Interestingly, the proof of unconditional uniqueness also yields a new proof of local well-posedness in high dimensions $d \geq 5$ (see Remark 4.3).

We should also stress that the unconditional uniqueness in $d = 3$ is still open due to the failure of the endpoint Strichartz estimates except the radial case (see however [23] for an interesting result concerning uniqueness of weak solutions to defocusing NLW in $d = 3$ under a local energy inequality assumption on the light cone).

We remark that the stability result of this paper combined with a modification of the profile decomposition for the linear wave equation, that was for $d = 3$ obtained by Bahouri and Gérard [1] and extended to high dimensions $d > 3$ by Bulut [3], implies that the dichotomy result of Kenig and Merle [17] is valid in all dimensions $d \geq 3$. Hence the stability result of this paper is a technical tool that can be applied directly to understand the dynamics of the focusing (NLW) below the energy threshold.

Another application of the stability result obtained in this paper is in studying the dynamics of the focusing (NLW) at the energy threshold $E\,(u_0, u_1) = E\,(W, 0)$ in high dimensions. Such dynamics were analyzed by Duyckaerts and Merle [5] for $3 \leq d \leq 5$, and recently by Li and Zhang [20] in high dimensions $d \geq 6$ (see also [6] and [21] for the NLS case).

[2] In Theorem 3.6 we do not assume smallness in exotic Strichartz spaces, as it was the case with the stability result for the NLS in [32].

Organization of the paper. In Section 2 we introduce the notations and present various estimates that will be used throughout the paper. Main results of this paper: the local well-posedness Theorem 3.3, the unconditional uniqueness Theorem 3.4, the standard blow-up criterion Lemma 3.5 and the stability result Theorem 3.6 are stated in Section 3. Theorems 3.3 and 3.4 are proved in Section 4. In Section 5 we present the proof of the main stability result, by first presenting a short-term perturbation result followed by the main long-term perturbation result.

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2. Notation and Preliminaries

2.1. **Notations.** In what follows, we write $X \lesssim Y$ or $Y \gtrsim X$ to indicate that there exists a constant $C > 0$ such that $X \leq CY$. We also use the symbol $O(Y)$ to denote any quantity X with the property $|X| \lesssim Y$ and ∇ for the derivative operator in the space variable.

For any time interval $I \subset \mathbb{R}$, we write $L_t^q L_x^r(I \times \mathbb{R}^d)$ to denote the Banach space of functions $u : I \times \mathbb{R}^d \to \mathbb{R}$ with the norm

$$\|u\|_{L_t^q L_x^r(I\times\mathbb{R}^d)} := \left(\int_I \left(\int_{\mathbb{R}^d} |u|^r dx\right)^{\frac{q}{r}} dt\right)^{\frac{1}{q}} < \infty,$$

with the standard definitions when q or r is equal to infinity. When $q = r$, we abbreviate $L_t^q L_x^q$ as $L_{t,x}^q$.

We define the Fourier transform on $\mathbb{R}^d$ by

$$\hat{f}(\xi) := (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)dx,$$

and, for $s \in \mathbb{R}$, the fractional differentiation operator $|\nabla|^s$ by

$$\widehat{|\nabla|^s f}(\xi) := (4\pi^2|\xi|^2)^{\frac{s}{2}} \hat{f}(\xi),$$

which allows us to define the homogeneous Sobolev norm,

$$\|f\|_{\dot{H}_x^{s,p}} := \||\nabla|^s f\|_{L_x^p(\mathbb{R}^d)}.$$

In the case, $p = 2$, we abbreviate $\dot{H}_x^{s,2}$ as $\dot{H}_x^s$.

For any constant $C > 0$, we define

$$\phi_{\leq C}(x) := \phi\big(\tfrac{x}{C}\big) \quad \text{and} \quad \phi_{>C} := 1 - \phi_{\leq C}, \tag{2.1}$$

where $\phi \in C^\infty(\mathbb{R}^d)$ is a radial bump function supported in the ball $\{x \in \mathbb{R}^d : |x| \leq \frac{25}{24}\}$ with $\phi(x) = 1$ on $\{x \in \mathbb{R}^d : |x| \leq 1\}$.

For each number $j \in \mathbb{Z}$, we define the following standard Littlewood-Paley Fourier multipliers

$$\widehat{\Delta_{\leq j} f}(\xi) := \phi_{\leq 2^j}(\xi)\hat{f}(\xi),$$
$$\widehat{\Delta_{> j} f}(\xi) := \phi_{> 2^j}(\xi)\hat{f}(\xi),$$
$$\widehat{\Delta_j f}(\xi) := (\phi_{\leq 2^j} - \phi_{\leq 2^{j-1}})(\xi)\hat{f}(\xi),$$

with similar definitions for $\Delta_{<j}$ and $\Delta_{\geq j}$. Moreover, we define

$$\Delta_{j<\cdot\leq l} := \Delta_{\leq l} - \Delta_{\leq j} = \sum_{j<m\leq l} \Delta_m$$

whenever $j < l$.

We will use Bernstein estimate:

$$\|\Delta_j u\|_{L^q(\mathbb{R}^d)} \lesssim 2^{j(\frac{d}{p}-\frac{d}{q})} \|\Delta_j u\|_{L^p(\mathbb{R}^d)}, \tag{2.2}$$

where $1 \leq p \leq q \leq \infty$.

We recall the definition of the homogenous Besov spaces $\dot{B}^s_{p,q}$ (see for instance [2]). For each $s \in \mathbb{R}$ and $1 \leq p \leq \infty$, $1 \leq q < \infty$, we define

$$\|u\|_{\dot{B}^s_{p,q}} = \left(\sum_{j\in\mathbb{Z}} (2^{sj}\|\Delta_j u\|_{L^p(\mathbb{R}^d)})^q\right)^{\frac{1}{q}},$$

and

$$\dot{B}^s_{p,q}(\mathbb{R}^d) = \{u \in \mathcal{S}'(\mathbb{R}^d) : \|u\|_{\dot{B}^s_{p,q}(\mathbb{R}^d)} < \infty\}.$$

Another equivalent characterization of Besov space will also be used in this paper (see [2]). Namely, for $0 < s < 1$, $1 \leq p \leq \infty$, $1 \leq q < \infty$,

$$\|f\|_{\dot{B}^s_{p,q}} \sim \left(\int_{\mathbb{R}^d} \frac{\|f(x+t) - f(x)\|_p^q}{|t|^{d+sq}} dt\right)^{\frac{1}{q}}. \tag{2.3}$$

And

$$\|f\|_{\dot{B}^s_{p,\infty}} \sim \sup_{t\in\mathbb{R}^d} |t|^{-s}\|f(x+t) - f(x)\|_p. \tag{2.4}$$

The following lemma is a simple consequence of the definition of Besov norms:

Lemma 2.1. *Let $0 < s, \alpha < 1$ such that $\frac{s}{\alpha} < 1$. Let $1 < p \leq \infty$ such that $1 < p\alpha \leq \infty$. Let $f(z)$ be Hölder continuous of order α. Then,*

$$\|f(u)\|_{\dot{B}^s_{p,\infty}} \lesssim \|u\|^\alpha_{\dot{B}^{\frac{s}{\alpha}}_{p\alpha,\infty}}.$$

Proof. Let $u \in \dot{B}^{\frac{s}{\alpha}}_{p\alpha,\infty}$ be given. Then by (2.4) we have the inequality,

$$\begin{aligned}
\|f(u)\|_{\dot{B}^s_{p,\infty}} &\lesssim \sup_{t\in\mathbb{R}^d}\left[|t|^{-s}\left(\int_{\mathbb{R}^d}|f(u(x+t))-f(u(x))|^p\,dx\right)^{\frac{1}{p}}\right]\\
&\lesssim \sup_{t\in\mathbb{R}^d}\left[|t|^{-s}\left(\int_{\mathbb{R}^d}|u(x+t)-u(x)|^{\alpha p}dx\right)^{\frac{1}{p}}\right]\\
&= \sup_{t\in\mathbb{R}^d}\left(|t|^{-\frac{s}{\alpha}}\|u(x+t)-u(x)\|_{L^{p\alpha}}\right)^{\alpha}\\
&\le \left(\sup_{t\in\mathbb{R}^d}|t|^{-\frac{s}{\alpha}}\|u(x+t)-u(x)\|_{L^{p\alpha}}\right)^{\alpha}\\
&\sim \|u\|^{\alpha}_{\dot{B}^{\frac{s}{\alpha}}_{p\alpha,\infty}}
\end{aligned}$$

where in the second inequality we have used the Hölder continuity of f. □

2.2. **Function Spaces.** For dimensions $d \ge 6$ and any time interval $I \subset \mathbb{R}$, we introduce the following norms:

$$\|u\|_{S(I)} = \|u\|_{L_{t,x}^{\frac{2(d+1)}{d-2}}(I\times\mathbb{R}^d)},$$

$$\|u\|_{\dot{S}^1(I)} = \sup\left\{\|u\|_{L_t^q\dot{B}^{1-\beta(r)}_{r,2}(I\times\mathbb{R}^d)}, \|\partial_t u\|_{L_t^q\dot{B}^{-\beta(r)}_{r,2}(I\times\mathbb{R}^d)} : (q,r)\text{ wave-admissible}\right\}$$

$$\begin{aligned}
\|u\|_{W(I)} &= \|u\|_{L_t^{\frac{2(d+1)}{d-1}}\dot{B}^{\frac{1}{2}}_{\frac{2(d+1)}{d-1},2}(I\times\mathbb{R}^d)},\\
\|u\|_{W'(I)} &= \|u\|_{L_t^{\frac{2(d+1)}{d+3}}\dot{B}^{\frac{1}{2}}_{\frac{2(d+1)}{d+3},2}(I\times\mathbb{R}^d)},\\
\|u\|_{X(I)} &= \|u\|_{L_t^{\frac{d^2+d}{d+2}}\dot{H}^{\frac{2}{d},\frac{2(d+1)}{d-1}}(I\times\mathbb{R}^d)},\\
\|u\|_{X'(I)} &= \|u\|_{L_t^{\frac{d^2+d}{3d+2}}\dot{H}^{\frac{2}{d},\frac{2(d+1)}{d+3}}(I\times\mathbb{R}^d)},\\
\|u\|_{Y(I)} &= \|u\|_{L_t^{\frac{2d^3-7d^2-9d}{d^3-6d^2+7d-2}}\dot{H}^{\frac{d^2-4d-2}{2d^2-9d},\frac{4d^3-14d^2-18d}{2d^3-11d^2+11d-8}}(I\times\mathbb{R}^d)}.
\end{aligned} \tag{2.5}$$

Remark 2.2. We stress here that the Strichartz space $\dot{S}^1$ is defined in terms of Besov spaces. Choosing the working space as a Besov space allows us to bound the fractional derivative of the difference of the nonlinear term (see Lemma 2.10). Although Besov spaces are stronger than Sobolev spaces when $p > 2$, Lemma 5.5 shows that the boundedness of the Sobolev norms of near solutions implies the boundedness of the Besov norms. Therefore with the help of Lemma 5.5, our main theorem (Theorem 3.6) can be proved in the pure Sobolev setting.

As a consequence of interpolation, we identify the following relationships between the norms defined above in (2.5) and the standard Strichartz spaces.

Lemma 2.3 (Interpolations)**.** *Let $d \ge 6$ and $I \subset \mathbb{R}$ be any time interval. Then we have the following inequalities:*

(a)

$$\begin{aligned}\|u\|_X &\lesssim \|u\|^{\theta_1}_{L_{t,x}^{\frac{2(d+1)}{d-2}}} \cdot \|u\|^{1-\theta_1}_{L_t^\infty \dot{H}^{\frac{2d-4}{d^2-4d-4}, \frac{2d^2-8d-8}{d^2-6d+8}}} \\ &\lesssim \|u\|^{\theta_1}_S \cdot \|u\|^{1-\theta_1}_{L_t^\infty \dot{H}^1},\end{aligned}$$

where $\theta_1 = \frac{2d+4}{d^2-2d}$.

(b)

$$\begin{aligned}\|u\|_S &\lesssim \|u\|^{\theta_2}_X \cdot \|u\|^{1-\theta_2}_{L_t^{\frac{2(d+1)}{d-1}} \dot{H}^{\frac{1}{2}, \frac{2(d+1)}{d-1}}} \\ &\lesssim \|u\|^{\theta_2}_X \cdot \|u\|^{1-\theta_2}_W\end{aligned}$$

where $\theta_2 = \frac{d}{d^2-3d-4}$.

(c)

$$\begin{aligned}\|u\|_{L_t^{\frac{4(d+1)}{d-2}} L_x^{\frac{4(d^2+d)}{2d^2-3d-2}}} &\lesssim \|u\|^{\theta_3}_X \cdot \|u\|^{1-\theta_3}_{L_t^{\frac{2(d+1)}{d-1}} \dot{H}^{\frac{1}{2}, \frac{2(d+1)}{d-1}}} \\ &\lesssim \|u\|^{\theta_3}_X \cdot \|u\|^{1-\theta_3}_W\end{aligned}$$

where $\theta_3 = \frac{d^2}{2(d^2-3d-4)}$.

(d)

$$\begin{aligned}\|u\|_{L_t^{\frac{2(d+1)}{d-2}} \dot{H}^{\frac{1}{2}, \frac{2(d^2+d)}{d^2-d+1}}} &\lesssim \|u\|^{\theta_4}_X \cdot \|u\|^{1-\theta_4}_Y \\ &\lesssim \|u\|^{\theta_4}_X \cdot \|u\|^{1-\theta_4}_{\dot{S}^1},\end{aligned}$$

where $\theta_4 = \frac{1}{2(d-4)}$.

(e) *We also have the embedding*

$$\dot{S}^1 \hookrightarrow L_t^{\frac{d^2+d}{d+2}} \dot{H}^{\frac{d^2-2d-2}{d^2-d}, \frac{2d^3-2d}{d^3-5d-8}} \hookrightarrow X.$$

2.3. **Strichartz Estimates.** We state the Strichartz estimates for the wave equation, which we frequently use throughout the paper (see for instance [10], [14], [22]).

Lemma 2.4 (Strichartz). *Let the pairs* (q_i, r_i), $i = 1, 2$, *satisfy*

$$\frac{2}{q_i} = (d-1)(\frac{1}{2} - \frac{1}{r_i}),$$
$$2 \le q_i,\, r_i \le \infty, \quad \text{and} \quad (q_i, r_i, d) \neq (2, \infty, 3),$$

and let u *satisfy*

$$\begin{cases} u_{tt} - \Delta u = f, \\ u(0) = u_0 \in \dot{H}^1, \\ u_t(0) = u_1 \in L^2. \end{cases}$$

Then

$$\|u\|_{L_t^{q_1} \dot{B}^{1-\beta(r_1)}_{r_1,2}} + \|\partial_t u\|_{L_t^{q_1} \dot{B}^{-\beta(r_1)}_{r_1,2}} \lesssim \|u_0\|_{\dot{H}^1} + \|u_1\|_{L^2} + \|f\|_{L_t^{q_2'} \dot{B}^{\beta(r_2)}_{r_2',2}},$$

where $\beta(r_i) = \frac{d+1}{2}(\frac{1}{2} - \frac{1}{r_i})$ *and* $\frac{1}{q_2} + \frac{1}{q_2'} = \frac{1}{r_2} + \frac{1}{r_2'} = 1$.

Now, we record the following decay estimate (see [10]).

Lemma 2.5 (Decay estimate for $\frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}$)**.** *We have*

$$\left\|\frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}f\right\|_{\dot{B}^{1-\beta(r)}_{r,2}} \lesssim |t|^{-\gamma(r)}\|f\|_{\dot{B}^{\beta(r)}_{r',2}},$$

where $0 \leq \gamma(r) = (d-1)(\frac{1}{2}-\frac{1}{r}) \leq 1$.

As a consequence of Lemma 2.5, we prove a Strichartz estimate establishing a connection between the spaces $X(I)$ and $X'(I)$. This estimate will be essential for obtaining appropriate estimates of the nonlinear term.

Lemma 2.6 (Exotic Strichartz in X)**.** *Let* $0 \in I$ *be a time interval. Then,*

$$\left\|\int_0^t \frac{\sin((t-\tau)\sqrt{-\Delta})}{\sqrt{-\Delta}}f(\tau)d\tau\right\|_{X(I)} \lesssim \|f\|_{X'(I)}. \tag{2.6}$$

Proof. The inequality (2.6) follows directly from the decay estimate (Lemma 2.5) and the Hardy-Littlewood-Sobolev inequality in time. □

2.4. **Nonlinear Estimates.** In many of our arguments, we will require estimates on the nonlinearity. To obtain these estimates, our main tools will be several facts from fractional calculus.

Lemma 2.7 (Fractional Leibniz rule [4])**.** *Let* $s \in (0,1]$ *and* $1 < r, p_1, p_2, q_1, q_2 < \infty$ *be given such that* $\frac{1}{r} = \frac{1}{p_i} + \frac{1}{q_i}$ *for* $i = 1,2$. *Then there exists* $C > 0$ *such that,*

$$\||\nabla|^s (fg)\|_{L^r} \leq C\|f\|_{L^{p_1}}\||\nabla|^s g\|_{L^{q_1}} + \||\nabla|^s f\|_{L^{p_2}}\|g\|_{L^{q_2}}.$$

Lemma 2.8 (C^1 fractional chain rule [4])**.** *Suppose* $G \in C^1(\mathbb{C})$*,* $s \in (0,1]$*, and* $1 < q, q_1, q_2 < \infty$ *are such that* $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$. *Then*

$$\||\nabla|^s G(u)\|_{L^q} \lesssim \|G'(u)\|_{L^{q_1}}\||\nabla|^s u\|_{L^{q_2}}.$$

When G fails to be C^1, but remains Hölder continuous, we have the following version of the chain rule.

Lemma 2.9 (C^α fractional chain rule [33])**.** *Let* G *be a Hölder continuous function of order* $0 < \alpha < 1$. *Then for every* $0 < s < \alpha$*,* $1 < p < \infty$ *and* $\frac{s}{\alpha} < \sigma < 1$*, there exists* $C > 0$ *such that*

$$\||\nabla|^s G(u)\|_{L^p(\mathbb{R}^d)} \leq C\||u|^{\alpha-\frac{s}{\sigma}}\|_{L^{p_1}(\mathbb{R}^d)}\||\nabla|^\sigma u\|^{\frac{s}{\sigma}}_{L^{\frac{s}{\sigma}p_2}(\mathbb{R}^d)}$$

provided $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ *and* $(1-\frac{s}{\alpha\sigma})p_1 > 1$.

We now prove the following lemma, which is an essential tool in obtaining the Hölder continuity of the nonlinearity in Strichartz spaces of Besov type (see Section 5 for more details).

Lemma 2.10. *Let* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}$. $1 < p_i < \infty$, $i = 1,2,3,4$. *Assume the function* $F \in C^{1,\alpha}(\mathbb{R},\mathbb{R})$*,* $0 < \alpha < 1$. *Then*

$$\|F(u)-F(v)\|_{\dot{B}^{\frac{1}{2}}_{p,2}} \lesssim \|u-v\|_{\dot{B}^{\frac{1}{2}}_{p_1,2}} \cdot \||u|^\alpha\|_{p_2} + \||u-v|^\alpha\|_{p_3} \cdot \|v\|_{\dot{B}^{\frac{1}{2}}_{p_4,2}}. \tag{2.7}$$

Proof. This is a simple consequence of the definition of Besov space and the Hölder inequality. Recall that for $0 < s < 1$, $1 < p < \infty$,

$$\|f\|_{\dot{B}^s_{p,2}(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} \frac{\|f(x+t) - f(x)\|_p^2}{|t|^{d+2s}} dt \right)^{\frac{1}{2}}. \tag{2.8}$$

By using the fundamental theorem of calculus, we have

$$\begin{aligned} &F(u(x+t)) - F(u(x)) \\ &\quad = (u(x+t) - u(x)) \cdot \int_0^1 F'(\lambda u(x+t) + (1-\lambda)u(x)) d\lambda \end{aligned}$$

and

$$\begin{aligned} &F(v(x+t)) - F(v(x)) \\ &\quad = (v(x+t) - v(x)) \cdot \int_0^1 F'(\lambda v(x+t) + (1-\lambda)v(x)) d\lambda \end{aligned}$$

Subtracting the above two identities and rearranging terms, we obtain

$$\begin{aligned} &F(u(x+t)) - F(v(x+t)) - F(u(x)) + F(v(x)) \\ &\quad = ((u-v)(x+t) - (u-v)(x)) \int_0^1 F'(\lambda u(x+t) + (1-\lambda)u(x)) d\lambda \\ &\qquad + (v(x+t) - v(x)) \int_0^1 (F'(\lambda u(x+t) + (1-\lambda)u(x)) \\ &\qquad - F'(\lambda v(x+t) + (1-\lambda)v(x))) d\lambda. \end{aligned}$$

Therefore by Hölder continuity of F' and translation invariance of L^p norms in $\mathbb{R}^d$, we get

$$\begin{aligned} &\|F(u(x+t)) - F(v(x+t)) - F(u(x)) + F(v(x))\|_p \\ &\quad \le \|(u-v)(x+t) - (u-v)(x)\|_{p_1} \cdot \||u|^\alpha\|_{p_2} \\ &\qquad + \|v(x+t) - v(x)\|_{p_3} \cdot \||u-v|^\alpha\|_{p_4}, \end{aligned} \tag{2.9}$$

where $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}$. Now clearly (2.7) follows from (2.8) and (2.9). □

With these estimates in hand, we now prove some further inequalities that will help us to bound the nonlinear term.

Lemma 2.11 (Nonlinear estimates)**.** *We have*

$$\|F(u)\|_{W'(I)} \lesssim \|u\|_{X(I)}^{\theta_2\frac{4}{d-2}} \|u\|_{\dot{S}^1(I)}^{(1-\theta_2)\frac{4}{d-2}+1} \tag{2.10}$$

$$\|F(u)\|_{X'(I)} \lesssim \|u\|_{X(I)}^{\theta_2\frac{4}{d-2}+1} \|u\|_{\dot{S}^1(I)}^{(1-\theta_2)\frac{4}{d-2}} \tag{2.11}$$

$$\||\nabla|^{\frac{2}{d}}F'(u)\|_{L_t^{\frac{d+1}{2}}L_x^{\frac{d^3+d^2}{2d^2+2d+2}}(I)} \lesssim \|u\|^{\frac{4}{d}}_{L_t^{\frac{2(d+1)}{d-2}}\dot{H}^{\frac{1}{2},\frac{2(d^2+d)}{d^2-d+1}}(I)} \cdot \|u\|_{S(I)}^{\frac{8}{d(d-2)}}. \tag{2.12}$$

$$\begin{aligned}\|F(u)-F(w)\|_{X'(I)} \lesssim{} & \|u-w\|_{X(I)}\cdot(\|u-w\|_{S(I)}^{\frac{4}{d-2}}+\|w\|_{S(I)}^{\frac{4}{d-2}})\\ &+\|u-w\|_{X(I)}\cdot(\|u-w\|_{S(I)}+\|w\|_{S(I)})^{\frac{8}{d(d-2)}}\\ &\cdot(\|u-w\|_{X(I)}^{\theta_4}\cdot\|u-w\|_{\dot{S}^1(I)}^{1-\theta_4}\\ &+\|w\|_{X(I)}^{\theta_4}\cdot\|w\|_{Y(I)}^{1-\theta_4})^{\frac{4}{d}}\end{aligned} \tag{2.13}$$

$$\begin{aligned}\|F(u)-F(w)\|_{W'(I)} \lesssim{} & \|u-w\|_{W(I)}\cdot(\|u-w\|_{S(I)}^{\frac{4}{d-2}}+\|w\|_{S(I)}^{\frac{4}{d-2}})\\ &+\|u-w\|_{S(I)}^{\frac{4}{d-2}}\cdot\|w\|_{W(I)}.\end{aligned} \tag{2.14}$$

Proof. First (2.10) and (2.11) follow from Lemma 2.8, Hölder in time, Lemma 2.3, and $\dot{S}^1 \hookrightarrow W(I)$. (2.12) follows from Lemma 2.9 and Hölder in time. Next we establish (2.13). By the fundamental theorem of calculus,

$$F(u)-F(w) = (u-w)\int_0^1 F'(\lambda u+(1-\lambda)w)d\lambda.$$

Therefore by Lemma 2.7 and Hölder in time,

$$\|F(u)-F(w)\|_{X'(I)}$$

$$\lesssim \sup_{0\le\lambda\le1}\Big\|\||\nabla|^{\frac{2}{d}}(u-w)\|_{L_x^{\frac{2(d+1)}{d-1}}}\cdot\|F'(\lambda u+(1-\lambda)w)\|_{L_x^{\frac{d+1}{2}}}\Big\|_{L_t^{\frac{d^2+d}{3d+2}}} \tag{2.15}$$

$$+\sup_{0\le\lambda\le1}\Big\|\||\nabla|^{\frac{2}{d}}F'(\lambda u+(1-\lambda)w)\|_{L_x^{\frac{d^3+d^2}{2d^2+2d+2}}}\cdot\|u-w\|_{L_x^{\frac{2d^3+2d^2}{d^3-d^2-4d-4}}}\Big\|_{L_t^{\frac{d^2+d}{3d+2}}} \tag{2.16}$$

For (2.15), by Hölder we have

$$(2.15) \lesssim \|u-w\|_{X(I)}\cdot(\|u-w\|_{S(I)}^{\frac{4}{d-2}}+\|w\|_{S(I)}^{\frac{4}{d-2}}). \tag{2.17}$$

Similarly, for (2.16), by Hölder, (2.12), Sobolev and Lemma 2.3, we have

$$\begin{aligned}(2.16) \lesssim{} & \sup_{0\le\lambda\le1}\||\nabla|^{\frac{2}{d}}F'(\lambda u+(1-\lambda)w)\|_{L_t^{\frac{d+1}{2}}L_x^{\frac{d^3+d^2}{2d^2+2d+2}}(I)}\cdot\|u-w\|_{L_t^{\frac{d^2+d}{d+2}}L_x^{\frac{2d^3+2d^2}{d^3-d^2-4d-4}}(I)}\\ \lesssim{} & \|u-w\|_{X(I)}\cdot(\|u-w\|_{X(I)}^{\theta_4}\cdot\|u-w\|_{\dot{S}^1(I)}^{1-\theta_4}+\|w\|_{X(I)}^{\theta_4}\cdot\|w\|_{Y(I)}^{1-\theta_4})^{\frac{4}{d}}\\ &\cdot(\|u-w\|_{S(I)}+\|w\|_{S(I)})^{\frac{8}{d(d-2)}}.\end{aligned} \tag{2.18}$$

Clearly now (2.13) follows from (2.17) and (2.18). Finally, (2.14) follows directly from Lemma 2.10 and Hölder in time. □

3. Statements of main results

In this section we state the main results of this paper. We begin by recalling the definition of a strong solution to the Cauchy problem (NLW).

Definition 3.1 (Strong solution)**.** We call u a *strong solution* to (NLW) on a time interval I if $u \in C(I, \dot{H}^1)$ and satisfies the Duhamel formula

$$u(t) = K(t)(u_0, u_1) + \int_0^t \frac{\sin(\sqrt{-\Delta}(t-\tau))}{\sqrt{-\Delta}} F(u(\tau)) d\tau$$

in the sense of tempered distributions for every $t \in I$.

Remark 3.2. We stress here that the definition of a strong solution only requires the fact that $u \in C(I, \dot{H}^1)$. In particular, Strichartz space is not involved in the definition of the solution.

As discussed in the introduction, the local theory for (NLW) has been extensively studied. We now formulate Theorem 3.3 resembling the statement in [17]. The proof combines the ideas from [17] with the ideas used in the proof of local existence in [32]. As a result we obtain the local existence in the space $\dot{S}^1$ and local well-posedness in X.

Theorem 3.3. *Let $d \geq 6$, $(u_0, u_1) \in \dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$, and $I \subset \mathbb{R}$ be an interval with $t_0 = 0 \in I$ such that*

$$\|(u_0, u_1)\|_{\dot{H}^1 \times L^2} \leq A.$$

Then there exists $\eta = \eta(A)$ such that

$$\|K(t)(u_0, u_1)\|_{S(I)} < \eta$$

implies that there exists a unique solution u to (NLW) with $(u, \partial_t u) \in C(I; \dot{H}^1 \times L^2)$, and

$$\|u\|_{W(I)} + \|\partial_t |\nabla|^{-1} u\|_{W(I)} < \infty,$$
$$\|u\|_{\dot{S}^1(I)} < \infty,$$
$$\|u\|_{X(I)} \leq 2\delta,$$

where $\delta = C\eta^{\theta_1} A^{1-\theta_1}$, where θ_1 is as in Lemma 2.3.

We prove the unconditional uniqueness of strong solutions as stated in the following theorem:

Theorem 3.4 (Unconditional uniqueness of strong solutions)**.** *Let u, v be two strong solutions of (NLW) on I. Suppose $u(t_0) = v(t_0)$, $u_t(t_0) = v_t(t_0)$ for some $t_0 \in I$, then $u(t) = v(t)$, $\forall\, t \in I$.*

We also prove the following lemma which gives the standard blow-up criterion, that was formulated for the (NLW) in $\mathbb{R} \times \mathbb{R}^d$ for $d = 3, 4, 5$ by Kenig and Merle in [17]. Here we extend this blow-up criterion to higher dimensions $d \geq 6$ by following the ideas of the proof of the blow-up criterion for the NLS in high dimensions [32].

Lemma 3.5 (Standard Blow-Up Criterion)**.** *Let $(u_0, u_1) \in \dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ and $u \in C([t_0, T_0], \dot{H}^1)$ be given such that u is a strong solution to (NLW) on $[t_0, T_0]$ and*

$$\|u\|_{S([t_0,T_0])} < \infty.$$

Then there exists $\delta = \delta(u_0, u_1)$ *such that* u *extends to a strong solution to (NLW) on* $[t_0, T_0 + \delta]$.

The main result of this paper is the following long term perturbation theorem, the proof of which we present in Section 5.

Theorem 3.6 (Long time perturbation, Sobolev version)**.** *Assume* $\tilde{u}$ *is a near solution on* $I \times \mathbb{R}^d$

$$\partial_{tt}\tilde{u} - \Delta\tilde{u} = F(\tilde{u}) + e$$

such that

(a)

$$\|\tilde{u}\|_{L_t^\infty \dot{H}^1(I)} + \|\partial_t\tilde{u}\|_{L_t^\infty L^2(I)} + \|\tilde{u}\|_{L_t^{\frac{2(d+1)}{d-1}} \dot{H}^{\frac{1}{2},\frac{2(d+1)}{d-1}}(I)} \le E, \tag{3.1}$$

(b)

$$\|\tilde{u}_0 - u_0\|_{\dot{H}^1} + \|\tilde{u}_1 - u_1\|_{L^2} \le E'.$$

(c) *Smallness:*

$$\|K(t)(\tilde{u}_0 - u_0, \tilde{u}_1 - u_1)\|_{S(I)} \le \epsilon \tag{3.2}$$

$$\||\nabla|^{\frac{1}{2}} e\|_{L_{t,x}^{\frac{2(d+1)}{d+3}}(I)} \le \epsilon. \tag{3.3}$$

Then there exists $\epsilon_0 = \epsilon_0(d, E', E)$ *such that if* $0 < \epsilon < \epsilon_0$*, for (NLW) with initial data* (u_0, u_1)*, there exists a unique solution* u *on* $I \times \mathbb{R}^d$ *with the properties*

$$\|\tilde{u} - u\|_{\dot{S}^1(I)} \le C(d, E', E) \cdot E',$$

$$\|\tilde{u} - u\|_{S(I)} \le C(d, E', E) \cdot \epsilon^c. \tag{3.4}$$

Here $0 < c < 1$ *is a constant depending only on the dimension* d.

4. Local well-posedness

In this section we prove that the Cauchy problem (NLW) is locally wellposed. Let $t_0 \in \mathbb{R}$ be given. By time translation invariance, we may assume $t_0 = 0$.

4.1. **The proof of the local existence Theorem 3.3.** First we observe that by Lemma 2.3

$$\begin{aligned}\|K(t)(u_0, u_1)\|_{X(I)} &\le C\,\|K(t)(u_0, u_1)\|_{S(I)}^{\theta_1}\,\|K(t)(u_0, u_1)\|_{L_t^\infty \dot{H}^1}^{1-\theta_1} \\ &\le C\eta^{\theta_1} A^{1-\theta_1}\end{aligned} \tag{4.1}$$

Let

$$\delta \equiv C\eta^{\theta_1} A^{1-\theta_1}.$$

Next, we define the sequence of iterates by

$$\begin{aligned}&u^{-1} = 0,\\ &u^0 = K(t)(u_0, u_1),\\ &u^{n+1} = K(t)(u_0, u_1) + \int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}} F(u^n)(s)ds,\end{aligned}$$

We show that the sequence is bounded in $\dot{S}^1(I)$ and in $X(I)$. By (4.1) and the Strichartz inequality, we have

$$\|u^0\|_{X(I)} \le \delta \quad \text{and} \quad \|u^0\|_{\dot{S}^1(I)} \le CA. \tag{4.2}$$

Let $a = 2\delta$ and $b = 2CA$, and suppose for $n \ge 1$

$$\|u^n\|_{X(I)} \le a \quad \text{and} \quad \|u^n\|_{\dot{S}^1(I)} \le b.$$

Then, by the Strichartz inequality and (2.10), we obtain

$$\begin{aligned} \|u^{n+1}\|_{\dot{S}^1(I)} &\le CA + C\|u^n\|_{X(I)}^{\theta_2 \frac{4}{d-2}} \|u^n\|_{\dot{S}^1(I)}^{(1-\theta_2)\frac{4}{d-2}} \|u^n\|_{\dot{S}^1(I)} \\ &\le \frac{b}{2} + Ca^{\theta_2 \frac{4}{d-2}} b^{(1-\theta_2)\frac{4}{d-2}} b \\ &\le b, \end{aligned}$$

if we choose a small enough so that $Ca^{\theta_2 \frac{4}{d-2}} b^{(1-\theta_2)\frac{4}{d-2}} \le \frac{1}{2}$. Similarly, by (4.2), Lemma 2.6, and (2.11), we get

$$\begin{aligned} \|u^{n+1}\|_{X(I)} &\le \delta + C\|u^n\|_{X(I)}^{\theta_2 \frac{4}{d-2}} \|u^n\|_{\dot{S}^1(I)}^{(1-\theta_2)\frac{4}{d-2}} \|u^n\|_{X(I)} \\ &\le \frac{a}{2} + Ca^{\theta_2 \frac{4}{d-2}} b^{(1-\theta_2)\frac{4}{d-2}} a \\ &\le a, \end{aligned}$$

assuming that a is chosen such that it satisfies the same smallness condition as above. Hence, by induction we have

$$\|u^n\|_{X(I)} \le a \quad \text{and} \quad \|u^n\|_{\dot{S}^1(I)} \le b, \quad n \ge 0.$$

Next we show the sequence is Cauchy in $X(I)$. To that end, we note that applying Lemma 2.6 and (2.13) allows us to obtain

$$\begin{aligned} &\|u^{n+1} - u^n\|_{X(I)} \\ &\quad \lesssim \|F(u^n) - F(u^{n-1})\|_{X'(I)} \\ &\quad \lesssim \|u^n - u^{n-1}\|_{X(I)} \cdot (\|u^n - u^{n-1}\|_{S(I)}^{\frac{4}{d-2}} + \|u^{n-1}\|_{S(I)}^{\frac{4}{d-2}}) \qquad (4.3) \\ &\qquad + \|u^n - u^{n-1}\|_{X(I)} \cdot (\|u^n - u^{n-1}\|_{X(I)}^{\theta_4} \cdot \|u^n - u^{n-1}\|_{\dot{S}^1(I)}^{1-\theta_4} \\ &\qquad\quad + \|u^{n-1}\|_{X(I)}^{\theta_4} \cdot \|u^{n-1}\|_{Y(I)}^{1-\theta_4})^{\frac{4}{d}} \\ &\qquad \cdot (\|u^n - u^{n-1}\|_{S(I)} + \|u^{n-1}\|_{S(I)})^{\frac{8}{d(d-2)}}. \qquad (4.4) \end{aligned}$$

Then by Lemma 2.3 we get

$$(4.3) \lesssim \|u^n - u^{n-1}\|_{X(I)} a^{\theta_2 \frac{4}{d-2}} b^{(1-\theta_2)\frac{4}{d-2}},$$

and using $\dot{S}^1(I) \hookrightarrow Y(I)$ we get

$$\begin{aligned} (4.4) &\lesssim \|u^n - u^{n-1}\|_{X(I)} \cdot (\|u^n - u^{n-1}\|_{X(I)}^{\theta_4} \cdot \|u^n - u^{n-1}\|_{\dot{S}^1(I)}^{1-\theta_4} \\ &\qquad + \|u^{n-1}\|_{X(I)}^{\theta_4} \cdot \|u^{n-1}\|_{\dot{S}^1(I)}^{1-\theta_4})^{\frac{4}{d}} \cdot (\|u^n - u^{n-1}\|_{S(I)} + \|u^{n-1}\|_{S(I)})^{\frac{8}{d(d-2)}} \\ &\lesssim \|u^n - u^{n-1}\|_{X(I)} \cdot (a^{\theta_4} b^{1-\theta_4})^{\frac{4}{d}} \cdot (a^{\theta_2} b^{1-\theta_2})^{\frac{8}{d(d-2)}}. \end{aligned}$$

It follows that if a is small enough, the sequence converges to u in $X(I)$. Since u^n are bounded in $\dot{S}^1$, they are in particular bounded in $W(I)$, which is reflexive, so u^n

converge weakly to u in $W(I)$. Then, by the Strichartz inequality, we conclude $u \in \dot{S}^1(I)$. Also standard arguments using the nonlinear estimate (2.13) and essentially repeating the calculations above show u solves (NLW) as needed.

4.2. **Unconditional uniqueness.** Having proved the existence of solutions stated in Theorem 3.3, we now prove Theorem 3.4, which gives the unconditional uniqueness of strong solutions.

We first recall the following fact about Besov norms, which can be proved using basic properties of Littlewood-Paley operators.

Lemma 4.1 (Equivalence of Besov norms). *Let $1 < p < \infty$ and $1 \le q \le \infty$. Let $s > 0$. Then*

$$\|f\|_{\dot{B}^s_{p,q}} \approx \left(\sum_{j\in\mathbb{Z}} \left(2^{js} \|\Delta_{\ge j} f\|_{L^p_x} \right)^q \right)^{\frac{1}{q}},$$

and

$$\|f\|_{\dot{B}^{-s}_{p,q}} \approx \left(\sum_{j\in\mathbb{Z}} \left(2^{-js} \|\Delta_{\le j} f\|_{L^p_x} \right)^q \right)^{\frac{1}{q}}.$$

In our proof of Theorem 3.4 we will use the following fact regarding paraproducts. For any two functions f and g, we may decompose the product fg into the sum of a low frequency piece and a high frequency piece. Indeed, by frequency localization, we write

$$\begin{aligned} fg &= \sum_{j\in\mathbb{Z}} \Delta_j(fg) \\ &= \sum_{j\in\mathbb{Z}} \Delta_j(\Delta_{\le j+3} f g) + \sum_{j\in\mathbb{Z}} \Delta_j(\Delta_{>j+3} f \Delta_{\ge j+1} g) \\ &=: G_1(f,g) + G_2(f,g). \end{aligned} \tag{4.5}$$

We shall estimate G_1 and G_2 separately using the following lemma.

Lemma 4.2 (Paraproduct estimates). *Let $s > 0$, $\sigma > 0$, $1 < p_i < \infty$, $i = 1, \cdots, 6$. Then*

$$\|G_1(f,g)\|_{\dot{B}^{-s}_{p,2}} \lesssim \|f\|_{\dot{B}^{-s}_{p_1,2}} \cdot \|g\|_{p_2}, \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \tag{4.6}$$

$$\|G_2(f,g)\|_{\dot{B}^{-s}_{p,2}} \lesssim \|f\|_{\dot{B}^{-s}_{p_3,2}} \cdot \|g\|_{\dot{B}^{s_1}_{p_4,\infty}}, \quad s_1 > s, \ \frac{1}{p_3} + \frac{1}{p_4} = \frac{1}{p} + \frac{s_1}{d}, \tag{4.7}$$

$$\|G_2(f,g)\|_{\dot{B}^{\sigma}_{p,2}} \lesssim \|f\|_{\dot{B}^{-s}_{p_5,2}} \cdot \|g\|_{\dot{B}^{s+\sigma}_{p_6,\infty}}, \quad \frac{1}{p_5} + \frac{1}{p_6} = \frac{1}{p}. \tag{4.8}$$

Proof. In the proof of (4.6), we use Hölder and Lemma 4.1. We have

$$\begin{aligned} \|G_1(f,g)\|_{\dot{B}^{-s}_{p,2}} &\lesssim \left(\sum_{j\in\mathbb{Z}} \left(2^{-js} \|\triangle_{\le j+3} f g\|_p \right)^2 \right)^{\frac{1}{2}} \\ &\lesssim \left(\sum_{j\in\mathbb{Z}} \left(2^{-js} \|\triangle_{\le j+3} f\|_{p_1} \right)^2 \right)^{\frac{1}{2}} \|g\|_{p_2} \\ &\lesssim \|f\|_{\dot{B}^{-s}_{p_1,2}} \|g\|_{p_2}. \end{aligned}$$

We now prove (4.7). From the definition and the Bernstein estimate we have

$$\begin{aligned}
\|G_2(f,g)\|_{\dot B^{-s}_{p,2}} &\lesssim \Big(\sum_{j\in\mathbb{Z}}\Big(2^{-js}\|\triangle_j(\triangle_{>j+3}f\triangle_{>j+1}g)\|_p\Big)^2\Big)^{\frac12}\\
&\lesssim \Big(\sum_{j\in\mathbb{Z}}\Big(2^{-js}2^{js_1}\|\triangle_j(\triangle_{>j+3}f\triangle_{>j+1}g)\|_{\frac{pd}{d+ps_1}}\Big)^2\Big)^{\frac12}\\
&\lesssim \Big(\sum_{j\in\mathbb{Z}}\Big(2^{j(s_1-s)}\sum_{k>j+3,|k-k'|\le 2}\|\triangle_k f\|_{p_3}\|\triangle_{k'}g\|_{p_4}\Big)^2\Big)^{\frac12}\\
&\lesssim \Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k>j+3,|k-k'|\le 2}2^{(j-k)(s_1-s)}2^{-ks}\|\triangle_k f\|_{p_3}2^{ks_1}\|\triangle_{k'}g\|_{p_4}\Big)^2\Big)^{\frac12}\\
&\lesssim \Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k>j+3}2^{(j-k)(s_1-s)}2^{-ks}\|\triangle_k f\|_{p_3}\Big)^2\Big)^{\frac12}\|g\|_{\dot B^{s_1}_{p_4,\infty}}\\
&\lesssim \|f\|_{\dot B^{-s}_{p_3,2}}\|g\|_{\dot B^{s_1}_{p_4,\infty}}.
\end{aligned}$$

Here in the last line we have used the Young's inequality and the fact that $s_1 > s$.

Next we estimate (4.8). We have

$$\begin{aligned}
\|G_2(f,g)\|_{\dot B^{\sigma}_{p,2}} &\lesssim \Big(\sum_{j\in\mathbb{Z}}\Big(2^{j\sigma}\|\triangle_j(\triangle_{>j+3}f\triangle_{>j+1}g)\|_p\Big)^2\Big)^{\frac12}\\
&\lesssim \Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k>j+3,|k-k'|\le 2}2^{j\sigma}\|\triangle_k f\|_{p_5}\|\triangle_{k'}g\|_{p_6}\Big)^2\Big)^{\frac12}\\
&\lesssim \Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k>j+3,|k-k'|\le 2}2^{(j-k)\sigma}2^{-ks}\|\triangle_k f\|_{p_5}2^{k(s+\sigma)}\|\triangle_{k'}g\|_{p_6}\Big)^2\Big)^{\frac12}\\
&\lesssim \|f\|_{\dot B^{-s}_{p_5,2}}\|g\|_{\dot B^{s+\sigma}_{p_6,\infty}}.
\end{aligned}$$

□

We are now ready to turn our attention to the proof of Theorem 3.4.

Proof of Theorem 3.4. By the existence component of the local well-posedness result, we can construct a strong solution in $\dot S^1$. Therefore, without loss of generality, we assume u is a strong solution and satisfies

$$\|u\|_{\dot S^1(I)} \lesssim C(\|u_0\|_{\dot H^1}, \|u_1\|_{L^2}).$$

As before, we may also assume $t_0 = 0$. Now let $\delta = u - v$. Clearly, δ satisfies the equation

$$\delta_{tt} - \Delta\delta = F(u) - F(v), \qquad \delta(0) = 0,\ \delta_t(0) = 0.$$

By the fundamental theorem of calculus, we write

$$\begin{aligned} F(u)-F(v) &= \delta\int_0^1 F'(\lambda u+(1-\lambda)v)d\lambda \\ &= \delta\int_0^1 (F'((1-\lambda)\delta-u)-F'(u))d\lambda+\delta F'(u) \\ &= \delta\cdot H+\delta\cdot F'(u), \end{aligned}$$

where in the second equality we have used the fact that F' is an even function. Also due to the Hölder continuity of $F'(z)$, the function H has the pointwise bound

$$|H(x)| \lesssim |\delta(x)|^{\frac{4}{d-2}}. \tag{4.9}$$

Let I_0 be a small time interval containing 0. We shall choose I_0 sufficiently small later. Using the Strichartz inequality and the Duhamel formula, we estimate

$$\begin{aligned} \|\delta\|_{L_t^2\dot B^{-\frac{1}{d-1}}_{\frac{2(d-1)}{d-3},2}(I_0\times\mathbb{R}^d)} &\lesssim \|G_1(\delta,F'(u))\|_{L_t^{\frac{2(d+1)}{d+5}}\dot B^{-\frac{1}{d-1}}_{\frac{2(d^2-1)}{d^2+2d-7},2}(I_0\times\mathbb{R}^d)} &(4.10)\\ &\quad+\|G_2(\delta,F'(u))\|_{L_t^{\frac{2(d+1)}{d+5}}\dot B^{-\frac{1}{d-1}}_{\frac{2(d^2-1)}{d^2+2d-7},2}(I_0\times\mathbb{R}^d)} &(4.11)\\ &\quad+\|G_1(\delta,H)\|_{L_t^2\dot B^{\frac{1}{d-1}}_{\frac{2(d-1)}{d+1},2}(I_0\times\mathbb{R}^d)} &(4.12)\\ &\quad+\|G_2(\delta,H)\|_{L_t^2\dot B^{\frac{1}{d-1}}_{\frac{2(d-1)}{d+1},2}(I_0\times\mathbb{R}^d)}, &(4.13) \end{aligned}$$

where $G_1(\cdot,\cdot)$, $G_2(\cdot,\cdot)$ are defined in (4.5).

To estimate (4.10), we use (4.6) with

$$s=\frac{1}{d-1},\ p=\frac{2(d^2-1)}{d^2+2d-7},\ p_1=\frac{2(d-1)}{d-3},\ p_2=\frac{d+1}{2}$$

and Hölder in time to get

$$\begin{aligned} (4.10) &\lesssim \|\delta\|_{L_t^2\dot B^{-\frac{1}{d-1}}_{\frac{2(d-1)}{d-3},2}(I_0\times\mathbb{R}^d)}\|F'(u)\|_{L_{t,x}^{\frac{d+1}{2}}(I_0\times\mathbb{R}^d)} \\ &\lesssim \|\delta\|_{L_t^2\dot B^{-\frac{1}{d-1}}_{\frac{2(d-1)}{d-3},2}(I_0\times\mathbb{R}^d)}\|u\|^{\frac{4}{d-2}}_{L_{t,x}^{\frac{2(d+1)}{d-2}}(I_0\times\mathbb{R}^d)} \\ &\lesssim \|\delta\|_{L_t^2\dot B^{-\frac{1}{d-1}}_{\frac{2(d-1)}{d-3},2}(I_0\times\mathbb{R}^d)}\|u\|^{\frac{4}{d-2}}_{L_t^{\frac{2(d+1)}{d-2}}\dot B^{\frac{d}{2(d-1)}}_{\frac{2(d^2-1)}{d^2-2d+3},2}(I_0\times\mathbb{R}^d)} \end{aligned}$$

To estimate (4.11), we use (4.7) with the same s, p and

$$s_1=\frac{2}{d-1},\ p_3=\frac{2(d-1)}{d-3},\ p_4=\frac{d(d^2-1)}{2(d^2+1)}$$

in the space variable. In the time variable, we use the Hölder inequality. We also use Lemma 2.1 with $f = F'$. This gives us

$$
\begin{aligned}
(4.11) &\lesssim \|\delta\|_{L_t^2 \dot B^{-\frac{1}{d-1}}_{\frac{2(d-1)}{d-3},2}(I_0\times\mathbb{R}^d)} \|F'(u)\|_{L_t^{\frac{d+1}{2}} \dot B^{\frac{2}{d-1}}_{\frac{d(d^2-1)}{2(d^2+1)},\infty}(I_0\times\mathbb{R}^d)} \\
&\lesssim \|\delta\|_{L_t^2 \dot B^{-\frac{1}{d-1}}_{\frac{2(d-1)}{d-3},2}(I_0\times\mathbb{R}^d)} \|u\|^{\frac{4}{d-2}}_{L_t^{\frac{2(d+1)}{d-2}} \dot B^{\frac{d-2}{2(d-1)}}_{\frac{2d(d^2-1)}{(d-2)(d^2+1)},\infty}(I_0\times\mathbb{R}^d)} \\
&\lesssim \|\delta\|_{L_t^2 \dot B^{-\frac{1}{d-1}}_{\frac{2(d-1)}{d-3},2}(I_0\times\mathbb{R}^d)} \|u\|^{\frac{4}{d-2}}_{L_t^{\frac{2(d+1)}{d-2}} \dot B^{\frac{d}{2(d-1)}}_{\frac{2(d^2-1)}{d^2-2d+3},2}(I_0\times\mathbb{R}^d)}
\end{aligned}
$$

To estimate (4.13), we use (4.8) with $\sigma = \frac{1}{d-1}$, $p = \frac{2(d-1)}{d+1}$, $p_5 = \frac{2(d-1)}{d-3}$, $p_6 = \frac{d-1}{2}$ and Hölder in time to get [3]

$$
\begin{aligned}
(4.13) &\lesssim \|\delta\|_{L_t^2 \dot B^{-\frac{1}{d-1}}_{\frac{2(d-1)}{d-3},2}(I_0\times\mathbb{R}^d)} \|H\|_{L_t^\infty \dot B^{\frac{2}{d-1}}_{\frac{d-1}{2},\infty}(I_0\times\mathbb{R}^d)} \\
&\lesssim \|\delta\|_{L_t^2 \dot B^{-\frac{1}{d-1}}_{\frac{2(d-1)}{d-3},2}(I_0\times\mathbb{R}^d)} \|H\|^{\frac12}_{L_t^\infty \dot B^0_{\frac{d}{2},\infty}(I_0\times\mathbb{R}^d)} \|H\|^{\frac12}_{L_t^\infty \dot B^{\frac{4}{d-1}}_{\frac{d(d-1)}{2(d+1)},\infty}(I_0\times\mathbb{R}^d)} && (4.14) \\
&\lesssim \|\delta\|_{L_t^2 \dot B^{-\frac{1}{d-1}}_{\frac{2(d-1)}{d-3},2}(I_0\times\mathbb{R}^d)} \left\| |\delta(x)|^{\frac{4}{d-2}} \right\|^{\frac12}_{L_t^\infty L^{\frac{d}{2}}} && (4.15) \\
&\quad \cdot \left(\int_0^1 \|F'((1-\lambda)\delta - u) - F'(u)\|_{L_t^\infty \dot B^{\frac{4}{d-1}}_{\frac{d(d-1)}{2(d+1)},\infty}(I_0\times\mathbb{R}^d)} d\lambda \right)^{\frac12} \\
&\lesssim \|\delta\|_{L_t^2 \dot B^{-\frac{1}{d-1}}_{\frac{2(d-1)}{d-3},2}(I_0\times\mathbb{R}^d)} \|\delta\|^{\frac{2}{d-2}}_{L_t^\infty \dot H_x^1} \left(\|\delta\|^{\frac{2}{d-2}}_{L_t^\infty \dot H_x^1} + \|u\|^{\frac{2}{d-2}}_{L_t^\infty \dot H_x^1} \right),
\end{aligned}
$$

where to obtain (4.14) we use interpolation, to obtain (4.15) we use the pointwise bound (4.9) and the definition of H. In the last line we have used Lemma 2.1 and Sobolev embedding.

Finally we estimate (4.12). By frequency localization, we further decompose $G_1(\delta, H)$ as

$$
\begin{aligned}
G_1(\delta, H) = &\sum_{j\in\mathbb{Z}} \Delta_j(\Delta_{\le j+3}\delta \Delta_{\ge j-3} H) && (4.16) \\
&+ \sum_{j\in\mathbb{Z}} \Delta_j(\Delta_{j-2\le \cdot \le j+3}\delta \Delta_{< j-3} H). && (4.17)
\end{aligned}
$$

[3] A key point of the following estimate is to separate a portion of "δ" when estimating H. Since H is only bounded pointwise by $|\delta|^{\frac{4}{d-2}}$, instead of estimating $\|H\|_{\dot B^{\frac{2}{d-1}}_{\frac{d-1}{2},\infty}}$ directly, we have to use the interpolation inequality to extract a portion of $L_x^{\frac{d}{2}}$ norm of H which in turn can be bounded by $L_x^{\frac{2d}{d-2}}$-norm of δ. We thank F. Planchon for the correction on the previous text regarding this point.

A quick observation shows that (4.16) can be estimated in a similar way as (4.13). Therefore we have

$$(4.16) \lesssim \|\delta\|_{L_t^2 \dot B^{-\frac{1}{d-1}}_{\frac{2(d-1)}{d-3},2}(I_0\times\mathbb{R}^d)} \cdot \|\delta\|^{\frac{2}{d-2}}_{L_t^\infty \dot H_x^1(I_0\times\mathbb{R}^d)} \cdot (\|\delta\|^{\frac{2}{d-2}}_{L_t^\infty \dot H_x^1(I_0\times\mathbb{R}^d)} + \|u\|^{\frac{2}{d-2}}_{L_t^\infty \dot H_x^1(I_0\times\mathbb{R}^d)}).$$

Now we turn to estimating (4.17). To simplify notation, observe that $\Delta_{j-2\le\cdot\le j+3}\delta$ essentially behaves as $\Delta_j\delta$. Therefore in the estimate below we write $\Delta_j\delta$ in place of $\Delta_{j-2\le\cdot\le j+3}\delta$. With this convention, we have

$$\begin{aligned}(4.17) &\lesssim \left\| \left(\sum_{j\in\mathbb{Z}} (2^{\frac{1}{d-1}j} \|\Delta_j\delta\|_{\frac{2(d^3-d^2)}{d^3-3d^2+6d-2}} \cdot \|\Delta_{<j-3}H\|_{\frac{d^2}{2d-1}})^2 \right)^{\frac12} \right\|_{L_t^2(I_0)} \\ &\lesssim \left\| \|\delta\|_{\dot B^{\frac{1}{d-1}+\frac1d}_{\frac{2(d^3-d^2)}{d^3-3d^2+6d-2},2}} \cdot \|H\|_{\dot B^{-\frac1d}_{\frac{d^2}{2d-1},\infty}} \right\|_{L_t^2(I_0)}. \qquad (4.18)\end{aligned}$$

By embedding we have

$$\begin{aligned}\|H\|_{\dot B^{-\frac1d}_{\frac{d^2}{2d-1},\infty}} &\lesssim \|H\|_{L_x^{\frac d2}} \lesssim \|\delta^{\frac{4}{d-2}}\|_{L_x^{\frac d2}} \\ &\lesssim \|\delta\|^{\frac{4}{d-2}}_{\dot B^{\frac{1}{d-1}+\frac1d}_{\frac{2(d^3-d^2)}{d^3-3d^2+6d-2},2}}.\end{aligned}$$

By interpolation we have

$$\|\delta\|_{\dot B^{\frac{1}{d-1}+\frac1d}_{\frac{2(d^3-d^2)}{d^3-3d^2+6d-2},2}} \lesssim \|\delta\|^{1+\frac{1}{d^2}-\frac3d}_{\dot B^{-\frac{1}{d-1}}_{\frac{2(d-1)}{d-3},2}} \cdot \|\delta\|^{\frac3d-\frac{1}{d^2}}_{\dot H^1}.$$

Therefore

$$\begin{aligned}(4.18) &\lesssim \left\| \|\delta\|^{\frac{d+2}{d-2}}_{\dot B^{\frac{1}{d-1}+\frac1d}_{\frac{2(d^3-d^2)}{d^3-3d^2+6d-2},2}} \right\|_{L_t^2(I_0)} \\ &\lesssim \left\| \|\delta\|^{(1+\frac{1}{d^2}-\frac3d)\cdot\frac{d+2}{d-2}}_{\dot B^{-\frac{1}{d-1}}_{\frac{2(d-1)}{d-3},2}} \cdot \|\delta\|^{(\frac3d-\frac{1}{d^2})\cdot\frac{d+2}{d-2}}_{\dot H^1} \right\|_{L_t^2(I_0)} \\ &\lesssim \|\delta\|_{L_t^2\dot B^{-\frac{1}{d-1}}_{\frac{2(d-1)}{d-3},2}(I_0\times\mathbb{R}^d)} \cdot \|\delta\|^{\frac{4}{d-2}}_{L_t^\infty \dot H^1(I_0\times\mathbb{R}^d)}.\end{aligned}$$

Collecting all the estimates, we get

$$\begin{aligned}&\|\delta\|_{L_t^2\dot B^{-\frac{1}{d-1}}_{\frac{2(d-1)}{d-3},2}(I_0\times\mathbb{R}^d)} \\ \lesssim&\|\delta\|_{L_t^2\dot B^{-\frac{1}{d-1}}_{\frac{2(d-1)}{d-3},2}(I_0\times\mathbb{R}^d)} \cdot \Bigg(\|\delta\|^{\frac{4}{d-2}}_{L_t^\infty \dot H^1(I_0\times\mathbb{R}^d)} + \|u\|^{\frac{4}{d-2}}_{L_t^{\frac{2(d+1)}{d-2}} \dot B^{\frac{d}{2(d-1)}}_{\frac{2(d^2-1)}{d^2-2d+3},2}(I_0\times\mathbb{R}^d)} \\ &+ \|\delta\|^{\frac{2}{d-2}}_{L_t^\infty \dot H_x^1(I_0\times\mathbb{R}^d)} \cdot \|u\|^{\frac{2}{d-2}}_{L_t^\infty \dot H_x^1(I_0\times\mathbb{R}^d)} \Bigg).\end{aligned}$$

Observing that $\delta \in C(I_0, \dot{H}^1)$, $\delta(0) = 0$ and noting the boundedness of

$$\|u\|^{\frac{4}{d-2}}_{L_t^{\frac{2(d+1)}{d-2}} \dot{B}^{\frac{d}{2(d-1)}}_{\frac{2(d^2-1)}{d^2-2d+3},2}(I_0\times\mathbb{R}^d)},$$

we conclude that for I_0 sufficiently small, $\delta = 0$ on I_0. A simple bootstrap argument then yields that $\delta = 0$ on the whole interval I. The theorem is proved. □

Remark 4.3. Interestingly, the proof of unconditional uniqueness also provides a proof of local well-posedness in high dimensions $d \geq 5$. We briefly sketch the argument as follows. Define the map

$$\phi(u) = K(t)(u_0, u_1) + \int_0^t \frac{\sin((t-\tau)\sqrt{-\Delta})}{\sqrt{-\Delta}} F(u(\tau)) d\tau.$$

Let $\delta > 0$ (to be fixed later) and choose the time interval I sufficiently small such that

$$\begin{aligned} \|K(t)(u_0,u_1)\|_{L_t^2 \dot{B}^{-\frac{1}{d-1}}_{\frac{2(d-1)}{d-3},2}(I)} + \|K(t)(u_0,u_1)\|_{L_t^{\frac{2(d+1)}{d-1}} \dot{B}^{\frac{1}{2}}_{\frac{2(d+1)}{d-1},2}(I)} \\ + \|K(t)(u_0,u_1)\|_{L_t^{\frac{2(d+1)}{d-2}} \dot{B}^{\frac{d}{2(d-1)}}_{\frac{2(d^2-1)}{d^2-2d+3},2}(I)} \leq \delta. \end{aligned}$$

Then consider the ball

$$\begin{aligned} B_1 = \Big\{ u \in \dot{S}^1(I) : \ \|u\|_{L_t^2 \dot{B}^{-\frac{1}{d-1}}_{\frac{2(d-1)}{d-3},2}(I)} \leq 2\delta, \|u\|_{L_t^{\frac{2(d+1)}{d-2}} \dot{B}^{\frac{d}{2(d-1)}}_{\frac{2(d^2-1)}{d^2-2d+3},2}(I)} \leq 2\delta, \\ \text{and} \quad \|u\|_{L_t^{\frac{2(d+1)}{d-1}} \dot{B}^{\frac{1}{2}}_{\frac{2(d+1)}{d-1},2}(I)} \leq 2\delta \Big\}. \end{aligned}$$

It is not difficult to check that ϕ maps B_1 into B_1 for δ sufficiently small. Furthermore by using estimates similar to (4.10), (4.11), we have

$$\begin{aligned} &\|\phi(u) - \phi(v)\|_{L_t^2 \dot{B}^{-\frac{1}{d-1}}_{\frac{2(d-1)}{d-3},2}(I)} \\ \lesssim &\|u - v\|_{L_t^2 \dot{B}^{-\frac{1}{d-1}}_{\frac{2(d-1)}{d-3},2}(I)} \cdot \Big(\|u\|^{\frac{4}{d-2}}_{L_t^{\frac{2(d+1)}{d-2}} \dot{B}^{\frac{d}{2(d-1)}}_{\frac{2(d^2-1)}{d^2-2d+3},2}(I)} + \|v\|^{\frac{4}{d-2}}_{L_t^{\frac{2(d+1)}{d-2}} \dot{B}^{\frac{d}{2(d-1)}}_{\frac{2(d^2-1)}{d^2-2d+3},2}(I)}\Big) \\ \lesssim &\delta^{\frac{4}{d-2}} \cdot \|u - v\|_{L_t^2 \dot{B}^{-\frac{1}{d-1}}_{\frac{2(d-1)}{d-3},2}(I)}, \end{aligned}$$

for all u, $v \in B_1$. This shows that ϕ is a contraction on B_1 if δ is sufficiently small and therefore we can find a unique solution in B_1.

We conclude this section by giving the proof of Lemma 3.5.

Proof of Lemma 3.5. Denote $L = \|u\|_{S([t_0,T_0])}$. We divide the proof into two steps.

Step 1. We show that

$$\|u\|_{\dot{S}^1([t_0,T_0])} \leq A := C(L,d) \cdot (\|u_0\|_{\dot{H}^1} + \|u_1\|_{L^2}). \tag{4.19}$$

let $\xi > 0$ be given (to be fixed later in the argument). First, we partition $[t_0, T_0]$ into $N = N(L, \xi, d)$ intervals $I_j = [t_j, t_{j+1}]$ such that

$$\|u\|_{S(I_j)} \leq \xi.$$

Then by the Strichartz inequality, we get

$$\begin{aligned} \|u\|_{\dot{S}^1(I_j)} &\lesssim \|u(t_j)\|_{\dot{H}^1} + \|\partial_t u(t_j)\|_{L^2} + \|u\|_{S(I_j)}^{\frac{4}{d-2}} \|u\|_{\dot{S}^1(I_j)} \\ &\lesssim \|u(t_j)\|_{\dot{H}^1} + \|\partial_t u(t_j)\|_{L^2} + \xi^{\frac{4}{d-2}} \|u\|_{\dot{S}^1(I_j)} \end{aligned}$$

Thus,

$$\|u\|_{\dot{S}^1(I_j)} \lesssim \|u(t_j)\|_{\dot{H}^1} + \|\partial_t u(t_j)\|_{L^2}$$

for ξ sufficiently small. A simple induction then shows that

$$\|u\|_{\dot{S}^1([t_0, T_0])} \leq C(L, d) \cdot (\|u_0\|_{\dot{H}^1} + \|u_1\|_{L^2}).$$

Step 2. By the local well-posedness Theorem 3.3, it is enough to show the existence of ϵ and δ such that

$$\|K(t - (T_0 - \epsilon))(u(T_0 - \epsilon), (\partial_t u)(T_0 - \epsilon))\|_{S([T_0 - \epsilon, T_0 + \delta])} \leq \eta, \tag{4.20}$$

where $\eta = \eta(A)$ is sufficiently small (specified by Theorem 3.3). We first estimate the piece on $[T_0 - \epsilon, T_0]$, i.e.

$$\|K(t - (T_0 - \epsilon))(u(T_0 - \epsilon), (\partial_t u)(T_0 - \epsilon))\|_{S([T_0 - \epsilon, T_0])} \tag{4.21}$$

Using Duhamel, Strichartz and (4.19), we get

$$\begin{aligned} (4.21) &\lesssim \|u\|_{S([T_0 - \epsilon, T_0])} + \||\nabla|^{\frac{1}{2}} F(u)\|_{L_{t,x}^{\frac{2(d+1)}{d+3}}([T_0 - \epsilon, T_0])} \\ &\lesssim \|u\|_{S([T_0 - \epsilon, T_0])} + \|u\|_{\dot{S}^1([T_0 - \epsilon, T_0])} \cdot \|u\|_{S([T_0 - \epsilon, T_0])}^{\frac{4}{d-2}} \\ &\lesssim \|u\|_{S([T_0 - \epsilon, T_0])} + A \cdot \|u\|_{S([T_0 - \epsilon, T_0])}^{\frac{4}{d-2}}. \end{aligned}$$

Clearly we can choose ϵ sufficiently small to obtain

$$(4.21) \leq \frac{\eta}{2}.$$

Now since ϵ is fixed, by Lebesgue monotone convergence, there exists δ sufficiently small, such that

$$\|K(t - (T_0 - \epsilon))(u(T_0 - \epsilon), (\partial_t u)(T_0 - \epsilon))\|_{S([T_0, T_0 + \delta])} \leq \frac{\eta}{2}.$$

Therefore by adding the two pieces together we have proved (4.20). By Theorem 3.3, it follows that there exists a unique solution v to (NLW) on $[T_0 - \epsilon, T_0 + \delta]$ with $v(T_0 - \epsilon) = u(T_0 - \epsilon)$. We then use unconditional uniqueness, Theorem 3.4, to see that $u = v$ on $[t_0, T_0]$ and thus v gives the desired extension. □

5. Long time perturbation

In this section, we prove a long-time perturbation result for (NLW). We start with the following short-time perturbation theorem.

Theorem 5.1 (Short time perturbation)**.** *Let* $(u_0, u_1) \in \dot{H}^1 \times L^2$*. Let* $\tilde{u}$ *be a near solution in the following sense*

$$\partial_{tt}\tilde{u} - \Delta\tilde{u} = F(\tilde{u}) + e$$

such that

(a)

$$\|\tilde{u}_0 - u_0\|_{\dot{H}^1} + \|\tilde{u}_1 - u_1\|_{L^2} \leq A'$$

(b) *Smallness:*

$$\|K(t)(u_0 - \tilde{u}_0, u_1 - \tilde{u}_1)\|_{X(I)} \leq \epsilon,$$

$$\||\nabla|^{\frac{1}{2}} e\|_{L_{t,x}^{\frac{2(d+1)}{d+3}}(I)} \leq \epsilon,$$

$$\|\tilde{u}\|_{X(I)} + \|\tilde{u}\|_{W(I)} + \|\tilde{u}\|_{S(I)} + \|\tilde{u}\|_{Y(I)} \leq \delta.$$

Then for $0 < \delta \leq \delta_0(d)$*,* $0 < \epsilon \leq \epsilon_0(A')$*, (NLW) with initial data* (u_0, u_1)*, there exists unique* u *on* $I \times \mathbb{R}^d$ *such that*

$$\|u - \tilde{u}\|_{\dot{S}^1} \leq C(d) \cdot A', \tag{5.1}$$

$$\|u - \tilde{u}\|_{X(I)} \leq C(d, A') \cdot \epsilon, \tag{5.2}$$

$$\|F(u) - F(\tilde{u})\|_{X'(I)} \leq C(d, A') \cdot \epsilon. \tag{5.3}$$

Proof. Assume u exists on I. Then

$$\partial_{tt}(\tilde{u} - u) - \Delta(\tilde{u} - u) = F(\tilde{u}) - F(u) + e.$$

Now

$$\|\tilde{u} - u\|_{X(I)} \lesssim \|K(t)(u_0 - \tilde{u}_0, u_1 - \tilde{u}_1)\|_{X(I)} \tag{5.4}$$

$$+ \left\| \int_0^t \frac{\sin((t-\tau)\sqrt{-\Delta})}{\sqrt{-\Delta}} e(\tau) d\tau \right\|_{X(I)} \tag{5.5}$$

$$+ \left\| \int_0^t \frac{\sin((t-\tau)\sqrt{-\Delta})}{\sqrt{-\Delta}} (F(\tilde{u}) - F(u)) d\tau \right\|_{X(I)}. \tag{5.6}$$

The estimate of (5.4) follows from the assumption and we have

$$(5.4) \lesssim \epsilon. \tag{5.7}$$

To estimate (5.5), we use Lemma 2.3 and Strichartz,

$$\begin{aligned}(5.5) &\lesssim \left\| \int_0^t \frac{\sin((t-\tau)\sqrt{-\Delta})}{\sqrt{-\Delta}} e(\tau) d\tau \right\|_{\dot{S}^1(I)} \\ &\lesssim \left\| |\nabla|^{\frac{1}{2}} e \right\|_{L_{t,x}^{\frac{2(d+1)}{d+3}}(I)} \\ &\lesssim \epsilon. \end{aligned} \tag{5.8}$$

To estimate (5.6), we use Lemma 2.6, Lemma 2.11 and Lemma 2.3 to get

$$
\begin{aligned}
(5.6) &\lesssim \|F(u)-F(\tilde u)\|_{X'(I)} \\
&\lesssim \|u-\tilde u\|_{X(I)} \cdot \big(\|u-\tilde u\|_{S(I)}^{\frac{4}{d-2}} + \|\tilde u\|_{S(I)}^{\frac{4}{d-2}}\big) \\
&\quad + \|u-\tilde u\|_{X(I)} \cdot \big(\|u-\tilde u\|_{X(I)}^{\theta_4} \cdot \|u-\tilde u\|_{\dot S^1(I)}^{1-\theta_4} + \|\tilde u\|_{X(I)}^{\theta_4} \cdot \|\tilde u\|_{Y(I)}^{1-\theta_4}\big)^{\frac{4}{d}} \\
&\qquad \cdot \big(\|u-\tilde u\|_{S(I)} + \|\tilde u\|_{S(I)}\big)^{\frac{8}{d(d-2)}} \\
&\lesssim \|u-\tilde u\|_{X(I)} \cdot \big(\|u-\tilde u\|_{X(I)}^{\frac{4}{d-2}\theta_2} \cdot \|u-\tilde u\|_{W(I)}^{\frac{4}{d-2}(1-\theta_2)} + \delta^{\frac{4}{d-2}}\big) \\
&\quad + \|u-\tilde u\|_{X(I)} \cdot \big(\|u-\tilde u\|_{X(I)}^{\theta_4} \cdot \|u-\tilde u\|_{\dot S^1(I)}^{1-\theta_4} + \delta\big)^{\frac{4}{d}} \\
&\qquad \cdot \big(\|u-\tilde u\|_{X(I)}^{\theta_2} \cdot \|u-\tilde u\|_{W(I)}^{1-\theta_2} + \delta\big)^{\frac{8}{d(d-2)}}.
\end{aligned} \tag{5.9}
$$

Collecting the estimates (5.7), (5.8) and (5.9) and using the fact that $\dot S^1 \hookrightarrow W$, we obtain

$$
\begin{aligned}
&\|\tilde u - u\|_{X(I)} \\
&\quad \lesssim \epsilon + \|u-\tilde u\|_{X(I)} \cdot \big(\|u-\tilde u\|_{X(I)}^{\frac{4}{d-2}\theta_2} \cdot \|u-\tilde u\|_{\dot S^1(I)}^{\frac{4}{d-2}(1-\theta_2)} + \delta^{\frac{4}{d-2}}\big) \\
&\qquad + \|u-\tilde u\|_{X(I)} \cdot \big(\|u-\tilde u\|_{X(I)}^{\theta_4} \cdot \|u-\tilde u\|_{\dot S^1(I)}^{1-\theta_4} + \delta\big)^{\frac{4}{d}} \\
&\qquad \cdot \big(\|u-\tilde u\|_{X(I)}^{\theta_2} \cdot \|u-\tilde u\|_{\dot S^1(I)}^{1-\theta_2} + \delta\big)^{\frac{8}{d(d-2)}}.
\end{aligned} \tag{5.10}
$$

This is the first estimate we need. Next we estimate $\|\tilde u - u\|_{\dot S^1(I)}$. By the Strichartz inequality and Lemma 2.11, we have

$$
\begin{aligned}
&\|\tilde u - u\|_{\dot S^1(I)} \\
&\quad \lesssim A' + \epsilon + \|F(\tilde u) - F(u)\|_{W'(I)} \\
&\quad \lesssim A' + \epsilon + \|\tilde u - u\|_{W(I)} \cdot \big(\|\tilde u - u\|_{S(I)}^{\frac{4}{d-2}} + \|\tilde u\|_{S(I)}^{\frac{4}{d-2}}\big) + \|\tilde u - u\|_{S(I)}^{\frac{4}{d-2}} \cdot \|\tilde u\|_{W(I)} \\
&\quad \lesssim A' + \epsilon + \|\tilde u - u\|_{\dot S^1(I)} \cdot \big(\|\tilde u - u\|_{X(I)}^{\frac{4}{d-2}\theta_2} \|\tilde u - u\|_{\dot S^1(I)}^{\frac{4}{d-2}(1-\theta_2)} + \delta^{\frac{4}{d-2}}\big) \\
&\qquad + \|\tilde u - u\|_{X(I)}^{\frac{4}{d-2}\theta_2} \cdot \|\tilde u - u\|_{\dot S^1(I)}^{\frac{4}{d-2}(1-\theta_2)} \cdot \delta.
\end{aligned} \tag{5.11}
$$

Now by (5.10), (5.11) and a continuity argument, we get (5.1), (5.2) for sufficiently small $\epsilon \le \epsilon_0(A')$ and $\delta \le \delta_0(d)$. We stress here that δ can be chosen to depend only on the dimension d. Plugging the estimates (5.1), (5.2) into (5.9), we also obtain (5.3). The theorem is proved. □

Theorem 5.1 treats the case when $\epsilon \ll A'$. In such a case all the constants in (5.2)–(5.3) depend on A'. One may wonder what happens when A' is of the same order as ϵ. In that case, similar arguments as in the proof of Theorem 5.1 give the following corollary.

Corollary 5.2 (Short time perturbation, ϵ-perturbation version). *Let $(u_0, u_1) \in \dot H^1 \times L^2$. Let $\tilde u$ be a near solution in the following sense*

$$\partial_{tt}\tilde u - \Delta \tilde u = F(\tilde u) + e$$

such that

(a)

$$\|\tilde{u}_0 - u_0\|_{\dot{H}^1} + \|\tilde{u}_1 - u_1\|_{L^2} \le \epsilon$$

(b) *Smallness:*

$$\||\nabla|^{\frac{1}{2}} e\|_{L_{t,x}^{\frac{2(d+1)}{d+3}}(I)} \le \epsilon,$$

$$\|\tilde{u}\|_{X(I)} + \|\tilde{u}\|_{W(I)} + \|\tilde{u}\|_{S(I)} + \|\tilde{u}\|_{Y(I)} \le \delta.$$

Then for $0 < \delta \le \delta_0(d)$, $0 < \epsilon \le \epsilon_0(d)$, *(NLW) with initial data* (u_0, u_1), *there exists unique* u *on* $I \times \mathbb{R}^d$ *such that*

$$\|u - \tilde{u}\|_{\dot{S}^1} \le C(d) \cdot \epsilon^c, \tag{5.12}$$

$$\|u - \tilde{u}\|_{X(I)} \le C(d) \cdot \epsilon, \tag{5.13}$$

$$\|F(u) - F(\tilde{u})\|_{X'(I)} \le C(d) \cdot \epsilon. \tag{5.14}$$

Here $0 < c < 1$ *is a constant depending only on the dimension* d.

Proof. One only needs to repeat the derivation of (5.10) and (5.11) as in the proof of Theorem 5.1. We omit the details. □

Next we establish the long time perturbation in Besov spaces by using the short time perturbation result, Theorem 5.1.

Theorem 5.3 (Long time perturbation, Besov version). *Assume* $\tilde{u}$ *is a near solution on* $I \times \mathbb{R}^d$

$$\partial_{tt}\tilde{u} - \Delta\tilde{u} = F(\tilde{u}) + e$$

such that

(a)

$$\|\tilde{u}\|_{\dot{S}^1(I)} \le E.$$

(b)

$$\|\tilde{u}_0 - u_0\|_{\dot{H}^1} + \|\tilde{u}_1 - u_1\|_{L^2} \le E'.$$

(c) *Smallness:*

$$\|K(t)(\tilde{u}_0 - u_0, \tilde{u}_1 - u_1)\|_{X(I)} \le \epsilon$$

$$\||\nabla|^{\frac{1}{2}} e\|_{L_{t,x}^{\frac{2(d+1)}{d+3}}(I)} \le \epsilon.$$

Then there exists $\epsilon_0 = \epsilon_0(d, E', E)$ *such that if* $0 < \epsilon < \epsilon_0$, *for (NLW) with initial data* (u_0, u_1), *there exists a unique solution* u *on* $I \times \mathbb{R}^d$ *with the properties*

$$\|\tilde{u} - u\|_{\dot{S}^1(I)} \le C(d, E', E) \cdot E',$$

$$\|\tilde{u} - u\|_{X(I)} \le C(d, E', E) \cdot \epsilon.$$

Proof. Let $\delta_0 = \delta_0(d)$ be chosen in the way as in Theorem 5.1. Denote $t_0 = 0$. Partition the time interval I into $I = \bigcup_{j=1}^k I_j = \bigcup_{j=1}^k [t_{j-1}, t_j]$ such that on each subinterval I_j

$$\|\tilde{u}\|_{S(I_j)} + \|\tilde{u}\|_{W(I_j)} + \|\tilde{u}\|_{X(I_j)} + \|\tilde{u}\|_{Y(I_j)} < \delta_0.$$

One can choose $k = O((E/\delta_0)^{C(d)})$ such intervals, where $C(d)$ is a constant depending only on the dimension d. Consider the first subinterval $I_1 = [0, t_1]$. By Theorem 5.1, for ϵ sufficiently small depending only on (d, E'), we have

$$\begin{aligned}
\|\tilde{u} - u\|_{\dot{S}^1(I_1)} &\le C(d, E') \cdot E', \\
\|\tilde{u} - u\|_{X(I_1)} &\le C(d, E') \cdot \epsilon, \\
\|F(\tilde{u}) - F(u)\|_{X'(I_1)} &\le C(d, E') \cdot \epsilon.
\end{aligned}$$

Next for $1 \le i \le k-1$, make the inductive assumption that

$$\|\tilde{u} - u\|_{\dot{S}^1(I_i)} \le C_i(d, E', E) \cdot E', \tag{5.15}$$

$$\|\tilde{u} - u\|_{X(I_i)} \le C_i(d, E', E) \cdot \epsilon, \tag{5.16}$$

$$\|F(\tilde{u}) - F(u)\|_{X'([0,t_i])} \le C_i(d, E', E) \cdot \epsilon. \tag{5.17}$$

Then for $I_{i+1} = [t_i, t_{i+1}]$ we will apply Theorem 5.1 with a time shift t_i. For this we have to check the hypotheses of Theorem 5.1. To this end, by (5.15), we have

$$\|\tilde{u}(t_i) - u(t_i)\|_{\dot{H}^1} + \|(\partial_t \tilde{u})(t_i) - (\partial_t u)(t_i)\|_{L^2_x} \le 2C_i(d, E', E) \cdot E'.$$

Next by using Duhamel's formula and (5.17), we have

$$\begin{aligned}
&\|K(t - t_i)(\tilde{u}(t_i) - u(t_i), (\partial_t \tilde{u})(t_i) - (\partial_t u)(t_i))\|_{X(I_{i+1})} \\
&\quad \lesssim \|K(t)(\tilde{u}_0 - u_0, \tilde{u}_1 - u_1)\|_{X(I)} + \|F(\tilde{u}) - F(u)\|_{X'([0,t_i])} + \||\nabla|^{\frac{1}{2}} e\|_{L_{t,x}^{\frac{2(d+1)}{d+3}}(I)} \\
&\quad \lesssim (1 + c_i(d, E', E)) \cdot \epsilon.
\end{aligned}$$

Then for ϵ sufficiently small depending only on (d, E', E), Theorem 5.1 and (5.17) give

$$\begin{aligned}
\|\tilde{u} - u\|_{\dot{S}^1(I_{i+1})} &\le C_{i+1}(d, E', E) \cdot E', \\
\|\tilde{u} - u\|_{X(I_{i+1})} &\le C_{i+1}(d, E', E) \cdot \epsilon, \\
\|F(\tilde{u}) - F(u)\|_{X'([0,t_{i+1}])} &\le \|F(\tilde{u}) - F(u)\|_{X'([0,t_i])} + \|F(\tilde{u}) - F(u)\|_{X'([t_i,t_{i+1}])} \\
&\le C_{i+1}(d, E', E) \cdot \epsilon.
\end{aligned}$$

Consequently we have verified (5.15)–(5.17) for all $1 \le i \le k$. We stress that the choice of ϵ is consistent since $k = O((E/\delta_0)^{C(d)})$ is finite and we only need to adjust ϵ at most k times. The theorem now follows by summing (5.15)–(5.16). □

Similar to the derivation of Corollary 5.2, the same arguments as in the proof of Theorem 5.3 give the following result.

Corollary 5.4 (Long time perturbation, Besov ϵ-perturbation version)**.** *Assume $\tilde{u}$ is a near solution on $I \times \mathbb{R}^d$*

$$\partial_{tt}\tilde{u} - \Delta\tilde{u} = F(\tilde{u}) + e$$

such that

(a)

$$\|\tilde{u}\|_{\dot{S}^1(I)} \le E.$$

(b)

$$\|\tilde{u}_0 - u_0\|_{\dot{H}^1} + \|\tilde{u}_1 - u_1\|_{L^2} \le \epsilon.$$

(c) *Smallness:*

$$\||\nabla|^{\frac{1}{2}} e\|_{L_{t,x}^{\frac{2(d+1)}{d+3}}(I)} \leq \epsilon.$$

Then there exists $\epsilon_0 = \epsilon_0(d, E)$ *such that if* $0 < \epsilon < \epsilon_0$*, for (NLW) with initial data* (u_0, u_1)*, there exists a unique solution* u *on* $I \times \mathbb{R}^d$ *with the properties*

$$\begin{aligned} \|\tilde{u} - u\|_{\dot{S}^1(I)} &\leq C(d, E) \cdot \epsilon^{c_1}, \\ \|\tilde{u} - u\|_{X(I)} &\leq C(d, E) \cdot \epsilon. \end{aligned} \tag{5.18}$$

Here $0 < c_1 < 1$ *is a constant depending on* (d, E)*.*

Proof. This is essentially a repetition of the proof of Theorem 5.3. Note that in (5.18) the constant c_1 depends both on the dimension d and E. This is a consequence of the short time perturbation theory (Corollary 5.2) where we lose a power of c due to the Hölder continuity of the nonlinearity. The additional dependence on E comes from the fact that we have to apply the short time theory $O(E^{C(d)})$ times. □

To obtain the usual Sobolev space version of Theorem 5.3, we need the following lemma, which shows that the $\dot{S}^1$ norm of the solution of the perturbed equation is bounded.

Lemma 5.5 (Boundedness of near solutions in Besov spaces)**.** *Let* $\tilde{u}$ *be a near solution on* $I \times \mathbb{R}^d$

$$\partial_{tt}\tilde{u} - \Delta\tilde{u} = F(\tilde{u}) + e,$$

such that

$$\|\tilde{u}\|_{L_t^\infty \dot{H}^1(I)} + \|\partial_t \tilde{u}\|_{L_t^\infty L^2(I)} + \|\tilde{u}\|_{L_t^{\frac{2(d+1)}{d-1}} \dot{H}^{\frac{1}{2}, \frac{2(d+1)}{d-1}}(I)} \leq E, \tag{5.19}$$

$$\|e\|_{L_t^{\frac{2(d+1)}{d+3}} \dot{H}^{\frac{1}{2}, \frac{2(d+1)}{d+3}}(I)} \leq E, \tag{5.20}$$

Then

$$\|\tilde{u}\|_{\dot{S}^1(I)} \leq C(d, E). \tag{5.21}$$

Proof. We first have the interpolation inequality

$$\|\tilde{u}\|_{L_{t,x}^{\frac{2(d+1)}{d-2}}} \lesssim \|\tilde{u}\|_{L_t^\infty \dot{H}^1}^{\frac{1}{d-1}} \|\tilde{u}\|_{L_t^{\frac{2(d+1)}{d-1}} \dot{H}^{\frac{1}{2}, \frac{2(d+1)}{d-1}}}^{\frac{d-2}{d-1}}. \tag{5.22}$$

Now by (5.19), (5.20), Strichartz and (5.22), we have

$$\begin{aligned} \|\tilde{u}\|_{\dot{S}^1(I)} &\lesssim E + \|\tilde{u}\|_{L_{t,x}^{\frac{2(d+1)}{d-2}}(I)}^{\frac{4}{d-2}} \cdot \|\tilde{u}\|_{L_t^{\frac{2(d+1)}{d-1}} \dot{H}^{\frac{1}{2}, \frac{2(d+1)}{d-1}}(I)} \\ &\lesssim E + E^{1+\frac{4}{d-2}}. \end{aligned}$$

This immediately gives us (5.21). □

We are now ready to prove the main perturbation result stated in Theorem 3.6.

Proof. By (3.1), (3.3) and taking $\epsilon < E$, Lemma 5.5 gives us

$$\|\tilde{u}\|_{\dot{S}^1(I)} \leq C(d, E).$$

By (3.2) and Lemma 2.3, we have

$$\begin{aligned}
&\|K(t)(\tilde{u}_0 - u_0, \tilde{u}_1 - u_1)\|_{X(I)} \\
&\quad \lesssim \|K(t)(\tilde{u}_0 - u_0, \tilde{u}_1 - u_1)\|_{S(I)}^{\theta_1} \cdot \|K(t)(\tilde{u}_0 - u_0, \tilde{u}_1 - u_1)\|_{L_t^\infty \dot{H}^1(I)}^{1-\theta_1} \\
&\quad \lesssim \epsilon^{\theta_1} \cdot (E')^{1-\theta_1}.
\end{aligned}$$

Now for ϵ sufficiently small depending on (d, E', E), we can apply Theorem 5.3 to obtain that

$$\|\tilde{u} - u\|_{\dot{S}^1(I)} \leq C(d, E', E) \cdot E', \tag{5.23}$$

$$\|\tilde{u} - u\|_{X(I)} \leq C(d, E', E) \cdot \epsilon. \tag{5.24}$$

Finally (3.4) follows from (5.24), Lemma 2.3 and (5.23). The theorem is proved. □

Finally we have the ϵ-perturbation version of Theorem 3.6 similar to Corollary 5.4. We omit the proof and leave the details to interested readers.

Corollary 5.6 (Long time perturbation, Sobolev ϵ-perturbation version)**.** *Assume $\tilde{u}$ is a near solution on $I \times \mathbb{R}^d$*

$$\partial_{tt}\tilde{u} - \Delta\tilde{u} = F(\tilde{u}) + e$$

such that

(a)

$$\|\tilde{u}\|_{L_t^\infty \dot{H}^1(I)} + \|\partial_t \tilde{u}\|_{L_t^\infty L^2(I)} + \|\tilde{u}\|_{L_t^{\frac{2(d+1)}{d-1}} \dot{H}^{\frac{1}{2}, \frac{2(d+1)}{d-1}}(I)} \leq E.$$

(b)

$$\|\tilde{u}_0 - u_0\|_{\dot{H}^1} + \|\tilde{u}_1 - u_1\|_{L^2} \leq \epsilon.$$

(c) *Smallness:*

$$\||\nabla|^{\frac{1}{2}} e\|_{L_{t,x}^{\frac{2(d+1)}{d+3}}(I)} \leq \epsilon.$$

Then there exists $\epsilon_0 = \epsilon_0(d, E)$ such that if $0 < \epsilon < \epsilon_0$, for (NLW) with initial data (u_0, u_1), there exists a unique solution u on $I \times \mathbb{R}^d$ with the properties

$$\begin{aligned}
\|\tilde{u} - u\|_{\dot{S}^1(I)} &\leq C(d, E) \cdot \epsilon^{c_3}, \\
\|\tilde{u} - u\|_{S(I)} &\leq C(d, E) \cdot \epsilon^{c_4}.
\end{aligned} \tag{5.25}$$

Here $0 < c_3 < 1$ is a constant depending on (d, E) and $0 < c_4 < 1$ depends only on the dimension d.

References

[1] H. Bahouri and P. Gérard, *High frequency approximation of solutions to critical nonlinear wave equations*, Amer. J. Math **121** (1999), 131–175.

[2] J. Bergh and J. Löfström, *Interpolation spaces–An introduction* Springer-Verlag, 1976.

[3] A. Bulut, *Maximizers for the Strichartz Inequalities for the Wave Equation*, Differential and Integral Equations (2010).

[4] F. M. Christ and M. I. Weinstein, Dispersion of small amplitude solutions of the generalized Korteweg-de Vries equation. *J. Funct. Analysis*, **100** No. 1, (1991), 87–109.

[5] T. Duyckaerts and F. Merle, *Dynamic of threshold solutions for energy-critical wave equation*, Int. Math. Res. Pap. IMRP 2008, Art ID rpn002

[6] T. Duyckaerts and F. Merle, *Dynamic of threshold solutions for energy-critical NLS*. Geom. Funct. Anal. 18 (2009), no. 6, 1787–1840.

[7] G. Furioli and E. Terraneo. Besov spaces and unconditional well-posedness for the nonlinear Schrödinger equation in $\dot{H}^s(\mathbb{R}^n)$. Commun. Contemp. Math. 5 (2003), no. 3, 349–367.

[8] G. Furioli, F. Planchon, E. Terraneo, Unconditional well-posedness for semilinear Schrödinger and wave equations in H^s. (English summary) Harmonic analysis at Mount Holyoke (South Hadley, MA, 2001), 147–156, Contemp. Math., 320, Amer. Math. Soc., Providence, RI, 2003.

[9] J. Ginibre, A. Soffer and G. Velo, *The global Cauchy problem for the critical nonlinear wave equation*, J. Func. Analysis, **110** (1992), pp. 96–130.

[10] J. Ginibre and G. Velo, *Generalized Strichartz inequalities for the wave equation*, J. Funct. Anal. **133** No. 1 (1995), 50–68.

[11] M. Grillakis, *Regularity and asymptotic behaviour of the wave equation with a critical nonlinearity*, Ann. of Math. **132** (1990), 485–509.

[12] M. Grillakis, *Regularity for the wave equation with a critical nonlinearity*, Comm. Pure Appl. Math. **45** (1992), 749–774.

[13] L. Kapitanski, *Global and unique weak solutions of nonlinear wave equations*, Math. Res. Lett., **1** (1994), No. 2, 211–223.

[14] M. Keel and T. Tao, *Endpoint Strichartz estimates*, Amer. J. Math., **120** No. 5 (1998), 955–980.

[15] C. E. Kenig, *Global well-posedness, scattering and blow-up for the energy critical, focusing, non-linear Schrödinger and wave equations*, Lecture notes available at http://www.math.uchicago.edu/∼cek/

[16] C. E. Kenig and F. Merle, *Global well-posedness, scattering and blow-up for the energy critical, focusing, non-linear Schrödinger equation in the radial case*, To appear in Invent. Math.

[17] C. E. Kenig and F. Merle, *Global well-posedness, scattering and blow-up for the energy critical focusing non-linear wave equation*, To appear in Acta Mathematica

[18] R. Killip and M. Visan, *Nonlinear Schrödinger equations at critical regularity* Preprint (2008).

[19] H. Levine, *Instability and nonexistence of global solutions to nonlinear wave equations of the form $Pu_{tt} = -Au + \mathcal{F}(u)$*, Trans. Amer. Math. Soc. **192** (1974), 1–21.

[20] D. Li and X. Zhang, *Dynamics for the energy critical nonlinear wave equation in high dimensions*, To appear in Transactions of AMS (2009).

[21] D. Li and X. Zhang, *Dynamics for the energy critical nonlinear Schrödinger equation in high dimensions*. J. Funct. Anal. 256 (2009), no. 6, 1928–1961.

[22] H. Lindblad and C. Sogge, *On existence and scattering with minimal regularity for semilinear wave equations*, J. Funct. Anal. **130** (1995), 357–426.

[23] N. Masmoudi and F. Planchon, *On uniqueness for the critical wave equation*. Comm. Partial Differential Equations **31** (2006), no. 7-9, 1099–1107.

[24] K. Nakanishi, *Scattering theory for nonlinear Klein-Gordon equation with Sobolev critical power*, Internat. Math. Res. Notices **1** (1999), 31–60.

[25] H. Pecher, *Nonlinear small data scattering for the wave and Klein-Gordon equation*, Math. Z. **185** (1984), 261–270.

[26] F. Planchon, *On uniqueness for semilinear wave equations*, Math. Z. **244** (2003), no. 3, 587–599.

[27] J. Shatah and M. Struwe, *Regularity results for nonlinear wave equations*, Ann. of Math. **138** (1993), 503–518.

[28] J. Shatah and M. Struwe, *Well-posedness in the energy space for semilinear wave equations with critical growth*, Internat. Math. Res. Notices **7** (1994), 303–309.

[29] J. Shatah and M. Struwe, "Geometric wave equations," Courant Lecture Notes in Mathematics, **2** (1998).

[30] C. Sogge, "Lectures on nonlinear wave equations," Monographs in Analysis II, International Press, 1995.

[31] M. Struwe, *Globally regular solutions to the u^5 Klein-Gordon equation*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **15** (1988), 495–513.

[32] T. Tao and M. Visan, *Stability of energy-critical nonlinear Schrödinger equations in high dimensions*, Electron. J. Diff. Eqns. **118** (2005), 1–28.

[33] M. Visan. The defocusing energy-critical nonlinear Schrödinger equation in higher dimensions. *Duke Math. J.*, 138(2):281–374, 2007.

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