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ERROR ESTIMATES FOR DISCRETE APPROXIMATIONS OF GAME OPTIONS WITH MULTIVARIATE DIFFUSION ASSET PRICES

YURI KIFER*

Dedicated to the memory of Professor Hiroshi Kunita

Abstract. We obtain error estimates for strong approximations of a diffusion with a diffusion matrix σ and a drift b by the discrete time process defined recursively

$X_N((n+1)/N) = X_N(n/N)+N^{-1/2}\sigma(X_N(n/N))\xi(n+1)+N^{-1}b(X_N(n/N)),$

where $\xi(n)$, $n \geq 1$ are i.i.d. random vectors, and apply this in order to approximate the fair price of a game option with a diffusion asset price evolution by values of Dynkin's games with payoffs based on the above discrete time processes. This provides an effective tool for computations of fair prices of game options with path dependent payoffs in a multi asset market with diffusion evolution.

1. Introduction

I encountered for the first time with the name Hiroshi Kunita around 1970 when as an undergraduate student at Moscow University I was suggested by E.B. Dynkin to translate into Russian together with M. Taksar the seminal paper by H. Kunita and S. Watanabe on square integrable martingales published in Nagoya Mathematical Journal in 1967. Still, I met Hiroshi for the first time in person only in 1995 on the 5th Gregynog Symposium when I was interested in random dynamical systems and processes in random environment, the interest which was motivated partially by the works on stochastic flows. Knowing about Hiroshi's book on stochastic flows I felt that we can find common interests and I discussed with him the question about random positive semigroups and their random generators related to the stochastic Kolmogorov-Fokker-Planck equation which yielded our joint paper [16] in the proceedings of the symposium. Later Hiroshi continued working on this topic publishing [18] and [19]. In 1997 I visited Hiroshi for a month at Kyushu University in the framework of the Japan Society for the Promotion of Sciences.

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I met Hiroshi for the last time in December 2004 on a financial mathematics conference at Nanzan University in Nagoya were he moved after his retirement from Kyushu University. At the conference I presented the paper [12] about approximations of prices of game options (see [14]) in the Black-Scholes market by prices of game options in the corresponding Cox-Ross-Rubinstein markets. This work relied on the Skorokhod embedding of martingales into the Brownian motion which is possible, in general, only in the one dimensional situation (see [23]) while the multivariate situation requires another approach (see [13]). Around the time of Nanzan conference Hiroshi also became interested in game options and together with S.Seko he wrote the paper [20].

In the present paper we continue the line of research in [12] and [13] approximating game options whose stocks evolutions are described by multidimensional diffusion processes. This is done constructing first strong approximations of the diffusion by a sequence of discrete time processes estimating L^2 errors of these approximations. These processes are discrete time processes obtained recursively for $n = 0, 1, \ldots, N-1$ by

$$X_N((n+1)/N) = X_N(n/N) + N^{-1/2}\sigma(X_N(n/N))\xi(n+1) + N^{-1}b(X_N(n/N)),$$

where $X_N(0) = x_0$, σ and b is a matrix and a vector functions, respectively and $\xi(n)$, $n \geq 1$ is a sequence of i.i.d. random vectors with $E\xi(1) = 0$. The strong approximation method enables us to redefine both the sequence $\xi(n)$, $n \geq 1$ and the limiting diffusion $d\Xi(t) = \sigma(\Xi(t))dW(t) + b(\Xi(t))dt$ preserving their distributions on a same sufficiently rich probability space so that the L^2-distance between them have the order $N^{-\delta}$ for some $\delta > 0$. In the second step we compare fair prices of options with payoffs based on these discrete approximations with the fair price of the option with the diffusion asset evolution. This is not straightforward since this prices are given by values of the corresponding Dynkin games which depend on sets of stopping times involved and the latter are different for the approximations and for the limiting diffusion. The payoffs of the above game options are supposed to be path dependent, and so free boundary partial differential equations methods cannot help here and discrete time approximations is the only possible approach in this situation to fair prace computations taking into account that in the discrete time case we can employ the dynamical programming (backward recursion) algorithm.

The setup of this paper is the special case of a more general discrete time setup in our recent paper [15] but the latter paper provides a detailed proof mostly in the continuous time averaging setup while here we deal with the more specific discrete time setup which can be described in the more transparent way. The motivation both for [15] and for the present paper comes, in particular, from the series of papers [11], [1], [2] and [7] on the weak diffusion limit in averaging and from the series of papers [4], [17], [22] and [8] on strong approximations (see [15] for more details). The former papers yielded only weak convergence results while the latter dealt only with approximations of the Brownian motion. We observe that in the one dimensional case it is still possible to use an extended version of the Skorokhod embedding into martingales which also yields error estimates for approximations (see [3]). Approximation similar to ours appeared previously in [10] but only the weak convergence to a diffusion was established there which, in principle, could not provide any error estimates. The reader may compare our approach with the well known Euler–Maruyama approximation of solutions of stochastic differential equations (see, for instance, [21]) where $\xi(n)$'s are increments of the Brownian motion. In our setup $\xi(n)$'s are quite general and, in particular, we can take i.i.d. random vectors taking on only few values which can be useful in applications since they are easier to simulate and compute than Gaussian random vectors.

2. Preliminaries and Main Results

We start with a complete probability space $(\Omega, \mathcal{F}, P)$, a sequence of independent identically distributed (i.i.d.) random vectors $\xi(n)$, $n \geq 1$ and a diffusion process Ξ solving the stochastic differential equation

$$d\Xi(t) = \sigma(\Xi(t))dW(t) + b(\Xi(t))dt \tag{2.1}$$

where W is the d-dimensional continuous Brownian motion while σ and b are bounded Lipschitz continuous $d \times d$ matrix and d-dimensional vector functions, respectively. Namely, we assume that for some constant $L \geq 1$ and all $x, y \in \mathbb{R}^d$,

$$|\sigma(x)| \leq L,\ |b(x)| \leq L,\ |\sigma(x) - \sigma(y)| \leq L|x-y|,\ |b(x) - b(y)| \leq L|x-y| \tag{2.2}$$

where $|\cdot|$ denotes the Euclidean norm of a vector or of a matrix. We assume also that

$$E\xi(1) = 0,\ E(\xi_i(1)\xi_j(1)) = \delta_{ij} \text{ and } |\xi(1)| \leq L \text{ almost surely (a.s.)} \tag{2.3}$$

where $\xi(n) = (\xi_1(n), \ldots, \xi_d(n))$ and δ_{ij} is the Kronecker delta. Next, we consider the sequence of discrete time processes X_N, $N \geq 1$ on $\mathbb{R}^d$ defined recursively for $n = 0, 1, \ldots, N-1$ by

$$X_N((n+1)/N) = X_N(n/N) + N^{-1/2}\sigma(X_N(n/N))\xi(n+1) + N^{-1}b(X_N(n/N)) \tag{2.4}$$

where $X_N(0) = \Xi(0) = x_0$ is fixed. We extend X_N to the continuous time setting

$$X_N(t) = X_N(n/N) \quad \text{if} \quad n/N \leq t < (n+1)/N \tag{2.5}$$

and without loss of generality we will assume that all our processes evolve on the time interval $[0, 1]$ so that n runs in (2.5) from 0 to N. The following result will be proved in Sections 3 and 4.

Theorem 2.1. *Suppose that the conditions (2.2) and (2.3) hold true and that the probability space $(\Omega, \mathcal{F}, P)$ is rich enough so that there exist a sequence of i.i.d. uniformly distributed random variables defined on it. Then for each integer $N \geq N_0 = ((10^8 d)^{24d} + 1)^4$ there exists a d-dimensional Brownian motion $W = W_N$ such that the strong solution $\Xi = \Xi_N$ of the stochastic differential equation (2.1) with such W and the initial condition $X_N(0) = \Xi(0) = x_0$ satisfies*

$$E \sup_{0 \leq t \leq 1} |X_N(t) - \Xi(t)|^2 \leq C_0 [N^{\frac{1}{4}}]^{-\frac{1}{50d}} \tag{2.6}$$

where $C_0 = C_3 e^{C_4} + 2L^2(L^2+1) + 40L^2$ with C_3 and C_4 defined at the end of Section 4. In particular, the Prokhorov distance between the path distributions of X_N and of Ξ is bounded by $C_0^{1/3}[N^{\frac{1}{4}}]^{-\frac{1}{150d}}$.

We observe that though we may have to redefine the Brownian motion W for each N separately the path distribution of the diffusion Ξ remains the same since it is continuous and the coefficients of the stochastic differential equation (2.1) do not change, and so we have all the time the same Kolmogorov equation and the same martingale problem (see [24]). Clearly, the estimate (2.6) is meaningful only for large N and we provide it for all $N \geq N_0$ though, of course, an explicit estimate in (2.6) can be obtained also when $1 \leq N \leq N_0$ taking into account that then $|X_N(t)| \leq L(\sqrt{N_0}L+1)$ and $E\sup_{0\leq t\leq 1}|\Xi(t)|^2 \leq 5L^2$. Theorem 2.1 can also be derived under some moment boundedness conditions rather than the uniform bounds in (2.2) which are assumed to reduce technicalities in our exposition. Observe also that if we consider time dependent coefficients $\sigma(t,x)$ and $b(t,x)$ Lipschitz continuous in both variables and take $\sigma(n/N, X_N(n/N))$ and $b(n/N, X_N(n/N))$ in (2.4) in place of $\sigma(X_N(n/N))$ and $b(X_N(n/N))$, then we will obtain $X_N(t)$ which approximates a time inhomogeneous diffusion with coefficients $\sigma(t,x)$ and $b(t,x)$ having essentially the same error estimates as in (2.6).

Next, we will describe an application of our results to computations of values of Dynkin's optimal stopping games and fair prices of game options with the payoff function having the form

$$R^\Xi(s,t) = G_s(\Xi)\mathbb{I}_{s<t} + F_t(\Xi)\mathbb{I}_{t\leq s} \tag{2.7}$$

where Ξ is a diffusion solving the stochastic differential equation (2.1). Here, $G_t \geq F_t$ and both are functionals on paths for the time interval $[0,t]$ satisfying certain regularity conditions specified below. Thus, if the first player stops at the time s and the second one at the time t then the former pays to the latter the amount $R^\Xi(s,t)$. The game runs until the termination time 1 when the game stops automatically, if it was not stopped before, and then the first player pays to the second one the amount $G_1(\Xi) = F_1(\Xi)$. Clearly, the first player tries to minimize the payment while the second one tries to maximize it. Under the conditions below this game has the value (see, for instance, Section 6.2.2 in [14]),

$$V^\Xi = \inf_{\sigma\in\mathcal{T}_{01}^\Xi}\sup_{\tau\in\mathcal{T}_{01}^\Xi} ER^\Xi(\sigma,\tau) \tag{2.8}$$

where $\mathcal{T}_{01}^\Xi$ is the set of all stopping times $0 \leq \tau \leq 1$ with respect to the filtration $\mathcal{F}_t^\Xi$, $t \geq 0$ generated by the diffusion Ξ or, which is the same, generated by the Brownian motion W.

When we are talking about asset prices then usually it is assumed that they are nonnegative, and so a diffusion with bounded coefficients maybe not a good model for a description of evolution of these prices. It maybe more appropriate to assume that the asset prices evolve according to the vector process described by exponents $(\exp(\Xi^{(i)}(t)),\ i = 1,\dots,d)$ where $\Xi^{(1)},\dots,\Xi^{(d)}$ are components of the vector Ξ. Nevertheless, it will be more convenient for us to speak about the diffusion Ξ itself and to impose conditions on the payoff functionals F and G such that exponential functionals will be allowed which will amount to the same effect as exponents describing the evolution of asset prices. Another important point that the fair price of a game option equals the value of the corresponding Dynkin optimal stopping game considered with respect to the equivalent martingale measure, i.e. with respect to the probability for which the assets evolution is described by a martingale (see [14]) provided that the interest rate is supposed to be zero. When the asset prices are given by the above exponential formula the probability P itself will be a martingale measure provided $b_i(x) = -\frac{1}{2}\sum_{j=1}^d \sigma_{ij}^2(x)$, $i = 1, \ldots, d$ for each $x \in \mathbb{R}^d$. Otherwise, we have to perform all estimates with respect to a martingale measure Q and, in particular, $\xi(1), \xi(2), \ldots$ in (2.4) should be an i.i.d. sequence satisfying (2.3) with respect to the probability Q. According to the Girsanov theorem (see, for instance, Section 7.4.3 in [14]),

$$\frac{dQ}{dP} = \Lambda \text{ where } \Lambda = \exp\left(-\int_0^1 \langle \zeta(s), dW(s)\rangle - \frac{1}{2}\int_0^1 |\zeta(s)|^2 ds\right)$$

where $\langle \cdot, \cdot \rangle$ is the inner product and the vector process $\zeta(s)$ satisfies

$$\sigma(\Xi(s))\zeta(s) = b(\Xi(s)) + \frac{1}{2}\eta(\Xi(s)) \text{ and } \eta = (\eta_1, \ldots, \eta_d),\ \eta_i(x) = \sum_{i=1}^d \sigma_{ij}^2(x).$$

Since our estimates do not depend explicitly on the probability measure once the setup above is preserved and since there is no one preferable stock evolution model here, we will not discuss this point further, and so, strictly speaking, we will deal with the approximation of the Dynkin game value V^Ξ and not of the fair price of the corresponding game option, i.e. we will make estimates with respect to the probability P and not with respect to an equivalent martingale measure which depends on a choice of the stock evolution model.

We assume that F_t and G_t, $t \in [0,1]$ are continuous functionals on the space $M_d[0,t]$ of bounded Borel measurable maps from $[0,t]$ to $\mathbb{R}^d$ considered with the uniform metric $d_{0t}(\upsilon, \tilde{\upsilon}) = \sup_{0\le s\le t} |\upsilon_s - \tilde{\upsilon}_s|$ and there exists a constant $K > 0$ such that

$$\begin{aligned} &|F_t(\upsilon) - F_t(\tilde{\upsilon})| + |G_t(\upsilon) - G_t(\tilde{\upsilon})| \\ &\quad \le K(d_{0t}(\upsilon, \tilde{\upsilon}) + \mathbb{I}_{\sup_{0\le u\le t}|\upsilon_u - \tilde{\upsilon}_u|>1}) \exp(K \sup_{0\le u\le t}(|\upsilon_u| + |\tilde{\upsilon}_u|)) \end{aligned} \tag{2.9}$$

and

$$|F_t(\upsilon) - F_s(\upsilon)| + |G_t(\upsilon) - G_s(\upsilon)| \le K(|t-s| + \sup_{u\in[s,t]} |\upsilon_u - \upsilon_s|) \exp\left(K \sup_{0\le u\le t} |\upsilon_u|\right). \tag{2.10}$$

Next, we will consider Dynkin's games with payoffs based on the process X_N,

$$R_N(s,t) = G_s(X_N)\mathbb{I}_{s<t} + F_t(X_N)\mathbb{I}_{t\le s}. \tag{2.11}$$

Denote by $\mathcal{F}^\xi_{mn}$, $m \le n$ the σ-algebra generated by $\xi(m), \ldots, \xi(n)$ and let $\mathcal{T}^\xi_{mn}$ be the set of all stopping times with respect to the filtration $\mathcal{F}^\xi_{0k}$, $k \ge 0$ taking on values $m, m+1, \ldots, n$. We allow also any stopping time to take on the value ∞, i.e. we allow players not to stop the game at all, but anyway the game is stopped automatically at the termination time 1 and then the first player pays to the second one the amount $G_1(X_N) = F_1(X_N)$. Now the game value of the Dynkin game in this setup is given by

$$V_N = \inf_{\zeta\in\mathcal{T}^\xi_{0N}} \sup_{\eta\in\mathcal{T}^\xi_{0N}} ER_N(\zeta/N, \eta/N). \tag{2.12}$$

Theorem 2.2. *Suppose that the conditions (2.9) and (2.10) as well as the conditions of Theorem 2.1 hold true. Then for each $\delta > 0$ there exists $C_\delta > 0$ such that for any integer $N \geq N_0$,*

$$|V^\Xi - V_N| \leq C_\delta [N^{\frac{1}{4}}]^{\delta - \frac{1}{100d}} \tag{2.13}$$

where C_δ does not depend on N and for each δ it can be estimated explicitly from the proof in Section 5.

Since we use in Theorem 2.2 a specific construction of the diffusion Ξ from Theorem 2.1 it is important to note that the game value V^Ξ depends only on the path distribution of Ξ, i.e. only on the diffusion coefficients σ and b, and not on a choice of the Brownian motion in the stochastic differential equation (2.1) (see [9]). We observe also that the main advantage in computation V_N in comparison to V^Ξ is the possibility to use the dynamical programming (backward recursion) algorithm. Namely, set $V_{NN} = F_1(X_N)$ and recursively for $n = N-1, \dots, 1, 0$,

$$V_{Nn} = \min\big(G_{n/N}(X_N),\, \max(F_{n/N}(X_N),\, E(V_{N,n+1}|\mathcal{F}^\xi_{0,n}))\big). \tag{2.14}$$

Then $V_{N0} = V_N$ (see, for instance, Section 6.2.2 in [14]). Of course, the computation of conditional expectations above becomes complicated if the σ-algebras $\mathcal{F}^\xi_{0n}$ are big but if we choose independent random vectors $\xi(n)$ in (2.4) taking on only few values then these σ-algebras contain not so many sets and the conditional expectations can be computed easily. Observe also that in the particular case when the diffusion Ξ is just a multidimensional Brownian motion, a result similar to Theorem 2.2 was obtained in [13] where it was sufficient to consider the standard normalized sums of random vectors $\xi(n)$ rather than the more subtle case of difference equations (2.4).

3. Auxiliary Estimates

Set $n_k = k[N^{\frac{1}{4}}]$, $k = 0, 1, \dots, k_N$ where $k_N = [N/[N^{\frac{1}{4}}]]$ where $[\cdot]$ denotes the integral part. Define

$$\begin{aligned}\hat{X}_N(t) = x_0 + N^{-1/2} \textstyle\sum_{0\leq k\leq k_N(t)} \big(\sigma(X_N(\frac{n_k}{N})) \sum_{n_k<l\leq n_{k+1}\wedge[Nt]} \xi(l) \\ +N^{-1/2} b(X_N(\frac{n_k}{N}))(n_{k+1}\wedge[Nt] - n_k)\big)\end{aligned} \tag{3.1}$$

where $k_N(t) = \max\{k :\, n_k \leq Nt\}$.

Lemma 3.1. *For any $N \geq 1$,*

$$E \sup_{0\leq t\leq 1} |X_N(t) - \hat{X}_N(t)|^2 \leq 136 L^8 N^{-1/2}. \tag{3.2}$$

Proof. First, we write

$$|X_N(t) - \hat{X}_N(t)|^2 \leq 2|M(t)|^2 + 2|J(t)|^2 \tag{3.3}$$

where

$$M(t) = N^{-1/2} \sum_{0\leq k\leq k_N(t)} \sum_{n_k<l\leq n_{k+1}\wedge[Nt]} \left(\sigma\left(X_N\left(\frac{l}{N}\right)\right) - \sigma\left(X_N\left(\frac{n_k}{N}\right)\right)\right)\xi(l+1)$$

and

$$J(t)=N^{-1}\sum_{0\le k\le k_N(t)}\sum_{n_k<l\le n_{k+1}\wedge[Nt]}\left(b\left(X_N\left(\frac{l}{N}\right)\right)-b\left(X_N\left(\frac{n_k}{N}\right)\right)\right).$$

Recall that if $h=h(x,y)$ is a bounded Borel function, $\mathcal{G}\subset\mathcal{F}$ is a σ-algebra and Y,Z are random variables such that Y is $\mathcal{G}$-measurable and Z is independent of $\mathcal{G}$, then $E(h(Y,Z)|\mathcal{G})=g(Y)$ where $g(x)=Eh(x,Z)$. It follows from here and from (2.3) that $M(t)$, $0\le t\le 1$ is a martingale. Hence, by (2.3) and the Doob martingale inequality (see, for instance, Section 6.1.2 in [14]),

$$E\sup_{0\le t\le 1}|M(t)|^2\le 4E|M(1)|^2 \tag{3.4}$$

$$=4N^{-1}\sum_{0\le k\le k_N(t)}\sum_{n_k<l\le n_{k+1}\wedge[Nt]}E\left|\left(\sigma\left(X_N\left(\frac{l}{N}\right)\right)-\sigma\left(X_N\left(\frac{n_k}{N}\right)\right)\right)\xi(l+1)\right|^2.$$

By (2.2)–(2.4) for $n_k<l\le n_{k+1}$,

$$E\left|\left(\sigma\left(X_N\left(\frac{l}{N}\right)\right)-\sigma\left(X_N\left(\frac{n_k}{N}\right)\right)\right)\xi(l+1)\right|^2\le L^4E\left|X_N\left(\frac{l}{N}\right)-X_N\left(\frac{n_k}{N}\right)\right|^2 \tag{3.5}$$

$$\le 2L^4\left(N^{-1}E\left|\sum_{n_k\le m<l}\sigma(X_N(m/N))\xi(m+1)\right|^2\right.$$

$$\left.+N^{-2}E\left|\sum_{n_k\le m<l}b(X_N(m/N))\xi(m+1)\right|^2\right)\le 16L^8N^{-1/2}.$$

Now, by (2.2) and (2.3),

$$E\sup_{0\le t\le 1}|J(t)|^2\le\sum_{0\le k\le k_N}\sum_{n_k<l\le n_{k+1}\wedge N}E\left|X_N\left(\frac{l}{N}\right)-X_N\left(\frac{n_k}{N}\right)\right|^2$$

and for $n_k<l\le n_{k+1}$,

$$\left|X_N\left(\frac{l}{N}\right)-X_N\left(\frac{n_k}{N}\right)\right|\le N^{-1/2}(L^2+N^{-1/2}L)[N^{\frac14}]\le 2L^2N^{-\frac14}.$$

These together with (3.3)–(3.5) yield (3.2). □

Next, we estimate the characteristic function of a sum of independent random vectors which is well known but for completeness and in order to provide explicit constants we provide the details.

Lemma 3.2. *For any integer $n\ge 1$ and $x\in\mathbb{R}^d$,*

$$|f_n(x,w)-\exp(-\frac{1}{2}\langle A(x)w,w\rangle)|\le C_1n^{-\wp} \tag{3.6}$$

for all $w\in\mathbb{R}^d$ *with* $|w|\le n^{\wp/2}$ *where* $A(x)=\sigma(x)\sigma^*(x)$,

$$f_n(x,w)=E\exp(i\langle w,\, n^{-1/2}\sigma(x)\sum_{0<l\le n}\xi(l)\rangle),$$

$\wp=1/6$ *and* $C_1=\frac{3}{2}L^6$.

Proof. Set $m_j=j[\sqrt{n}]$, $j=0,1,\dots,m(n)$, $m(n)=\max\{j:\, j[\sqrt{n}]\le n\}$, $y_j=\sigma(x)\sum_{m_j<l\le m_{j+1}\wedge n}\xi(l)$ and $\eta_j=\langle w,\, n^{-1/2}y_j\rangle$. Now we have

$$|f_n(x,w)-\exp(-\frac{1}{2}\langle A(x)w,w\rangle)|\le I_1+I_2 \tag{3.7}$$

where

$$I_1=\left|E\exp\left(i\sum_{0\le j\le m(n)}\eta_j\right)-\prod_{0\le j\le m(n)}Ee^{i\eta_j}\right|=0, \tag{3.8}$$

since η_j, $j=1,\dots,m(n)+1$ are independent random variables, and

$$\begin{aligned}I_2=&\left|\prod_{0\le j\le m(n)}Ee^{i\eta_j}-\exp\left(-\frac{1}{2}\langle A(x)w,w\rangle\right)\right| \\ \le&\sum_{0\le j\le m(n)}\left|Ee^{i\eta_j}-\exp\left(-\frac{(m_{j+1}\wedge n-m_j)}{2n}\langle A(x)w,w\rangle\right)\right|,\end{aligned} \tag{3.9}$$

where we use that

$$\left|\prod_{1\le j\le l}a_j-\prod_{1\le j\le l}b_j\right|\le\sum_{1\le j\le l}|a_j-b_j|$$

whenever $0\le|a_j|,|b_j|\le 1$, $j=1,\dots,l$.

Using (2.3) and the inequalities

$$\left|e^{ia}-1-ia+\frac{a^2}{2}\right|\le|a|^3 \text{ and } |e^{-a}-1+a|\le a^2 \text{ if } a\ge 0,$$

we obtain that

$$\begin{aligned}&\left|Ee^{i\eta_k}-\exp\left(-\frac{(m_{j+1}\wedge n-m_j)}{2n}\langle A(x)w,w\rangle\right)\right| \\ &\le\frac{1}{2}\left|E\eta_j^2-\frac{(m_{j+1}\wedge n-m_j)}{n}\langle A(x)w,w\rangle\right|+E|\eta_j|^3+\frac{1}{4n}|\langle A(x)w,w\rangle|^2.\end{aligned} \tag{3.10}$$

Now, by (2.3) and the independency of $\xi(l)$'s,

$$E\eta_j^2=n^{-1}\sum_{m_j<l\le m_{j+1}\wedge n}E\langle w,\sigma(x)\xi(l)\rangle^2=n^{-1}(m_{j+1}\wedge n-m_j)\langle A(x)w,w\rangle. \tag{3.11}$$

Hence,

$$I_2\le(\sqrt{n}+1)(n^{-3/2}L^6|w|^6+\frac{1}{4}n^{-1}L^4|w|^4)$$

and (3.6) follows. □

Set $Y_{N,k}(x) = \sigma(x)\sum_{n_k<l\le n_{k+1}}\xi(l)$ for $k = 0,1,\dots,k_N-1$ and $Y_{N,k_N}(x) = \sigma(x)\sum_{n_{k_N}<l\le N}\xi(l)$. As a corollary of Lemma 3.2 we obtain:

Lemma 3.3. *For any integer $N \ge 1$ and $k = 0,1,\dots,k_N-1$,*

$$\left|E\left(\exp\left(i\left\langle w,\,(n_{k+1}-n_k)^{-1/2}Y_{N,k}\left(X_N\left(\frac{n_k}{N}\right)\right)\right\rangle\right)\,\middle|\,\mathcal{F}^{\xi}_{0n_k}\right)-g_{X_N(n_k/N)}(w)\right| \tag{3.12}$$
$$\le C_1(n_{k+1}-n_k)^{-\wp}$$

for all $w\in\mathbb{R}^d$ with $|w|\le(n_{k+1}-n_k)^{\wp/2}$, where $g_x(w)=\exp(-\frac{1}{2}\langle A(x)w,w\rangle)$ and, recall, $\mathcal{F}^{\xi}_{0n}=\sigma\{\xi(1),\dots,\xi(n)\}$.

Proof. Since $X_N(\frac{n_k}{N})$ is $\mathcal{F}^{\xi}_{0n_k}$-measurable and $\sum_{n_k<j\le n_{k+1}}\xi(l)$ is independent of $\mathcal{F}^{\xi}_{0n_k}$, it follows that

$$E\left(\exp\left(i\left\langle w,\,(n_{k+1}-n_k)^{-1/2}Y_{N,k}\left(X_N\left(\frac{n_k}{N}\right)\right)\right\rangle\right)\,\middle|\,\mathcal{F}^{\xi}_{0n_k}\right)$$
$$= f_{n_{k+1}-n_k}\left(X_N\left(\frac{n_k}{N}\right),w\right),$$

where $f_n(x,w)$ was defined in Lemma 3.2, and so (3.12) follows from (3.6). □

4. Strong Approximation

The strong approximations here will be based on the following result which is a slight variation of Theorem 3 and Remark 2.6 from [22] with the additional feature from Theorem 4.6 of [8] that we enrich the probability space by a sequence of i.i.d. uniformly distributed random variables and not just by one such random variable and this result follows by essentially the same proofs as in the cited above papers.

Theorem 4.1. *Let $\{V_m,\,m\ge1\}$ be a sequence of random vectors with values in $\mathbb{R}^d$ defined on some probability space $(\Omega,\mathcal{F},P)$ and such that V_m is measurable with respect to $\mathcal{F}_m$, $m=1,2,\dots$ where $\mathcal{F}_m$, $m\ge1$ is a filtration of sub-σ-algebras of $\mathcal{F}$. Let $\mathcal{G}_m$ and $\mathcal{H}_m$, $m=0,1,\dots$ be two increasing sequences of countably generated sub-σ-algebras of $\mathcal{F}$ such that $\mathcal{H}_m\subset\mathcal{G}_m\subset\mathcal{F}_m$ for each $m\ge1$. Assume that the probability space is rich enough so that there exists on it a sequence of uniformly distributed on $[0,1]$ independent random variables U_m, $m\ge1$ independent of $\vee_{m\ge0}\mathcal{G}_m$. For each $m\ge1$, let $G_m(\cdot|\mathcal{H}_{m-1})$ be a regular conditional distribution on $\mathbb{R}^d$, measurable with respect to $\mathcal{H}_{m-1}$ and with the conditional characteristic function*

$$g_m(w|\mathcal{H}_{m-1})=\int_{\mathbb{R}^d}\exp(i\langle w,x\rangle)G_m(dx|\mathcal{H}_{m-1}),\ w\in\mathbb{R}^d.$$

Suppose that for some non-negative numbers ν_m,δ_m and $K_m\ge10^8d$,

$$\int_{|w|\le K_m}E\big|E(\exp(\langle w,V_m\rangle)|\mathcal{G}_{m-1})-g_m(w|\mathcal{H}_{m-1})\big|dw\le\nu_m(2K_m)^d \tag{4.1}$$

and that

$$E\big(G_m(\{x:\,|x|\ge\frac{1}{2}K_m\}|\mathcal{H}_{m-1})\big)<\delta_m. \tag{4.2}$$

Then there exists a sequence $\{W_m,\, m\geq 1\}$ *of* $\mathbb{R}^d$*-valued random vectors defined on* $(\Omega,\mathcal{F},P)$ *with the properties*

(i) W_m *is* $\mathcal{G}_m\vee\sigma\{U_m\}$*-measurable for each* $m\geq 1$*;*

(ii) $G_m(\cdot|\mathcal{H}_{m-1})$ *is conditional distribution of* W_m *given* $\sigma\{U_1,\dots,U_{m-1}\}\vee\mathcal{G}_{m-1}$*, in particular,* W_m *is conditionally independent of* $\sigma\{U_1,\dots,U_{m-1}\}\vee\mathcal{G}_{m-1}$ *(and so also of* $W_1,\dots,W_{m-1}$*) given* $\mathcal{H}_{m-1}$*,* $m\geq 1$*;*

(iii) Let $\varrho_m=16K_m^{-1}\log K_m+2\nu_m^{1/2}K_m^d+2\delta_m^{1/2}$*. Then*

$$P\{|V_m-W_m|\geq\varrho_m\}\leq\varrho_m \tag{4.3}$$

and, in particular, the Prokhorov distance between the distributions $\mathcal{L}(V_m)$ *and* $\mathcal{L}(W_m)$ *of* V_m *and* W_m*, respectively, does not exceed* ϱ_m*.*

Now, in the notations of Theorem 4.1 we set

$$V_k=(n_k-n_{k-1})^{-1/2}Y_{N,k-1}(X_N(\frac{n_{k-1}}{N})),$$

$\mathcal{F}_k=\mathcal{G}_k=\mathcal{F}^\xi_{0n_k}$, $\mathcal{H}_k=\sigma\{X_N(\frac{n_k}{N})\}$ and $g_k(w|\mathcal{H}_{k-1})=g_{X_N(\frac{n_{k-1}}{N})}(w)$ where g_x was defined in Lemma 3.3. Thus, $G_k(\cdot|\mathcal{H}_{k-1})=G_{X_N(\frac{n_{k-1}}{N})}(\cdot)$ where G_x is the mean zero d-dimensional Gaussian distribution with the covariance matrix $A(x)$ and the characteristic function g_x. By Lemma 3.3,

$$\begin{aligned}&\int_{|w|\leq K_k}E\big|E\big(\exp(i\langle w,V_k\rangle)|\mathcal{G}_{k-1}\big)-g_k(w|\mathcal{H}_{k-1})\big|dw\\&\leq C_1(n_k-n_{k-1})^{-\wp}(2K_k)^d\leq 2^dC_1[N^{\frac14}]^{-1/8}\end{aligned} \tag{4.4}$$

where we take $K_k=[N^{\frac14}]^{\frac{1}{24d}}<(n_k-n_{k-1})^{\wp/2}$. Next, for each $x\in\mathbb{R}^d$ let Θ_x be a mean zero Gaussian random variable with the covariance matrix $A(x)$. Then by (2.2) and the Chebyshev inequality,

$$\begin{aligned}&E\left(G_k\left(\left\{y\in\mathbb{R}^d:\,|y|\geq\frac12K_k\right\}\,\Big|\,\mathcal{H}_{k-1}\right)\right)\\&\leq\sup_{y\in\mathbb{R}^d}P\left\{|\Theta_y|\geq\frac12[N^{\frac14}]^{\frac{1}{24d}}\right\}\leq 4L^2d[N^{\frac14}]^{-\frac{1}{12d}}.\end{aligned} \tag{4.5}$$

In order to use Theorem 4.1 we need that $K_k\geq 10^8d$ and this will hold true if $N\geq N_0=((10^8d)^{24d}+1)^4$ which is the assumption of Theorem 2.1. Now, Theorem 4.1 provides us with random vectors $\{W_k,\,k\geq 1\}$ satisfying the properties (i)–(iii), in particular, given $X_N(\frac{n_{k-1}}{N})$, the random vector W_k has the mean zero Gaussian distribution with the covariance matrix $A(X_N(\frac{n_{k-1}}{N}))$ and it is conditionally independent of $\mathcal{G}_{k-1}$ and of $W_1,\dots,W_{k-1}$ while in view of (4.4) and (4.5) the property (iii) holds true with

$$\begin{aligned}\varrho_k=&\frac{2}{3d}[N^{\frac14}]^{-\frac{1}{24d}}\log([N^{\frac14}])+2\sqrt{C_1}[N^{\frac14}]^{-\frac{1}{24}}\\&+4L\sqrt{d}[N^{\frac14}]^{-\frac{1}{24d}}\leq[N^{\frac14}]^{-\frac{1}{24d}}(\log N+2\sqrt{C_1}+4L\sqrt{d}).\end{aligned} \tag{4.6}$$

Next, we obtain the uniform L^2-bound for the difference between the sums of $(n_k - n_{k-1})^{1/2} V_k$'s and of $(n_k - n_{k-1})^{1/2} W_k$'s. Set

$$I(t) = \sum_{0 \le k \le k_N(t)} (n_k - n_{k-1})^{1/2} (V_k - W_k).$$

Lemma 4.2. *For any integer $N \ge N_0$,*

$$E \max_{0 \le t \le 1} |I(t)|^2 \le C_2 N [N^{\frac{1}{4}}]^{-\frac{1}{50d}} \tag{4.7}$$

where

$$C_2 = \sup_{N \ge 1} \big([N^{\frac{1}{4}}]^{-\frac{1}{480d}} \sqrt{\log N} \big)(1 + 4L^2(L^2 + d) + 2L^2 d)(1 + \sqrt{2\sqrt{C_1}} + 2\sqrt{L\sqrt{d}}.$$

Proof. Set

$$M_k = \sum_{0 \le l \le k} (n_l - n_{l-1})^{1/2} (V_l - W_l).$$

Then

$$\max_{0 \le t \le 1} |I(t)|^2 = \max_{1 \le k \le k_N} |M_k|^2$$

and by the properties (i) and (ii) of Theorem 4.1 together with the conditional independence of each $V_l - W_l$ of $\mathcal{F}_{l-1} \vee \sigma\{U_1, \ldots, U_{l-1}\}$ given $X_N(\frac{n_{l-1}}{N})$, it is easy to see that M_k, $k = 1, 2, \ldots, k_N$ is a martingale with respect to the filtration $\mathcal{F}_k \vee \sigma\{U_1, \ldots, U_k\}$, $k = 1, \ldots, k_N$. Hence, by the Doob martingale inequality

$$E \max_{1 \le k \le k_N} |M_k|^2 \le 4E|M_{k_N}|^2 = 4[N^{\frac{1}{4}}] \sum_{1 \le k \le k_N} E|V_k - W_k|^2 \tag{4.8}$$

where we use also that $(V_k - W_k)$, $k = 1, \ldots, k_N$ are uncorrelated for different k's.

Next, by the Cauchy-Schwarz inequality

$$\begin{aligned} E|V_k - W_k|^2 &= E(|V_k - W_k|^2 \mathbb{I}_{|V_k - W_k| \le \varrho_k}) \\ &+ E(|V_k - W_k|^2 \mathbb{I}_{|V_k - W_k| > \varrho_k}) \\ &\le \varrho_k^2 + (E|V_k - W_k|^4)^{1/2} (P\{|V_k - W_k| > \varrho_k\})^{\frac{1}{2}} \\ &\le \varrho_k^2 + 4\varrho_k^{\frac{1}{2}} ((E|V_k|^4)^{1/2} + (E|W_k|^4)^{1/2}). \end{aligned} \tag{4.9}$$

Now, by (2.2) and (2.3),

$$\begin{aligned} E|V|^4 &\le [N^{\frac{1}{4}}]^{-2} L^4 E| \sum_{n_{k-1} < l \le n_k} \xi(l)|^4 \\ &\le [N^{\frac{1}{4}}]^{-2} L^4 \big([N^{\frac{1}{4}}] E|\xi(1)|^4 + [N^{\frac{1}{4}}]^2 (E|\xi(1)|)^2 \big) \le L^4 (L^2 + d^2). \end{aligned} \tag{4.10}$$

Since W_k is distributed as $\sigma(X_N(\frac{n_{k-1}}{N}))\mathcal{N}$, where $\mathcal{N}$ is the d-dimensional Gaussian random vector with the identity covariance matrix, we obtain that

$$E|W_k|^4 \le 3L^4 d^2. \tag{4.11}$$

Finally, (4.7) follows from (4.8)–(4.11). □

Next, let $W(t)$, $t\geq 0$ be a d-dimensional Brownian motion such that the increments $W(n_k)-W(n_{k-1})$ are independent of $X_N(\frac{n_{k-1}}{N})$ for any $k=1,\dots,k_N$. Then, given $X_N(\frac{n_{k-1}}{N})$, the sequences of random vectors

$$\tilde{W}_k=\sigma\left(X_N\left(\frac{n_{k-1}}{N}\right)\right)(W(n_k)-W(n_{k-1}))$$

and $(n_k-n_{k-1})^{1/2}W_k$, $k=1,\dots,k_N$ have the same distributions. Moreover, we can redefine the process $\xi(n)$, $1\leq n<\infty$ and choose a Brownian motion $W(s)$, $s\geq 0$ preserving their distributions so that the joint distribution of the sequences of pairs (V_k,W_k) and of $(V_k,\tilde{W}_k)$ will be the same and, in particular, that (4.7) will hold true with $\tilde{W}_k$ in place of W_k. Indeed, by the Kolmogorov extension theorem (see, for instance, [24]) such pair of processes exists if we impose consistent restrictions on their joint finite dimensional distributions. But since the pair of processes ξ and W_k, $1\leq k\leq k_N$ satisfying our conditions exist by Theorem 4.1 and Lemma 4.2, these restrictions are consistent and the required pair of processes exists. From now on we will drop tilde and denote $\sigma(X_N(\frac{n_{k-1}}{N}))(W(n_k)-W(n_{k-1}))$ by W_k which is supposed to satisfy (4.7).

Now, using the Brownian motion $W(t)$, $t\geq 0$ constructed above we consider the new Brownian motion $W_N(t)=N^{-1/2}W(tN)$, $0\leq t\leq 1$ and introduce the diffusion process $\Xi_N(t)$, $t\geq 0$ solving the stochastic differential equation (2.1) which we write now with W_N,

$$d\Xi_N(t)=\sigma(\Xi_N(t))dW_N(t)+b(\Xi_N(t))dt,\ \Xi_N(0)=x_0.$$

Now, we introduce the auxiliary process $\hat{\Xi}_N$ with coefficients frozen at times n_k,

$$\begin{aligned}\hat{\Xi}_N(t)=&x_0+\sum_{1\leq k\leq k_N(tN)}\left(\sigma\left(\Xi_N\left(\frac{n_{k-1}}{N}\right)\right)\left(W_N\left(\frac{n_k}{N}\right)-W_N\left(\frac{n_{k-1}}{N}\right)\right)\right.\\&\left.+N^{-1}b\left(\Xi_N\left(\frac{n_{k-1}}{N}\right)\right)(n_k-n_{k-1})\right).\end{aligned}$$

Lemma 4.3. *For any integer $N\geq 1$,*

$$E\max_{0\leq k\leq k_N}|\Xi_N(n_k/N)-\hat{\Xi}_N(n_k/N)|^2\leq 32\Delta(N)\tag{4.12}$$

where $\Delta(N)=N^{-1}[N^{\frac{1}{4}}]$.

Proof. First, we write

$$\begin{aligned}&E\max_{0\leq k\leq k_N}|\Xi_N(n_k/N)-\hat{\Xi}_N(n_k/N)|^2\\&\leq 2(E\max_{0\leq k\leq k_N}|J_1(n_k/N)|^2+E\max_{0\leq k\leq k_N}|J_2(n_k/N)|^2)\end{aligned}\tag{4.13}$$

where

$$J_1(t)=\int_0^t\big(\sigma(\Xi_N(s))-\sigma(\Xi_N([s/\Delta(N)]\Delta(N)))\big)dW_N(s)$$

and

$$J_2(t)=\int_0^t\big(b(\Xi_N(s))-b(\Xi_N([s/\Delta(N)]\Delta(N)))\big)ds.$$

By the Doob martingale inequality and the Itô isometry for stochastic integrals (see, for instance, [14], Sections 6.1.2 and 7.2.1),

$$E \max_{0\le k\le k_N)} |J_1(n_k/N)|^2 \tag{4.14}$$
$$\le 4\int_0^{[T/\Delta(N)]\Delta(N)} E|\sigma(\Xi_N(s)) - \sigma(\Xi_N([s/\Delta(N)]\Delta(N)))|^2 ds$$
$$\le 4L^2 \sum_{1\le k\le k_N} \int_{n_{k-1}/N}^{n_k/N} E|\Xi_N(s) - \Xi_N(n_{k-1}/N)|^2 ds.$$

By (2.2) and the Cauchy–Schwarz inequality,

$$E \max_{0\le k\le k_N} |J_2(n_k/N)|^2 \le L^2 \int_0^1 |\Xi_N(s) - \Xi_N([s/\Delta(N)]\Delta(N))|^2 ds. \tag{4.15}$$

Again, by (2.2), (2.3) and the moment inequalities for stochastic integrals

$$E|\Xi_N(s) - \Xi_N(n_{k-1}/N)|^2 \tag{4.16}$$
$$\le 2\left(E\left|\int_{n_{k-1}/N}^{s} \sigma(\Xi_N(u))dW_N(u)\right|^2 + L^2(s-n_{k-1}/N)^2\right)$$
$$\le 2L^2\Delta(N)(1+\Delta(N)) \le 4L^2\Delta(N)$$

since $s \in [n_{k-1}/N, n_k/N]$ here, and so $s - n_{k-1}/N \le \Delta(N)$. Now, (4.12) follows from (4.13)–(4.16). □

Next, we introduce another auxiliary process Ξ_N^X defined by

$$\Xi_N^X(t) = x_0 + \sum_{1\le k\le k_N(t)} \big(\sigma(X_N(\frac{n_{k-1}}{N}))(W_N(\frac{n_k}{N}) - W_N(\frac{n_{k-1}}{N}))$$
$$+ N^{-1} b(X_N(\frac{n_{k-1}}{N}))(n_k - n_{k-1})\big).$$

Then we can write

$$E \sup_{0\le s\le 1} |\hat{X}_N(s) - \hat{\Xi}_N(s)|^2 = E \max_{0\le k<k_N(TN)} |\hat{X}_N(n_k/N) \tag{4.17}$$
$$- \hat{\Xi}_N(n_k/N)|^2 \le 2(E \max_{0\le k<k_N(TN)} |\hat{X}_N(n_k/N) - \Xi_N^X(n_k/N)|^2$$
$$+ E \max_{0\le k<k_N(T)} |\Xi_N^X(n_k/N) - \hat{\Xi}_N(n_k/N)|^2).$$

By Lemma 4.2,

$$E \max_{0\le k\le n} |\hat{X}_N(n_k/N) - \Xi_N^X(n_k/N)|^2 \tag{4.18}$$
$$= E \max_{0\le k\le n} \left|\sum_{0\le l\le k} \sigma\left(X_N\left(\frac{n_l}{N}\right)\right)\left(N^{-\frac{1}{2}} \sum_{n_l<m\le n_{l+1}} \xi(m) - \left(W_N\left(\frac{n_{l+1}}{N}\right) - W_N\left(\frac{n_l}{N}\right)\right)\right)\right|^2$$
$$\le N^{-1} E \sup_{0\le t\le 1} |I(t)|^2 \le C_2[N^{\frac{1}{4}}]^{-\frac{1}{50d}}.$$

In order to estimate the second term in the right hand side of (4.17) introduce the σ-algebras $\mathcal{Q}_n=\mathcal{F}_{0n}\vee\sigma\{W(u),\,0\le u\le n\}$ and observe that by our construction for each k the increment $W(n_{k+1})-W(n_k)$ is independent of $\mathcal{Q}_{n_k}$. On the other hand, for any $k\ge n$ both $X_N(n_k/N)$ and $\Xi_N(n_k/N)$ are $\mathcal{Q}_{n_k}$-measurable. Hence,

$$\mathcal{I}_1(n_k)=\sum_{0\le l\le k-1}\left(\sigma\left(X_N\left(\frac{n_l}{N}\right)\right)-\sigma\left(\Xi_N\left(\frac{n_l}{N}\right)\right)\right)\left(W_N\left(\frac{n_{l+1}}{N}\right)-W_N\left(\frac{n_l}{N}\right)\right)$$

is a martingale in k with respect to the filtration $\mathcal{Q}_k, k=1,2,\dots,k_N-1$. Thus, by (2.2) and the Doob martingale inequality,

$$\begin{aligned}&E\max_{1\le k\le m}|\mathcal{I}_1(n_k)|^2\le 4E|\mathcal{I}_1(n_m)|^2 \qquad (4.19)\\ &\le 4\sum_{0\le l\le m-1}E|(\sigma(X_N(\frac{n_l}{N}))-\sigma(\Xi_N(\frac{n_l}{N})))(W_N(\frac{n_{l+1}}{N})-W_N(\frac{n_l}{N}))|^2\\ &\le 4dL^2\Delta(N)\sum_{0\le l<m}E|X_N(\frac{n_l}{N})-\Xi_N(\frac{n_l}{N})|^2.\end{aligned}$$

Next, observe that

$$\begin{aligned}&\max_{0\le k\le k_N}|\Xi_N(n_k/N)-\hat{\Xi}_N(n_k/n)|^2 \qquad (4.20)\\ &\le 2(E\max_{0\le k\le k_N}|\mathcal{I}_1(n_k)|^2+E\max_{0\le k\le k_N}|\mathcal{I}_2(n_k)|^2),\end{aligned}$$

where

$$\mathcal{I}_2(n_k)=N^{-1}\textstyle\sum_{0\le l\le k-1}\big(b(X_N(n_l))-b(\Xi_N(n_l))\big)(n_{l+1}-n_l).$$

By (2.2) we have

$$\begin{aligned}|\mathcal{I}_2(n_k)|^2&\le L^2(\Delta(N))^2\left(\sum_{0\le l<k}\left|X_N\left(\frac{n_l}{N}\right)-\Xi_N\left(\frac{n_l}{N}\right)\right|\right)^2 \qquad (4.21)\\ &\le L^2(\Delta(N))^2k\sum_{0\le l<k}\left|X_N\left(\frac{n_l}{N}\right)-\Xi_N\left(\frac{n_l}{N}\right)\right|^2\\ &\le L^2\Delta(N)\sum_{0\le l<k}\left|X_N\left(\frac{n_l}{N}\right)-\Xi_N\left(\frac{n_l}{N}\right)\right|^2.\end{aligned}$$

Now, denote

$$Q_k=E\max_{0\le l\le k}|X_N(n_l/N)-\Xi_N(n_l/N)|^2.$$

Then we obtain from (3.2), (4.12) and (4.17)–(4.21) that for $n\le k_N$,

$$Q_n\le C_3[N^{\frac14}]^{-\frac{1}{50d}}+C_4\Delta(N)\sum_{0\le k\le n-1}Q_k \qquad (4.22)$$

where $C_3=408L^8+6C_2+96$ and $C_4=L^2(16d+4)$. By the discrete (time) Gronwall inequality (see, for instance, [6]),

$$Q_{k_N}\le C_3[N^{\frac14}]^{-\frac{1}{50d}}\exp(C_4). \qquad (4.23)$$

It remains to estimate deviations of our continuous time processes within intervals of time $(n_k/N, n_{k+1}/N)$ which where not taken into account in previous estimates, i.e. we have to deal now with

$$\mathcal{J}_1 = E\sup_{0\le t\le 1}|X_N(t) - X_N(n_{k_N(tN)})|^2$$
$$\text{and } \mathcal{J}_2 = E\sup_{0\le t\le 1}|\Xi_N(t) - \Xi_N(n_{k_N(tN)})|^2.$$

By the straightforward estimates using (2.2) and (2.4) we obtain

$$\mathcal{J}_1 \le 2\Delta(N)L^2(L^2+1) \tag{4.24}$$

and

$$\mathcal{J}_2 \le 4(\mathcal{J}_3 + (2L)^2(\Delta(N))^2) \tag{4.25}$$

where

$$\mathcal{J}_3 = E\max_{0\le k\le k_N}\sup_{0\le s\le\Delta(N)}\left|\int_{n_k/N}^{N^{-1}n_k+s}\sigma(\Xi_N(u))dW_N(u)\right|^2.$$

By the Jensen (or Cauchy-Schwarz) inequality and the uniform moment estimates for stochastic integrals

$$\begin{aligned}\mathcal{J}_3 &\le \left(E\max_{0\le k\le k_N}\sup_{0\le s\le\Delta(N)}\left|\int_{n_k/N}^{N^{-1}n_k+s}\sigma(\Xi_N(u))dW_N(u)\right|^4\right)^{1/2} \\ &\le \left(\sum_{0\le k\le k_N}E\sup_{0\le s\le\Delta(N)}\left|\int_{n_k/N}^{N^{-1}n_k+s}\sigma(\Xi_N(u))dW_N(u)\right|^4\right)^{1/2} \\ &\le (\frac{4}{3})^2\left(\sum_{0\le k\le k_N}E\left|\int_{n_k/N}^{N^{-1}n_{k+1}}\sigma(\Xi_N(u))dW_N(u)\right|^4\right)^{1/2} \\ &\le 6L^2(\Delta(N))^{1/2}.\end{aligned} \tag{4.26}$$

Combining (4.23)–(4.26) we complete the proof of Theorem 2.1. □

5. Dynkin Games

In view of the form of our regularity conditions (2.9) and (2.10) on the payoff functionals F and G we will need the following exponential estimates.

Lemma 5.1. *(i) For any $M > 0$ and an integer $N \ge 1$,*

$$\max_{0\le n\le N}E\exp(M|X_N(n/N)|) \le D_M^X e^{M|x|} \tag{5.1}$$

and

$$\max_{0\le n\le N}E\exp(M|\hat{X}_N(n/N)|) \le D_M^X e^{M|x|} \tag{5.2}$$

where $x = X_N(0) = \hat{X}_N(0)$ and $D_M^X = 2d\exp(\frac{1}{2}d^4L^4 + L + \frac{1}{6}M^3d^6L^6e^{Md^2L^2})$ does not depend on N;

(ii) For any $\delta, M > 0$ and an integer $N \ge 1$,

$$E\exp\left(M\max_{0\le n\le N}|X_N(n/N)|\right) \le D_M^X e^{M|x|}N^\delta \tag{5.3}$$

and

$$\max_{0\le n\le N} E\exp(M|\hat X_N(n/N)|)\le D^X_{M,\delta}e^{M|x|}N^\delta \tag{5.4}$$

where $D^X_{M,\delta}=1+(D^X_{2M/\delta}D^X_{2M})^{1/2}$ *also does not depend on* N*;*

(iii) For any $M>0$*,*

$$E\exp(M\sup_{0\le t\le 1}|\Xi(t)|)\le D^\Xi_M e^{M|x|}\text{ and }E\exp(M\sup_{0\le t\le 1}|\hat\Xi_N(t)|)\le D^\Xi_M e^{M|x|} \tag{5.5}$$

where $x=\Xi(0)$ *and* $D^\Xi_M=2\exp(L+\frac12 ML^2d^2)$.

Proof. (i) Writing

$$X_N(n/N)=x+\sum_{k=0}^{n-1}\big(N^{-1/2}\sigma(X_N(k/N))\xi(k+1)+N^{-1}b(X_N(k/N))\big)$$

we obtain

$$\begin{aligned}&E\exp(M|X_N(n/N)|) \\ &\le e^{M(|x|+L)}E\exp\left(MN^{-1/2}\left|\sum_{k=0}^{n-1}\sigma(X_N(k/N))\xi(k+1)\right|\right)\\ &\le E\max_{1\le i\le d}\exp\left(MdN^{-1/2}\left|\sum_{k=0}^{n-1}\sum_{j=1}^d\sigma_{ij}(X_N(k/N))\xi_j(k+1)\right|\right)\\ &\le\sum_{1\le i\le d}\left(E\exp\left(MdN^{-1/2}\sum_{k=0}^{n-1}\sum_{j=1}^d\sigma_{ij}(X_N(k/N))\xi_j(k+1)\right)\right.\\ &\left.+E\exp\left(-MdN^{-1/2}\sum_{k=0}^{n-1}\sum_{j=1}^d\sigma_{ij}(X_N(k/N))\xi_j(k+1)\right)\right).\end{aligned} \tag{5.6}$$

To shorten a bit notations we set for this proof $g(x)=(g_1(x),\dots,g_d(x))$ where $g_j(x)=\pm Md\sigma_{ij}(x)$. Then we have to estimate

$$\begin{aligned}&E\exp\big(N^{-1/2}\sum_{k=0}^{n-1}\langle g(X_N(k/N)),\,\xi(k+1)\rangle\big)\\ &=E\left(\exp\left(N^{-1/2}\sum_{k=0}^{n-2}\langle g(X_N(k/N)),\,\xi(k+1)\rangle\right)\right.\\ &\left.\times E\Big(\exp(N^{-1/2}\langle g(X_N(n-1/N)),\,\xi(n)\rangle)|\mathcal F^\xi_{0,n-1}\Big)\right).\end{aligned} \tag{5.7}$$

Since $|g_j(x)|\le MdL$, $j=1,\dots,d$, it follows that

$$\begin{aligned}&|\exp(N^{-1/2}\langle g(X_N(n-1/N)),\,\xi(n)\rangle)-1-N^{-1/2}\langle g(X_N(n-1/N)),\,\xi(n)\rangle\\ &-\tfrac12N^{-1}\langle g(X_N(n-1/N)),\,\xi(n)\rangle^2|\le\textstyle\sum_{l=3}^\infty\frac{(Md^2L^2)^l}{N^{l/2}l!}\le\tilde DN^{-3/2}\end{aligned} \tag{5.8}$$

where $\tilde D=\frac16M^3d^6L^6e^{Md^2L^2}$. Hence,

$$E\big(\exp(N^{-1/2}\langle g(X_N(n-1/N)),\,\xi(n)\rangle)|\mathcal F_{0,n-1}\big) \tag{5.9}$$

$$\leq 1+\frac{1}{2}N^{-1}\langle g(X_N(n-1/N)),\,\xi(n)\rangle^2 \leq 1+\frac{1}{2}N^{-1}d^4L^4+\tilde{D}N^{-3/2}$$

where we used that $E(\xi(n)|\mathcal{F}_{0,n-1})=E\xi(n)=0$. Continuing in the same way with the sums in the exponent till $n-2,n-3,\dots,1$ we obtain that

$$E\exp\big(N^{-1/2}\sum_{k=0}^{n-1}\langle g(X_N(k/N)),\,\xi(k+1)\rangle\big) \tag{5.10}$$
$$\leq \left(1+\frac{1}{2}N^{-1}d^4L^4+N^{-3/2}\tilde{D}\right)^N$$
$$\leq \left(1+\frac{1}{2}N^{-1}d^4L^4\right)^N(1+\tilde{D}N^{-3/2})^N \leq \exp\left(\tilde{D}+\frac{1}{2}d^4L^4\right)$$

proving (5.1) while (5.2) follows in the same way.

(ii) Set $\Gamma(y)=\{|X_N(n/N)-x|\geq y\}$. By (i) and the exponential Chebyshev inequality for any $n\leq N$, $y\geq 0$ and $\delta>0$,

$$P\left\{\Gamma\left(\frac{\delta y}{2M}\right)\right\}\leq D_{\frac{2M}{\delta}}e^{-y}.$$

Then, taking $y=2\log N$ we have

$$E\exp\left(M\max_{0\leq n\leq N}|X_N(n/N)|\right) \tag{5.11}$$
$$\leq e^{M|x|}E\exp\left(M\max_{1\leq n\leq N}|X_N(n/N)-x|\right)$$
$$\leq e^{M|x|}\left(N^\delta+\sum_{n=1}^N E(\mathbb{I}_{\Gamma_n(\frac{\delta}{M}\log N)}\exp(M|X_N(n/N)-x|))\right)$$
$$\leq e^{M|x|}\left(N^\delta+\sum_{n=1}^N\left(P\left\{\Gamma_n\left(\frac{\delta}{M}\log N\right)\right\}\right)^{1/2}\big(E\exp(2M|X_N(n/N)-x|)\big)\right)^{1/2}$$
$$\leq e^{M|x|}\big(N^\delta+(D_{2M/\delta}D_{2M})^{1/2}\big)$$

proving (5.3) while (5.4) follows in the same way.

For (iii) we have

$$E\exp(M\sup_{0\leq t\leq 1}|\Xi(t)|) \tag{5.12}$$
$$\leq e^{(M(|x|+L)}E\exp(M\sup_{0\leq t\leq 1}|\int_0^t\sigma(\Xi(s))dW(s)|)$$
$$\leq e^{(M(|x|+L)}\sum_{i=1}^d\left(E\sup_{0\leq t\leq 1}\exp\left(Md\sum_{j=1}^d\int_0^t\sigma_{ij}(\Xi(s))dW_j(s)\right)\right.$$
$$\left.+E\sup_{0\leq t\leq 1}\exp\left(-Md\sum_{j=1}^d\int_0^t\sigma_{ij}(\Xi(s))dW_j(s)\right)\right).$$

Since

$$\exp\left(\pm Md\sum_{j=1}^{d}\int_0^t\sigma_{ij}(\Xi(s))dW_j(s)-\frac{M^2d^2}{2}\int_0^t\sum_{j=1}^{d}\sigma^2_{ij}(\Xi(s))ds\right)$$

is a martingale with the expectation equal one, it follows from (2.2) and the Doob martingale inequality that

$$E\sup_{0\le t\le 1}\exp\left(\pm Md\sum_{j=1}^{d}\int_0^t\sigma_{ij}(\Xi(s))dW_j(s)\right)\le e^{\frac{1}{2}M^2L^2d^2},$$

and so the first inequality in (5.5) follows while we obtain the second one in the same way. □

Let $\mathcal{T}^\Delta$ be the set of all stopping times with respect to the filtration $\mathcal{F}^\xi_{0,n_k}$, $k\ge 0$ taking on values n_k, $k=0,1,\dots,k_{\max}$ where $k_{\max}=k_N$ if $k_N=N/[N^{\frac14}]$ and $k_{\max}=k_N+1$ and $n_{k_{\max}}=N$ if $n_{k_N}<N$. Denote by $\mathcal{Q}_{n_k}$ the σ-algebra $\mathcal{F}^\xi_{0,n_k}\vee\sigma\{U_i,\,1\le i\le k\}$ where, recall, $U_1,U_2,\dots$ is a sequence of i.i.d. uniformly distributed random variables appearing in Theorem 4.1. Let $\mathcal{T}^\mathcal{Q}$ be the set of all stopping times with respect to the filtration $\mathcal{Q}_{n_k}$, $k\ge 0$ taking on values n_k, $k=0,1,\dots,k_{\max}$. Next, introduce the payoffs based on $\hat X_N$ (the same as in Lemma 3.1),

$$\hat R_N(s,t)=G_s(\hat X_N)\mathbb{I}_{s<t}+F_t(\hat X_N)\mathbb{I}_{t\le s}$$

and the Dynkin game values corresponding to the sets of stopping times $\mathcal{T}^\Delta$ and $\mathcal{T}^\mathcal{Q}$,

$$V_N^\Delta=\inf_{\sigma\in\mathcal{T}^\Delta}\sup_{\tau\in\mathcal{T}^\Delta}ER_N(\sigma/N,\tau/N),$$

$$\hat V_N^\Delta=\inf_{\sigma\in\mathcal{T}^\Delta}\sup_{\tau\in\mathcal{T}^\Delta}E\hat R_N(\sigma/N,\tau/N),$$

$$\text{and }\ \hat V_N^\mathcal{Q}=\inf_{\sigma\in\mathcal{T}^\mathcal{Q}}\sup_{\tau\in\mathcal{T}^\mathcal{Q}}E\hat R_N(\sigma/N,\tau/N).$$

Lemma 5.2. *For any $\delta>0$ and an integer $N\ge 1$,*

$$|V_N-V_N^\Delta|\le D^X_{K,\delta}Ke^{K|x|}N^{\delta-\frac14}(1+L+L^2),\tag{5.13}$$

where $x=X_N(0)$, and

$$|V_N^\Delta-\hat V_N^\Delta|\le 24\sqrt{D^X_{4K,\delta}}e^{K|x|}L^4N^{\frac12(\delta-\frac12)}.\tag{5.14}$$

Proof. For any $\zeta\in\mathcal{T}^\xi_{0N}$ set $\zeta^\Delta=\min\{n_k:\,n_k\ge\zeta\}$ which defines a stopping time from $\mathcal{T}^\Delta$ satisfying

$$N^{-1}\zeta+\Delta(N)\ge N^{-1}\zeta^\Delta\ge N^{-1}\zeta.\tag{5.15}$$

Since $\mathcal{T}^\xi_{0N}\supset\mathcal{T}^\Delta$ we see that

$$V_N\ge\inf_{\zeta\in\mathcal{T}^\xi_{0N}}\sup_{\eta\in\mathcal{T}^\Delta}ER(\zeta/N,\eta/N).$$

Then for any $\vartheta > 0$ there exists $\zeta_\vartheta \in \mathcal{T}^\xi_{0N}$ such that

$$V_N \geq \sup_{\eta\in\mathcal{T}^\Delta} ER_N(\zeta_\vartheta/N, \eta/N) - \vartheta,$$

and so

$$\begin{aligned} V_N &\geq \sup_{\eta\in\mathcal{T}^\Delta} ER_N(\zeta^\Delta_\vartheta/N, \eta/N) - \vartheta \\ &\quad - \sup_{\eta\in\mathcal{T}^\Delta} E(R_N(\zeta^\Delta_\vartheta/N, \eta/N) - R_N(\zeta_\vartheta/N, \eta/N)) \\ &\geq V_N^\Delta - \vartheta - \sup_{\eta\in\mathcal{T}^\Delta} J_1(\zeta_\vartheta/N, \eta/N) \end{aligned} \tag{5.16}$$

where for any $\zeta \in \mathcal{T}^\xi_{0N}$ and $\eta \in \mathcal{T}^\Delta$,

$$J_1(\zeta/N, \eta/N) = E(R_N(\zeta^\Delta/N, \eta/N) - R_N(\zeta/N, \eta/N)).$$

Since $\zeta^\Delta \geq \zeta$,

$$R_N(\zeta/N, \eta/N) = G_{\zeta/N}(X_N) \text{ whenever } R_N(\zeta^\Delta/N, \eta/N) = G_{\zeta^\Delta/N}(X_N).$$

Hence, by (2.10) and (5.15),

$$\begin{aligned} &R_N(\zeta^\Delta/N, \eta/N) - R_N(\zeta/N, \eta/N) \leq \max\big(|G_{\zeta^\Delta/N}(X_N) - G_{\zeta/N}(X_N)|, \\ &|F_{\zeta^\Delta/N}(X_N) - F_{\zeta/N}(X_N)|\big) \\ &\leq K\big(\Delta(N)(1+L) \\ &+ \frac{1}{N^{1/2}} \max_{0\leq k\leq k_{\max}} \max_{1\leq l\leq N^{\frac{1}{4}}} \left| \sum_{n_k+l\leq j\leq n_{k+1}} \sigma(X_N(j/N)\xi(j)\right| \exp\left(K \max_{0\leq n\leq N} |X_N(n/N)|\right) \\ &\leq K(\Delta(N)(1+L) + L^2N^{-1/4}) \exp\left(K \max_{0\leq n\leq N} |X_N(n/N)|\right). \end{aligned} \tag{5.17}$$

Taking here ζ_ϑ in place of ζ we obtain from (5.16), (5.17) and Lemma 5.1(ii) that

$$V_N \geq V_N^\Delta - \vartheta - D^X_{K,\delta} K e^{K|x|} N^{\delta-\frac{1}{4}}(1 + L + L^2)$$

and since $\vartheta > 0$ is arbitrary we have that

$$V_N \geq V_N^\Delta - D^X_{K,\delta} K e^{K|x|} N^{\delta-\frac{1}{4}}(1 + L + L^2). \tag{5.18}$$

On the other hand, since the Dynkin game here has a value (see, for instance, [14], Section 6.2.2) we can write also that

$$V_N = \sup_{\eta\in\mathcal{T}^\xi_{0N}} \inf_{\zeta\in\mathcal{T}^\xi_{0N}} ER_N(\zeta/N, \eta/N) \leq \inf_{\zeta\in\mathcal{T}^\Delta} ER(\zeta/N, \eta_\vartheta/N) + \vartheta \tag{5.19}$$

for each $\vartheta > 0$ and some $\eta_\vartheta \in \mathcal{T}^\xi_{0N}$. Introducing η^Δ_ϑ and arguing as above we obtain that

$$V_N \leq V_N^\Delta + D^X_{K,\delta} K e^{K|x|} N^{\delta-\frac{1}{4}}(1 + L + L^2)$$

which together with (5.18) completes the proof of (5.13).

In order to prove (5.14) we observe that by (2.9), Lemma 3.1, Lemma 5.1(ii), the Chebyshev and the Cauchy-Schwarz inequalities

$$|V_N^\Delta - \hat V_N^\Delta| \le \sup_{\zeta\in\mathcal{T}^\Delta}\sup_{\eta\in\mathcal{T}^\Delta} E|R_N(\zeta/N,\eta/N) - \hat R_N(\zeta/N,\eta/N)| \tag{5.20}$$
$$\le \max\left(E\sup_{0\le t\le 1}|F_t(X_N)-F_t(\hat X_N)|,\, |G_t(X_N)-G_t(\hat X_N)|\right)$$
$$\le KE\left(\left(\sup_{0\le t\le 1}|X_N(t)-\hat X_N(t)| + \mathbb{I}_{\sup_{0\le t\le 1}|X_N(t)-\hat X_N(t)|>1}\right)\right.$$
$$\left.\times\exp\left(K\sup_{0\le t\le 1}(|X_N(t)|+|\hat X_N(t)|)\right)\right)$$
$$\le 2K\left(E\sup_{0\le t\le 1}|X_N(t)-\hat X_N(t)|^2\right)^{1/2}$$
$$\times\left(E\exp(4K\sup_{0\le t\le 1}|X_N(t)|)\right)^{1/4}\left(E\exp(4K\sup_{0\le t\le 1}|\hat X_N(t)|)\right)^{1/4}$$
$$\le 24\sqrt{D^X_{4K,\delta}}e^{K|x|}L^4N^{\frac{1}{2}(\delta-\frac{1}{2})}.$$

yielding (5.14). □

Lemma 5.3. *For any integer $N\ge 1$,*

$$\hat V_N^\Delta = \hat V_N^{\mathcal{Q}}. \tag{5.21}$$

Proof. We prove (5.21) obtaining both $\hat V_N^\Delta$ and $\hat V_N^{\mathcal{Q}}$ by the standard dynamical programming (backward recursion) procedure (see, for instance, Section 1.3.2 in [14]). Namely, we have $\hat V_N^\Delta = \hat V_{N,0}^\Delta$ and $\hat V_N^{\mathcal{Q}} = \hat V_{N,0}^{\mathcal{Q}}$ where

$$\hat V^\Delta_{N,k_{\max}} = F_T(\hat X) = \hat V^{\mathcal{Q}}_{N,k_{\max}} \tag{5.22}$$

proceeding recursively

$$\hat V^\Delta_{N,k} = \min\big(G_{n_k/N}(\hat X_N),\, \max(F_{n_k/N}(\hat X_N),\, E(\hat V^\Delta_{N,k+1}|\mathcal{F}^\xi_{0,n_k}))\big)$$

and

$$\hat V^{\mathcal{Q}}_{N,k} = \min\big(G_{n_k/N}(\hat X_N),\, \max(F_{n_k/N}(\hat X_N),\, E(\hat V^{\mathcal{Q}}_{N,k+1}|\mathcal{Q}_{n_k}))\big).$$

Since each σ-algebra $\sigma\{U_1,\dots,U_k\}$ is independent of $\xi_1,\xi_2,\dots$ by the construction, i.e. it is independent of all σ-algebras $\mathcal{F}^\xi_{0,l}$, $l=0,\pm1,\dots$, and so it is independent of X_N, it follows (see, for instance, [5], p.323 or [13], Remark 4.3) that

$$E(\hat V^\Delta_{N,k+1}|\mathcal{F}^\xi_{0,n_k}) = E(\hat V^\Delta_{N,k+1}|\mathcal{Q}_{n_k}),$$

and so starting from (5.22) we proceed recursively to $\hat V^\Delta_{N,0} = \hat V^{\mathcal{Q}}_{N,0}$ proving (5.21). □

Next, we turn our attention to the diffusion Ξ constructed in Theorem 2.1 and consider the corresponding Dynkin game value V^Ξ given by (2.8). Set $\mathcal{G}^\Xi_{n_k} = \sigma\{W_N(u/N):\, u\le n_k\}$ and observe that by the construction

$$\mathcal{G}^\Xi_{n_k} \subset \mathcal{Q}_{n_k} = \mathcal{F}^\xi_{0,n_k}\vee\sigma\{U_i,\, 1\le i\le k\} \tag{5.23}$$

where W_N is the Brownian motion constructed in Section 4. Let $\mathcal{T}^{\Xi}_{\Delta}$ be the set of all stopping times with respect to the filtration $\mathcal{G}^{\Xi}_{n_k}$, $k \geq 0$ and $\mathcal{T}^{\mathcal{Q}}_{\Delta}$ be the set of all stopping times with respect to the filtration $\mathcal{Q}_{n_k}$, $k \geq 0$, both taking values n_k when k runs from 0 to $k_{\max}$. Set

$$\hat{R}^{\Xi}(s,t) = G_s(\hat{\Xi})\mathbb{I}_{s<t} + F_t(\hat{\Xi})\mathbb{I}_{t\leq s}$$

where the process $\hat{\Xi}$ is the same as in Lemma 4.3. Set

$$V^{\Xi}_{\Delta} = \inf_{\zeta\in\mathcal{T}^{\Xi}_{\Delta}} \sup_{\eta\in\mathcal{T}^{\Xi}_{\Delta}} ER^{\Xi}(\zeta/N, \eta/N),$$

$$\hat{V}^{\Xi}_{\Delta} = \inf_{\zeta\in\mathcal{T}^{\Xi}_{\Delta}} \sup_{\eta\in\mathcal{T}^{\Xi}_{\Delta}} E\hat{R}^{\Xi}(\zeta/N, \eta/N)$$

and

$$\hat{V}^{\Xi}_{\mathcal{Q}} = \inf_{\zeta\in\mathcal{T}^{\mathcal{Q}}} \sup_{\eta\in\mathcal{T}^{\mathcal{Q}}} E\hat{R}^{\Xi}(\zeta/N, \eta/N).$$

Lemma 5.4. *For any integer $N \geq 1$,*

$$|V^{\Xi} - V^{\Xi}_{\Delta}| \leq K\left(\Delta(N)D^{\Xi}_{K} + 18L\right)\sqrt{2D^{\Xi}_{2K}}\sqrt{\Delta(N)}, \tag{5.24}$$

where $x = \Xi(0)$, and

$$|V^{\Xi}_{\Delta} - \hat{V}^{\Xi}_{\Delta}| \leq (4\sqrt{2} + 2L)Ke^{K|x|}\sqrt{D^{\Xi}_{4K}}\sqrt{\Delta(N)}. \tag{5.25}$$

Proof. The proof is similar to Lemma 5.2 but here in place of estimates for X_N we have to use moment estimates for diffusions. Set $\mathcal{T}^{\Xi,N}_{01} = \{\zeta : \zeta/N \in \mathcal{T}^{\Xi}_{01}\}$ where, recall, $\mathcal{T}^{\Xi}_{01}$ is the set of stopping times with respect to the filtration $\mathcal{F}^{\Xi}_t = \sigma\{W_N(s),\, s \leq t\}$ having values in $[0,1]$. For any $\xi \in \mathcal{T}^{\Xi,N}_{01}$ define $\zeta^{\Delta} = \min\{n_k : n_k \geq \zeta\}$ which yields a stopping time from $\mathcal{T}^{\Xi}_{\Delta}$ satisfying (5.15). Since $\mathcal{T}^{\Xi}_{\Delta} \subset \mathcal{T}^{\Xi,N}_{0,1}$ we have that

$$V^{\Xi} \geq \inf_{\zeta\in\mathcal{T}^{\Xi,N}_{01}} \sup_{\eta\in\mathcal{T}^{\Xi}_{\Delta}} ER^{\Xi}(\zeta/N, \eta/N).$$

In the same way as in (5.16) we obtain that for some $\zeta_{\vartheta} \in \mathcal{T}^{\Xi,N}_{01}$,

$$V^{\Xi} \geq V^{\Xi}_{\Delta} - \vartheta - \sup_{\eta\in\mathcal{T}^{\Xi}_{\Delta}} J_2(\zeta_{\vartheta}/N, \eta/N) \tag{5.26}$$

where for any $\zeta \in \mathcal{T}^{\Xi,N}_{01}$ and $\eta \in \mathcal{T}^{\Xi}_{\Delta}$,

$$J_2(\zeta/N, \eta/N) = E(R^{\Xi}(\zeta^{\Delta}/N, \eta/N) - R^{\Xi}(\zeta/N, \eta/N)).$$

As in (5.17) we obtain from (2.10) and (5.15) that

$$\begin{aligned} &R^{\Xi}(\zeta^{\Delta}/N, \eta/N) - R^{\Xi}(\zeta/N, \eta/N) \\ &\quad\leq K\left(\Delta(N) + \max_{0\leq k\leq k_{\max}} \sup_{n_k/N\leq s\leq n_{k+1}/N} |\Xi(n_{k+1}/N) - \Xi(s)|\right) \\ &\qquad\times \exp\left(K \sup_{0\leq t\leq 1} |\Xi(t)|\right). \end{aligned} \tag{5.27}$$

By the Cauchy-Schwarz inequality,

$$|E(R^\Xi(\zeta^\Delta/N,\eta/N)-R^\Xi(\zeta/N,\eta/N))|\leq K\Delta(N)E\exp(K\sup_{0\leq t\leq 1}|\Xi(t)|) \quad (5.28)$$
$$+K\big(E\max_{0\leq k\leq k_{\max}}\sup_{n_k/N\leq s\leq n_{k+1}/N}|\Xi(n_{k+1}/N)-\Xi(s)|^2\big)^{1/2}$$
$$\times\left(E\exp\left(2K\sup_{0\leq t\leq 1}|\Xi(t)|\right)\right)^{1/2}.$$

Next, we write

$$\big(E\max_{1\leq k\leq k_{\max}}\sup_{n_{k+1}/N\geq s\geq n_k/N}|\Xi(n_{k+1}/N)-\Xi(s)|^2\big)^{1/2} \quad (5.29)$$
$$\leq(\sum_{1\leq k\leq k_{\max}}E\sup_{n_{k+1}/N\geq s\geq n_k/N}|\Xi(n_{k+1}/N)-\Xi(s)|^4)^{1/4}$$

and

$$E\sup_{n_{k+1}/N\geq s\geq n_k/N}|\Xi(n_{k+1}/N)-\Xi(s)|^4 \quad (5.30)$$
$$\leq 8E|\Xi(n_{k+1}/N)-\Xi(n_k/N)|^4+8E\sup_{n_{k+1}/N\geq s\geq n_k/N}|\Xi(s)-\Xi(n_k/N)|^4.$$

By the standard moment estimates for stochastic integrals

$$E|\Xi(n_{k+1}/N)-\Xi(n_k)/N|^4\leq 8E|\int_{n_k/N}^{n_{k+1}/N}\sigma(\Xi(u))dW_N(u)|^4 \quad (5.31)$$
$$+8E(\int_{n_k/N}^{n_{k+1}/N}b(\Xi(u))du)^4\leq 288\Delta(N)\int_{n_k/N}^{n_{k+1}/n}E|\sigma(\Xi(u))|^4du$$
$$+8L^4(\Delta(N))^4\leq 8L^4(\Delta(N))^2(36+(\Delta(N))^2)$$

and

$$E\sup_{n_{k+1}/N\geq s\geq n_k/N}|\Xi(s)-\Xi(n_k/N)|^4 \quad (5.32)$$
$$\leq 8(4/3)^4E\left|\int_{n_k/N}^{n_{k+1}/N}\sigma(\Xi(u))dW_N(u)\right|^4$$
$$+8E\left(\int_{n_k/N}^{n_{k+1}/N}b(\Xi(u))du\right)^4\leq 8L^4(\Delta(N))^2(36(4/3)^4+(\Delta(N))^2).$$

Combining (5.26)–(5.32) together with Lemma 5.1(iii)we obtain the required lower bound for $V^\Xi-V^\Xi_\Delta$ taking into account that $\vartheta>0$ is arbitrary. On the other hand, since the Dynkin game has a value under our conditions (see, for instance, [14], Section 6.2.2) we can write that

$$V^\Xi=\sup_{\eta\in\mathcal{T}_{01}^{\Xi,N}}\inf_{\zeta\in\mathcal{T}_{01}^{\Xi,N}}ER^\Xi(\zeta/N,\eta/N)\leq\inf_{\zeta\in\mathcal{T}_\Delta^\Xi}ER^\Xi(\zeta/N,\eta_\vartheta/N)+\vartheta$$

for any $\vartheta>0$ and some $\eta_\vartheta\in\mathcal{T}_{01}^{\Xi,N}$. Introducing η_ϑ^Δ and relying on the same arguments as above we obtain the corresponding upper bound for $V^\Xi-V^\Xi_\Delta$ and complete the proof of (5.24).

Next, we obtain (5.25) by (2.9), Lemma 4.3, Lemma 5.1(iii), the Chebyshev and the Cauchy-Schwarz inequalities,

$$|V_\Delta^\Xi - \hat{V}_\Delta^\Xi| \le \sup_{\zeta\in\mathcal{T}_\Delta^\Xi}\sup_{\eta\in\mathcal{T}_\Delta^\Xi} E|R^\Xi(\zeta/N,\eta/N) - \hat{R}^\Xi(\zeta/N,\eta/N)| \tag{5.33}$$

$$\le K\left(E\left(\max_{0\le k\le k_{\max}}|\Xi(k/N)-\hat{\Xi}(k/N)| + \mathbb{I}_{\max_{0\le k\le k_{\max}}|\Xi(k/N)-\hat{\Xi}(k/N)|>1}\right)^2\right)^{1/2}$$

$$\times\left(E\exp(4K\max_{0\le k\le k_{\max}}|\Xi(k/N)|\right)^{1/4}\left(E\exp(4K\max_{0\le k\le k_{\max}}|\hat{\Xi}(k/N)|\right)^{1/4}$$

$$\le 2K\sqrt{D_{4K}^\Xi}\left(E\max_{0\le k\le k_N}|\Xi(k/N)-\hat{\Xi}(k/N)|^2 + E|\Xi(1)-\Xi(k_N/N)|^2\right)^{1/2}$$

$$\le (4\sqrt{2}+2L)Ke^{K|x|}\sqrt{D_{4K}^\Xi}\sqrt{\Delta(N)}$$

completing the proof of the lemma. □

Next, we introduce the new process Ψ_N, first recursively at the times $N^{-1}n_k$ and then extending it for all $t\in[0,T]$ in the piece-wise constant fashion. Namely, we set $\Psi_N(0)=x$ and (with $n_0=0$),

$$\begin{aligned}\Psi_N(N^{-1}n_{k+1}) = \Psi_N(N^{-1}n_k)&\\ &+\sigma(\Psi_N(N^{-1}n_k))(W_N(N^{-1}n_{k+1})-W_N(N^{-1}n_k))\\ &+N^{-1}b(\Psi_N(N^{-1}n_k))(n_{k+1}-n_k)\end{aligned}$$

for $k=0,1,\dots,k_{\max}-1$. Set also $\Psi_N(t)=\Psi_N(N^{-1}n_k)$ if $N^{-1}n_k\le t<N^{-1}n_{k+1}$.

Lemma 5.5. *For any integer $N\ge 1$,*

$$E\max_{0\le k\le k_{\max}}|\Xi(N^{-1}n_k)-\Psi_N(N^{-1}n_k)|^2 \le 96\Delta(N)\exp(24L^2d). \tag{5.34}$$

Proof. We have

$$\begin{aligned}\left|\Xi(N^{-1}n_k)-\Psi_N(N^{-1}n_k)\right|^2 \le 3\Bigg(&|\Xi(N^{-1}n_k)-\hat{\Xi}(N^{-1}n_k)|^2\\ &+\left|\sum_{0\le l<k}(\sigma(\Xi(N^{-1}n_l))-\sigma(\Psi_N(N^{-1}n_l)))(W_N(N^{-1}n_{l+1})-W_N(N^{-1}n_l))\right|^2\\ &+\left(N^{-1}\sum_{0\le l<k}|b(\Xi(N^{-1}n_l))-b(\Psi_N(N^{-1}n_l))|(n_{l+1}-n_l)\right)^2\Bigg),\end{aligned}$$

and so

$$\begin{aligned}&\max_{0\le k\le n}|\Xi(N^{-1}n_k)-\Psi_N(N^{-1}n_k)|^2 \\ &\quad\le 3\left(\max_{0\le k\le n}|\Xi(N^{-1}n_k)-\hat{\Xi}(N^{-1}n_k)|^2 + \max_{0\le k\le n}|M_k|^2\right.\end{aligned} \tag{5.35}$$

$$+4k_{\max}(\Delta(N))^2\sum_{0\leq l<n}|b(\Xi(N^{-1}n_l))-b(\Psi_N(N^{-1}n_l))|^2\Bigg)$$

where

$$M_k=\sum_{0\leq l<k}(\sigma(\Xi(N^{-1}n_l))-\sigma(\Psi_N(N^{-1}n_l)))(W_N(N^{-1}n_{l+1})-W_N(N^{-1}n_l))$$

is a martingale with respect to the filtration $\{\mathcal{G}^\Xi_{n_k},\, k\geq 0\}$ since $\sigma(\Xi(N^{-1}n_l))-\sigma(\Psi_N(N^{-1}n_l))$ is $\mathcal{G}^\Xi_{n_l}$-measurable while $W_N(N^{-1}n_{l+1})-W_N(N^{-1}n_l)$ is independent of $\mathcal{G}^\Xi_{n_l}$.

Hence, by the Doob martingale moment inequality and by the Lipschitz continuity of σ (with the constant L),

$$E\max_{0\leq k\leq n}|M_k|^2\leq 4E|M_n|^2\leq 4L^2dN^{-1}\sum_{0\leq k<n}Q_k(n_{k+1}-n_k) \tag{5.36}$$

where

$$Q_n=E\max_{0\leq k\leq n}|\Xi(N^{-1}n_k)-\Psi_N(N^{-1}n_k)|^2.$$

By (5.35), (5.36) and Lemma 4.3 we obtain that

$$Q_n\leq 96\Delta(N)+24L^2d\Delta(N)\sum_{0\leq k<n}Q_k.$$

Thus, by the discrete (time) Gronwall inequality (see [6]),

$$Q_n\leq 96\Delta(N)\exp(24L^2d\Delta(N)n)$$

and since $n\leq k_{\max}$, (5.34) follows. □

Next, we introduce the values of Dynkin games with payoffs based on the process Ψ_N. Namely, we set

$$R^\Psi_N(s,t)=G_s(\Psi_N)\mathbb{I}_{s<t}+F_t(\Psi_N)\mathbb{I}_{t\leq s},$$
$$V^\Psi_\Delta=\inf_{\zeta\in\mathcal{T}^\Xi_\Delta}\sup_{\eta\in\mathcal{T}^\Xi_\Delta}ER^\Psi_N(N^{-1}\zeta,N^{-1}\eta)$$
$$\text{and } V^\Psi_{\mathcal{Q}}=\inf_{\zeta\in\mathcal{T}^{\mathcal{Q}}}\sup_{\eta\in\mathcal{T}^{\mathcal{Q}}}ER^\Psi_N(N^{-1}\zeta,N^{-1}\eta).$$

Lemma 5.6. *For any $\varepsilon>0$,*

$$V^\Psi_\Delta=V^\Psi_{\mathcal{Q}}. \tag{5.37}$$

Proof. As in Lemma 5.3 we will prove (5.37) obtaining both V^Ψ_Δ and $V^\Psi_{\mathcal{Q}}$ by the dynamical programming procedure. Again, we have $V^\Psi_\Delta=V^\Psi_{\Delta,0}$ and $V^\Psi_{\mathcal{Q}}=V^\Psi_{\mathcal{Q},0}$ where $V^\Psi_{\Delta,k_{\max}}=F_T(\Psi_N)=V^\Psi_{\mathcal{Q},k_{\max}}$ and for $k=k_{\max}-1,k_{\max}-2,\dots,0$,

$$V^\Psi_{\Delta,k}=\min\big(G_{N^{-1}n_k}(\Psi_N),\,\max(F_{N^{-1}n_k}(\Psi_N),\,E(V^\Psi_{\Delta,k+1}|\mathcal{G}^\Xi_{n_k}))\big)$$

and

$$V^\Psi_{\mathcal{Q},k}=\min\big(G_{N^{-1}n_k}(\Psi_N),\,\max(F_{N^{-1}n_k}(\Psi_N),\,E(V^\Psi_{\mathcal{Q},k+1}|\mathcal{Q}_{n_k}))\big).$$

For any vectors $x_0,x_1,x_2,\dots,x_{k_{\max}}\in\mathbb{R}^d$ set $x(0)=x_0$, $x(t)=x_k$ if $N^{-1}n_k\leq t<N^{-1}n_{k+1}$ and define the functions

$$q_{k_N(t)}(x_1,\dots,x_{k_N(t)})=F_t(x)\text{ and } r_{k_N(t)}(x_1,\dots,x_{k_N(t)})=G_t(x).$$

Introduce

$$\Phi_l(x_1,\dots,x_l)=\min\big(r_l(x_1,\dots,x_l),\,\max(q_l(x_1,\dots,x_l),\,h(x_1,\dots,x_l))\big)$$

where

$$h(x_1,\dots,x_l)=E\Phi_{l+1}\big(x_1,\dots,x_l,\,x_l+\sigma(x_l)(W_N(N^{-1}n_{l+1})-W_N(N^{-1}n_l))\big).$$

Since $\Psi_N(N^{-1}n_l)$ is both $\mathcal{G}_{n_l}$ and $\mathcal{Q}_{n_l}$-measurable while $W_N(N^{-1}n_{l+1}) - W_N(N^{-1}n_l)$ is independent of both $\mathcal{G}_{n_l}$ and $\mathcal{Q}_{n_l}$ we see by induction that

$$V^{\Psi}_{\mathcal{Q},l}=\Phi_l(\Psi_N(N^{-1}n_1),\Psi_N(N^{-1}n_2),\dots,\Psi_N(N^{-1}n_l))=V^{\Psi}_{\Delta,l},$$

for all $l=k_{\max},k_{\max}-1,\dots,0$ where $\Phi_0=\min(F_0(x_0),\max(G_0(x_0),E\Phi_1(x_0+\sigma(x_0)W_N(N^{-1}n_1)))$, and (5.37) follows. □

Now we can complete the proof of Theorem 2.2 writing first,

$$|V^{\Xi}-V_N|\le|V_N-V^{\Delta}_N|+|V^{\Delta}_N-\hat V^{\mathcal{Q}}_N|+|\hat V^{\mathcal{Q}}_N-V^{\Psi}_{\mathcal{Q}}| \tag{5.38}$$
$$+|V^{\Psi}_{\mathcal{Q}}-\hat V^{\Xi}_{\Delta}|+|\hat V^{\Xi}_{\Delta}-V^{\Xi}_{\Delta}|+|V^{\Xi}_{\Delta}-V^{\Xi}|.$$

It remains to estimate $|\hat V^{\mathcal{Q}}_N-V^{\Psi}_{\mathcal{Q}}|$ and $|V^{\Psi}_{\mathcal{Q}}-\hat V^{\Xi}_{\Delta}|=|V^{\Psi}_{\Delta}-\hat V^{\Xi}_{\Delta}|$ since all other terms in the right hand side of (5.38) are dealt with by Lemmas 5.2–5.4. In both remaining estimates we use the fact that the game values there are defined with respect to the same sets of stopping times which will allow us to rely on uniform bounds on distances between the corresponding processes. By (2.9) and the Cauchy-Schwarz inequality,

$$|\hat V^{\mathcal{Q}}_N-V^{\Psi}_{\mathcal{Q}}|\le\sup_{\zeta\in\mathcal{T}^{\mathcal{Q}}}\sup_{\eta\in\mathcal{T}^{\mathcal{Q}}}E|\hat R(N^{-1}\zeta,N^{-1}\eta)-R^{\Psi}_N(N^{-1}\zeta,N^{-1}\eta)| \tag{5.39}$$
$$\le\max\left(E\sup_{0\le t\le1}|F_t(\hat X_N)-F_t(\Psi_N)|,\,E\sup_{0\le t\le1}|G_t(\hat X_N)-F_t(\Psi_N)|\right)$$
$$\le\sqrt{2}K\left(E\max_{0\le k\le k_{\max}}\left|\hat X_N(N^{-1}n_k)-\Psi_N(N^{-1}n_k)\right|^2\right.$$
$$\left.+P\left\{\max_{0\le k\le k_{\max}}|\hat X_N(N^{-1}n_k)-\Psi_N(N^{-1}n_k)|>1\right\}\right)^{1/2}$$
$$\times\left(E\exp\left(2K\left(\max_{0\le k\le k_{\max}}(|\hat X_N(N^{-1}n_k)|+|\Psi_N(N^{-1}n_k)|)\right)\right)\right)^{1/2}.$$

Next, by Lemmas 3.1, 5.5 and Theorem 2.1,

$$E\max_{0\le k\le k_{\max}}|\hat X_N(N^{-1}n_k)-\Psi_N(N^{-1}n_k)|^2 \tag{5.40}$$
$$\le3E\max_{0\le k\le k_{\max}}|\hat X_N(N^{-1}n_k)-X_N(N^{-1}n_k)|^2$$
$$+3E\max_{0\le k\le k_{\max}}|X_N(N^{-1}n_k)-\Xi(N^{-1}n_k)|^2$$
$$+3E\max_{0\le k\le k_{\max}}|\Xi(N^{-1}n_k)-\Psi_N(N^{-1}n_k)|^2$$
$$\le408L^8N^{-1/2}+3C_0[N^{\frac14}]^{-\frac{1}{50d}}+96\exp(24L^2d)\Delta(N).$$

In view of the Chebyshev inequality the probability in (5.39) is also estimated by the right hand side of (5.40).

Similarly, by (2.9) and by Lemmas 4.3, 5.1 and 5.5,

$$|V^{\Psi}_{\Delta}-\hat{V}^{\Xi}_{\Delta}|\leq\sup_{\zeta\in\mathcal{T}^{\Delta}}\sup_{\eta\in\mathcal{T}^{\Delta}}E|R^{\Psi}_{N}(N^{-1}\zeta,N^{-1}\eta)\tag{5.41}$$
$$-\hat{R}^{\Xi}(N^{-1}\zeta,N^{-1}\eta)|\leq 2K\big(E\max_{0\leq k\leq k_{\max}}|\Psi_N(N^{-1}n_k)-\hat{\Xi}(N^{-1}n_k)|^2\big)^{1/2}$$
$$\times\big(E\exp(2K(\max_{0\leq k\leq k_{\max}}(|\Psi_N(N^{-1}n_k)|+|\hat{\Xi}(N^{-1}n_k)|)))\big)^{1/2}$$
$$\leq 2\sqrt{2}K\big(E\max_{0\leq k\leq k_{\max}}|\Psi_N(N^{-1}n_k)-\Xi(N^{-1}n_k)|^2$$
$$+E\max_{0\leq k\leq k_{\max}}|\Xi(N^{-1}n_k)-\hat{\Xi}(N^{-1}n_k)|^2\big)^{1/2}(D^{\Xi}_{4K})^{1/2}$$
$$\leq 16K\sqrt{\Delta(N)}(1+3\exp(24L^2d))^{1/2}(D^{\Xi}_{4K})^{1/2}.$$

Combining (5.38) together with (5.39)–(5.41) and Lemmas 5.1–5.4 we complete the proof of Theorem 2.2. □

References

1. Borodin, A. N.: *A limit theorem for solutions of differential equations with random right-hand side*, Theory Probab. Appl. 22 (1977), 482–497.
2. Borodin, A. N. and Freidlin, M. I.: *Fast oscillating random perturbations of dynamical systems with conservation laws*, Annales de l'I.H.P., sec. B, 31 (1995), 485–525.
3. Bayraktar, E., Dolinsky, Ya., and Guo, J.: *Recombining tree approximations for optimal stopping for diffusions*, SIAM J. Financial Math. 9 (2018), 602–633.
4. Berkes, I. and Philipp, W.: *Approximation theorems for independent and weakly dependent random vectors*, Annals Probab. 7 (1979), 29–54.
5. Chung, K.-L.: *A Course in Probability*, 3d edition, Acad. Press, San Diego, Ca., 2001.
6. Clark, D. S.: *A short proof of a discrete Gronwall inequality*, Discrete Appl. Math. 16 (1987), 279–281.
7. Cogburn, R. and Ellison, J. A.: *A stochastic theory of adiabatic invariance*, Commun. Math. Phys. 149 (1992), 97–126.
8. Dehling, H. and Philipp, W.: *Empirical process technique for dependent data*, In: H.G. Dehling, T. Mikosch and MSorenson (Eds.), *Empirical Process Technique for Dependent Data*, p.p. 3–113, Birkhäuser, Boston, 2002.
9. Dolinsky, Y.: *Applications of weak convergence for hedging of game options*, Ann. Appl. Probab. 20 (2010), 1891–1906.
10. He, H.: *Convergence from discrete-to continuous-time contingent claims prices*, Review Financial Studies 3 (1990), 523–546.
11. Khasminskii, R. Z.: *A limit theorem for the solution of differential equations with random rand-hand sides*, Theory Probab. Appl. 11 (1966), 390–406.
12. Kifer, Yu.: *Error estimate for binomial approximation of game options*, Annals of Appl. Probab. 16 (2006), 984–1033.
13. Kifer, Yu.: *Optimal stopping and strong approximation theorems*, Stochastics 79 (2007), 253–273.
14. Kifer, Yu.: *Lectures on Mathematical Finance and Related Topics*, World Scientific, Singapore, 2020.
15. Kifer, Yu.: *Strong diffusion approximation in averaging and value computation in Dynkin's games*, arXiv: 2011.07907.

16. Kifer, Yu. and Kunita, H.: *Random positive semigroups and their infinitesimal generators*, in: Stochastic Analysis and Appl., eds. I.M.Davis, A.Truman, K.D.Elworthy, p.p. 270–285, World Scientific, Singapore, 1996.
17. Kuelbs, J. and Philipp, W.: *Almost sure invariance principles for partial sums of mixing B-valued random variables*, Annals Probab. 8 (1980), 1003–1036.
18. Kunita, H.: *Infinitesimal generators of random positive semigroups*, Taiwanese J. Math. 1 (1997), 371–387.
19. Kunita, H.: *Ergodic properties of random positive semigroups*, Acta Applicandae Math. 63 (2000), 185–2001.
20. Kunita, H. and Seko, S.: *Game call options and their exercise regions,* Tech. Report, NANZAN-TR-2004-06.
21. Mao, X.: *Stochastic Differential Equations and Applications*, 2nd. ed., Woodhead, Oxford, 2010.
22. Monrad, D. and Philipp, W.: *Nearby variables with nearby laws and a strong approximation theorem for Hilbert space valued martingales*, Probab. Th. Rel. Fields 88 (1991), 381–404.
23. Monrad, D. and Philipp, W.: *The problem of embedding vector-valued martingales in a Gaussian process*, Theory Probab. Appl. 35 (1991), 374–377.
24. Stroock, D. W. and Varadhan, S. R. S.: *Multidimensional Diffusion processes*, Springer-Verlag, Berlin, 1997.

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