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TWO OF KUNITA'S PAPERS ON STOCHASTIC FLOWS IN EARLY 1980S

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Dedicated to the memory of Professor Hiroshi Kunita

ABSTRACT. Two of Kunita's papers in early 1980s on diffeomorphic property of stochastic flows are revisited, and corresponding results by the author are presented.

1. Introduction

H. Kunita wrote in his last book [6, p.109]:

> The diffeomorphic property of maps $x \to X_t^x$ received a lot of attention around 1980. We refer Elworthy [K25], Malliavin [K78], Bismut [K8], Le Jan [K75], Harris [K36], Ikeda-Watanabe [K41] and Kunita [K59]. A method of proving the diffeomorphic property is to approximate SDEs by a sequence of stochastic ordinary differential equations. (Omission)
>
> Kunita [K59] presented another method for proving the diffeomorphic property through a skillful use of the Kolmogorov-Totoki theorem. In this monograph, we took another method by constructing a backward flow $\check{\Psi}_{s,t}(x)$ which satisfies Lemma 3.7.2. An advantage of the new method is that it can be applied for proving the diffeomorphic property of solutions of SDE on a manifold as in Sect. 7.1. [1]

While he referred his book [5] in 1990 as [K59], both methods cited in the second paragraph were established by him in early 1980s; in 1981 ([3]) and in 1982 ([4]).

Recall that showing the diffeomorphic property consists of two ingredients: one is proving the smoothness, and the other is proving the bijectivity. The above two methods mentioned by him in the second paragraph were employed to show the bijectivity. The first method was introduced in [3] and uses a homotopy theoretical observation. The second one was used in [4] for the first time and takes advantage of the newly introduced backward stochastic differential equation (SDE in short).

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[1] The numbers of Lemma and Section are same as in his book [6]. To avoid a confusion between his bibliography and ours, the letter "K" is added to the numeric labels of his bibliography.

It should be remarked that the word "backward SDE" is completely different from the one used in the recent contexts of the Mathematical Finance. In this paper, we revisit both methods and present corresponding results by the author.

The construction of this paper is as follows. In Section 2, we shall discuss the homotopy theoretical approach. After reviewing Kunita's result, we will present the result by the author on stochastic flows of diffeomorphisms on bounded domains in $\mathbb{R}^d$. Section 3 is devoted to backward SDEs. In the section, we first recall Kunita's result and then give another application of backward SDEs to estimating the Malliavin covariance.

2. A Homotopy Theoretical Approach to Surjectivity

2.1. The paper in 1981. In this subsection we give a brief review on a part of Kunita's result achieved in [3].

Take Lipschitz continuous $X_j : \mathbb{R}^d \to \mathbb{R}^d$ and continuous semi-martingales $\{M_t^j\}_{t\geq 0}$ defined on a filtered complete probability space $(\Omega, \mathcal{F}, \mathbf{P}, \{\mathcal{F}_t\}_{t\geq 0})$, where $1 \leq j \leq r$ and the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfies that $\bigcap_{\varepsilon>0} \mathcal{F}_{t+\varepsilon} = \mathcal{F}_t$ for $t \geq 0$. Denote by $\{N_t^j\}_{t\geq 0}$ and $\{A_t^j\}_{t\geq 0}$ the martingale part and the bounded variation part of $\{M_t^j\}_{t\geq 0}$, respectively, and assume that

$$|A_t^j - A_s^j| + |\langle N^j\rangle_t - \langle N^j\rangle_s| \leq |t-s| \quad \text{for } s,t \in [0,\infty), 1 \leq j \leq r,$$

where $\{\langle N^j\rangle_t\}_{t\geq 0}$ stands for the quadratic variation process of $\{N_t^j\}_{t\geq 0}$. Consider the Itô type SDE on $\mathbb{R}^d$:

$$d\xi_t = \sum_{j=1}^r X_j(\xi_t) dM_t^j.$$

For $x \in \mathbb{R}^d$, denote by $\{\xi_t(x)\}_{t\geq 0}$ the solution with initial condition $\xi_0 = x$. Kunita showed that, for $T > 0$, $p \geq 2$, and $q \in \mathbb{R}$, there exist constants $K_{p,T}^{(1)}$, $K_{q,T}^{(2)}$, and $K_{q,T}^{(3)}$ such that

$$\mathbb{E}[|\xi_t(x) - \xi_s(y)|^p] \leq K_{p,T}^{(1)}\Big(|x-y|^p + |t-s|^{p/2}\Big), \tag{2.1}$$

$$\mathbb{E}[|\xi_t(x) - \xi_t(y)|^q] \leq K_{q,T}^{(2)}|x-y|^q, \tag{2.2}$$

$$\mathbb{E}[(1+|\xi_t(x)|^2)^q] \leq K_{q,T}^{(3)}(1+|x|^2)^q \quad \text{for } s,t \in [0,T], x,y \in \mathbb{R}^d. \tag{2.3}$$

Combined with the Kolmogorov-Totoki continuity theorem ([5]), he showed:

- (2.1) yields the continuity of the mapping $(t,x) \mapsto \xi_t(x)$.
- (2.2) implies the continuity of the mapping

  $$[0,\infty) \times \Big((\mathbb{R}^d \times \mathbb{R}^d) \setminus \{(x,x) \mid x \in \mathbb{R}^d\}\Big) \ni (t,x,y) \mapsto \frac{1}{|\xi_t(x) - \xi_t(y)|},$$

  from which the injectivity of $x \mapsto \xi_t(x)$ follows.
- (2.3) implies the continuity of the mapping

  $$[0,\infty) \times \widehat{\mathbb{R}}^d \ni (t,x) \mapsto \zeta_t(x) = \begin{cases} \dfrac{1}{1+|\xi_t(x)|} & \text{if } x \in \mathbb{R}^d, \\ 0 & \text{if } x = \infty, \end{cases}$$

where $\widehat{\mathbb{R}}^d = \mathbb{R}^d \cup \{\infty\}$ is the one-point compactification of $\mathbb{R}^d$. This yields the continuity of the mapping

$$\widehat{\mathbb{R}}^d \ni x \mapsto \widehat{\xi}_t(x) = \begin{cases} \xi_t(x) & \text{if } x \in \mathbb{R}^d, \\ \infty & \text{if } x = \infty. \end{cases}$$

Remember an elementary, homotopy theoretical fact that if a continuous mapping $\phi : \mathbb{S}^d \to \mathbb{S}^d$ is not surjective, where $\mathbb{S}^d$ is the d-dimensional unit sphere, then it is homotopic to a constant mapping. Identifying $\widehat{\mathbb{R}}^d$ with $\mathbb{S}^d$, we then see that $\widehat{\xi}_t : \widehat{\mathbb{R}}^d \to \widehat{\mathbb{R}}^d$ is surjective. Since ∞ is invariant, $\xi_t : \mathbb{R}^d \to \mathbb{R}^d$ is also surjective.

Consequently $\{\xi_t(\cdot)\}_{t\geq 0}$ determines a stochastic flow of homeomorphisms on $\mathbb{R}^d$.

2.2. Flows on a bounded domain in $\mathbb{R}^d$. Kunita continued to apply the same method of using the homotopy theory to showing the surjectivity of stochastic flows in his book [5] in 1990. In this subsection, we shall see the topological argument is also applicable to stochastic flows of diffeomorphisms on bounded domains in $\mathbb{R}^d$ ([9]).

Let D be a bounded domain in $\mathbb{R}^d$ with C^∞-boundary, that is, for each boundary point, there exist a neighborhood U and a function $\phi \in C^\infty(U)$ such that the gradient $\nabla\phi(x) \neq 0$ for any $x \in \partial D \cap U$ and $D \cap U = \{x \in U \mid \phi(x) < 0\}$. Such a ϕ is called a local defining function of ∂D. For C^∞-vector fields $V_0, \dots, V_r$ with compact supports, consider the Stratonovich type SDE

$$d\xi_t = \sum_{j=1}^r V_j(\xi_t) \circ dB_t^j + V_0(\xi_t)dt,$$

where $\{B_t = (B_t^1, \dots, B_t^r)\}_{t\geq 0}$ is an r-dimensional Brownian motion on $(\Omega, \mathcal{F}, \mathbf{P})$. Denote by $\{\xi_t(x)\}_{t\geq 0}$ the solution with the initial condition $\xi_0(x) = x$ as before, and by $\{\xi_t(\cdot)\}_{t\geq 0}$ the associated stochastic flow of diffeomorphisms on $\mathbb{R}^d$.

Proposition 2.1. *Assume that, for every neighborhood U of a boundary point with a local defining function ϕ,*

$$V_j\phi = 0 \quad \text{on } \partial D \cap U \text{ for } 0 \leq j \leq r. \tag{2.4}$$

Denote by $\xi_t(\cdot)|_D$ the restriction of $\xi_t(\cdot)$ to D. Then, $\{\xi_t(\cdot)|_D\}_{t\geq 0}$ determines a stochastic flow of diffeomorphisms on D.

Proof. By the assumption (2.4), V_js are all tangential to ∂D. Then, for any connected component S of ∂D, $\xi_t(\cdot)|_S$ is a submersion onto an open subset $\xi_t(S)$ of S. Since S is compact, so is $\xi_t(S)$. In particular, $\xi_t(S)$ is closed. Therefore, $\xi_t(S) = S$, i.e., $\xi_t(\cdot)|_S : S \to S$ is surjective. Thus ∂D is an invariant set of $\xi_t(\cdot)$ and hence $\{\xi_t(\cdot)|_D\}_{t\geq 0}$ determines a stochastic flow of diffeomorphisms on D. □

3. Backward SDE

3.1. The paper in 1982. In this subsection we give a brief review on backward SDEs introduced by Kunita in [4]. While he treated more general settings, we shall review in the form, which is suitable for the use in the next subsection.

Let $\mathbb{M}$ be a σ-compact d-dimensional C^∞-manifold. Denote by $\mathcal{X}(\mathbb{M})$ the space of all C^∞-vector fields on $\mathbb{M}$. Given $X_0,\dots,X_r\in\mathcal{X}(\mathbb{M})$, consider the Stratonovich type SDE on $\mathbb{M}$:

$$d\xi_t=\sum_{j=1}^r X_j(\xi_t)\circ dB_t^j+X_0(\xi_t)\circ dt, \tag{3.1}$$

where $\{B_t=(B_t^1,\dots,B_t^r)\}_{t\ge0}$ is an r-dimensional Brownian motion on $(\Omega,\mathcal{F},\mathbf{P})$ as in Subsection 2.1. For $x\in\mathbb{M}$, denote by $\{\xi_{s,t}(x)\}_{t\ge s}$ the solution of the SDE (3.1) with $\xi_{s,s}(x)=x$. Suppose that under an imbedding of $\mathbb{M}$ into $\mathbb{R}^{d'}$, X_js extend to C_b^∞-vector fields [2] on $\mathbb{R}^{d'}$. Then, the SDE (3.1) is exactly conservative, i.e., $\mathbf{P}(\tau(s,x)=\infty$ for all $x\in\mathbb{M})=1$ for every $s\ge0$, where $\tau(s,x)$ is the explosion time of $\{\xi_{s,t}(x)\}_{t\ge s}$. Then $\{\xi_{s,t}\}_{0\le s\le t}$ determines a stochastic flow of diffeomorphisms on $\mathbb{M}$.

To see the surjectivity, Kunita introduced the backward stochastic integrals as follows. Put $\mathcal{F}_s^t=\overline{\sigma[\{B_u-B_v\in A\mid s\le u,v\le t,A\in\mathcal{B}(\mathbb{R}^r)\}]}^{\mathbf{P}}$, where $\mathcal{B}(\mathbb{R}^r)$ is the Borel σ-field of $\mathbb{R}^r$. Given $t>0$, for $\{\mathcal{F}_u^t\}_{u\le t}$-predictable and continuous process $\{f(u)\}_{s\le u\le t}$ with $\mathbb{E}[\int_s^t||f(u)|^2du]<\infty$, define the **backward Itô integral** $\int_s^t f(u)\hat{d}B_u^j$ by

$$\int_s^t f(u)\hat{d}B_u^j=\lim_{|\Delta|\to0}\sum_{k=0}^{n-1}f(u_{k+1})(B_{u_{k+1}}^j-B_{u_k}^j),$$

where $\Delta=\{s=u_0<u_1<\dots<u_n=t\}$ and $|\Delta|=\max_k|u_{k+1}-u_k|$. If $\{f(u)\}_{s\le u\le t}$ is a backward semi-martingale, then the **backward Stratonovich integral** $\int_s^t f(u)\circ\hat{d}B_u^j$ is also defined. Using the backward SDE, Kunita showed the bijectivity of $\xi_{s,t}$ as follows.

Theorem 3.1 ([4, Theorem 2]). *Let $\eta_{s,t}(x)$ be the unique solution of the backward SDE:*

$$\hat{d}\eta_{s,t}(x)=X_0(\eta_{s,t}(x))ds+\sum_{j=1}^r X_j(\eta_{s,t}(x))\circ\hat{d}B_s^j \tag{3.2.13}$$

such that $\eta_{t,t}(x)=x$. Then $\eta_{s,t}=\xi_{s,t}^{-1}$.

3.2. Malliavin calculus on manifolds. In this section, we will give another application of inverse flow $\{\eta_{s,t}\}_{s\le t}$. We first review on the Malliavin calculus on manifolds by referencing and modifying the results in [10]. Take and fix a Riemannian metric g on $\mathbb{M}$.

Let $T>0$, $\mathcal{W}$ be the space of continuous $\mathbb{R}^r$-valued functions w on $[0,T]$ with $w(0)=0$, and μ be the Wiener measure on it. The Cameron-Martin subspace H of $\mathcal{W}$ is the space of absolutely continuous $h\in\mathcal{W}$ with square integrable derivative $\dot{h}$. The inner product of H is defined ty

$$\langle h_1,h_2\rangle_H=\int_0^T\langle\dot{h}_1(t),\dot{h}_2(t)\rangle_{\mathbb{R}^r}dt\quad\text{for } h_1,h_2\in H,$$

---

[2] The subscript "b" of C_b^∞ means that derivatives of all orders are bounded.

where $\langle\cdot,\cdot\rangle_{\mathbb{R}^r}$ is the inner product of $\mathbb{R}^r$. Denote by $\mathfrak{D}^\infty$ the space of all $\mathbb{R}$-valued Wiener functionals, which and whose H-derivatives of all orders are pth integral for any $p\in(1,\infty)$. For definition, see [1, 7, 8]. The H-derivative of $G\in\mathfrak{D}^\infty$ is denoted by DG.

A Wiener functional $F:\mathcal{W}\to\mathbb{M}$ is said to be in $\mathcal{D}^\infty_{\mathrm{loc}}(\mathbb{M})$ if $F^*f:=f(F)\in\mathfrak{D}^\infty$ for any $f\in C_0^\infty(\mathbb{M})$ [3] . For $F\in\mathcal{D}^\infty_{\mathrm{loc}}(\mathbb{M})$ and μ-a.s. $w\in\mathcal{W}$, define the linear mapping $(F_*)_w$ from H to $T_{F(w)}\mathbb{M}$ ($\equiv$the tangent space of $\mathbb{M}$ at $F(w)$) by

$$((F_*)_w h)f=\langle D(F^*f)(w),h\rangle_H \quad \text{for } h\in H \text{ and } f\in C_0^\infty(\mathbb{M}).$$

$(F_*)_w$ is also called the H-derivative of F at w. Further, under the identification between H and its dual space, the dual operator of $(F_*)_w$ determines a linear operator $(F_*)_w^\dagger$ from $T^*_{F(w)}\mathbb{M}$ ($\equiv$ the cotangent space of $\mathbb{M}$ at $F(w)$) to H.

The H-derivative $(F_*)_w$ and the composition $(F_*)_w(F_*)_w^\dagger$ admit local expressions as follows. Let $(U;(x^1,\dots,x^d))$ be a local coordinate neighborhood of $\mathbb{M}$, i.e., U is an open subset of $\mathbb{M}$ and $(x^1,\dots,x^d)$ is a local coordinate system on U. For relatively compact open sets U_1 and U_2 with $\overline{U_1}\subset U_2\subset\overline{U_2}\subset U$, take $\varphi^i\in C_0^\infty(\mathbb{M})$ such that $\varphi^i(x)=x^i$ on U_2 and put $F^i=\varphi^i(F)$ for $1\le i\le d$. Then the H-derivative $(F_*)_w:H\to T_{F(w)}\mathbb{M}$ is expressed as

$$(F_*)_w=\sum_{i=1}^d\langle DF^i(w),\cdot\rangle_H\Big(\frac{\partial}{\partial x^i}\Big)_{F(w)} \quad \text{for } \mu\text{-a.s. } w\in\{F\in U_1\}.$$

Hence the composition $(F_*)_w(F_*)_w^\dagger$ satisfies

$$(F_*)_w(F_*)_w^\dagger\Big((dx^i)_{F(w)}\Big)=\sum_{j=1}^d\langle DF^i(w),DF^j(w)\rangle_H\Big(\frac{\partial}{\partial x^j}\Big)_{F(w)} \tag{3.2}$$
$$\text{for } 1\le i\le d \text{ and } \mu\text{-a.s. } w\in\{F\in U_1\}.$$

Let $\mathfrak{i}_x:T_x\mathbb{M}\to T_x^*\mathbb{M}$ be the identification of the two spaces via the Riemannian metric g, i.e., $(\mathfrak{i}_x(u))(v)=g_x(u,v)$ for $u,v\in T_x\mathbb{M}$. Then $(F_*)_w(F_*)_w^\dagger\circ\mathfrak{i}_{F(w)}$ determines a $(1,1)$-tensor, and we can define $\det[(F_*)_w(F_*)_w^\dagger\circ\mathfrak{i}_{F(w)}]$.

Definition 3.2. Let $\widetilde{\mathbb{M}}$ be a σ-compact Riemannian manifold, O be an open subset of $\mathbb{M}$, and $\widetilde{F}\in\mathcal{D}^\infty_{\mathrm{loc}}(\widetilde{\mathbb{M}})$. $F\in\mathcal{D}^\infty_{\mathrm{loc}}(\mathbb{M})$ is said to be non-degenerate on O under the control of $\widetilde{F}$ if it holds

$$\frac{1}{\det[(F_*)_w(F_*)_w^\dagger\circ\mathfrak{i}_{F(w)}]}\mathbf{1}_O(F)\mathbf{1}_K(\widetilde{F})\in\bigcap_{p\in(1,\infty)}L^p(\mu) \tag{3.3}$$

for any compact $K\subset\widetilde{\mathbb{M}}$.

For a local coordinate neighborhood $(U;(x^1,\dots,x^d))$ with $U\subset O$ as above, by (3.2), it holds

$$\det[(F_*)_w(F_*)_w^\dagger\circ\mathfrak{i}_{F(w)}]=\det\Big(\Big(\langle DF^i,DF^j\rangle_H\Big)_{1\le i,j\le d}\Big)\det g_{F(w)}$$

[3]The subscript "0" of C_0^∞ means that the support is compact.

for μ-a.s. $w \in \{F \in U_1\}$. Thus Definition 3.2 is an extension of the well-known concept of non-degeneracy of $\mathbb{R}^r$-valued Wiener functional $G = (G^1, \dots, G^d) \in (\mathfrak{D}^\infty)^d$ to manifolds.

By using the dual operator D^* of the H-derivative D, we can then show an integration by parts formula associated with non-degenerate $F \in \mathcal{D}^\infty_{\text{loc}}(\mathbb{M})$.

Theorem 3.3. *Suppose $F \in \mathcal{D}^\infty_{\text{loc}}(\mathbb{M})$ is non-degenerate on O under the control of $\widetilde{F}$. For $V \in \mathcal{X}(\mathbb{M})$, put*

$$K_F[V](w) = \begin{cases} [(F_*)_w^\dagger \circ \mathfrak{l}_{F(w)} \circ ((F_*)_w (F_*)_w^\dagger \circ \mathfrak{l}_{F(w)})^{-1}](V(F(w)) & \text{if } F(w) \in O, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\phi \in C_0^\infty(O)$ and $\psi \in C_0^\infty(\widetilde{\mathbb{M}})$. Then,

(1) *$\phi(F)\psi(\widetilde{F})K_F[V] \in \mathfrak{D}^\infty(H)$ ($\equiv$ the space of H-valued $\mathfrak{D}^\infty$-Wiener functionals).*
(2) *Define $\Phi_{F,\phi,\psi;V} : \mathfrak{D}^\infty \to \mathfrak{D}^\infty$ by*

$$\Phi_{F,\phi,\psi;V}(G) = D^*(G\phi(F)\psi(\widetilde{F})K_F[V]) \quad \text{for } G \in \mathfrak{D}^\infty.$$

Then it holds

$$\int_{\mathcal{W}} (Vf)(F)\phi(F)\psi(\widetilde{F})G d\mu = \int_{\mathcal{W}} f(F)\Phi_{F,\phi,\psi;V}(G) d\mu$$

for any $f \in C_0^\infty(\mathbb{M})$ and $G \in \mathfrak{D}^\infty$.

Observing

$$\Phi_{F,\phi,\psi;V}(G) = \Phi_{F,\phi,\psi;V}(G)\mathbf{1}_{\text{supp}\,\phi}(F)\mathbf{1}_{\text{supp}\,\psi}(\widetilde{F}),$$

where $\text{supp}\,\phi$ is the support of ϕ, we can show

Corollary 3.4. *Let F be as in Theorem 3.3. Take $\overline{\phi} = \{\phi_i\}_{i\in\mathbb{N}} \subset C_0^\infty(O), \overline{\psi} = \{\psi_i\}_{i\in\mathbb{N}} \subset C_0^\infty(\widetilde{\mathbb{M}})$ such that $\phi_{i+1} = 1$ on $\text{supp}\,\phi_i$ and $\psi_{i+1} = 1$ on $\text{supp}\,\psi_i$ for every $i \in \mathbb{N}$. For $\{V_i\}_{i\in\mathbb{N}} \subset \mathcal{X}(\mathbb{M})$, define $\Phi_{F,\overline{\phi},\overline{\psi};V_1} = \Phi_{F,\phi_1,\psi_1;V_1}$ and*

$$\Phi_{F,\overline{\phi},\overline{\psi};V_1,\dots,V_n} = \Phi_{F,\phi_n,\psi_n;V_n} \circ \Phi_{F,\overline{\phi},\overline{\psi};V_1,\dots,V_{n-1}} \quad \text{for } n \geq 2.$$

Then it holds

$$\int_{\mathcal{W}} (V_1 \cdots V_n f)(F)\phi_1(F)\psi_1(\widetilde{F})G d\mu = \int_{\mathcal{W}} f(F)\Phi_{F,\overline{\phi},\overline{\psi};V_1,\dots,V_n}(G) d\mu$$

for any $f \in C_0^\infty(\mathbb{M})$, $G \in \mathfrak{D}^\infty$, and $n \in \mathbb{N}$.

The existence of smooth density function of F on O can be shown from the above integration by parts formula.

Theorem 3.5. *Let $G \in \mathfrak{D}^\infty$, $F \in \mathcal{D}^\infty_{\text{loc}}(\mathbb{M})$, and O is an open subset of $\mathbb{M}$. Suppose F is non-degenerate on O under the control of $\widetilde{F}$. Moreover, assume that for every relatively compact open set U with $\overline{U} \subset O$, there exists a compact subset $K \subset \widetilde{\mathbb{M}}$ such that $\{F \in U\} \subset \{\widetilde{F} \in K\}$. Then there exists $p_{F;G} \in C^\infty(O)$ such that*

$$\int_{\mathcal{W}} f(F)G d\mu = \int_{\mathbb{M}} f p_{F;G} d\nu \quad \text{for any } f \in C_0^\infty(O),$$

where ν is the volume measure on $\mathbb{M}$.

We now consider the sufficient condition for non-degeneracy. For the sake of simplicity, we shall assume that $\mathbb{M}$ is compact. Let $\xi_{s,t}(x)$ be as in Subsection 3.1. Then $\xi_{0,t}(x)$ is in $\mathcal{D}^{\infty}_{\text{loc}}(\mathbb{M})$. Let $\xi_{0,t}(x)_*$ be its H-derivative.

For a smooth mapping f from $\mathbb{M}$ to another C^∞-manifold $\mathbb{M}'$, denote by $[f_*]_x : T_x\mathbb{M} \to T_{f(x)}\mathbb{M}'$ its differential at $x \in \mathbb{M}$. In particular, $[\xi_{0,t*}]_x$ and $[\xi_{0,s}^{-1}{}_*]_x$ stands for the derivatives at x of the smooth mappings $\xi_{0,t}(\cdot)$ and $\xi_{0,s}^{-1}(\cdot)$ from $\mathbb{M}$ to itself, respectively. Then it holds

$$(\xi_{0,t}(x)_*)h = \sum_{j=1}^{r} \int_0^t [\xi_{0,t*}]_x [\xi_{0,s}^{-1}{}_*]_{\xi_{0,s}(x)} X_j(\xi_{0,s}(x)) \dot{h}^j(s) ds$$

for $h = (h^1, \dots, h^d) \in H$. Hence we have

$$(\xi_{0,t}(x)_*)(\xi_{0,t}(x)_*)^\dagger = \sum_{j=1}^{r} \int_0^t \{[\xi_{0,t*}]_x [\xi_{0,s}^{-1}{}_*]_{\xi_{0,s}(x)} X_j(\xi_{0,s}(x))\}^{\otimes 2} ds. \tag{3.4}$$

Put

$$A_0(t,x) = \sum_{j=1}^{r} \int_0^t \{[\xi_{0,s}^{-1}{}_*]_{\xi_{0,s}(x)} X_j(\xi_{0,s}(x))\}^{\otimes 2} ds.$$

Due to the stochastic Taylor expansion coming out of the identity

$$\begin{aligned}[\xi_{0,s}^{-1}{}_*]_{\xi_{0,s}(x)} X(\xi_{0,s}(x)) = &\sum_{j=1}^{r} \int_0^s [\xi_{0,u}^{-1}{}_*]_{\xi_{0,u}(x)} [X_j, X](\xi_{0,u}(x)) \circ dB^j(u) \\ &+ \int_0^s [\xi_{0,u}^{-1}{}_*]_{\xi_{0,u}(x)} [X_0, X](\xi_{0,u}(x)) du \quad \text{for } X \in \mathcal{X}(\mathbb{M}),\end{aligned}$$

assuming a Hölmander type condition we can show the integrability of any order of $(\det[A_0(t,x) \circ \mathfrak{l}_x])^{-1}$. For example, see [1, 8]. Due to the integrability of $[\xi_{0,t*}]_x$, we see that (3.3) holds with $F = \xi_{0,t}(x)$ without any control by $\widetilde{F}$.

We now proceed to an application of the inverse flow $\{\eta_{s,t}\}_{s \le t}$ in Theorem 3.1. Let $\mathbb{M}'$ be a σ-compact Riemann manifold and $\pi : \mathbb{M} \to \mathbb{M}'$ be a proper submersion. Put $\zeta_t(x) = \pi(\xi_{0,t}(x))$. Then $\zeta_t(x) \in \mathcal{D}^{\infty}_{\text{loc}}(\mathbb{M}')$. By (3.4), we obtain

$$(\zeta_t(x)_*)(\zeta_t(x)_*)^\dagger = \sum_{j=1}^{r} \int_0^t \{[\pi_*]_{\xi_{0,t}(x)} [\xi_{0,t*}]_x [\xi_{0,s}^{-1}{}_*]_{\xi_{0,s}(x)} X_j(\xi_{0,s}(x))\}^{\otimes 2} ds.$$

On account of the observation in the previous paragraph, one may expect that the estimation of $(\zeta_t(x)_*)(\zeta_t(x)_*)^\dagger$ comes down to that of $\pi_* A_0(t,x) \pi_*^\dagger$. This idea is too naïve, since $[\xi_{0,t*}]_x$ stirs tangent spaces. However, using the inverse stochastic flow, the idea works. In fact, the flow property implies that $\xi_{s,t} \circ \xi_{0,s} = \xi_{0,t}$, and hence that

$$\xi_{0,s} = \eta_{s,t} \circ \xi_{0,t}.$$

This implies

$$[\xi_{0,s}^{-1}{}_*]_{\xi_{0,s}} X_j(\xi_{0,s}(x)) = [\xi_{0,t}^{-1}{}_*]_x [\eta_{s,t}^{-1}{}_*]_{\xi_{0,s}(x)} X_j(\eta_{s,t}(\xi_{0,t}(x))).$$

Hence putting

$$\widehat{A}_0(t,x)=\sum_{j=1}^{r}\int_0^t\{[\eta_{s,t}^{-1}{}_*]_{\xi_{0,s}(x)}X_j(\eta_{s,t}(x))\}^{\otimes 2}ds,$$

we have

$$(\zeta_t(x)_*)(\zeta_t(x)_*)^\dagger=\pi_{\xi_{0,t}(x)}\widehat{A}_0(t,\xi_{0,t}(x))\pi_{\xi_{0,t}(x)}^\dagger.$$

Thus we come down to a similar estimation as that for $A_0(t,x)$, this time we need to control $\xi_{0,t}(x)$ in addition. This argument works well, and we can conclude

Theorem 3.6 ([10]). *Set* $\mathcal{A}^{(0)}=\{X_1,\dots,X_r\}$, $\mathcal{A}^{(n)}=\{[X_j,X]\mid 0\le j\le r, X\in\mathcal{A}^{(n-1)}\}$, *and* $\mathcal{A}^{(\infty)}=\bigcup_{n=0}^\infty\mathcal{A}^{(n)}$. *For* $x\in\mathbb{M}$, *put*

$$\mathcal{A}_x=\left\{\sum_{i=1}^n a_iA_i(x)\,\middle|\, a_i\in\mathbb{R}, A_i\in\mathcal{A}^{(\infty)}, 1\le i\le n, n\in\mathbb{N}\right\}.$$

Suppose that

$$[\pi_*]_x\mathcal{A}_x=T_{\pi(x)}\mathbb{M}' \quad \textit{for any } x\in\mathbb{M}.$$

Then $\pi(\xi_{0,t}(x))$ *is non-degenerate on any open subset of* $\mathbb{M}'$ *without any control of* $\widetilde{F}$. *In particular, its distribution possesses a smooth density function with respect to the volume measure on* $\mathbb{M}'$.

Remark 3.7. Let $\mathcal{O}(\mathbb{M})$ be the orthonormal frame bundle over $\mathbb{M}$ and $\pi:\mathcal{O}(\mathbb{M})\to\mathbb{M}$ be the bundle projection. It is known ([1]) that the Brownian motion on $\mathbb{M}$, the diffusion process generated by the half of the Laplacian on $\mathbb{M}$, is realized as the projection $\pi(\xi_{0,t}(x))$ of the solution $\{\xi_{0,t}(x)\}_{t\ge 0}$ of the SDE on $\mathcal{O}(\mathbb{M})$. More generally, diffusion processes generated by sub-Laplacians on sub-Riemannian manifolds are realized as images of solutions of SDEs on principle bundles associated with sub-Riemannian structure. For details, see [2]. In both cases, the above theorem is applicable.

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References

1. Ikeda, N. and Watanabe, S.: *Stochastic Differential Equations and Diffusion Processes*, North Holland/Kodansha, Amsterdam/Tokyo, 1981.
2. Inahama, Y. and Taniguchi, S.: Heat trace asymptotics on equiregular sub-Riemannian manifolds, *J. Math. Soc. Japan*, **72** (2020), 1049–1096.
3. Kunita, H.: On the decomposition of solutions of stochastic differential equations: in *Stochastic Integrals, Proceedings of the LMS Durham Symposium, July 7–17, 1980*, Lecture Notes in Math., **851**, (1981) 213–255, Springer, Berlin.
4. Kunita, H.: On backward stochastic differential equations, *Stochastics*, **6** (1982), 293–313.
5. Kunita, H.: *Stochastic Flows and Stochastic Differential Equations*, Cambridge Univ. Press, Cambridge, 1990.
6. Kunita, H.: *Stochastic Flows and Jump-diffusions*, Springer Nature Singapore Pte Ltd., Singapore, 2019.
7. Matsumoto, H. and Taniguchi, S.: *Stochastic Analysis —Itô and Malliavin calculus in tandem*, Cambridge Univ. Press, Cambridge, 2017.

8. Shigekawa, I.: *Stochastic Analysis*, Amer. Math. Soc., Providence, 2004.
9. Taniguchi, S.: Stochastic flows of diffeomorphisms on an open set in $\boldsymbol{R}^n$, *Stochastics Stoc. Rep.*, **28** (1989), 301–315.
10. Taniguchi, S.: Malliavin's stochastic calculus of variations for manifold-valued Wiener functionals and its applications, *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, **65** (1983), 269–290.

K8. Bismut, J. M.: *Mécanique Aléatoire*, Lecture Notes in Math., **866**, Springer, Berlin, 1981.
K25. Elworthy, K. D.: *Stochastic Differential Equations on Manifolds*, LMS Lecture Note Series, vol.70, Cambridge University Press, Cambridge, 1982.
K36. Harris, T. E.: Coalescing and noncoalescing stochastic flows in $\mathbb{R}_1$, *Stoch. Proc. Appl.*, **17** (1984), 187–210.
K41. Ikeda, N. and Watanabe, S.: [1]
K59. Kunita, H.: [5]
K75. Le Jan, Y.: Flots de diffusions dans $\mathbb{R}^d$, *C.R. Acad. Sci. Paris*, Ser. I, **294** (1982), 679–699.
K78. Malliavin, P.: *Géometrie Différentielle Stochastique*, Le Presses de I'Université, Montréal, 1978.

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