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HESSIAN FORMULAS AND ESTIMATES FOR PARABOLIC SCHRÖDINGER OPERATORS

XUE-MEI LI*

Dedicated to the memory of Professor Hiroshi Kunita

ABSTRACT. We study the Cauchy problem for the parabolic equation $\frac{\partial}{\partial t} = \mathcal{L}$ and the h -Brownian motion which is the Markov process with the weighted Laplacian $\frac{1}{2}\Delta^h := \frac{1}{2}\Delta + \nabla h$ where Δ the Laplace-Beltrami operator on M , and h, V real valued functions on M and \mathcal{L} is the weighted Schrödinger operator $\mathcal{L} = \frac{1}{2}\Delta + \nabla h - V$.

We first obtain new geometric criteria for the gradient stochastic differential equation (SDE) with generator $\frac{1}{2}\Delta^h$: non-explosion, strong 1-completeness, moment bounds, and exponential integrability. We then study the linearisation problem associated with the gradient SDE, introduce also a doubly damped stochastic parallel transport on tensors, involving only geometric quantities. Together with the stochastic damped transport this allows to obtain a new Hessian formula for the weighted heat semi-group, obtained with a hybrid formula $\text{Hess}(P_t^h f)(v_2, v_1) = \mathbb{E}[\nabla df(W_t(v_2), W_t(v_1))] + \mathbb{E}[df(W_t^{(2)}(v_1, v_2))]$, and a corresponding formula for $e^{t\mathcal{L}}$. These formulae are then used for obtaining path integration formula for $\text{Hess} P_t^{h,V} f(v_1, v_2)$, a 2nd order Feynman-Kac formula, based on path integration, not involving any derivatives of f or V .

With these intrinsic second order Feynman-Kac formula, global estimates are obtained for these semi-groups, their derivatives, and that of their fundamental solutions are obtained. These estimates are in terms of bounds on $\text{Ric} - 2\text{Hess} h$, on the curvature operator, and on the cyclic sum of the gradient of the Ricci tensor.

Finally, for manifolds with a pole, we prove that the Hessian of the fundamental solution is the product of an exact Gaussian term with a term involving the semi-classical bridge, the latter is further estimated to lead to Hessian estimates. Precise estimates are then obtained for the derivatives of the logarithmic heat kernels.

1. Introduction

Many years ago, after I solved the open problem on the existence of a global smooth solution flow for stochastic differential equations on a non-compact manifold, Professor Kunita wrote to me that he was writing a survey on stochastic flows and he would be pleased if I would send him a paragraph detailing my new results on this, to which I replied naïvely that I did not obtain any new results – assuming Professor Kunita was

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expecting me to have worked out significant following ups, having started to work on other fields, I was too ashamed that I did not have further work on this subject that was worth mentioning. Belatedly and years later, I learnt that he was asking for a summary of my previous work. Although this article contains an array of results focusing on the Hessian of heat semigroups, we do include new results on the strong 1-completeness. The latter is a generalisation for the global smooth solution flow problem, it is concerned with continuous dependence on the initial data restricted to the image of a curve, which has application to the intertwining of semi-groups. Professor Kunita 's celebrated contribution to the stochastic flow problem is immortalised in his beautiful monograph [57], we think it befitting to dedicate this article to his memory. This article was first posted in the Arxiv [66], see also [69].

1.1. The Parabolic Schrödinger Operator. Let M be a complete connected smooth Riemannian manifold of dimension n greater than 1, Δ the Laplace-Beltrami operator on M ,

$$\Delta^h = \Delta + 2L_{\nabla h}$$

the Bismut-Witten (or weighted) Laplacian, where h is a real valued smooth function on M , ∇ denotes the gradient operator and also stands for the Levi-Civita connection. For a real valued bounded and Hölder continuous function V on M , let $P_t^{h,V} f$ denotes the solution to the equation

$$\begin{aligned} \frac{\partial}{\partial t} f_t &= \left(\frac{1}{2} \Delta^h - V \right) f_t, \quad \text{for } t > 0; \\ \lim_{t \downarrow 0} f_t &= f. \end{aligned} \tag{1.1}$$

where V is a real-valued bounded and locally Hölder continuous function on M . We denote by $p_t^{h,V}$ the fundamental solution of (1.1). If $h = 0$ or $V = 0$ the corresponding subscripts will be dropped. There is a vast literature on parabolic Schrödinger operators on a complete Riemannian manifold, see the books Cycon, Froese, Kirsh and Simon [25], Davies and Safanov [26], and Simon [83] and the reference therein. We mention in particular Molchanov, Azencott, Donnelly, Elworthy and Truman, Aizenman and Simon, P. Li and Yau [73, 14, 28, 33, 6, 59]. Our aim is to understand the Hessian of the weighted heat kernels and the fundamental solutions of the weighted Schrödinger operators, on complete Riemannian manifolds, and obtain global Hessian estimates with precise asymptotics. In particular we study the second order derivative of the logarithmic kernel $\log p_t^h$, the latter plays a role in the L^2 analysis of the space of continuous paths on a Riemannian manifold and are also related to parabolic Harnack inequalities, and Talagrand's conjecture. On \mathbb{R}^n , one has

$$\begin{aligned} p_t(x, y) &= (2\pi t)^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{2t}}, & \nabla \cdot \log p_t(\cdot, y_0) &= -\frac{d\nabla d(x_0, y)}{t} \\ \frac{\nabla d p_t}{p_t} &= -\frac{1}{t} + \frac{d^2(x_0, y)}{t^2}. \end{aligned}$$

The heat kernel enjoys Gaussian type upper and lower bounds. How closely the derivative bounds follow the Euclidean case? We will see they are mirrored very well by manifolds with a pole. We also give global estimates for more general manifolds.

1.2. The fundamental solution. In order to obtain information on the fundamental solution $p_t^{h,V}$, one might wish to use the Brownian bridge. This does not work very well. For example, the usual formula for ∇p_t using the Brownian bridge is circular as the Brownian bridge is a Markov process with Markov generator $\frac{1}{2}\Delta + \nabla \log p_{T-t}(\cdot, y_0)$ where T being the terminal time and y_0 the terminal value. The extension of the operator to differential 1-forms is $\frac{1}{2}\Delta + \nabla^2 \log p_{T-t}(\cdot, y_0)$. However it is possible to obtain information on the Hessian of the Schrödinger kernel from that on p_t^h . Although our aim is to obtain precise estimates for the weighted Schrödinger kernel on manifolds with a pole, we did obtain Hessian estimates for $P_t^{h,V} f$ and on $\text{Hess } p_t^h$ on general complete Riemannian manifolds, generalising those in Sheu, Norris, Hsu, Aida, Stroock-Turetsky and XD Li [80, 76, 55, 1, 84, 61]. These are proved under conditions **C2** and **C1(c)** presented in §3.

Our second aim is to study the stochastic system with generator $\frac{1}{2}\Delta^h$, this will allow us to obtain path integration formulas of first and second order, which in turn allow to obtain the estimates for the derivatives of the logarithmic kernels.

Definition 1.1. An h -Brownian motion $(x_t, t \geq 0)$ we mean a Markov process with generator $\frac{1}{2}\Delta^h$. An h -Brownian bridge $(z_s^t, s \geq 0)$ is the h -Brownian motion conditioned to reach a point y_0 at a given time t . (Gradient SDEs are defined and studied in §4.)

The h -Brownian motion gives a probabilistic representation for P_t^h and for the Feynman - Kac semigroup via Feynman - Kac's path integration formula,

$$P_t^{h,V} f(x_0) = \mathbb{E} \left[f(x_t) e^{-\int_0^t V(x_s) ds} \right]; \quad (1.2)$$

while the h -Brownian bridge plays the role of the δ -measure at y_0 leading to the following formula for the Feynman - Kac kernel

$$p_t^{h,V}(x_0, y_0) = p_t(x_0, y_0) \mathbb{E} \left[e^{-\int_0^t V(z_s^t) ds} \right]. \quad (1.3)$$

Feynman-Kac formulae and Feynman-path integrals are effective tools in a range of studies. For their use in connection with stochastic processes see the book Freidlin [45] and more recently, Sturm, Albeverio, Mazzucchi and Gueneyusu, Albeverio, Kawabi and Rockner, Li and Thompson, and Rincon [85, 8, 52, 7, 71, 79].

In this article, we obtain second order Feynman-Kac formula, they are path integration formulas for the derivatives $\nabla^2 P_t^{h,V}$, allowing to obtain global estimates for $\nabla^2 \log p_t(x, y)$. We strive to obtain these on the most general manifolds.

1.3. Motivation. One motivation for this is to seek and study a probability measure on loop spaces over manifolds that plays the role of Lebesgue measure in finite dimensions and the role of Gaussian measures on linear spaces. A candidate for this is the probability distributions of Brownian bridge from x to y at time 1. The Brownian bridge on $[0, t]$ for $t < 1$ is equivalent to that of the Brownian motion measure. It is the Brownian motions conditioned to reach a point y at time 1, a Markov process with generator

$$\frac{1}{2}\Delta^h + \nabla \log p_{1-t}^h(x, y), \quad t < 1.$$

It is essential to show that $\lim_{t \rightarrow 1} y_t = 1$, which involves estimating $\nabla \log p_t^h$.

It is desirable to study the property of the Brownian bridge measure. For example, is there an L^2 Hodge-De Rham theory? What is its tail behaviour? Indeed it was shown in [3] that Poincaré inequalities implies exponential integrability of 'Lipschitz continuous

functions'. Despite the collective effort from the community, surprisingly very little is known on the validity of Poincaré inequalities. For the Euclidean space this was the celebrated work by L. Gross [51], for Lie groups see [5, 30, 43] and for the hyperbolic space see [1, 20]. See also [46, 11, 47] for defective inequalities. It appears that a key ingredient for the inequality to hold for all smooth cylindrical functions is estimates on the derivatives of the heat kernel.

1.4. On Manifolds with a pole. A good illustration for what we seek is for manifolds diffeomorphic to \mathbb{R}^n . If y_0 is a pole, the heat kernel and its derivatives are expected to be also Gaussian like, but we will need to discount the growth of the volume at a point away from y_0 . It is reasonable to base the study on an approximate ansatz :

$$k_t(\cdot, y_0) = (2\pi t)^{-\frac{n}{2}} e^{-\frac{d^2(\cdot, y_0)}{2t}} J_{y_0}^{-\frac{1}{2}},$$

where J_{y_0} is the Jacobian determinant of the exponential map at y_0 , also called the Ruse invariant, Walker and Elworthy[89, 34]. For the hyperbolic space,

$$J = \left(\frac{1}{d} \sinh d\right)^{n-1} \quad \Phi = -\frac{(n-1)^2}{8} + \frac{(n-1)(n-3)}{8} (d^{-2} - \sinh^{-2}(d)).$$

Upper and lower bounds of the following form $C \frac{1}{V(x, \sqrt{t})} e^{-\frac{d^2(x, y)}{Ct}}$ where $V(x, \sqrt{t})$ is the volume of the geodesic ball at x of radius \sqrt{t} , was first proven for manifolds of bounded geometry, see Donnelly [28] and also Cheeger, Gromov and Taylor [18]. It is remarkable that this extends to manifolds whose curvature is bounded between two constants and to manifolds of non-negative curvature Cheng, P. Li and Yau, P. Li and Yau [22, 59].

Our ansatz for the Hessian of p_t on manifolds with a pole y_0 is of the form,

$$\nabla d p_T^{h, V} = e^{h(y_0) - h(x_0)} k_t(x_0, y_0) A_T. \quad (1.4)$$

Definition 1.2. A point y_0 is a pole for a manifold M if the exponential map at y_0 is a diffeomorphism from the tangent space $T_{y_0}M$ at y_0 to M . We denote by J_{y_0} its Jacobian determinant of the exponential map, and let $\Phi = \frac{1}{2} J_{y_0}^{\frac{1}{2}} \Delta J_{y_0}^{-\frac{1}{2}}$.

Manifold with negative sectional curvature are manifolds with every point a pole. Take for example the constant negative curvature hyperbolic space is in dimension 3, then

$$p_t(x, y_0) = (2\pi t)^{-\frac{3}{2}} e^{-\frac{t}{2}} e^{-\frac{d^2(x, y_0)}{2t}} J_{y_0}^{-\frac{1}{2}}(x),$$

$$J_{y_0}(x) = \left(\frac{\sinh d(x, y_0)}{d(x, y_0)}\right)^2.$$

Even for the constant curvature hyperbolic space, computations for the derivatives of the heat kernel on are already messy. Fortunately Elworthy and Truman [33] obtained an 'elementary formula' of for the heat kernel:

$$p_t(x_0, y_0) = k_t(x_0, y_0) \mathbb{E} \left(e^{\int_0^t \Phi(\tilde{x}_s) ds} \right),$$

which we can extend to its derivatives. Here y_0 is a pole and \tilde{x}_t is a strong Markov process with time dependent Markov generator $\frac{1}{2}\Delta + \nabla \log k_{T-t}(\cdot, y_0)$ where $0 < t \leq T$. The semi-classical bridge can be considered as an approximation to the delta measure. It

enjoys the excellent property: its radial part is the n -dimensional Bessel bridge. The estimates will crucially depend on that of the Bessel bridge. Gaussian upper bound follows readily from the ‘elementary formula’,

$$p_t^V(x_0, y_0) = k_t(x_0, y_0) \mathbb{E} \left(e^{\int_0^t (\Phi - V)(\tilde{x}_s) ds} \right). \quad (1.5)$$

Unlike the standard representation by the Brownian bridge, (1.3), this representation becomes amenable for the heat kernel estimates as soon as the Riemannian metric on M is not trivial.

The elementary formula was generalised in Watling [91] and used by M. Ndumu [74] to obtain very nice heat kernel estimates. Using elementary formulas for the study of the heat kernel were pioneered by M. Ndumu [75] and S. Aida [2]. In [2] the estimates are obtained under assumptions on the derivatives of Ruse invariant and on the derivatives of the distance function to the pole, also the gradient of the Ricci curvature are assumed to be bounded. Very recently the semi-classical measure on the pinned path space is studied in Li [68] and gradient estimates for the Schrödinger operator $\frac{1}{2}\Delta^h - V$ are obtained in Li and Thompson [71], where the Ruse invariant is not differentiated.

The Hessian formula we obtain is of a similar form:

$$\nabla dp(t, v_1, v_2) = (2\pi t)^{-\frac{n}{2}} e^{-\frac{d^2(\cdot, y_0)}{2t}} J_{y_0}^{-\frac{1}{2}} \mathbb{E} \left[e^{\int_0^t \Phi(\tilde{x}_s) ds} (\tilde{N}_t + \tilde{Q}_t) \right].$$

To obtain estimates on $\nabla \log p_t$, we use this together with similar expressions for the the gradient from [71], it is then sufficient to estimate $\tilde{N}_t + \tilde{Q}_t$ and assume a bound on Φ .

By Lemmas 9.5, 9.7, we see the following hold under **C4(a)(b)**.

$$\begin{aligned} \left| \tilde{N}_t \right|_{L^p(\Omega)} &\leq c(p, n) \left(a_1(\alpha, p, T) + e^{|K|t} \frac{d^2(x_0, y_0)}{t^2} + a_2(K, t) \frac{1}{t} \right), \\ \left| \tilde{Q}_t \right|_{L^p(\Omega)} &\leq c(p, n) \left(b_4(2p) \frac{d(x_0, y_0)}{t} + b_4(2p) \frac{1}{\sqrt{t}} + b_3(2) \frac{1}{\sqrt{t}} + A \right). \end{aligned}$$

We observe that $\frac{d(x_0, x_t)}{\sqrt{t}}$, where x_t is a Brownian motion, is of order \sqrt{t} . The constants in these are explicitly given in terms of J_{y_0} , bound on the Ricci curvature Ric and the curvature operator $\|\mathcal{R}\|$, and also on a curvature term $\|\Theta\|$ which we define shortly.

Our method departs, slightly, from that of Elworthy and Truman [33]. We take advantage of the equivalence of the semi-classical bridge with the h -Brownian motion before the terminal time and navigating the singularity at the terminal time, this latter requires careful analysis. For the second order derivative of P_t^h the singularity problems becomes pronounced. We are also able to avoid differentiating the damped stochastic parallel translations along the semi-classical bridge. Instead we use the damped stochastic parallel translation equations for the h Brownian motion, and run them along semi-classical bridge paths.

The ansatz (1.4), which we will deduce under condition **C4**, would lead easily to an exact Gaussian bound for the Hessian of p_t , and we will prove the correction term A_t has the appropriate short time asymptotics. We explain now the geometric formulas appeared earlier.

1.5. Intrinsic geometric quantities. The results in this article are obtained under conditions on the bilinear form $\text{Ric} - 2\text{Hess } h$ where Ric denotes the Ricci curvature and

Hess h denotes the second derivative of h on the curvature operator \mathcal{R} , and on a curvature Θ^h which we define below, the cyclic sum of ∇Ric^\sharp . If M is a symmetric space, the gradient of the curvature vanishes, hence Θ describes a symmetry property of the manifold.

Definition 1.3. For $v_1, v_2, v_3 \in T_{x_0}M$,

$$\begin{aligned} \langle \Theta(v_2)v_1, v_3 \rangle &= \left(\nabla_{v_3} \text{Ric}^\sharp \right) (v_1, v_2) - \left(\nabla_{v_1} \text{Ric}^\sharp \right) (v_3, v_2) - \left(\nabla_{v_2} \text{Ric}^\sharp \right) (v_1, v_3), \\ \Theta^h(v_2, v_1) &= \frac{1}{2} \Theta(v_2)(v_1) + \nabla^2(\nabla h)(v_2, v_1) + \mathcal{R}(\nabla h, v_2)(v_1). \end{aligned} \tag{1.6}$$

We will first obtain a second order Feynman-Kac formula for general manifold, this is a formula for the Hessians of the semigroups, they are denoted by $\text{Hess } P_t^{h,V} f$ and $\text{Hess } p_t^{h,V}$. In such formulas neither the initial function f nor the potential function is differentiated, and then use them for estimates.

1.6. Doubly damped stochastic parallel translation equation. We are in a position to introduce a new concept: the doubly damped stochastic parallel translation equation. This is a stochastic equation driven by $\frac{1}{2} \text{Ric}^\sharp - \nabla^2 h$, \mathcal{R} and Θ^h , and use it to develop a probabilistic representation for the Hessian of the solution without potential. Such a probabilistic representation for the hessian was proved in Elworthy and Li [37], where the solutions of a gradient stochastic differential equation (SDE) together with its linearised and the twice linearised equations are used; a local Hessian formula was presented in Arnaudon, Plank and Thalmaier [10] where $V = 0$; also a formula for $\Delta P_t f$ was given in Elworthy and Li [38] not involving the derivative of the Ricci curvature. Differentiation formulas of all orders for heat kernels was obtained in Norris [76], generalising a formula in Bismut [17] for $\nabla \log p_t$ in terms of the Brownian bridge, see also Driver and Thalmaier [31].

The stochastic damped parallel translation process (W_t) is given by an ODE driven by the zero order part of the weighted Laplacian on differential 1-forms, $\Delta^h = -\nabla^* \nabla - (\frac{1}{2} \text{Ric}^\sharp - \nabla^2 h)$. They are also the local conditional expectations of the derivative flows, with respect to the filtration of (x_t) , the latter are solutions of the linearised gradient SDE. Thus $dP_t f(v) = \mathbb{E} df(W_t(v))$. Using the solutions $W_t^{(2)}$ to the doubly damped stochastic parallel translation equation, we prove that

$$\nabla df(v_1, v_2) = \mathbb{E} \nabla df(W_t(v_2), W_t(v_1)) + \mathbb{E} df(W_t^{(2)}(v_2, v_1)),$$

holds under conditions **C1(a)**, **C3**, and **C5(g)** in §3.

1.7. Commutative relations. The Hessian of P_t^h does not satisfy the commutative relations enjoyed by d and Δ^h , $de^{\frac{1}{2}\Delta^h} f = e^{\frac{1}{2}\Delta^h} (df)$ and $\Delta^h e^{\frac{1}{2}\Delta^h} f = e^{\frac{1}{2}\Delta^h} (\Delta^h f)$, explaining the involvement of Θ^h in the doubly damped stochastic parallel translation whose solutions are in fact the local conditional expectations of the spatial derivatives of W_t .

To avoid differentiating the potential term, we use the variation of constant formula and has the following formal expression:

$$\text{Hess } P_t^{h,V} f(v_1, v_2) = \text{Hess } P_t^h f(v_1, v_2) + \int_0^t \text{Hess} (P_{t-r}^h (V P_r^{h,V} f)) (v_1, v_2) dr.$$

Every time the differentiation is shifted from the initial value f to the heat kernel, we see a loss of integrability in time of the order of $\frac{1}{\sqrt{t}}$. In particular $|\text{Hess } P_{t-r}^h|$ is of order $\frac{1}{t-r}$ and the integral given above has a formal singularity at $r = t$. See Theorem 7.6 which is proved under condition **C1**. Another type of singularity will present itself when we work on the Feynman-Kac kernel $p_t^{h,V}$.

We outline the paper. In §2 we introduce the notation, discuss the intuitive ideas, motivations and related work. The main results and *geometric conditions* will be presented in §3. In §4 moment estimates and the strong 1-completeness for gradient SDEs are obtained. In §5 we prove the primitive Hessian formula, define and study the doubly damped stochastic parallel translation equation estimating the norm of its solutions and proving their exponential integrability, and estimate the relevant terms in the Hessian formulas given in §7. The immediate consequences of the Hessian formulas are the basic Hessian estimates for general manifolds §8. In §9 we justify the ‘exact Gaussian formulas’ for the Hessians in terms of the semi-classical bridge, leading to exact Gaussian estimates, §10, for $\nabla^2 p_t$.

2. Preliminaries

We begin with explaining the intrinsic ideas behind the damped stochastic parallel translation, with heat equations on differential one forms and the derivative flow of a gradient SDE. We then discuss estimates on the moments of the derivative flows and the continuous dependence of the solution flow of an SDE on its initial value. Finally we remarks on Hessian estimates for the logarithms of the heat kernels.

Throughout this paper $\{\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}\}$ denotes the underlying probability space with filtration \mathcal{F}_t satisfying the usual hypothesis. Let $\{B_t^i\}$ be a family of independent one-dimensional Brownian motions and we write $B_t = (B_t^1, \dots, B_t^m)$. The notation $\circ dB_s$ indicates Stratonovich integration.

2.1. Heat equation for differential 1- form and damped parallel translation. Let $\text{Ric}_x : T_x M \times T_x M \rightarrow \mathbb{R}$ denote the Ricci curvature, $\text{Ric}_x^\sharp : T_x M \rightarrow T_x M$ be the linear map given by $\langle \text{Ric}_x^\sharp u, v \rangle = \text{Ric}_x(u, v)$. Similarly $(\nabla^2 h)^\sharp = \nabla \cdot \nabla h$. We also denote by Ric_x^\sharp the corresponding operator on differential 1-forms. Let δ^h denote the L^2 adjoint of the differential d with respect to the weighted volume measure $e^{2h} dx$, then $\Delta^h = -(d + \delta^h)^2$. Furthermore $d + \delta^h$ is essentially self-adjoint on the space of smooth functions (and on smooth differential forms) with compact supports, on which d and Δ^h commute, Li [62]. In particular $dP_t^h f$ solves the following equation on differential 1-forms with initial condition the exact differential 1-form df ,

$$\frac{\partial}{\partial t} \phi_t = \frac{1}{2} (\nabla)^{h,*} \nabla \phi_t + \left(-\frac{1}{2} \text{Ric}_x^\sharp + (\nabla^2 h)^\sharp \right) \phi_t,$$

where $(\nabla)^{h,*}$ is the adjoint of ∇ with respect to $e^{2h} dx$.

It is therefore not surprising that probability representations for $dP_t^h f$ involves the solution to the following damped stochastic parallel equation

$$\frac{DW_t}{dt} = \left(-\frac{1}{2} \text{Ric}_{x_t}^\sharp + (\nabla^2 h)^\sharp \right) (W_t), \quad (2.1)$$

Its solution with initial value v_0 will be denoted by $W_t(v_0)$. It is a stochastic process of finite variation. Here the equation is defined path by path, for each ω , $\frac{DW_t(\omega)}{dt} = //t \frac{d}{dt} //t^{-1}$ and $//t \equiv //t(x, \omega)$ is the stochastic parallel translation along the the h -Brownian paths. The stochastic process W_t takes values in $\mathbb{L}(T_{x_0}M; T_{x_t}M)$.

The following probabilistic representation,

$$e^{\frac{1}{2}t\Delta}df(v) = \mathbb{E}df(W_t(v)),$$

was given in H. Airault [4], initially for $h = 0$ and for compact manifolds. The formula holds in fact for Brownian motion with a general drift.

2.2. Linearised SDE, the derivative flow, and strong 1-completeness. For $r \geq 2$, let $\{X_i\}_{i=1}^m$ be C^{r+1} vector fields, X_0 a C^r vector field. Let $(F_t(x), t < \xi(x))$ denote the maximal solution to the SDE

$$dx_t = \sum_{i=1}^m X_i(x_t) \circ dB_t^i + X_0(x_t)dt = X(x_t) \circ dB_t + X_0(x_t)dt. \quad (2.2)$$

with initial value x . For a fixed point $x_0 \in M$ we write $x_t = F_t(x_0)$ for simplicity. The SDE is said to be *complete or conservative* if for every x , $\xi(x) = \infty$ almost surely, which does not necessarily mean that for a common set ω of measure zero, $F_t(x)$ has infinite life time for every x . The completeness property is determined by the generator of the SDE. Assume the SDE is complete. Another useful commutation relation, should it hold, is

$$dP_t f(v) = \mathbb{E}df(TF_t(v)),$$

where $TF_t(v)$ is the *derivative flows* of (2.2),

Definition 2.1. The derivative flow is solution to the linearised SDE:

$$Dv_t = \sum_{i=1}^m \nabla X_i(v_t) \circ dB_t^i + \nabla X_0(v_t)dt.$$

Let $\mathcal{F}_t^{x_0} = \sigma\{x_s : s \leq t\}$ and $v \in T_{x_0}M$. Then $W_t(v)$ is the local conditional expectation of $TF_t(v)$, and if the latter is integrable,

$$W_t(v) = \mathbb{E}\{TF_t(v) | \mathcal{F}_t^{x_0}\}.$$

Definition 2.2. We say an SDE is *strongly complete* or it has a *global smooth solution flow* if $(t, x) \mapsto F_t(x, \omega)$ is continuous on $\mathbb{R}_+ \times M$ almost surely.

For one dimensional manifolds, the two concepts are equivalent. It is common to believe that well posedness implies strong completeness, or the first exit time of the solution from a set with smooth boundary is continuous with respect to the initial value. See e.g. Li and Scheutzw [70] for discussions on counter examples. Consider $\dot{z} = z^2$ on the complex plane, a solution of which has finite life time if and only if its initial value is a non-zero real number. Indeed for $\delta \neq 0$ and $x_0 \neq 0$, consider $z(t) = \frac{x_0(1+i\delta)}{1-x_0(1+i\delta)t}$. Its norm is bounded in time and so for sufficiently large $a > 0$, its first exit time from the ball of radius a is infinity, unless $\delta = 0$; while the solution from x_0 will exit the ball in finite time. For an SDE, even if the solutions from any given point does not explode, the solution must restart from random initial points for which the uncountability of the exceptional zero measure sets must be taken into consideration. If the system is strongly

complete then for every t , $x \mapsto F_t(x, \omega)$ is C^r . It is sufficient to use a weaker notion of strong completeness.

Definition 2.3. [64] The SDE is strongly 1-complete if for any smooth curves $\sigma : [0, 1] \rightarrow M$, $F_t(\sigma(s), \omega)$ is continuous in (s, t) almost surely for all $t \in [0, T]$ and $s \in [0, 1]$ for some $T > 0$.

This is a weaker concept than the existence of a global smooth solution flow. Between strong 1-completeness and strong completeness there is a range of notions: the strong p -completeness for $p = 2, \dots, n - 1$. Roughly speaking the flow takes a p -dimensional sub-manifold to a p -dimensional sub-manifold. If $p = n - 1$, this is equivalent to strong completeness. If $M = \mathbb{R}^n$, X_i are globally Lipschitz continuous and the stochastic integral are in Itô form, then the strong completeness follows from the fixed point theorem. For compact manifolds, the complete SDE can be lifted to a complete SDE on the group of diffeomorphism over M and the strong completeness follows. Because of the difficulty in localising, we introduce the derivative flow method to obtain strong completeness for non-compact manifolds and for SDEs on \mathbb{R}^n without Lipschitz conditions.

We illustrate the use of this in the lemma below is taken from Li [64, Cor 9.2], we make a slight improvement on this, replacing the continuity of $|T_{\sigma(s)}F_t|$ in L^1 there by a more intrinsic condition.

Lemma 2.4. *Suppose that (2.2) is strongly 1-complete with $\mathbb{E}|TF_t|$ finite. Suppose that $s \mapsto |W_t(\dot{\sigma}(s))|$ is continuous in L^1 . If f is BC^1 then for $v \in T_{x_0}M$ and $x_0 \in M$, $d(P_t^h f)(v) = \mathbb{E}[df(W_t(v))]$.*

Proof. Take a normal geodesic curve $\sigma : [0, \ell] \rightarrow M$ with initial value x_0 and initial velocity v . By the strong 1-completeness, for almost all ω , $F_t(\sigma(s))$ is differentiable with respect to s and $d(P_t f)(v) = \lim_{s \rightarrow 0} \mathbb{E} \left(\frac{1}{s} \int_0^s df(T_{\sigma(r)}F_t(\dot{\sigma}(r), \omega)) dr \right)$. If $T_{\sigma(r)}F_t(\dot{\sigma}(r))$ is integrable, then $\mathbb{E}\{TF_t(\dot{\sigma}(r)) | \mathcal{F}_t^{\sigma(r)}\}$ is $W_t(\dot{\sigma}(r))$, from which we see that

$$d(P_t f)(v) = \lim_{s \rightarrow 0} \frac{1}{s} \int_0^s \mathbb{E}[df(W_t(\dot{\sigma}(r)))] dr.$$

By the strong 1-completeness $W_t(\dot{\sigma}(r))$ is continuous in r . If it is continuous in L^1 and $f \in BC^1$ we may take $r \rightarrow 0$ to obtain $\mathbb{E}df(W_t)$ on the right hand side. \square

Remark 2.5. If $\sup_{s \in [0, \ell]} \mathbb{E}|W_t(\dot{\sigma}(s))|^{1+\delta}$ is finite for some $\delta > 0$, then $s \mapsto |W_t(\dot{\sigma}(s))|$ is continuous in L^1 , which holds in particular under one of the conditions from **C5(a)** to **C5(d)**. See Lemma 4.2.

The derivative flow describes the evolution of the distance, $d(F_t(x), F_t(y))$, between two solutions with initial values x and y respectively. We denote by $v_t := TF_t(v_0)$ its solution with initial value $v_0 \in T_{x_0}M$. Then, c.f [63],

$$|v_t|^p = |v_0|^p + p \sum_{i=1}^m \int_0^t |v_s|^{p-2} \langle \nabla_{v_s} X^i, v_s \rangle dB_s^i + \frac{p}{2} \int_0^t |v_s|^{p-2} H_p(x_s)(v_s, v_s) ds, \quad (2.3)$$

where for $v \in T_x M$,

$$H_p(x)(v, v) = \sum_{i=1}^m \langle \nabla^2 X^i(X^i, v), v \rangle + \sum_1^m \langle \nabla_{\nabla_v X^i} X^i, v \rangle + \sum_1^m |\nabla X^i(v)|^2 + (p-2) \frac{1}{|v|^2} \sum_1^m \langle \nabla_v X^i, v \rangle^2 + 2 \langle \nabla_v X_0, v \rangle. \quad (2.4)$$

set

$$\bar{H}_p(x) = \sup_{|v|=1, v \in T_x M} H_p(v, v) \quad (2.5)$$

If \bar{H}_1 is bounded above, then $\sup_{x \in D} \mathbb{E} \sup_{s \leq t} |T_x F_s|$ is finite for every compact set D and the SDE is strongly 1-complete. If for a number $p \geq 1$ and a function $f : M \rightarrow \mathbb{R}$ satisfying that

$$\sup_{x \in D} \mathbb{E} \exp \left(6p^2 \int_0^t f(F_s(x)) \chi_{s < \xi(x)} ds \right) < \infty, \quad (2.6)$$

the following two conditions hold, (a) $\sum_{i=1}^m |\nabla X_i|^2 \leq f$, (b) $\bar{H}_p \leq 6f$. Then

$$\sup_{x \in D} \mathbb{E} \sup_{s \leq t} |T_x F_s|^p < \infty$$

for every compact set K and the SDE is strongly 1-complete. For a weaker criterion see Li [64, Thm. 3.1 & 5.3]. For $\sup_{x \in D} \mathbb{E} |T_x F_t|^p$ to be finite, we may use localising techniques to remove condition (a). See also Chen and Li [19] for the case where the coefficients are not smooth.

2.3. Hessian Estimates. Finally we comment on $\log p_t$. The gradient and the Hessian of p_t have been considered in the study of parabolic Harnack inequalities, see Li and Yau, Hamilton, Aizenman and Simon [59, 53, 6], also the book by Jost [56]. For bounded strict elliptic second order differential operators on \mathbb{R}^n , they are obtained in Sheu [80]. For compact manifolds the relevant small time gradient estimate was given and used in Driver [29]. The remarkable Hessian estimates $|\nabla^2 \log p_t| \leq C(\frac{1}{\sqrt{t}} + \frac{d}{t})^2$ and indeed ‘off cut-locus’ estimates for derivatives of all order are given in Malliavin and Stroock [72] with generalisation to that in Hsu, Aida, Atroock-Turetsky-98 [55, 1, 84], and more recently, XD Li [61], see also Engoulatov, Coulhon and Zhang [41, 23].

These estimates are naturally relevant in the study of loop spaces (continuous loops), where the Brownian bridge measure is used as a reference measure playing the role of the Lebesgue measure on \mathbb{R}^n . The basic questions concerning the Brownian bridge measures include the tail behaviour of the measure, Poincare and Logarithmic Sobolev inequalities (LSI), which we discuss further later. One also wish to obtain a Hodge decomposition theorem of the space of L^2 spaces of differential forms and to seek a Hodge deRham theorem linking the topology of the infinite dimensional pinned path space with the cohomology defined by the Hodge decomposition, Shigekawa, Fang and Franchi, Elworthy and Li, Elworthy and Li, and Aida [81, 44, 39, 40, 1].

If $M = \mathbb{R}^n$, the Brownian bridge measure is Gaussian for which L. Gross [51] proved his celebrated logarithmic Sobolev inequality, this extends to pinned path spaces over Lie groups, Airault and Malliavin, Driver and Lohrenz, and Frang [5, 30, 43]. Although there are progress, for example see Gong and Rockner and Wu [46], Arnaudon and Simon [11], Gong and Ma [47], a counter example of a manifold for which Poincaré inequality does not hold on its loop space was given in Eberle [32]. Further positive results were established in Chen-Li-Wu and Aida [20, 1], the first is for the hyperbolic space and the latter was for asymptotically Euclidean manifolds. It is likely that the validity of the Poincaré inequality for the natural metric induced by stochastic parallel translation measures how far the Brownian bridge deviates from a Gaussian measure. In particular we have the conjecture.

Conjecture. Poincaré inequality on the loop space, with reference measure the Brownian bridge measure, does not hold on the spheres.

3. Main Results

Let (x_t) be an h -Brownian with initial value x_0 , $//_t \equiv //_t(x_\cdot) : T_{x_0}M \rightarrow T_{x_t}M$ the stochastic parallel translation, and $W_t : T_{x_0}M \rightarrow T_{x_t}M$ the damped stochastic parallel translation so that for almost every ω , $//_t^{-1}W_t(\omega)$ is a matrix valued stochastic process solving a linear ODE given by $-\frac{1}{2}//_t^{-1}Ric^\sharp//_t + //_t^{-1}\nabla^2h(//_t\cdot)$.

Let $v_1, v_2 \in T_{x_0}M$, fix $W_t(v_1), W_t(v_2)$. We introduce the doubly damped stochastic parallel translation equation, along the h -Brownian paths:

$$\begin{aligned} DW_t^{(2)}(v_1, v_2) &= \left(-\frac{1}{2}Ric^\sharp_{x_t} + \nabla \cdot \nabla h \right) \left(W_t^{(2)}(v_1, v_2) \right) dt \\ &\quad + \frac{1}{2}\Theta^h(W_t(v_2))(W_t(v_1))dt \\ &\quad + \mathcal{R}(d\{x_t\}, W_t(v_2))W_t(v_1), \\ W_0^{(2)}(v_1, v_2) &= v_2, \quad v_1, v_2 \in T_{x_0}M, \end{aligned} \tag{3.1}$$

where $d\{x_t\}$ denotes integration with respect to the martingale part of (x_t) , see Definition 7.1.

Definition 3.1. The solution to the doubly damped covariant stochastic integral equation, denoted by $W_t^{(2)}(v_1, v_2)$, is called the doubly damped stochastic parallel translation.

Unlike the damped stochastic parallel translation, which is a process of finite variation and whose norm has a pointwise upper bound $\|W_t\|^2 \leq e^{-\int_0^t \rho^h(x_s)ds}$, the doubly damped one has a non-trivial martingale part. The estimates on $\|W_t^{(2)}\|$ are given in §5. Its exponential integrability requires growth conditions on the curvature \mathcal{R} and Θ . These estimates are also essential for validating taking the time of the semi-classical Brownian bridge to the terminal time in the forthcoming formula for the hessian of the Feynman-Kac formula for manifolds with a pole, see e.g. §9.1, Lemmas 6.1 and 6.3.

3.1. Geometric Conditions and strong completeness. We state the set of conditions to which we refer throughout this paper. Let \mathcal{R} denote the curvature and Θ is the cyclic sum of the covariant derivatives of the Ricci curvature defined by (1.6). Let ST_xM denotes the unit tangent space at x , $v_i \in T_xM$, and let $\{E_i\}$ be an o.n.b. of the tangent space T_xM . Denote by $\|\mathcal{R}(\cdot, v_2)v_1\|$ the Hilbert-Schmidt norm of the linear map $\mathcal{R}(\cdot, v_2)v_1 : T_xM \rightarrow T_xM$,

$$\|\mathcal{R}(\cdot, v_2)v_1\| := \left(\sum_{i=1}^n |\mathcal{R}(E_i, v_2)(v_1)|^2 \right)^{\frac{1}{2}}.$$

Set

$$\|\mathcal{R}_x\| = \sup_{v_1, v_2 \in ST_xM} \|\mathcal{R}(\cdot, v_2)v_1\|, \quad \|\Theta^h\| = \sup_{v_1, v_2 \in ST_xM} \{|\Theta^h(v_2)v_1|\}. \tag{3.2}$$

We then fix an isometric immersion of the manifold in an Euclidean space \mathbb{R}^m and denote by α_x its second fundamental form. Set

$$\begin{aligned} H_1(v, v) &= -\text{Ric}(v, v) + 2\text{Hess}(h)(v, v) + |\alpha_x(v, \cdot)|_{\text{HS}}^2 - |\alpha_x(v, v)|^2 \\ H_2(v, v) &= -\text{Ric}(v, v) + 2\text{Hess}(h)(v, v) + |\alpha_x(v, \cdot)|_{\text{HS}}^2. \end{aligned} \quad (3.3)$$

We first observe that $|\alpha_x(v, \cdot)|_{\text{HS}}^2 \geq |\alpha_x(v, v)|^2$ and if $m = n + 1$, then

$$H_1 = -\text{Ric} + 2\text{Hess}(h).$$

Set for $p = 1, 2$,

$$\begin{aligned} \overline{H}_p(x) &= \sup_{|v|=1, x \in T_x M} H_p(v, v), \\ \rho^h(x) &= \inf_{v \in T_x M, |v|=1} \{\text{Ric}(v, v) - 2\text{Hess}(h)(v, v)\}. \end{aligned}$$

We remark that H_1 or H_2 does not appear in the Hessian formula nor in the estimates. The controls over \overline{H}^p leading to L^p boundedness of the derivative flows on any compact time interval and the bounds are locally uniform in space.

We are going to gather conditions together. Condition **C1** is used to obtain precise formulas; Condition **C2** is used to obtain exponential integrability of the doubly damped process; Conditions **C2** and **C1(c)** are used to obtain Hessian estimates on a general complete Riemannian manifold; **C4** is used to obtain estimates on manifolds with a pole.

Let $K \in \mathbb{R}$, $c, C, \delta, \delta_1, \delta_2$ are positive constants. Let α_2 be a sufficiently small positive number. Let r denotes the distance function from a given point. The following conditions, some of overlapping are used for a variety of results:

- Condition 3.2 (C1).* (a) $\rho^h \geq -K$;
 (b) $\sup_{s \leq t} \mathbb{E}(\|W_s^{(2)}\|^2) < \infty$;
 (c) for all $f \in BC^2(M; \mathbb{R})$, $v_1, v_2 \in T_{x_0} M$,

$$\text{Hess}(P_t f)(v_2, v_1) = \mathbb{E}[\nabla df(W_t(v_2), W_t(v_1))] + \mathbb{E}[df(W_t^{(2)}(v_2, v_1))]. \quad (3.4)$$

Condition 3.3 (C2).

$$|\rho^h| \leq K, \quad \|\mathcal{R}_x\| \leq \|\mathcal{R}\|_\infty, \quad \|\Theta^h\|^2 \leq c + \delta r^2$$

for δ sufficiently small.

Condition 3.4 (C3). $\|\Theta^h\| + \|\mathcal{R}\| \leq ce^{Cr}$ for $C < \alpha_2$.

The following is from **Proposition 5.6**.

Proposition 3.5. *Assume **C1(a)+C3**. If furthermore a gradient SDE is strongly one complete with square integrable derivative flow, locally uniform in time and in space, then **C1(c)** holds.*

Proposition 3.6. *By Lemma 5.5, Conditions **C1(a) + C3** imply **C1(b)** hold.*

Condition 3.7 (C4). Let y_0 be a pole.

- (a) $\rho^h \geq -K$, $|\nabla \log J| + |\nabla h| \leq ce^{\delta_1 r^2}$ where $\delta_1 < \alpha_2$, and **C3**.

(b) $\Phi^h \leq C + \delta_2 r^2$ where δ_2 is sufficiently small and

$$\Phi^h = -\frac{1}{2}|\nabla h|^2 - \frac{1}{2}\Delta h + \frac{1}{2}J_{y_0}^{\frac{1}{2}}\Delta J_{y_0}^{-\frac{1}{2}}.$$

(c) **C1(c)**.

Condition 3.8 (**C5**). (a)

$$\rho^h \geq -C(1+r^2), \quad \langle \nabla h, \nabla r \rangle \leq c(1+r);$$

(b) $\text{Ric} \geq -C(1+r^2)$ and $\langle \nabla h, \nabla r \rangle \leq c(1+r)$;

(c) There exist Borel measurable functions $f, g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ s.t.

$$|h| \leq f \circ r, \quad \rho^h \geq -g \circ r, \quad (fg)(s) \leq c(1+s^2).$$

(d) $\rho^h \geq -K$.

(e) $\overline{H}_1 \leq c(1+\ln r)$

(f) $\overline{H}_1 \leq c(1+r^2)$.

(g) $\overline{H}_2 \leq c + \delta r^2$, δ is sufficiently small.

The following is either known or proved in the article.

Proposition 3.9. • Any one of the conditions from **C5(a)** to **C5(d)** implies that the h -Brownian motions do not explode and finite moments of all order of its radial process.

- If in addition **C5(e)** holds, the gradient SDE (c.f. §4), with generator $\frac{1}{2}\Delta^h$, is strongly 1-complete. If **C5(d)**+**C5(f)** hold, the corresponding gradient SDE is strongly 1-complete and the square of its radial process is exponentially integrable for a small exponent. See Theorem 4.4.
- If **C5(g)** holds, the gradient SDE has square integrable derivative flow, with bound locally uniform in time and in space.

Corollary 3.10. By Proposition 5.6, **C1(a)** + **C3**+ **C5(g)** \implies **C1(c)**.

Remark 3.11. The assumption for the strong 1-completeness is purely technical, and so are conditions on \overline{H}^p , which validates differentiating the stochastic flow of the gradient SDE with respect to the initial data.

3.2. Feynman-Kac formula. Let V be a positive bounded Hölder continuous function. Set

$$\begin{aligned} N_t &= \frac{4}{t^2} \int_{\frac{t}{2}}^t \langle d\{x_s\}, W_s(v_1) \rangle \int_0^{\frac{t}{2}} \langle d\{x_s\}, W_s(v_2) \rangle, \\ Q_t &= \frac{2}{t} \int_0^{t/2} \langle d\{x_s\}, W_s^{(2)}(v_1, v_2) \rangle, \\ \mathbb{V}_{a,t} &= (V(x_a) - V(x_0))e^{-\int_a^t (V(x_s) - V(x_0))ds}, \quad \mathbb{V}_t := e^{-\int_0^t V(x_s)ds}. \end{aligned} \tag{3.5}$$

For a general manifold we have a Hessian formula in terms of the above terms, leading to estimates on $\nabla^2 \log p_t$.

Below we will present the version of the second order Feynman-Kac formula on manifolds with a pole. Let (x_t) denote the semi-classical bridge \tilde{x}_t with initial value x_0 . We define the processes just $\tilde{W}_t, \tilde{W}_t^{(2)}, \tilde{N}_t, \tilde{M}_t$, and $\tilde{V}_{a,t}$ as in (2.1) and (3.1) and (3.5) with x_t replaced by \tilde{x}_t .

Remark 3.12. If (x_t) is a Brownian motion with a drift, solving an SDE, then W_t is the conditional expectation of the solution to the linearised SDE, the derivative flow. The linearised SDE will involve the derivative of the driving vector fields which would have seemed to be, at first glance, the appropriate tangent processes along the semi-classical bridge. Our \tilde{W}_t process along (\tilde{x}_t) will not be deduced from the derivative flow of the corresponding SDE for the semi-classical bridge, in particular we do not differentiate $\nabla \log k_t$, nor do we differentiate the distance function. The same remark applies to $\tilde{W}_t^{(2)}$.

Theorem 9.8. Assume **C4**. Let $T > 0$ and $\beta_T^h = e^{\int_0^T \Phi^h(\tilde{x}_s) ds}$. Then

$$\begin{aligned} & e^{h(x_0) - h(y_0) + V(x_0)T} \frac{\nabla dp_T^{h,V}(v_1, v_2)}{k_T(x_0, y_0)} \\ &= \mathbb{E} \left[\beta_T (\tilde{N}_T + \tilde{Q}_T) \right] + \int_0^T \mathbb{E} \left[\beta_T \tilde{V}_{T-r, T} (\tilde{Q}_{t-r} + \tilde{N}_{T-r}) \right] dr. \end{aligned} \quad (3.6)$$

Estimates for the Hessians of p_t follows easily from Lemmas 9.5, 9.7, which states the following hold under **C4(a)(b)**.

$$\begin{aligned} \left| \tilde{N}_T \right|_{L^p(\Omega)} &\leq c(p, n) \left(a_1(\alpha, p, T) + e^{K|T|} \frac{d^2(x_0, y_0)}{T^2} + a_2(K, T) \frac{1}{T} \right), \\ \left| \tilde{Q}_T \right|_{L^p(\Omega)} &\leq c(p, n) \left(b_4(2p) \frac{d(x_0, y_0)}{T} + b_4(2p) \frac{1}{\sqrt{T}} + b_3(2) \frac{1}{\sqrt{t}} + A \right), \end{aligned}$$

where the constants are explicitly given in terms of J_{y_0} , $\|\Theta^h\|$, K and $\|\mathcal{R}\|$, see Definitions 9.4 and 9.6.

4. Gradient SDEs: Integrability and Strong 1-completeness

Definition 4.1. If $i : M \rightarrow \mathbb{R}^m$ is an isometric embedding we define $X(x)(e)$ to be the gradient of the function $\langle i(x), e \rangle$ where $e \in \mathbb{R}^m$ and $Y(x)$ the adjoint of $X(x) : \mathbb{R}^m \rightarrow T_x M$. If $\{e_i\}$ is an orthonormal basis of \mathbb{R}^m we define $X_i(x) = X(x)(e_i)$ then the following is called a gradient Brownian system,

$$dx_t = \sum_{i=1}^m X_i(x_t) \circ dB_t^i + \nabla h(x_t) dt, \quad (4.1)$$

In this case, $\nabla_v X^i = A_x(v, Y(x)e_i)$, $Y(x) : \mathbb{R}^m \rightarrow \nu_x$ is the orthogonal projection to the normal bundle, A_x is the shape operator given by the relation

$$\langle A_x(v_1, w), v_2 \rangle = \langle \alpha_x(v_1, v_2), w \rangle$$

and $\alpha_x : T_x M \times T_x M \rightarrow \nu_x$ is the second fundamental form of the embedding. If $|\alpha_x(v, \cdot)|_{HS}^2$ denotes the Hilbert-Schmidt norm of $\alpha_x(v, \cdot)$, i.e. $\sum_{j=1}^m |\alpha_x(v, f_j)|_{\nu_x}^2$ for $\{f_i\}$ an o.n.b. of $T_x M$. Then

$$\sum_1^m \langle \nabla_v X^i, v \rangle^2 = |\alpha_x(v, v)|_{\nu_x}^2, \quad \sum_1^m |\nabla_v X^i|^2 = \sum_{j=1}^m |\alpha_x(v, f_j)|_{\nu_x}^2.$$

The map $H_p : TM \times TM \rightarrow \mathbb{R}$, defined in (2.4), is given by geometric quantities:

$$H_p(v, v) = -\text{Ric}(v, v) + 2\text{Hess}(h)(v, v) + |\alpha_x(v, \cdot)|_{HS}^2 + (p-2)|\alpha_x(v, v)|^2.$$

In particular for a co-dimension one sub-manifold of an Euclidean space,

$$H_1 = -\text{Ric} + 2\text{Hess}(h).$$

(Note that H_1 is a classic object: D. Bakry showed that the h -gradient Brownian system is complete if ρ^h is bounded from below. If furthermore $\int_0^\infty \mathbb{E} e^{-\int_0^t \rho^h(x_s) ds} dt$ is finite, it was shown in [65] that the weighted measure e^{2h} is finite, consequently the gradient system has a finite invariant measure and the fundamental group $\pi_1(M)$ is finite.) Let x_0 be a point in the manifold, C, K are constants, and r denotes the distance function from a given point, e.g. y_0 . Recall \overline{H}_1 is the supremum of the bilinear form H_1 , c.f. (2.5).

Lemma 4.2. *Assume the dimension of M is greater than 1.*

- (1) *Under one of the following conditions,*
 - (a) $\text{Ric} \geq -C(1+r^2)$ and $dr(\nabla h) \leq C(1+r)$;
 - (b) $\rho^h \geq -C(1+r)$,*the h -Brownian motion (x_t) is complete and for any $t > 0$,*

$$\sup_{s \leq t} \mathbb{E}[d^p(x_s, y_0)] \leq c_1(p)[d^p(x_0, y_0) + t]e^{c_2(p)t}.$$

If furthermore $\overline{H}_1 \leq c(1 + \ln r)$, then the gradient Brownian system (4.1) is strongly 1-complete.

- (2) *Suppose that $\rho^h \geq -K$. There exists a number $\alpha_1 > 0$ such that for any compact set D , for any $t > 0$ and for any θ satisfying $\theta t < \alpha_1$,*

$$\sup_{y_0 \in D} \sup_{s \leq t} \mathbb{E} \left(e^{\theta d^2(x_s, y_0)} \right) < \infty.$$

Also, for any $p > 0$, $\mathbb{E}[\sup_{s \leq t} d(x_s, y_0)^p] \leq c_1(p)e^{c_2(p)t}t^{\frac{p}{2}}$ for some constants $c_1(p), c_2(p)$. If furthermore $\overline{H}_1 \leq C(1 + d^2(\cdot, y_0))$, the gradient SDE (4.1) is strongly 1-complete.

Proof. Fix $y_0 \in M$, we write $r_t = d(x_t, y_0)$ and $r(x) = d(x, y_0)$. For $p \geq 2$, we apply Itô's formula to r . On $\{t < \zeta(x_0)\}$, where $\zeta(x_0)$ is the life time of the h -Brownian motion from x_0 , the following holds,

$$\begin{aligned} r_t^p &= d^p(x_0, y_0) + p \sum_{i=1}^m \int_0^t r_s^{p-1} \langle \nabla r, X_i \rangle_{x_s} dB_s^i + \frac{1}{2} p(p-1) \int_0^t r_s^{p-2} ds \\ &\quad + \frac{1}{2} p \int_0^t r_s^{p-1} (\underline{\Delta} r + 2L_{\nabla h} r)(x_s) ds - L_t^{\text{Cut}} \end{aligned}$$

where L_t^{Cut} is a non-negative term, vanishing off the cut locus, $\underline{\Delta} r$ is the Laplacian of the distance function off the cut locus of y_0 and vanishes on the cut locus. See [24], especially for $h = 0$. This can be obtained also by taking a smooth approximation r_ϵ of r and applying to them Itô's formula, and the following distributional inequality [93]. For positive test function f ,

$$\int_M \Delta r f \leq \int_{M \setminus \{\text{Cut}(y_0)\}} r \Delta f,$$

and so the measure $\Delta r(1_{\{\text{Cut}(y_0)\}})$ is non-positive and $L_t^{\text{Cut}} \geq 0$. Note that Δr is of order $\frac{1}{r}$ near zero, and $r^{p-1} \Delta r$ vanishes for $p > 2$.

Let us take $p \geq 1$. For part (1a), we apply the standard Laplacian comparison theorem to Δr . On the whole manifold where $r(x)$ is smooth, $\Delta r(x)$ is less or equal to $\sqrt{(n-1)K} \cot\left(r\sqrt{\frac{K}{n-1}}\right)$ if $\text{Ric} \geq K$ and K is positive, $\frac{n-1}{r}$ for $K = 0$ and $\sqrt{-K(n-1)} \coth\left(r\sqrt{\frac{-K}{n-1}}\right)$ if $K < 0$. Taking the infimum of the lower bound of the Ricci curvature over an exhausting sequence of relatively compact sets, noting that Δr is of the order $\frac{1}{r}$ near $r = 0$ and otherwise it grows at the order of the square root of the infimum of the Ricci curvature, we see that there exists a constant c such that

$$r^{p-1}(\Delta r + 2L_{\nabla h}r) \leq c + cr^p.$$

We do not need to worry the exhausting sets where the lower bound of the Ricci curvature is non-negative. The conclusion of part (1a) follows from a localising procedure which removes the local martingale part in the formula for r_t^p , Young's inequality to bound $r_t^{p-2} \leq c(p)r_t^p + c(p)$, Gronwall's inequality, and Fatou's lemma to conclude finiteness of the moments of the radial process from any initial point: for some constants $c_i(p)$,

$$\sup_{s \leq t} \mathbb{E}[d^p(x_s, y_0)] \leq c_1(p)(d^p(x_0, y_0) + t)e^{c_2(p)t}. \quad (4.2)$$

In particular the SDE is complete.

For part (1b) we apply Bochner's formula to r . Denote ∂_r covariant differentiation w.r.t. ∇r . Since $|\nabla r| = 1$,

$$\partial_r(\Delta r) = -|\text{Hess } r|^2 - \text{Ric}(\nabla r, \nabla r).$$

Using the identity $\partial_r \langle \nabla h, \nabla r \rangle = 2 \text{Hess}(h)(\nabla r, \nabla r)$, we see that

$$\partial_r(\Delta^h r) = -|\text{Hess } r|^2 + (-\text{Ric} + 2 \text{Hess}(h))(\nabla r, \nabla r) \leq -\rho^h.$$

If $\rho^h \geq K$, along a geodesic segment γ from y_0 , $\Delta^h r \leq -Kr - \Delta^h r(\gamma(0))$. From the assumption that $\rho^h \geq -C(1+r)$, we have again $r^{p-1}\Delta^h r \leq c + cr^p$. The same argument as before leads to (4.2), concluding also non-explosion.

By Theorem 8.5 in [64] the gradient flow is strongly 1-complete if

$$\sup_{x \in K} \mathbb{E} \left(e^{c \int_0^t \overline{H}_1(F_s(x)) ds} \right) < \infty, \quad (4.3)$$

for c positive. Since $r(x_t)$ has uniform p th -moments for x_0 in a compact subset,

$$\mathbb{E} \left(e^{c \int_0^t \overline{H}_1(F_s(x)) ds} \right) \leq \mathbb{E} \frac{1}{t} \int_0^t e^{ct \overline{H}_1(F_s(x))} ds < \infty.$$

The finiteness follows from $\overline{H}_1(F_s(x)) \leq C + C \ln r(F_s(x))$, proving the strong 1-completeness.

Finally we assume that $\rho^h \geq K$. Let $2a = \inf_{x \in D} \text{inj}(x)$ where $\text{inj}(x)$ is the injectivity radius at x and D a compact set containing x_0 . Away from 0, we have seen in part (1b) that $\partial_r(\Delta^h r) \leq K$, and so along a geodesic γ from x_0 , $\Delta^h r(\gamma(s)) \leq -Kr(\gamma(s)) - \Delta^h r(\gamma(\epsilon))$. On a set close to x_0 , $\Delta^h r = \frac{n-1}{r} + dr(\nabla \log(Je^{2h}))$ where J is the Jacobian determinant of the exponential map \exp_{y_0} . Since $\sup_{x \in D} |\nabla \log(Je^{2h})|_x$ is bounded, there exists a constant c and a one dimensional Brownian motion β_t which may depend on x_0 , such that for all $x_0 \in D$,

$$r_t \leq r(x_0) + \beta_t + \int_0^t \left(\frac{n-1}{2r_s} + c \right) ds - L_t,$$

which we compare with the following equation

$$dR_t = d\beta_t + \frac{n-1}{2R_t}dt + cdt, \quad R_0 = r(x_0).$$

The process is the radial process of the SDE $dz_t = dB_t + c\frac{z_t}{|z_t|}dt$ on \mathbb{R}^n with z_0 a point such that $|z_0| = r(x_0)$. By comparing with the Bessel process we see that the paths of $|z_t|$ do not hit zero with probability 1. We make a Girsanov transform to remove the drift. Set

$$M_t = e^{-c \int_0^t \langle dB_s, \frac{B_s}{|B_s|} \rangle - \frac{c^2}{2}t},$$

Let $\alpha_1 > 0$ be a number such that $\mathbb{E}[e^{2\alpha_1|B_t|^2}]$ is finite. Since M_t has finite second moment, $\mathbb{E}(M_t)^2 \leq e^{c^2 t}$, we see that

$$\mathbb{E}[e^{\alpha_1 R_t^2}] = \mathbb{E}[e^{\alpha_1 |z_t|^2}] = \mathbb{E}\left(e^{\alpha_1 |z_0 + B_t|^2} e^{-c \int_0^t \langle dB_s, \frac{B_s}{|B_s|} \rangle - \frac{c^2}{2}t}\right)$$

is finite. By the comparison theorem, $\sup_{x \in D} \mathbb{E}[e^{\alpha_1 r_t^2}]$ is finite for any compact set D . Similarly,

$$\mathbb{E}\left[\sup_{s \leq t} r_s^p\right] = \mathbb{E}\left[\sup_{s \leq t} R_s^p\right] \leq \mathbb{E}\left[\sup_{s \leq t} |z_0 + B_s|^p\right] e^{c(p)t}.$$

Consequently, if we take $y_0 = x_0$ and $z_0 = 0$, and apply Burkholder-Davies-Gundy inequality to obtain that

$$\mathbb{E}\left[\sup_{s \leq t} d(x_s, x_0)^p\right] \leq c(p)e^{c(p)t}t^{\frac{p}{2}}.$$

Since $H_1 \leq C(1 + r^q)$, $\mathbb{E}e^{ct\bar{H}_1(F_s(x))} < \infty$ for sufficiently small t , say $t < t_0$. The strong 1-completeness follows from (4.3), first on a small time interval $[0, t_0]$, and then extended to all finite times by the strong Markov property. \square

For better estimates on $\Delta^h r$ we make use of $-|\text{Hess } r|^2$, observing that $-|\text{Hess } r|^2 \leq -\frac{(\Delta r)^2}{n-1}$, and compare $\Delta^h r$ with the explicit solution of the ODE $m' = -\frac{m^2}{n-1} - K$ with $m(0) = \infty$. This leads to the Laplacian comparison theorem in [92], generalising the standard Ricci comparison theorem. Let r denote the distance function from a given point y_0 .

Theorem 4.3. [92] *Suppose that $\rho^h \geq K$.*

- (1) *If $\langle \nabla h, \nabla r \rangle \leq 2a$ where a is a positive number. Then along a minimal geodesic segment from y_0 ,*

$$\Delta^h r \leq \begin{cases} a + \sqrt{(n-1)K} \cot\left(r\sqrt{\frac{K}{n-1}}\right), & K > 0 \text{ \& } r\sqrt{\frac{K}{n-1}} \leq \frac{\pi}{2}, \\ a + \frac{n-1}{r}, & K = 0, \\ a + \sqrt{(n-1)(-K)} \coth\left(r\sqrt{\frac{-K}{n-1}}\right), & K < 0. \end{cases}$$

- (2) *if $|h| \leq k/2$, then for an explicit number $r_0 > 0$,*

$$\Delta^h r \leq \begin{cases} \sqrt{(n+4k-1)K} \cot\left(r\sqrt{\frac{K}{n+4k-1}}\right), & K > 0, \text{ \& } r \leq r_0 \\ \frac{n+4k-1}{r}, & K = 0 \\ \sqrt{(n+4k-1)(-K)} \coth\left(r\sqrt{\frac{-K}{n+4k-1}}\right), & K < 0. \end{cases}$$

This, together with the earlier lemmas and its proof lead to our final theorem on strong 1-completeness.

Theorem 4.4. *Under one of the conditions from C5(a) to C5(d), the h -Brownian motion is complete and for any $t > 0$*

$$\sup_{s \leq t} \mathbb{E}[d^p(x_s, y_0)] \leq c_1(p)[d^p(x_0, y_0) + t]e^{c_2(p)t}.$$

- (1) *The gradient SDE is strongly 1-complete if furthermore C5(e) holds, in which case $\mathbb{E}[|T_x F_t|]$ is also finite.*
- (2) *Strong 1-completeness also holds under C5(d) + C5(f) in which case there exists a number $\alpha_1 > 0$ s.t. for any compact set D , for any $t > 0$ and for any θ satisfying $\theta t < \alpha_1$,*

$$\sup_{s \leq t} \sup_{x_0 \in D} \mathbb{E} \left(e^{\theta d^2(x_s, x_0)} \right) < \infty.$$

In both cases $|T_x F_t|$ has finite expectation.

Proof. The explosion problem is discussed earlier. We use Proposition 8.5 in [64] which states that

$$\sup_{x \in D} \mathbb{E} \exp \left(\frac{1}{2} \int_0^t \overline{H}_1(x_s) ds \right) < \infty$$

for all compact subset D implies that $\mathbb{E}|v_t|$ is finite and the gradient SDE is strongly 1-complete. If $\overline{H}_1 \leq c(1 + \ln r)$ and $\sup_{x \in D} \mathbb{E}[r(x_t)]^p$ is finite for all p , the inequality indeed holds. Similarly it holds if $\overline{H}_1 \leq c(1 + r^{q-1})$ and $\sup_{x \in D} \mathbb{E}e^{\delta r^2(x_t)} < \infty$ for some $\delta > 0$. \square

We will eventually assume that ρ^h is bounded from below, in which case $|W_t|$ is bounded. Otherwise, if for example $\rho^h \geq -C(1 + \ln r)$ where $C \geq 0$ and $\partial_r h \leq c(1 + r)$, then $r(x_t)$ has moments of all orders. From $|W_t|^2 \leq e^{-\int_0^t \rho^h(x_s) ds}$, we see that for all $p \geq 1$,

$$\sup_{x_0 \in D} \sup_{s \leq t} \mathbb{E}|W_s|^p \leq \sup_{x_0 \in D} \frac{1}{t} \int_0^t \mathbb{E} e^{-\frac{1}{2} p t \rho^h(x_u)} du,$$

the right hand side is finite for any compact set D and $t > 0$.

5. Doubly Damped Stochastic Parallel Translations

If $(x_t, t \leq T)$ is a continuous stochastic process on M , \mathcal{F}_t^x its filtration and $\mathcal{F} = \vee_t \mathcal{F}_t^x$, augmented as usual, a TM -valued stochastic process $(V_t, t \leq T)$ is said to be along (x_t) if the projection of V_t to M is x_t .

Definition 5.1. [35, Def 3.3.2] Given a stochastic parallel translation $//_t$ along the stochastic process (x_t) , we say that a stochastic process (\bar{V}_t) with values in TM is a local conditional expectation of V with respect to the σ -algebra \mathcal{F}^x if there exists a \mathcal{F}^x -measurable real valued process $(\alpha_t, t \leq T)$ and a family of \mathcal{F}_t^x -stopping times τ_n increasing to T such that $//_{t \wedge \tau_n}^{-1} V_{t \wedge \tau_n} \alpha_{t \wedge \tau_n}$ has finite expectation and

$$\mathbb{E}\{ //_{t \wedge \tau_n}^{-1} V_{t \wedge \tau_n} \alpha_{t \wedge \tau_n} | \mathcal{F}^x \} = //_{t \wedge \tau_n}^{-1} \bar{V}_{t \wedge \tau_n} \alpha_{t \wedge \tau_n}.$$

By Corollary 3.3.4 in [35], if $|V_t| \in L^1$ then the local conditional expectation is just the conditional expectation.

In this section $F_t(x)$ denotes the solution to the gradient SDE (4.1), and $TF_t(v)$ its derivative flow. Let $x_t = F_t(x_0)$ and define the σ -algebra $\mathcal{F}_t^{x_0} = \sigma\{x_s : s \leq t\}$, the σ -algebra generated by the solution flow with initial value x_0 , of the gradient SDE, up to t .

5.1. Doubly damped equation and a primitive formula. Let j be a parallel field, with $j(0) = v_2$, along the normalised geodesic γ with initial condition x_0 and initial velocity $\dot{\gamma}(0) = v_1$. If W_t is the stochastic damped parallel translation along the paths of $\{x_t\}$, we differentiate it w.r.t. the initial value of the path and take conditional expectations as following:

$$V_t := \mathbb{E} \left\{ \frac{D}{ds} \Big|_{s=0} W_t(j(s)) \mid \mathcal{F}_t^{x_0} \right\}.$$

Note that $W_0(j(s)) = j(s)$ is parallel, so its covariant derivative in s vanishes.

Lemma 5.2. *Suppose that $\text{Ric} - 2 \text{Hess}(h)$ is bounded from below.*

- (a) *the gradient SDE is strongly 1-complete.*
- (b) *for every s , $\mathbb{E}|T_{\gamma(s)}F_t|$ and $\mathbb{E}|\nabla_{TF_t(\gamma(s))}W_t|_{\gamma(s)}$ are finite.*
- (c) *$s \mapsto \mathbb{E}\left\{\frac{D}{ds}W_t(\gamma(s)) \mid \mathcal{F}_t^{\gamma(s)}\right\}$ is continuous in $L^1(\Omega)$;*

Then for all $f \in BC^2$,

$$\text{Hess}(P_t^h f)(v_2, v_1) = \mathbb{E}[\nabla df(W_t(v_2), W_t(v_1))] + \mathbb{E}[df(V_t)].$$

Proof. Since $dP_t^h f = e^{\frac{1}{2}\Delta^h}(df)$,

$$dP_t^h f(j(s)) = \mathbb{E}df(W_t(j(s))).$$

We observe that $W_t(j(s)) \in T_{F_t(\gamma(s))}M$ is a function of $F_t(\gamma(s))$. By the strong 1-completeness we see that both $F_t(\gamma(s))$ and $W_t(j(s))$ are differentiable in s , and the conditions of the theorem ensure that we may change the order of taking expectation with differentiation with respect to s . In fact,

$$\begin{aligned} \nabla d(P_t^h f)(v_1, v_2) &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int_0^\epsilon \mathbb{E} \frac{d}{ds} (df(W_t(j(s)))) ds \\ &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int_0^\epsilon \left(\mathbb{E}[\nabla df(TF_t(\dot{\gamma}(s)), W_t(j(s)))] - \mathbb{E} \left[df \left(\frac{D}{ds} W_t(j(s)) \right) \right] \right) ds \\ &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int_0^\epsilon \mathbb{E}[\nabla df(W_t(\dot{\gamma}(s)), W_t(j(s)))] ds \\ &\quad - \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int_0^\epsilon \mathbb{E} \left[df \left(\mathbb{E} \left\{ \frac{D}{ds} W_t(j(s)) \mid \mathcal{F}_t^{\gamma(s)} \right\} \right) \right] ds. \end{aligned}$$

We have used the fact that $|TF_t|$ is integrable so $\mathbb{E}\{TF_t(v_2) \mid \mathcal{F}_s^{x_0}\} = W_t(v_2)$, and that $|\frac{D}{ds}W_t(j(s))|$ is integrable. Since the SDE is strongly 1-complete,

$$\lim_{s \rightarrow 0} \nabla df(W_t(\dot{\gamma}(s)), W_t(j(s))) = \nabla df(W_t(v_2), W_t(v_1))$$

and $\frac{D}{ds}W_t(j(s))$ converges to $\nabla_{TF_t(v_2)}(v_1)$ as $s \rightarrow 0$. Since ρ^h is bounded from below, $|W_t|_x$ is uniformly bounded, and the first limit converges in L^2 . The L^1 continuity of $s \mapsto \frac{D}{ds}W_t(j(s))$ and the fact that $f \in BC^2$ lead to the required conclusion. \square

Definition 5.3. Given $W_t(v_1)$, let $W_t^{(2)}(v_1, v_2)$ (abbreviated as $W_t^{(2)}$) denote the solution to the following covariant differential equation

$$\begin{aligned} DW_t^{(2)}(v_1, v_2) &= \left(-\frac{1}{2}\text{Ric}^\# + (\nabla^2 h)^\#\right) \left(W_t^{(2)}(v_1, v_2)\right) dt + \frac{1}{2}\Theta^h(W_t(v_2))(W_t(v_1))dt \\ &\quad + \mathcal{R}(d\{x_t\}, W_t(v_2))W_t(v_1), \\ W_0^{(2)}(v_1, v_2) &= 0, \end{aligned} \tag{5.1}$$

where \mathcal{R} is the curvature tensor, $\{x_t\}$ denotes the martingale part of x_t .

Lemma 5.4. *Suppose that the gradient SDE is strongly 1-complete. Then $W_t^{(2)}(v_1, v_2)$ is the local conditional expectation of $\frac{D}{ds}|_{s=0}W_t(j(s))$. If the latter is integrable, then*

$$W_t^{(2)}(v_1, v_2) = \mathbb{E} \left\{ \frac{D}{ds}|_{s=0}W_t(j(s)) \mid \mathcal{F}_t^{x_0} \right\}.$$

Proof. Let γ be the normal geodesic defined above. We differentiate the vector field W_t along the surface $F_t(\gamma(s), \omega)$ in the s direction followed by stochastic covariant differential in t direction, using strong 1-completeness. Thus,

$$\begin{aligned} D \left(\frac{D}{ds}|_{s=0}W_t(j(s)) \right) &= \frac{D}{ds}|_{s=0} (DW_t(j(s))) + \mathcal{R}(X(x_t) \circ dB_t, TF_t(v_2))W_t(v_1) \\ &= \frac{D}{ds}|_{s=0} \left(\left(-\frac{1}{2}\text{Ric}^\# + (\nabla^2 h)^\#\right)(W_t(j(s))) \right) \\ &\quad + \mathcal{R}(X(x_t) \circ dB_t + \nabla h(x_t)dt, TF_t(v_2))W_t(v_1). \end{aligned}$$

The curvature term results from exchanging the derivative of $W_t(j(s))$ in the direction of the stochastic differential (because of W_t is also a function of $(F_t(\gamma(s)))$ in t and the derivative of $F_t(j(s))$ in s . We also used that $\frac{D}{ds}j(s) = 0$ and that the differential of $W_t(F_t(\gamma(s)))$ in t satisfies the stochastic damped parallel translation equation. Let us compute the term involving $\nabla^2 h$:

$$\frac{D}{ds}|_{s=0}(\nabla^2 h)^\#(W_t(j(s))) = \nabla_{TF_t(v_1)}(\nabla^2 h)^\#(W_t(v)) + (\nabla^2 h)^\# \left(\frac{D}{ds}|_{s=0}W_t(j(s)) \right).$$

The first term can also be written as $\nabla^3 h(\nabla_{TF_t(v_1)}, W_t(v), \cdot)$, and consequently,

$$\begin{aligned} & d//_t^{-1} \left(\frac{D}{ds}|_{s=0}W_t(j(s)) \right) \\ &= //_t^{-1} \left(-\frac{1}{2}\nabla_{TF_t(v_2)}\text{Ric}^\#(W_t(v_1)) + \nabla^2(\nabla h)(TF_t(v_2), W_t(v_1)) \right) dt \\ &\quad + \left(-\frac{1}{2}//_t^{-1}\text{Ric}^\# + (\nabla^2 h)^\# \right) \left(\frac{D}{ds}|_{s=0}(W_t(j(s))) \right) dt \\ &\quad + //_t^{-1}\mathcal{R}(X(x_t) \circ dB_t, TF_t(v_2))W_t(v_1) + //_t^{-1}\mathcal{R}(\nabla h(x_t), TF_t(v_2))W_t(v_1)dt. \end{aligned}$$

The notation $\nabla^2(\nabla h)$ should be clear, note the function $\langle \nabla^2(\nabla h)(v, u), w \rangle$ is symmetric in (u, w) . We unravel the Stratonovich stochastic integral into one with respect to $d\{x_t\}$, the martingale part of x_t :

$$//_t^{-1}\mathcal{R}(X(x_t) \circ dB_t, TF_t(v_2))W_t(v_1)$$

$$= //t^{-1} \mathcal{R}(d\{x_t\}, TF_t(v_2))W_t(v_1) + \frac{1}{2} \text{trace}(\nabla \cdot \mathcal{R})(\cdot, TF_t(v_2))W_t(v_1)dt,$$

where we have used the following fact for the gradient system: at any point $x \in M$ either $\nabla X_i(x)$ vanishes or $X_i(x)$ vanishes. We use the following formula, see e.g. [34], for an o.n.b. $\{f_i\}$ of $T_x M$, and $v_i \in T_x M$,

$$\sum_{i=1}^n \langle \nabla_{f_i} R(f_i, v_2)v_1, v_3 \rangle = (\nabla_{v_3} \text{Ric})(v_1, v_2) - (\nabla_{v_1} \text{Ric})(v_3, v_2).$$

We combine the three terms involving the covariant derivative of the Ricci curvature, and set:

$$\langle \Theta(v_2)(v_1), v_3 \rangle = \left(\nabla_{v_3} \text{Ric}^\# \right) (v_1, v_2) - \left(\nabla_{v_1} \text{Ric}^\# \right) (v_3, v_2) - \left(\nabla_{v_2} \text{Ric}^\# \right) (v_1, v_3),$$

which is symmetric in (v_1, v_2) . Setting,

$$\Theta^h(v_2)(v_1) = \frac{1}{2} \Theta(v_2, v_1) + \nabla^2(\nabla h)(v_2, v_1) + \mathcal{R}(\nabla h, v_2)(v_1),$$

to see that

$$\begin{aligned} & d//t^{-1} \left(\frac{D}{ds} \Big|_{s=0} W_t(j(s)) \right) \\ &= \frac{1}{2} //t^{-1} \Theta^h(TF_t(v_2))(W_t(v_1))dt + \left(-\frac{1}{2} //t^{-1} \text{Ric}^\# + \nabla^2 h \right) \left(\frac{D}{ds} \Big|_{s=0} (W_t(j(s))) \right) dt \\ & \quad + //t^{-1} \mathcal{R}(X(x_t) \circ dB_t, TF_t(v_2))W_t(v_1). \end{aligned}$$

With this preparation we condition the above stochastic equation with respect to $\mathcal{F}_t^{x_0}$ and use the fact that $W_t(v_2)$ is the local conditional expectation of $TF_t(v_2)$ and apply Lemma 3.3.1 in [35] to obtain:

$$\begin{aligned} d//t^{-1} W_t^{(2)} &= \left(-\frac{1}{2} //t^{-1} \text{Ric}^\# + //t^{-1} \nabla^2 h \right) \left(W_t^{(2)} \right) dt + \frac{1}{2} //t^{-1} \Theta^h(W_t(v_2))(W_t(v_1))dt \\ & \quad + //t^{-1} \mathcal{R}(d\{x_t\}, W_t(v_2))W_t(v_1). \end{aligned}$$

We have used the fact that $\int_0^s //t^{-1} \mathcal{R}(d\{x_t\}, \cdot)$ is adapted to the filtration of $\{x_t\}$, see [35, Theorem 3.1.2]. The conclusion about the local expectation follows. The rest follows from Corollarys 3.3.4 in [35]. \square

Let $p > 1$ and r denotes the Riemannian distance from a given point. Observe that for any $t > 0$ and for any compact set D , $\sup_{x \in D} \sup_{s \leq t} \mathbb{E}(|T_x F_s|^p)$ is finite if $\bar{H}_p \leq c_1 + c_2 r^2$ where c_2 , depending possibly on t , is sufficiently small. See (2.6). Below let $c > 0$ be a constant and $\alpha_2 > 0$ be a sufficiently small constant.

Lemma 5.5. *Suppose $\rho^h \geq -K$, $\|\Theta^h\| \leq ce^{Cr}$, $\|\mathcal{R}\| \leq ce^{Cr}$, where $C \leq \alpha_2$. Then,*

(a) *for any compact subset D ,*

$$\sup_{x_0 \in D} \sup_{s \leq t} \mathbb{E} \|W_t^{(2)}\|^2 < \infty.$$

(b) *If furthermore $\int_0^t (\mathbb{E}[|T_x F_s|^{2p}])^{\frac{1}{p}} ds < \infty$ for some $p > 1$. then*

$$\mathbb{E}(|V_t|^2) < \infty.$$

Proof. Take unit vectors v_1, v_2 in $T_{x_0}M$. For part (1) we observe that

$$\begin{aligned} d \left| W_t^{(2)}(v_1, v_2) \right|^2 &= (-\text{Ric} + 2\text{Hess } h)(W_t^{(2)}(v_1, v_2), W_t^{(2)}(v_1, v_2))dt \\ &+ \left\langle W_t^{(2)}(v_1, v_2), \Theta^h(W_t(v_2))(W_t(v_1)) \right\rangle dt \\ &+ 2 \left\langle \mathcal{R}(d\{x_t\}, W_t(v_2))W_t(v_1), W_t^{(2)}(v_1, v_2) \right\rangle + \sum_{i=1}^n |\mathcal{R}(u_i e_i, W_t(v_2))W_t(v_1)|^2 dt. \end{aligned}$$

The penultimate term, which we denote by (M_t) , in the above equation is a local martingale. By taking suitable stopping times $\{\tau_k\}$, this term vanishes. Since $\rho^h \geq -K$, $|W_t|^2 \leq e^{-Kt}$ for any $t \geq 0$, and we obtain the following estimates:

$$\begin{aligned} &\mathbb{E} \left| W_{t \wedge \tau_k}^{(2)}(v_1, v_2) \right|^2 \\ &\leq |v_2|^2 + \mathbb{E} \int_0^{t \wedge \tau_k} \left(\frac{1}{2} + \rho^h(x_s) \right) \left| W_s^{(2)}(v_1, v_2) \right|^2 ds \\ &\quad + \mathbb{E} \int_0^{t \wedge \tau_k} \frac{1}{2} \left| \Theta^h(W_s(v_2))W_s(v_1) \right|^2 ds + \mathbb{E} \int_0^{t \wedge \tau_k} \|\mathcal{R}(\cdot, W_s(v_2))W_s(v_1)\|^2 ds \\ &\leq |v_2|^2 + \mathbb{E} \int_0^{t \wedge \tau_k} \left(\frac{1}{2} + K \right) \left| W_s^{(2)}(v_1, v_2) \right|^2 ds \\ &\quad + \mathbb{E} \int_0^{t \wedge \tau_k} \frac{1}{2} \|\Theta^h\|_{x_s}^2 e^{2Ks} |v_1|^2 |v_2|^2 ds + \mathbb{E} \int_0^{t \wedge \tau_k} \|\mathcal{R}\|_{x_s}^2 e^{2Ks} |v_1|^2 |v_2|^2 ds. \end{aligned}$$

By Lemma 4.2, $\sup_{s \leq t} \mathbb{E} e^{\alpha_1 d^2(x_s, x_0)}$ is uniformly bounded in x_0 . Take $\alpha_2 = \frac{1}{2}\alpha_1$ to see both $\mathbb{E}[\|\Theta^h\|_{x_s}^2]$ and $\mathbb{E}[\|\mathcal{R}\|_{x_s}^2]$ are finite. We apply Gronwall's lemma followed by taking $k \rightarrow \infty$ and using Fatou's lemma to see that $\mathbb{E} \sup_{r \leq t} \left| W_r^{(2)}(v_1, v_2) \right|^2 \leq C(t)$ where $C(t)$ is a constant depending on t , and is locally uniform w.r.t. x_0 .

For part (2) we use the following equation from the proof of the last lemma.

$$\begin{aligned} d//_t^{-1}V_t &= \left(-\frac{1}{2}//_t^{-1}\text{Ric}^\sharp + \nabla^2 h \right) (V_t) dt + \frac{1}{2}//_t^{-1}\Theta^h(TF_t(v_2))(W_t(v_1))dt \\ &\quad + //_t^{-1}\mathcal{R}(X(x_t) \circ dB_t, TF_t(v_2))W_t(v_1). \end{aligned}$$

We may proceed as before, but replacing $W_t(v_2)$ be $TF_t(v_2)$. We finally have to take care of the following terms

$$\mathbb{E}[\|\Theta^h\|^2 |TF_t|^2] \leq (\mathbb{E}\|\Theta^h\|^{2q})^{\frac{1}{q}} (\mathbb{E}|TF_t|^{2p})^{\frac{1}{p}}.$$

If $2qC < \alpha_1$, $\mathbb{E}\|\Theta^h\|^{2q}$ is finite. The term involving \mathcal{R} can be treated similarly. \square

The following proposition follows immediately from the two pervious lemmas. Denote by c, δ positive constants, K a constant, and α_2 a sufficiently small constant.

Proposition 5.6. *Suppose that*

- (a) $\rho^h \geq -K$, $\|\Theta^h\| \leq ce^{Cr}$, $\|\mathcal{R}\| \leq ce^{Cr}$, where $C \leq \alpha_2$.

(b) the gradient SDE is strongly 1-complete and

$$\int_0^t (\mathbb{E}[|T_x F_s|^{2p}]^{\frac{1}{p}} ds < \infty$$

for any x and for some $p > 1$.

Then for all $f \in BC^2$,

$$\text{Hess}(P_t^h f)(v_2, v_1) = \mathbb{E}[\nabla df(W_t(v_2), W_t(v_1))] + \mathbb{E}[df(W_t^{(2)}(v_1, v_2))]. \quad (5.2)$$

An immediate consequence are the following primitive L_∞ estimates:

Corollary 5.7. *Suppose the conclusions of Proposition 5.6. Let $f \in BC^2$. Then*

$$|\text{Hess}(P_t^h f)(v_2, v_1)| \leq |\nabla df|_\infty \mathbb{E}[e^{-\rho^h(x_s)ds}] + |df|_\infty \mathbb{E}[|W_t^{(2)}(v_1, v_2)|].$$

6. Exponential Integrability

Let us fix a h -Brownian motion, i.e. fix a probability space and its sample paths, this could be the solution of a gradient SDE, or the projection of the canonical horizontal SDE, or any other pathwise representation. We define $(W_t, W_t^{(2)})$ to be the solution to the following system of equations along the chosen sample paths $\{x(\omega)\}$, so we have a triple of stochastic processes $(x_t, W_t, W_t^{(2)})$.

$$\begin{aligned} \frac{D}{dt} W_t &= -\frac{1}{2} \text{Ric}_{x_t}^\#(W_t) - (\nabla^2 h)_{x_t}^\#(W_t), \\ W_0 &= v_1 \\ W_t d(W_t^{-1} W_t^{(2)}) &= \frac{1}{2} \Theta^h(W_t(v_1)(W_t(v_2))) dt + \mathcal{R}(\{dx_t\}, W_t(v_2)) W_t(v_1), \\ W_0^{(2)} &= 0. \end{aligned} \quad (6.1)$$

We next investigate the exponential integrability of $|W_t^{(2)}|$. In this, we assume that the curvature operator is bounded for the simplicity of the exposition. In the lemma below, $\|\mathcal{R}\|_\infty$ and T are positive constants.

Lemma 6.1. *Suppose condition C2. Set $C_1(T, 0) = 1$,*

$$C_1(T, K) = \sup_{0 < s \leq 3KT} \frac{1}{s} (e^s - 1), \quad \alpha_2(T, K, \|\mathcal{R}\|_\infty) = \frac{1}{49n^2 \|\mathcal{R}\|_\infty^2 C_1(T, K)}.$$

Then there exists a universal constant c such that for $v_1, v_2 \in ST_{x_0} M$, and for any $\alpha \leq \alpha_2(T, K, \|\mathcal{R}\|_\infty)$,

$$\begin{aligned} \mathbb{E} \exp\left(\alpha \gamma |W_t^{(2)}(v_1, v_2)|^2\right) &\leq c e^{2\alpha \gamma} \sqrt{\mathbb{E} \exp\left(4t \gamma \alpha \int_0^t e^{3Ks} \|\Theta^h\|_{x_s}^2 ds\right)} \\ &\leq c e^{\frac{2\gamma}{49n^2 \|\mathcal{R}\|_\infty^2}} \sqrt{\mathbb{E} \exp\left(\frac{4t \gamma}{49n^2 \|\mathcal{R}\|_\infty^2 C_1(t, K)} \int_0^t e^{3Ks} \|\Theta^h\|_{x_s}^2 ds\right)}. \end{aligned}$$

Proof. Take $v_1, v_2 \in ST_x M$, and write $W_t^{(2)} = W_t^{(2)}(v_1, v_2)$ for simplicity. We first observe that

$$W_t d\left(W_t^{-1}W_t^{(2)}\right) = DW_t^{(2)} + \left(-\frac{1}{2}\text{Ric}^\# + (\nabla^2 h)^\#\right)\left(W_t^{(2)}\right) dt.$$

By the definition of $W_t^{(2)}$, we see that

$$\begin{aligned} & W_t^{-1}W_t^{(2)} - v_2 \\ &= \frac{1}{2} \int_0^t W_s^{-1}(\Theta^h(W_s(v_1))(W_s(v_2))) ds + \int_0^t W_s^{-1}\left(\mathcal{R}(u_s dB_s, W_s(v_2))W_s(v_1)\right). \end{aligned}$$

Denote by A_t and M_t respectively the first and the second term on the right hand side. If α is a positive number, we use elementary and Cauchy-Schwartz inequalities to obtain the following brutal estimate:

$$\begin{aligned} \mathbb{E} \exp\left(\alpha|W_t^{(2)}|^2\right) &= \mathbb{E} \exp\left(\alpha(v_2 + A_t + M_t)\right)^2 \\ &\leq e^{2\alpha|v_2|^2} \mathbb{E} \exp\left(4\alpha A_t^2 + 4\alpha M_t^2\right) \leq e^{2\alpha|v_2|^2} \sqrt{\mathbb{E} e^{8\alpha A_t^2}} \sqrt{\mathbb{E} e^{8\alpha M_t^2}}. \end{aligned}$$

Let us consider the j th component of the \mathbb{R}^n -valued local martingale (M_t) , where \mathbb{R}^n is identified with $T_{x_0}M$,

$$M_t^j := \int_0^t \left\langle W_s^{-1}\left(\mathcal{R}(u_s dB_s, W_s(v_2))W_s(v_1)\right), e_j \right\rangle.$$

This has quadratic variation

$$f_t^j = \sum_{i=1}^n \int_0^t \left\langle W_s^{-1}\left(\mathcal{R}(u_s e_i, W_s(v_2))W_s(v_1), e_j\right) \right\rangle^2 ds.$$

Since \mathcal{R} is bounded, ρ^h is bounded, the quadratic variations are uniformly bounded by a constant,

$$f_t^j \leq n \|\mathcal{R}\|_\infty^2 \int_0^t e^{3Ks} ds \leq 3tn \|\mathcal{R}\|_\infty^2 \frac{e^{3Kt} - 1}{3Kt}.$$

Thus M_t^j is a time changed Brownian motion of the form $\beta_{f_t^j}^j$ where each β^j is a one dimensional Brownian motion. Furthermore for a universal constant $c(n)$ depending on n ,

$$e^{8\alpha|M_t|^2} = e^{8\alpha \sum_{j=1}^n (M_t^j)^2} \leq c(n) \sum_{j=1}^n e^{8\alpha n (M_t^j)^2}.$$

In particular, $\mathbb{E} e^{8\alpha|M_t|^2}$ is bounded by a universal constant if $8\alpha n f_t^j < \frac{1}{2}$. Letting $C_1(t, K) = \sup_{0 < t \leq T} \frac{e^{3Kt} - 1}{3Kt}$, it is sufficient to take $\alpha < \frac{1}{48n^2 \|\mathcal{R}\|_\infty^2 C_1(t, K)}$. Since $\alpha_2(t) = \frac{1}{49n^2 \|\mathcal{R}\|_\infty^2 C_1(t, K)}$, for any $\alpha \leq \alpha(t)$, $\mathbb{E} e^{8\alpha|M_t|^2}$ is bounded by a universal constant.

We consider the first term A_t , which is easy to estimate. In fact,

$$|A_t|^2 \leq \frac{t}{2} \int_0^t |W_s^{-1}(\Theta^h(W_s(v_1))(W_s(v_2)))|^2 ds \leq \frac{t}{2} \int_0^t e^{3Ks} \|\Theta^h\|_{x_s}^2 ds.$$

By part (2) in Lemma 4.2, it is exponentially integrable to any given exponent if $\|\Theta^h\|^2 \leq c + \delta r^2$ where δ is sufficiently small (this is part of the assumption **C2**). Putting everything together we see that there exists a universal constant c such that for $\alpha \leq \alpha_2(t)$, $\gamma \in (0, 1)$,

$$\mathbb{E} \exp \left(\alpha \gamma |W_t^{(2)}|^2 \right) \leq c e^{2\gamma\alpha} \sqrt{\exp \left(4t\gamma\alpha \int_0^t e^{3Ks} \mathbb{E} \|\Theta^h\|_{x_s}^2 ds \right)}.$$

For the maximum value of α ,

$$\begin{aligned} \mathbb{E} \left[e^{8\gamma\alpha_2(t, K, \|\mathcal{R}\|_\infty) |A_t|^2} \right] &\leq \mathbb{E} \exp \left(4t \gamma \alpha_2(t, K, \|\mathcal{R}\|_\infty) \int_0^t e^{3Ks} \|\Theta^h\|_{x_s}^2 ds \right) \\ &\leq \mathbb{E} \exp \left(\frac{4t\gamma}{49n^2 \|\mathcal{R}\|_\infty^2 C_1(t, K)} \int_0^t e^{3Ks} \|\Theta^h\|_{x_s}^2 ds \right). \end{aligned}$$

Putting everything together we have, for any $\gamma \in (0, 1)$,

$$\mathbb{E} \exp \left(\alpha \gamma |W_t^{(2)}|^2 \right) \leq c e^{2\gamma\alpha} \sqrt{\mathbb{E} \exp \left(\frac{4t\gamma}{49n^2 \|\mathcal{R}\|_\infty^2 C_1(t, K)} \int_0^t e^{3Ks} \|\Theta^h\|_{x_s}^2 ds \right)}.$$

To conclude we use the equivalence of the two norms $|W_t^{-1} W_t^{(2)}|$ and $|W_t^{(2)}|$, which follows from the assumption that ρ^h is bounded and hence $e^{-Kt} |W_t^{-1} W_t^{(2)}| \leq |W_t^{(2)}| \leq e^{Kt} |W_t^{-1} W_t^{(2)}|$. \square

For further estimates we use the following basic well known entropy inequality.

Lemma 6.2. *Let ϕ be a non-negative measurable function on Ω s.t. $\mathbb{E}\phi \neq 0$. If Ψ is a measurable function on Ω such that $\phi\Psi$, $e^{-\Psi}$ and e^Ψ are integrable then*

$$-\mathbb{E} \left[\phi \log \frac{\phi}{\mathbb{E}\phi} \right] + (\mathbb{E}\phi) \log \mathbb{E}(e^{-\Psi}) \leq \mathbb{E}[\phi\Psi] \leq \mathbb{E} \left[\phi \log \frac{\phi}{\mathbb{E}\phi} \right] + (\mathbb{E}\phi) \log \mathbb{E}(e^\Psi). \quad (6.2)$$

Proof. The second inequality follows from Jensen's inequality,

$$\begin{aligned} \log \mathbb{E}[e^\Psi] &= \log \left[\frac{\mathbb{E}[(e^\Psi \phi^{-1} \phi)]}{\mathbb{E}\phi} \mathbb{E}\phi \right] \geq \frac{\mathbb{E}[\phi \log(e^\Psi \phi^{-1})]}{\mathbb{E}\phi} + \log(\mathbb{E}\phi) \\ &= \frac{\mathbb{E}[\phi(\Psi - \log \phi)]}{\mathbb{E}\phi} + \log(\mathbb{E}\phi). \end{aligned}$$

The first inequality follows from taking $-\Psi$ in place of Ψ . \square

Lemma 6.3. *Let (x_t) an h -Brownian motion. Suppose **C2**. Let f be a non-negative bounded and Borel measurable function normalised such that $P_t^h f(x_0) = 1$. Let v_1, v_2 from the unit tangent space $ST_{x_0}M$.*

- (1) *Let $c_1(n) = 14\sqrt{2}n\sqrt{C_1(t/2, K)}\|\mathcal{R}\|_\infty$. Then there exist numbers c_2 and c_3 , which depend on $n, K, \|\mathcal{R}\|_\infty, \Theta$ and are given explicitly in the proof, s.t.*

$$\begin{aligned} &\left| \mathbb{E} \left(f(x_t) \cdot \frac{2}{t} \int_0^{\frac{t}{2}} \langle u_s dB_s, W_s^{(2)}(v_1, v_2) \rangle \right) \right| \\ &\leq \frac{1}{\sqrt{t}} (c_1 \mathbb{E}[(f \log f)(x_t)] + c_2 + c_3 A(t/2, K, \|\mathcal{R}\|_\infty, \|\Theta^h\|)), \end{aligned}$$

where the constant $A(t/2, \Theta^h)$ is finite and is given by

$$A(t/2, K, \|\mathcal{R}\|_\infty, \|\Theta^h\|) = \log \left(\mathbb{E} \exp \left(2\alpha_2(t/2) \int_0^{\frac{t}{2}} e^{3Ks} \|\Theta^h\|_{x_s}^2 ds \right) \right).$$

(2) For a number $\delta_0 > 0$ set $C_2(t, K) = (4 + \delta_0) \sup_{0 < s \leq t} \left(\frac{e^{sK} - e^{-\frac{sK}{2}}}{Kt} \right)$. There exists a number $c_0(\delta)$ depending on δ_0 such that for N_t defined in Definition 7.3,

$$|\mathbb{E}(f(x_t) \cdot N_t)| \leq \frac{1}{t} C_2(t, K) \mathbb{E}(f \log f)(x_t) + \frac{1}{t} C_2(t, K) c(\delta_0).$$

Proof. For v_1, v_2 fixed we write $W_t^{(2)}$ for $W_t^{(2)}(v_1, v_2)$ to ease notation. Let $\alpha_2(t)$ be the number in Lemma 6.1:

$$\alpha_2(T, K, \|\mathcal{R}\|_\infty) = \frac{1}{49n^2 \|\mathcal{R}\|_\infty^2 C_1(T, K)}.$$

For any number $\gamma \neq 0$, we apply Lemma 6.2 to $\Psi = \gamma \sqrt{\frac{\alpha_2(t)}{2t}} \int_0^{\frac{t}{2}} \langle u_s dB_s, W_s^{(2)} \rangle$ and to the mean 1 random variable $\phi = f(x_t)$ to obtain:

$$\begin{aligned} & \gamma \mathbb{E} \left(f(x_t) \cdot \frac{2}{t} \int_0^{\frac{t}{2}} \langle u_s dB_s, W_s^{(2)} \rangle \right) \\ & \leq \frac{1}{\sqrt{t}} \frac{2\sqrt{2}}{\sqrt{\alpha_2(\frac{t}{2})}} \left[\mathbb{E}[(f \log f)(x_t)] + \log \mathbb{E} \exp \left(\gamma \sqrt{\frac{\alpha_2(\frac{t}{2})}{2t}} \left(\int_0^{\frac{t}{2}} \langle u_s dB_s, W_s^{(2)} \rangle \right) \right) \right]. \end{aligned}$$

Observe that

$$c_1 \equiv \frac{2\sqrt{2}}{\sqrt{\alpha_2(\frac{t}{2})}} \leq 14\sqrt{2}n \|\mathcal{R}\|_\infty \sqrt{C_1(t/2, K)}.$$

Next we use the following inequality, for a real valued continuous local martingale M_t and real number c , $\mathbb{E}e^{cM_t} \leq \sqrt{\mathbb{E}e^{2c^2(M, M)_t}}$. Let γ be a number with $0 < |\gamma| \leq 1$. Note that $\alpha_2(t) \equiv \alpha_2(t, K, \|\mathcal{R}\|_\infty)$ is a monotone decreasing function. By Lemma 6.1 we have the following estimate

$$\begin{aligned} & \log \mathbb{E} \exp \left(\gamma \sqrt{\frac{\alpha_2(t/2)}{2t}} \left(\int_0^{\frac{t}{2}} \langle u_s dB_s, W_s^{(2)} \rangle \right) \right) \\ & \leq \frac{1}{2} \log \mathbb{E} e^{\frac{\gamma^2 \alpha_2(t/2)}{t} \left(\int_0^{\frac{t}{2}} |W_s^{(2)}|^2 ds \right)} \leq \frac{1}{2} \log \left(\frac{2}{t} \int_0^{\frac{t}{2}} \mathbb{E} e^{\frac{1}{2} \gamma^2 \alpha_2(t/2) |W_s^{(2)}|^2} ds \right) \\ & \leq \frac{1}{2} \log \left(\frac{2}{t} \int_0^{\frac{t}{2}} c(n) e^{\gamma^2 \alpha_2(t/2)} \sqrt{\mathbb{E} \exp \left(2\gamma^2 s \alpha_2(t/2) \int_0^s e^{3Kr} \|\Theta^h\|_{x_r}^2 dr \right)} \right) \\ & \leq \frac{1}{2} \log c(n) + \frac{1}{2} \gamma^2 \alpha_2(t/2) + \frac{1}{4} \log \left(\mathbb{E} \exp \left(2\gamma^2 t \alpha_2(t/2) \int_0^t e^{3Kr} \|\Theta^h\|_{x_r}^2 dr \right) \right). \end{aligned}$$

Since $C_1(t, K) \geq 1$, $\alpha_2(s, K, \|\mathcal{R}\|_\infty^2) \leq \frac{1}{49n^2\|\mathcal{R}\|_\infty^2}$, we have,

$$\begin{aligned} & \frac{2\sqrt{2}}{\sqrt{t\alpha_2(t/2)}} \log \mathbb{E} \exp \left(\gamma \sqrt{\frac{\alpha_2(t/2)}{2t}} \left(\int_0^{\frac{t}{2}} \langle u_s dB_s, W_s^{(2)} \rangle \right) \right) \\ & \leq \frac{1}{\sqrt{t}} \frac{\sqrt{2} \log c(n)}{\sqrt{\alpha_2(t/2)}} + \frac{1}{\sqrt{t}} \sqrt{2} \gamma^2 \sqrt{\alpha_2(t/2)} \\ & \quad + \frac{1}{\sqrt{t}} \frac{\sqrt{2}}{2\sqrt{\alpha_2(t/2)}} \log \left(\mathbb{E} \exp \left(2\gamma^2 t \alpha_2(t/2) \int_0^t e^{3Kr} \|\Theta^h\|_{x_r}^2 dr \right) \right) \\ & \leq \frac{1}{\sqrt{t}} c_2 + c_3 \frac{1}{\sqrt{t}} \log \left(\mathbb{E} \exp \left(2\alpha_2(t/2) \int_0^{\frac{t}{2}} e^{3Ks} \|\Theta^h\|_{x_s}^2 ds \right) \right). \end{aligned}$$

where

$$c_2 = 7n \log c(n) \|\mathcal{R}\|_\infty \sqrt{2C_1(t/2, K)} + \frac{\sqrt{2}\gamma^2}{7n\|\mathcal{R}\|_\infty}, \quad c_3 = \frac{7}{\sqrt{2}} n \|\mathcal{R}\|_\infty \sqrt{C_1(t/2, K)}.$$

To see that $A(t/2, \Theta^h) = \log \left(\mathbb{E} \exp \left(2\alpha_2(t/2) \int_0^{\frac{t}{2}} e^{3Ks} \|\Theta^h\|_{x_s}^2 ds \right) \right)$ is finite we use Jensen's inequality to see that

$$A(t/2, K, \|\mathcal{R}\|_\infty, \|\Theta^h\|) \leq \log \left(\frac{2}{t} \int_0^{\frac{t}{2}} \mathbb{E} \left[\exp \left(t\alpha_2(t/2) e^{3Ks} \|\Theta^h\|_{x_s}^2 \right) ds \right] \right).$$

Since ρ^h is bounded above, by Lemma 4.2, $\alpha r^2(x_t)$ is exponentially integrable for a small number α . By assumption, $\|\Theta^h\|_{x_s}^2 \leq c + \delta r^2(x_s)$ and by a sufficiently small δ we meant that $\delta 2\alpha_2(t/2) \int_0^{\frac{t}{2}} e^{3Ks} < \alpha$. This completes the proof for part (1). For the second part recall that

$$N_t = \frac{4}{t^2} \int_{\frac{t}{2}}^t \langle u_s dB_s, W_s(v_1) \rangle \int_0^{\frac{t}{2}} \langle u_s dB_s, W_s(v_2) \rangle.$$

Set $c_1(t, K) = \frac{e^{tK}-1}{Kt}$ and $\alpha_3(t) = \frac{1}{4+\delta_0} \left(\sup_{0 < s \leq t} c_1(s/2, K) e^{\frac{Ks}{2}} \right)^{-1}$. By Lemma 6.2, for any number γ with $0 < |\gamma| \leq 1$,

$$\gamma \mathbb{E} (f(x_t) \cdot N_t) \leq \frac{1}{t} \frac{1}{\alpha_3(t)} \mathbb{E} (f \log f)(x_t) + \frac{1}{t} \frac{1}{\alpha_3(t)} \log \mathbb{E} e^{\gamma t \alpha_3(t) N_t}.$$

To estimate the exponential term we first apply Cauchy Schwartz's inequality to obtain

$$\mathbb{E} e^{\gamma t \alpha_3(t) N_t} \leq \sqrt{\mathbb{E} e^{\frac{2|\gamma|\alpha_3(t)}{t} \left(\int_{\frac{t}{2}}^t \langle u_s dB_s, W_s(v_1) \rangle \right)^2}} \sqrt{\mathbb{E} e^{\frac{2|\gamma|\alpha_3(t)}{t} \left(\int_0^{\frac{t}{2}} \langle u_s dB_s, W_s(v_2) \rangle \right)^2}}.$$

Denote by Γ_1 and Γ_2 respectively the quadratic variations of the following local martingales

$$\int_{\frac{t}{2}}^{\cdot} \langle u_s dB_s, W_s(v_1) \rangle \quad \int_0^{\cdot} \langle u_s dB_s, W_s(v_2) \rangle.$$

Then

$$\Gamma_1(t) \leq \int_{\frac{t}{2}}^t e^{Ks} ds = \frac{e^{Kt} - e^{\frac{Kt}{2}}}{K}, \quad \Gamma_2(t/2) \leq \frac{e^{\frac{Kt}{2}} - 1}{K}.$$

These local martingales are time changed one dimensional Brownian motions, and so their exponentials are integrable if

$$2|\gamma|\alpha_3(t)\frac{\Gamma_i(t)}{t} < \frac{1}{2}.$$

These hold by construction, hence $\mathbb{E}e^{t\alpha_3(t)N_t}$ is finite and bounded by a universal constant (depending on δ_0 and $|\gamma|$).

$$\gamma\mathbb{E}(f(x_t) \cdot N_t) \leq \frac{1}{t}\frac{1}{\alpha_3(t)}\mathbb{E}(f \log f)(x_t) + \frac{1}{t}\frac{1}{\alpha_3(t)}c(\delta_0, |\gamma|).$$

Finally we observe that $\frac{1}{\alpha_3(t)}$ is locally bounded to complete the proof. \square

7. 2nd Order Feynman-Kac Formulas

Let $B_t = (B_t^1, \dots, B_t^m)$. Set $u_s = TF_s(v_1)$ and $v_s = TF_s(v_2)$. In [37, Theorem 2.3], the following formula was given for the second order derivative of the heat semigroup $P_t f$ where f is a BC^2 function, and (x_t) is the solution to a gradient Brownian system.

$$\begin{aligned} \text{Hess } P_t f(x_0)(v_1, v_2) &= \frac{4}{t^2}\mathbb{E}\left\{f(x_t)\int_{\frac{t}{2}}^t\langle Y(x_s)u_s, dB_s\rangle\int_0^{\frac{t}{2}}\langle Y(x_s)v_s, dB_s\rangle\right\} \\ &\quad + \frac{2}{t}\mathbb{E}\left\{f(x_t)\int_0^{\frac{t}{2}}\langle DY(x_s)(u_s)(v_s), dB_s\rangle\right\} \\ &\quad + \frac{2}{t}\mathbb{E}\left\{f(x_t)\int_0^{\frac{t}{2}}\langle Y(x_s)\nabla TF_s(v_1, v_2), dB_s\rangle\right\}. \end{aligned}$$

This was proved using the martingale method developed in [62, X.-M. Li], using heat semigroup on differential forms. In this formula the derivatives of the initial function f is not involved, emphasizing the smoothing property of the Bismut-Witten Laplacian.

The conditions on the driving vector fields and their derivatives are presented in [37, Theorem 2.3]. Suppose that the curvature and the shape operator of the manifold and their first order derivatives are bounded, it is not difficult to see that these conditions are satisfied. It is possible to use the technique of filtering to remove the redundant noise in the formula for $\nabla dP_t f$ leading to an intrinsic formula. These computations are lengthy and will involve the second order derivatives of the solution to SDE with respect to the initial data, and so involving the derivatives of the driving vector fields up to order 3. For $\Delta P_t f$, the intrinsic formula is proven to hold under conditions on the Ricci curvature not their derivatives, which does not seem to be the case for the Hessian of $P_t f$ (see [38]), a local Hessian formula is presented in [10].

Let $(x_t, t \geq 0)$ be a h -Brownian motion. If f is a smooth function with compact support we define $M_t^{df} := f(x_t) - f(x_0) - \int_0^t \Delta^h f(x_s) ds$.

Definition 7.1 ([36]). If α_t is a predictable process with $\alpha_t \in T_{x_t}^* M$, there is a unique process which we denote by $\int_0^t \alpha_s d\{x_s\}$ satisfying that

$$\left\langle \int_0^\cdot \alpha_s d\{x_s\}, M_t^{df} \right\rangle_t = \int_0^t \alpha_s(\nabla f(x_s)) ds.$$

We say that $\int_0^\cdot \alpha_s d\{x_s\}$ is the integral of α_s w.r.t. the martingale part of x_s .

The h -Brownian motion has two notable representations. The first is given by the gradient SDE of an isometric embedding $i : M \rightarrow \mathbb{R}^m$:

$$dx_t = \sum_{i=1}^m X_i(x_t) \circ dB_t^i + \nabla h(x_t) dt,$$

see 4 for detail. For the gradient SDE,

$$\int_0^t \alpha_s d\{x_s\} = \sum_{i=1}^m \int_0^t \alpha_s (X_i(x_s)) dB_s^i.$$

Let OM denote the orthonormal frame bundle over M and $\pi : OM \rightarrow M$ the canonical projection taking $u : \mathbb{R}^n \rightarrow T_x M$ to x . Let $\{e_i\}$ be an o.n.b. of \mathbb{R}^n and $\{H_i\}$ the corresponding fundamental horizontal vector fields on OM , so if π is the natural projection taking a frame $u : \mathbb{R}^n \rightarrow T_x M$ to the base point x , $H_i(u)$ projects to ue_i , and $H_i(u)$ is the horizontal lift of ue from $T_{\pi(u)}M$ to $T_u OM$. We consider the canonical horizontal SDE:

$$du_t = \sum_{i=1}^n H_i(u_t) \circ dB_t^i + \mathfrak{h}_{u_t}(\nabla h(\pi(u_t))) dt, \quad (7.1)$$

whose initial value u_0 is taken from $\pi^{-1}x_0$ and where $\mathfrak{h}_u(v)$ denotes the horizontal lift of a vector on $T_{\pi(u)}M$ to $T_u OM$, through $u_0 \in \pi^{-1}(x_0)$. The solutions (u_t) are called horizontal Brownian motions. Then $\pi(u_t)$ is a h -Brownian motion with initial value x_0 . This is the canonical representation of the h -brownian motion. Furthermore, $u_t u_0^{-1}$ is the stochastic parallel translation along (x_t) , and

$$\int_0^t \alpha_s d\{x_s\} = \int_0^t \alpha_s (u_s dB_s).$$

In the rest of this section, (x_t) denotes a h -Brownian flow, given by one of the above two representations, and $(x_t, W_t, W_t^{(2)})$ solves equation (6.1).

Theorem 7.2. *Suppose C1. Then for any $f \in \mathcal{B}_b$, any $t > 0$,*

$$\begin{aligned} \text{Hess } P_t^h f(v_1, v_2) &= \frac{4}{t^2} \mathbb{E} \left[f(x_t) \int_{t/2}^t \langle d\{x_s\}, W_s(v_1) \rangle \int_0^{t/2} \langle d\{x_s\}, W_s(v_2) \rangle \right] \\ &\quad + \frac{2}{t} \mathbb{E} \left[f(x_t) \int_0^{t/2} \langle d\{x_s\}, W_s^{(2)} \rangle \right], \end{aligned}$$

If $x_t = \pi(u_t)$, see (7.1), then $d\{x_s\} = u_s dB_s$.

Proof. It is sufficient to prove this for $f \in C_K^\infty$, followed by a smooth approximation of the bounded measurable function by a uniformly bounded sequence of functions in C_K^∞ and use the upper bound on the stochastic damped parallel translation and the L^2 boundedness of $\|W_s^{(2)}\|$ to pass the limit in a neighbourhood of a given point. We will also need the gradient formula: for all $0 < t \leq T$ and $v \in T_{x_0}M$,

$$dP_T^h f(v) = \frac{1}{t} \mathbb{E} \left[f(x_T) \int_0^t \langle W_s(v), u_s dB_s \rangle \right]$$

holds if $\text{Ric} - 2\text{Hess}(h)$ is bounded from below. See e.g. Elworthy and Li [37], Elworthy, LeJan, and Li [35], Thalmaier and Wang [86], Arnaudon, Driver and Thalmaier [9], and Li and Thompson [71].

Let $f \in C_K^\infty$. We only need to prove this for $x_t = \pi(u_t)$, where u_t solves (7.1). An application of Weitzenböck formula $\Delta = \text{trace } \nabla^* \nabla - \text{Ric}^\sharp$ shows that $dP_t^h f = e^{\frac{1}{2}\Delta^h}(df)$ solves the heat equation on differential 1-forms with initial value df :

$$\frac{d}{dt} dP_t^h f = \frac{1}{2} \text{trace}(\nabla^{h,*} \nabla)(dP_t^h f) + \frac{1}{2} \left(-\text{Ric}^\sharp + 2(\nabla^2 h)^\sharp \right) (dP_t^h f),$$

where $\nabla^{h,*}$ is the adjoint of ∇ with respect to $e^{2h} dx$. Applying Itô's formula to (u_t, W_t) and $f \circ \pi$, a routine computation shows that

$$d(P_{T-t}^h f)(W_t(v_1)) = d(P_t^h f)(v_1) + \int_0^t \nabla_{u_s dB_s} (dP_{T-s}^h f)(W_s(v_1)).$$

Taking $t \uparrow T$ we see that,

$$(df)(W_T(v_1)) = d(P_T^h f)(v_1) + \int_0^T \nabla_{u_s dB_s} (d(P_{T-s}^h f)(W_s(v_1))).$$

If $\rho^h \geq K$ then $|W_t|^2 \leq e^{-Kt}$. Consequently we may use Itô's isometry to obtain:

$$\begin{aligned} & \mathbb{E}(df)(W_T(v_1)) \int_0^T \langle u_s dB_s, W_s(v_2) \rangle \\ &= \mathbb{E} \left[\int_0^T \text{Hess}(P_{T-s}^h f)(W_s(v_2), W_s(v_1)) ds \right]. \end{aligned} \quad (7.2)$$

Since f satisfies (5.2),

$$\text{Hess}(P_t^h f)(v_2, v_1) = \mathbb{E} [\text{Hess } f(W_t(v_2), W_t(v_1))] + \mathbb{E} df(W_t^{(2)}),$$

we replace t by s and f by $P_{T-s}^h f$ which is easily seen to be also a BC^2 function:

$$\mathbb{E} [\text{Hess } P_{T-s}^h f(W_s(v_1), W_s(v_2))] = \text{Hess}(P_s^h (P_{T-s}^h f))(v_1, v_2) - \mathbb{E} d(P_{T-s}^h f)(W_s^{(2)}).$$

With this, we return to equation (7.2)

$$\begin{aligned} & \mathbb{E} \left[(df)(W_T(v_1)) \int_0^T \langle u_s dB_s, W_s(v_2) \rangle \right] \\ &= T \text{Hess } P_T^h f(v_1, v_2) + \mathbb{E} \left[\int_0^T dP_{T-s}^h f(W_s^{(2)}) ds \right]. \end{aligned}$$

Now $T \text{Hess } P_T^h f(v_1, v_2)$ is given by

$$\mathbb{E} \left[(df)(W_T(v_1)) \int_0^T \langle u_s dB_s, W_s(v_2) \rangle \right] - \mathbb{E} \left[\int_0^T dP_{T-s}^h f(W_s^{(2)}) ds \right].$$

Setting $T = t/2$ and replacing f by $P_{t/2}^h f$, we see that

$$\left(\frac{t}{2}\right) \text{Hess } P_t^h f(v_1, v_2)$$

$$\begin{aligned}
&= \mathbb{E} \left[(dP_{t/2}^h f)(W_{t/2}(v_1)) \int_0^{t/2} \langle u_s dB_s, W_s(v_2) \rangle \right] - \mathbb{E} \left[\int_0^{t/2} dP_{t-s}^h f(W_s^{(2)}) ds \right] \\
&= \frac{2}{t} \mathbb{E} \left[f(x_t) \int_{t/2}^t \langle u_s dB_s, W_s(u_1) \rangle \int_0^{t/2} \langle u_s dB_s, W_s(v_2) \rangle \right] \\
&\quad + \mathbb{E} \left[f(x_t) \int_0^{t/2} \langle u_s dB_s, W_s^{(2)} \rangle \right].
\end{aligned}$$

In the above we have used the Markov property and the first order derivative formula for the first term which is obtained from considering $P_{\frac{t}{2}, t-s}^h(f \circ \pi)(\psi_{\frac{t}{2}, s}(u_{\frac{t}{2}}))$ where $\psi_{s,t}$ denotes the solution flow for the canonical SDE begins with s and ending at t and $\psi_{0,t} = u_t$. Apply Itô's formula to it, followed by taking s to t we see the following,

$$f(x_t) = P_{\frac{t}{2}, t}^h f(x_{\frac{t}{2}}) + \int_{\frac{t}{2}}^t (dP_{\frac{t}{2}, t-r}^h f)(u_r dB_r).$$

Multiply both sides by the suitable martingale we see

$$\mathbb{E} \left[f(x_t) \int_{t/2}^t \langle u_s dB_s, W_s(u_0) \rangle \right] = \mathbb{E} \left[\int_{t/2}^t (dP_{\frac{t}{2}, t-r}^h f)(W_r(v_1)) dr \right] = \frac{t}{2} dP_{\frac{t}{2}}^h f(v_1).$$

For the second term we observe that

$$P_{t/2}^h f(x_{t/2}) = P_t^h f(x_0) + \int_0^{t/2} d(P_{t-s}^h f)_{x_s}(u_s dB_s)$$

and multiply both sides by $\int_0^{t/2} \langle u_s dB_s, W_s^{(2)} \rangle$. Since $\int_0^t |W_s^{(2)}|^2 ds$ is finite, we obtain $\mathbb{E} \left[P_{\frac{t}{2}}^h f(x_{\frac{t}{2}}) \int_0^{t/2} \langle u_s dB_s, W_s^{(2)} \rangle \right] = \mathbb{E} \left[\int_0^{t/2} d(P_{t-s}^h f)(W_s^{(2)}) \right] ds$ and the desired formula follows. \square

Definition 7.3. Given $v_1, v_2 \in T_x M$ we define

$$N_t = \frac{4}{t^2} \int_{\frac{t}{2}}^t \langle d\{x_s\}, W_s(v_1) \rangle \int_0^{\frac{t}{2}} \langle d\{x_s\}, W_s(v_2) \rangle,$$

where $d\{x_s\}$ stands for the martingale part of x_s .

In case x_t is given by the projection of u_t ,

$$N_t = \frac{4}{t^2} \int_{\frac{t}{2}}^t \langle u_s dB_s, W_s(v_1) \rangle \int_0^{\frac{t}{2}} \langle u_s dB_s, W_s(v_2) \rangle. \quad (7.3)$$

Lemma 7.4. Assume ρ^h is bounded from below. If $V : M \rightarrow \mathbb{R}$ is bounded and Hölder continuous with $V(x_0) = 0$ and $f : M \rightarrow \mathbb{R}$ is a bounded measurable function, then

$$\int_0^t (\mathbb{E} [P_{t-r}^h (V P_r^h f) N_{t-r}])^{1+\epsilon} dr$$

is finite for some number $\epsilon > 0$.

Proof. It is sufficient to show that there exist positive numbers c and δ such that

$$\mathbb{E} [P_{t-r}^h (VP_r^{h,V} f) N_{t-r}] \leq \frac{c}{(t-r)^{1-\delta}}.$$

By Cauchy-Schwarz inequality and the bound on $|W_t|$ using the lower bound on ρ^h , and Burkholder-Davies-Gundy inequality we see that for any $q > 0$, $\|N_{t-r}\|_{L^q}$ is of the order of $\frac{1}{t-r}$, and so

$$\mathbb{E} [P_{t-r}^h (VP_r^{h,V} f) N_{t-r}] \leq C \frac{1}{t-r} \left(\mathbb{E} [P_{t-r}^h (VP_r^{h,V} f)^2] \right)^{\frac{1}{2}}.$$

To treat $P_{t-r}^h (VP_r^{h,V} f)$, we simply drop $P_r^{h,V} f$ which can be seen easily to be bounded, following from the Feynman-Kac's formula. We have,

$$\mathbb{E} [P_{t-r}^h (VP_r^h f)]^2 \leq (\|f\|_\infty)^2 \mathbb{E} V^2(x_t).$$

If V is globally Hölder continuous of order α ,

$$\mathbb{E} V^2(x_{t-r}) \leq \mathbb{E} d^{2\alpha}(x_{t-r}, x_0) \leq c(t-r)^\alpha.$$

giving enough to ensure the required integrability. We have used part (2) of Lemma 4.2.

Now we suppose that V is Hölder continuous of order α on a small geodesic ball $B(a)$ of radius a around V_0 . Denote by τ the first exit time of x_t from U . Then for a constant c depending on $B(a)$,

$$\begin{aligned} \mathbb{E} V^2(x_{t-r}) &\leq V^2(x_{t-r}) \mathbf{1}_{t-r \leq \tau} + \mathbb{E} V^2(x_{t-r}) \mathbf{1}_{t-r > \tau} \\ &\leq c(B(a))(t-r)^\alpha + |V|_\infty \mathbb{P}(\tau \leq t-r) \\ &\leq c(B(a))(t-r)^\alpha + |V|_\infty \frac{\mathbb{E} \sup_{s \leq t-r} d^2(x_s, x_0)}{a^2} \\ &\leq c(B(a))(t-r)^\alpha + |V|_\infty \frac{t-r}{a^2}. \end{aligned}$$

We used in line 3 Chebyshev inequality and in line 4 an estimate from Lemma 4.2. The proof is complete. \square

We present an elementary lemma.

Lemma 7.5. *Let $g : [0, \infty) \times M \rightarrow \mathbb{R}$ be a Borel measurable function. Define $Tg(t) = \int_0^t g(r, x) dr$. Then $d(Tg)(t)(v) = \int_0^t dg(r, \cdot)(v)$ for any $v \in T_x M$ provided that for any normal geodesic $(\gamma(t), t \in [0, a])$ with initial value x and initial velocity $\dot{\gamma}(0)$, $dg(r, \gamma(s))(\dot{\gamma}(s)) \in L^1([0, t] \times [0, a])$, and $s \mapsto \int_0^t dg(r, \gamma(s))(\dot{\gamma}(s))$ is continuous in L^1 .*

Proof. We turn the differential into an integral and use Fubini's theorem to exchange the order of integration and obtain $d(Tg)(t)(v) = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int_0^\epsilon \int_0^t dg(r, \gamma(s))(\dot{\gamma}(s)) ds dr$. Finally we use the L^1 continuity to conclude. \square

For $a < t$ and x_0 fixed we define

$$\mathbb{V}_{t-r,t} = (V(x_{t-r}) - V(x_0)) e^{-\int_{t-r}^t [V(x_s) - V(x_0)] ds}. \quad (7.4)$$

Thus we first remove $V(x_0)$ and then put it back.

Theorem 7.6 (Second Order Feynman-Kac Formula). *Assume C1. Let V be a bounded Hölder continuous function. Then for any $f \in \mathcal{B}_b(M; \mathbb{R})$,*

$$\begin{aligned} \text{Hess } P_t^{h,V} f(v_1, v_2) &= e^{-V(x_0)t} \mathbb{E} [f(x_t) N_t] + e^{-V(x_0)t} \mathbb{E} \left[f(x_t) \frac{2}{t} \int_0^{t/2} \langle d\{x_s\}, W_s^{(2)}(v_1, v_2) \rangle \right] \\ &\quad + e^{-V(x_0)t} \int_0^t \mathbb{E} \left[f(x_t) \frac{2\mathbb{V}_{t-r,t}}{t-r} \int_0^{(t-r)/2} \langle d\{x_s\}, W_s^{(2)}(v_1, v_2) \rangle \right] dr \\ &\quad + e^{-V(x_0)t} \int_0^t \mathbb{E} [f(x_t) \mathbb{V}_{t-r,t} N_{t-r}] dr. \end{aligned}$$

Proof. Let us assume that $V(x_0) = 0$. If $V(x_0) \neq 0$, we shift it to be zero at x_0 . If $U = V - V(x_0)$ then $P_t^{h,V} = e^{-V(x_0)t} P_t^{h,U}$, and

$$dP_t^{h,V} f = e^{-V(x_0)t} dP_t^{h,U} f, \quad \nabla dP_t^{h,V} f = e^{-V(x_0)t} \nabla dP_t^{h,U} f.$$

Let $f \in \mathcal{B}_b$ we may differentiate both sides of the variation of constant formula and obtain the following formula where the differentiation is w.r.t. the first variable at the point x_0 and $V(x_0) = 0$:

$$\text{Hess } P_t^{h,V} f(v_1, v_2) = \text{Hess } P_t^h f(v_1, v_2) + \int_0^t \text{Hess } P_{t-r}^h (V P_r^{h,V} f)(v_1, v_2) dr. \quad (7.5)$$

To justify the exchange of integral and differential we first check that

$$dP_t^{h,V} f(v) = dP_t^h f(v) + \int_0^t dP_{t-r}^h (V P_r^{h,V} f)(v) dr$$

holds for any $v \in T_{x_0}M$. Indeed let $x_t = \pi(u_t)$ where (u_t) is the canonical process, we apply the formula, $dP_t^h f = \frac{1}{2} \mathbb{E} f(x_t) \int_0^t \langle u_r dB_r, W_r(\dot{\sigma}(s)) \rangle$, which holds since ρ^h is bounded from below. Then,

$$\begin{aligned} &\int_0^t dP_{t-r}^h (V P_r^{h,V} f)(\dot{\sigma}(s)) \\ &= \int_0^t \frac{1}{t-r} \mathbb{E} \left[(V P_{t-r}^{h,V} f)(F_t(\sigma(s))) \int_0^{t-r} \langle u_r dB_r, W_r(\dot{\sigma}(s)) \rangle \right]. \end{aligned}$$

Let $\sigma : [0, a] \rightarrow M$ be a normal geodesic with initial conditions x_0 and v . Since V, f are bounded,

$$\left| \mathbb{E} \left[(V P_{t-r}^{h,V} f)(F_t(\sigma(s))) \int_0^{t-r} \langle u_r dB_r, W_r(\dot{\sigma}(s)) \rangle \right] \right| \leq C \frac{1}{\sqrt{t-r}}.$$

By Lemma 7.5 the exchange of order of integration and differentiation is justified.

For the second order derivative, we apply Theorem 7.2 to its integrand. Formally we have,

$$\begin{aligned} &\int_0^t \text{Hess } P_{t-r}^h (V P_r^{h,V} f)(v_1, v_2) dr \\ &= \int_0^t \frac{4}{(t-r)^2} \mathbb{E} \left[(V P_r^{h,V} f)(x_{t-r}) \int_{(t-r)/2}^{t-r} \langle u_s dB_s, W_s(v_1) \rangle \int_0^{(t-r)/2} \langle u_s dB_s, W_s(v_2) \rangle \right] dr \end{aligned}$$

$$+ \int_0^t \frac{2}{t-r} \mathbb{E} \left[(VP_r^{h,V} f)(x_{t-r}) \int_0^{(t-r)/2} \langle u_s dB_s, W_s^{(2)}(v_1, v_2) \rangle \right] dr.$$

The integrand has an obvious singularity at $r = t$. Since $|W_s^{(2)}|^2$ has second moment which is locally bounded, the norm of the integrand in the second integral is of order $\sqrt{t-r}$ and is integrable. For the first integral,

$$\int_{(t-r)/2}^{t-r} \langle u_s dB_s, W_s(v_1) \rangle \int_0^{(t-r)/2} \langle u_s dB_s, W_s(v_2) \rangle,$$

we use Lemma 7.4 to see the first integrand is uniformly integrable and hence we may exchange the order of integration and differentiation.

Finally, by the zero order Feynman-Kac formula, the first term on the right hand side is:

$$\int_0^t dr \mathbb{E} \left[V(x_{t-r}) f(x_t) e^{-\int_{t-r}^t V(x_r) dr} N_{t-r} \right] dr,$$

while the second term becomes,

$$- \int_0^t dr \frac{2}{t-r} \mathbb{E} \left[V(x_{t-r}) f(x_t) e^{-\int_{t-r}^t V(x_r) dr} \int_0^{(t-r)/2} \langle u_s dB_s, W_s^{(2)} \rangle \right].$$

This concludes the proof. \square

An immediate consequence of (7.5) is that the Hessian of the Feynman-Kac kernel $p_t^{h,V}$ can be expressed by p_t^h and its derivatives.

Corollary 7.7. *We assume $V(x_0) = 0$ for simplicity and the conditions of the earlier theorem.*

$$\text{Hess } p_t^{h,V}(x_0, y) = \text{Hess } p_t^h(x_0, y) + \int_0^t \int_M V(z) \text{Hess } p_{t-r}^h(x_0, z) p_r^{h,V}(z, y) dz dr;$$

$$\begin{aligned} & \text{Hess } p_t^{h,V}(x_0, y) \\ &= \text{Hess } p_t^h(x_0, y) + \int_0^t \int_M V(z) \text{Hess } p_{t-r}^h(x_0, z) p_r^h(z, y) \mathbb{E}[e^{-\int_0^r V(Y_s^{r,z,y})}] dz dr, \end{aligned}$$

where $Y_s^{r,z,y}$ is the h -Brownian bridge with terminal value r , initial value z and terminal value y .

The proof for these are straightforward. The integrals make sense by the previous estimates.

8. Hessian Estimates for Schrödinger Semi-groups and Feynman-Kac Kernels

If ϕ is a random function, set $\mathcal{H}(\phi) = \phi \log \phi$. The following first order estimates are given in [71] for a positive function normalised such that $P_t^{h,V} f(x_0) = 1$:

$$|\nabla P_t^{h,V} f|_{x_0} \leq \sqrt{\frac{C_1(t, K)}{t}} \left(\mathbb{E} \left[\left(\mathcal{H} \left(f(x_t) e^{\int_0^t V(x_s) ds} \right) \right)^+ \right] \right)^{\frac{1}{2}} + t |\nabla V|_{\infty} C_2(t, K),$$

Where C_1, C_2 are explicit constants. Choosing f to be the Feynman-Kac kernel in the above we have the following kernel estimates:

$$\begin{aligned} & |\nabla \log p_t^{h,V}|_{x_0} \\ & \leq \frac{\sqrt{2C_1}}{\sqrt{t}} \left(\left(\sup_{y \in M} \log \frac{p_t^h(y, y_0)}{p_{2t}^h(x_0, y_0)} + 2t(\sup V - \inf V) \right)^+ \right)^{\frac{1}{2}} + t|\nabla V|_{\infty} C_2. \end{aligned}$$

Since

$$\nabla d \log P_t^{h,V} f = \frac{\nabla d P_t^{h,V} f}{P_t^{h,V} f} - \nabla \log P_t^{h,V} f \otimes \nabla \log P_t^{h,V} f,$$

for Hessian estimates on $\log P_T^{h,V}$ it is sufficient to estimate the first term in the identity.

We make an Hessian estimate for the Schrödinger operator at the point where the potential function vanishes. (Shift the potential to zero otherwise).

Theorem 8.1. *Assume **C2+** **C1(c)**. Let V be a bounded Hölder continuous non-negative function with $V(x_0) = 0$. If f is a non-negative non-trivial function, then*

$$\begin{aligned} \left| \frac{\nabla d P_t^{h,V} f}{P_t^{h,V} f} \right|_{x_0} & \leq \frac{1}{\sqrt{t}} \left(c_1 \mathbb{E} \left[\frac{f(x_t)}{P_t^h f(x_0)} \log \left(\frac{f(x_t)}{P_t^h f(x_0)} \right) \right] + c_2 + c_3 A \right) \\ & \quad + \frac{1}{t} c_2(t, K) \mathbb{E} \left[\frac{f(x_t)}{P_t^h f(x_0)} \log \left(\frac{f(x_t)}{P_t^h f(x_0)} \right) \right] + \frac{1}{t} C_2(t, K) c(\delta_0) \\ & \quad + c|f|_{\infty} (|V|_{\infty})^{\frac{1}{2}} \int_0^t \frac{\sqrt{P_{t-r}|V|(x_0)}}{t-r} dr \\ & \quad + c|f|_{\infty} |V|_{\infty} \sqrt{t} \sup_{s \leq t} \mathbb{E} |W_s^{(2)}(v_1, v_2)|^2 \end{aligned}$$

Here the constants may depend on the time and on the potential V , the dependence are locally bounded.

Proof. Under the conditions of the theorem, the Hessian formula in Theorem 7.6 holds. For the first two terms on the right hand side, we divide it by $P_t^h f(x_0)$ and apply Lemma 6.3,

$$\begin{aligned} & \mathbb{E} [f(x_t) N_t] + \mathbb{E} \left[f(x_t) \frac{2}{t} \int_0^{t/2} \langle d\{x_s\}, W_s^{(2)}(v_1, v_2) \rangle \right] \\ & \leq \frac{P_t^h f(x_0)}{\sqrt{t}} \left(c_1 \mathbb{E} \left[\frac{f(x_t)}{P_t^h f(x_0)} \log \left(\frac{f(x_t)}{P_t^h f(x_0)} \right) \right] + c_2 + c_3 A \right) \\ & \quad + \frac{P_t^h f(x_0)}{t} \left(c_2(t, K) \mathbb{E} \left[\frac{f(x_t)}{P_t^h f(x_0)} \log \left(\frac{f(x_t)}{P_t^h f(x_0)} \right) \right] + \frac{1}{t} C_2(t, K) c(\delta_0) \right). \end{aligned}$$

These two terms are of order $\frac{1}{t}$ and $\frac{1}{\sqrt{t}}$ (small time) respectively. For the fourth term, we must use regularity of V to compensate for the singularity in time. Let us apply Hölder and Burkholder-Davis-Gundy inequality to see that

$$\left| \int_0^t \mathbb{E} [f(x_t) \nabla_{t-r,t} N_{t-r}] dr \right| \leq c|f|_{\infty} (|V|_{\infty})^{\frac{1}{2}} \int_0^t (|V(x_{t-r})|_{L^1(\Omega)})^{\frac{1}{2}} \frac{1}{t-r} dr.$$

for some constant c , and also,

$$\begin{aligned} & \int_0^t \mathbb{E} \left[f(x_t) \frac{2\mathbb{V}_{t-r,t}}{t-r} \int_0^{(t-r)/2} \langle d\{x_s\}, W_s^{(2)}(v_1, v_2) \rangle \right] dr \\ & \leq c|f|_\infty |V|_\infty \sqrt{t} \sup_{s \leq t} \mathbb{E} |W_s^{(2)}(v_1, v_2)|^2. \end{aligned}$$

The proof is complete. \square

This leads to estimates on the Hessian of the Feynman-Kac kernel.

Corollary 8.2. *Suppose **C2** and **C1(c)**. Then,*

$$\begin{aligned} \left| \frac{\nabla d p_{2t}^h}{p_{2t}^h} \right|_{x_0} & \leq \frac{1}{\sqrt{t}} \left(c_1 \sup_{y \in M} \left(\log \frac{p_t^h(y, y_0)}{p_{2t}^h(x_0, y_0)} \right) + c_2 + c_3 A \right) \\ & \quad + \frac{1}{t} c_2(t, K) \sup_{y \in M} \left(\log \frac{p_t^h(y, y_0)}{p_{2t}^h(x_0, y_0)} \right). \end{aligned}$$

Proof. In the theorem above, take $V = 0$ and $f = p_t^h(x, y_0)$ where y_0 is a fixed point. Set $\mathcal{H}_t(f, x_0)$. Note that $P_t^h(p_t^h(\cdot, y_0))(x) = p_{2t}^h(x, y_0)$ we see that

$$\begin{aligned} \frac{|\nabla d p_{2t}^h(\cdot, y_0)|_{x_0}}{p_{2t}^h(x_0, y_0)} & \leq \frac{1}{\sqrt{t}} \left(c_1 \mathbb{E} \left[\frac{p_t^h(x_t, y_0)}{p_{2t}^h(x_0, y_0)} \log \left(\frac{p_t^h(x_t, y_0)}{p_{2t}^h(x_0, y_0)} \right) \right] + c_2 + c_3 A \right) \\ & \quad + \frac{1}{t} c_2(t, K) \mathbb{E} \left[\frac{p_t^h(x_t, y_0)}{p_{2t}^h(x_0, y_0)} \log \left(\frac{p_t^h(x_t, y_0)}{p_{2t}^h(x_0, y_0)} \right) \right]. \end{aligned}$$

Observe that,

$$\begin{aligned} \mathcal{H}_t(p_t^h(\cdot, y_0), x_0) & \equiv \mathbb{E} \left[\frac{p_t^h(x_t, y_0)}{p_{2t}^h(x_0, y_0)} \log \left(\frac{p_t^h(x_t, y_0)}{p_{2t}^h(x_0, y_0)} \right) \right] \\ & \leq \sup_{y \in M} \left(\log \frac{p_t^h(y, y_0)}{p_{2t}^h(x_0, y_0)} \right) \mathbb{E} \left[\frac{p_t^h(x_t, y_0)}{p_{2t}^h(x_0, y_0)} \right] \leq \sup_{y \in M} \left(\log \frac{p_t^h(y, y_0)}{p_{2t}^h(x_0, y_0)} \right). \end{aligned} \quad (8.1)$$

This completes the proof. \square

If the fundamental solution p_t^h satisfies an on diagonal upper bound for the kernel of the following form: $p_t^h(y, y_0) \leq \lambda(t)$, where $\lambda(t)$ satisfies the doubling condition $\frac{\lambda(t)}{\lambda(2t)} \leq C$, and a lower bound of the form $p_t^h(x_0, y_0) \geq c_1 \lambda(t) e^{-d^2(x_0, y_0)/c_2 t}$, then for a constant c , we have

$$\sup_{y \in M} \left(\log \frac{p_t^h(y, y_0)}{p_{2t}^h(x_0, y_0)} \right) \leq c + c \frac{d^2(x_0, y_0)}{t},$$

and the familiar type estimates

$$\frac{|\nabla d p_{2t}^h(x_0, y_0)|}{p_{2t}^h(x_0, y_0)} \leq c \frac{d^2(x_0, y_0)}{t} + c \frac{1}{t}.$$

Such inequalities for strict elliptic operators on \mathbb{R}^n are known to Aronson [13], and to manifolds Li and Yau [59, Thm. 2.2]. See also Varopoulos, Fabes and Stroock, [88, 42]. In this paper we do not elaborate this in detail and we refer to the following papers on heat kernel and other related estimates: Grigor'yan, Hu and Lau, [50], Davies [27], Norris [77], Grigor'yan [49], Shubin [82], Hebisch and Saloff-Coste [54], Bakry and Qian [15],

Arnaudon, Thalmaier, and Wang, [12], XD Li [60], Norris and Stroock [78], Zhang [94]. See also Li and Xu [58] for differential Harnack inequalities and Barlow, Grigoryan and Kumagai [16] for the study in the context of Dirichlet form.

9. Manifolds with a Pole and Semi-classical Riemannian Bridges

Let y_0 be a pole for M . We denote by J_{y_0} the Jacobian determinant of the exponential map $\exp_{y_0} : T_{y_0}M \rightarrow M$ at y_0 :

$$J_{y_0} = |\det D_{\exp_{y_0}^{-1}(y)} \exp_{y_0}|.$$

The subscript y_0 will be omitted from time to time. Set $\Phi(y) = \frac{1}{2} J_{y_0}^{\frac{1}{2}}(y) \Delta J_{y_0}^{-\frac{1}{2}}(y)$. For $T > 0$ fixed, a semi-classical bridge (also called semi-classical Riemannian bridge) \tilde{x}_s is a time dependent diffusion with generator $\frac{1}{2} \Delta + \nabla \log k_{T-s}(\cdot, y_0)$ where, for d the Riemannian distance function,

$$k_t(x_0, y_0) := (2\pi t)^{-\frac{n}{2}} e^{-\frac{d^2(x_0, y_0)}{2t}} J^{-\frac{1}{2}}(x_0).$$

The radial process $r_t = d(\tilde{x}_t, y_0)$ of the semi-classical bridge is the n -dimensional Bessel bridge. On \mathbb{R}^n , the semi-classical bridge agrees with the Brownian motion conditioned to be at y_0 at time T . The semi-classical bridge are introduced in [33], K. D. Elworthy and A. Truman who proved the following formula,

$$p_T^V(x_0, y_0) = k_T(x_0, y_0) \mathbb{E} \left[e^{\int_0^T (\Phi - V)(\tilde{x}_s) ds} \right], \quad (9.1)$$

under the condition that $\Phi - V$ is bounded from above. Their consideration comes from classical mechanics. Their method to overcome the singularity at time T is to overshoot the target by a drift $\nabla \log k_{T+\epsilon-t}$. In Li [68] and Li and Thompson [71], we considered the semi-classical bridge on $[0, t]$ where $t < T$ on which the distributions of the semi-classical bridge process and the h -Brownian motion are equivalent, and use Girsanov transform to prove this formula and a gradient formula for the Feynman-Kac semi-groups. In those cases it is not difficult to take the limit t to T . A similar technique is used in [67].

Let us define

$$\Phi^h = -\frac{1}{2} |\nabla h|^2 - \frac{1}{2} \Delta h + \Phi.$$

Let \tilde{u}_s denote the solution to the following canonical horizontal SDE on the orthonormal frame bundle

$$d\tilde{u}_s = \sum_{i=1}^n H_i(\tilde{u}_s) \circ dB_s^i + \mathfrak{h}_{\tilde{u}_s}(\nabla \log k_{T-s}(\pi(\tilde{u}_s))) ds$$

with $\tilde{u}_0 = u_0$ and $\nabla \log k_{T-s}$ indicates the horizontal lift of $\nabla \log k_{T-s}$, then $\tilde{x}_s = \pi(\tilde{u}_s)$ is a semi-classical bridge with initial value $x_0 = \pi(u_0)$. Let (u_t) be the solution to the canonical horizontal SDE (7.1).

Lemma 9.1 (Lemma 3.2 [71]). *Suppose that $h \in C^2(M; \mathbb{R})$ and that the h -Brownian motion x_t is complete. Fix $t \in [0, T)$. Then the probability distributions of u_t and \tilde{u}_t are equivalent on \mathcal{F}_t . If F is a function on the path space $C([0, t]; M)$ then $\mathbb{E}F(u_\cdot) = \mathbb{E}F(\tilde{u}_\cdot | M_t)$ where*

$$M_t := e^{h(\tilde{x}_t) - h(x_0)} \frac{k_T(x_0, y_0)}{k_{T-t}(\tilde{x}_t, y_0)} \exp \left[\int_0^t \Phi^h(\tilde{x}_s) ds \right]. \quad (9.2)$$

This result follows from the identity $\frac{1}{2}\Delta^h + \nabla \log(k_{T-s}e^{-h}) = \frac{1}{2}\Delta + \nabla(\log k_{T-s})$, Girsanov's theorem for SDEs states that M_t equals to

$$\exp\left[-\sum_{i=1}^m \int_0^t \langle \nabla \log(k_{T-s}e^{-h})(\tilde{x}_s), \tilde{u}_s e_i \rangle dB_s^i - \frac{1}{2} \int_0^t |\nabla \log(k_{T-s}e^{-h})|_{(\tilde{x}_s)}^2 ds\right],$$

which reduced to (9.2) by an application of Itô's formula for $\log(e^{-h}k_{T-t})(\tilde{x}_t)$ and the following identities:

$$\begin{aligned} & \frac{1}{2}|\nabla \log(k_{T-s}e^{-h})|^2 + \left(\frac{\partial}{\partial s} + \frac{1}{2}\Delta^h + Z\right) \log(k_{T-s}e^{-h}) \\ &= \frac{1}{8}|\nabla \log J|^2 + \frac{1}{2}|\nabla h|^2 - \frac{1}{4}\Delta(\log J) - \frac{1}{2}\Delta h - |\nabla h|^2 + \langle Z, \nabla \log(k_{T-s}e^{-h}) \rangle \\ &= \frac{1}{8}|\nabla \log J|^2 - \frac{1}{4}\Delta(\log J) - \frac{1}{2}\Delta h - \frac{1}{2}|\nabla h|^2 + \langle Z, \nabla(\log k_{T-s}e^{-h}) \rangle. \end{aligned}$$

9.1. Basic Estimates. We intend to give derivative versions of the elementary formula (9.1), aiming to obtain Hessian estimates for $\log p_t^h$ and p_t^h . For this we set $\frac{D}{dt}\tilde{W}_t := //_t \frac{d}{dt} //_t^{-1} \tilde{W}_t$, where $//_t$ indicates stochastic parallel transport along \tilde{x} . If $\tilde{x}_t = \pi(\tilde{u}_t)$ then $//_t = \tilde{u}_t \tilde{u}_0^{-1}$. If \tilde{x}_t is defined to be the solution of the gradient SDE, then if we identify \mathbb{R}^n with $T_{x_0}M$, $//_t$ is given by the solution of the horizontal lift of the gradient SDE to the orthonormal frame bundle.

We denote by $\tilde{W}_t(v_0) : T_{x_0}M \rightarrow T_{\tilde{x}_t}M$ the solutions to the following equation along the semi-classical bridge \tilde{x}_t ,

$$\frac{D}{dt}\tilde{W}_t = -\frac{1}{2}\text{Ric}_{\tilde{x}_t}^\#(\tilde{W}_t) + (\nabla^2 h)_{\tilde{x}_t}^\#(\tilde{W}_t), \quad \tilde{W}_0 = \text{id}_{T_{x_0}M} \quad (9.3)$$

To obtain a formula for the Hessian, let us consider the triple, where \tilde{x}_t is a semi-classical Riemannian bridge and $(\tilde{x}_t, \tilde{W}_t, \tilde{W}_t^{(2)})$ is the solution to the following system of equations:

$$\begin{aligned} \frac{D}{dt}\tilde{W}_t &= \left(-\frac{1}{2}\text{Ric}_{\tilde{x}_t}^\# + (\nabla^2 h)_{\tilde{x}_t}^\#\right)(\tilde{W}_t), \quad \tilde{W}_0 = v_1. \\ D\tilde{W}_t^{(2)} &= \left(-\frac{1}{2}\text{Ric}_{\tilde{x}_t}^\# + (\nabla^2 h)_{\tilde{x}_t}^\#\right)(\tilde{W}_t^{(2)}) dt - \frac{1}{2}\Theta^h(\tilde{W}_t(v_2))(\tilde{W}_t(v_1))dt \\ &+ R(d\{\tilde{x}_t\}, \tilde{W}_t(v_2))\tilde{W}_t(v_1), \quad \tilde{W}_0^{(2)} = 0 \end{aligned} \quad (9.4)$$

where $\{\tilde{x}_s\}$ is the martingale part of \tilde{x}_t . The tilde signs over W_t and $W_t^{(2)}$ indicate that these transport processes are along the semi-classical bridge. We use $\tilde{W}_t(v_1)$ and $W_t^{(2)}(v_1, v_2)$ for the solution with initial value $v_1, v_2 \in T_{x_0}M$, and $W_t, W_t^{(2)}$ are linear maps from $T_{x_0}M$ to $T_{\tilde{x}_t}M$, where $\tilde{x}_0 = x_0$. Occasionally, when there is no risk of ambiguity, the solutions are also denoted as $W_t, W_t^{(2)}$ (the the initial values are suppressed) for simplicity.

Before proceeding to prove the Hessian formula, we make elementary estimations and justify taking the limit t to T in integrals, appearing in the Hessian formula, of functionals of \tilde{x}_t from 0 to t . The following elementary inequalities will be used frequently.

Lemma 9.2. *Let $p > 0$. For a constant $c(p)$, the following estimates hold.*

$$\sup_{0 \leq t \leq T} \frac{2}{t} \int_0^{\frac{t}{2}} \frac{s^{\frac{p}{2}}}{(T-s)^{\frac{p}{2}} T^{\frac{p}{2}}} ds \leq c(p) T^{-\frac{p}{2}}, \quad \sup_{0 \leq t \leq \frac{T}{2}} \frac{2}{t} \int_{\frac{t}{2}}^t \frac{s^{\frac{p}{2}}}{(T-s)^{\frac{p}{2}} T^{\frac{p}{2}}} ds \leq c(p) T^{-\frac{p}{2}},$$

$$\sup_{\frac{T}{2} \leq t \leq T} \frac{2}{t} \int_{\frac{t}{2}}^t \frac{s^{\frac{p}{2}}}{(T-s)^{\frac{p}{2}} T^{\frac{p}{2}}} ds \leq c(p) T^{-p/2}, \quad \text{for } p < 2.$$

Lemma 9.3. (1) *For every $p \geq 1$ and T there exists a constant $\delta > 0$ such that if $\Phi^h \leq C + \delta d^2(\cdot, y_0)$ for some C , then*

$$\sup_{t \leq T} \mathbb{E} \left(e^{p \int_0^t \Phi^h(\tilde{x}_s) ds} \right) < \infty.$$

(2) *For some number $c(p, n)$, depending only on p and n ,*

$$\mathbb{E} d^p(\tilde{x}_s, y_0) \leq c(p) (T-s)^p \left(\frac{d(x_0, y_0)}{T} \right)^p + c(p, n) \left(\frac{s(T-s)}{T} \right)^{\frac{p}{2}},$$

For any $\delta > 0$ and $s \in [0, T)$,

$$\begin{aligned} & \mathbb{E} |\nabla \log k_{T-s}(\tilde{x}_s, y_0)|^2 \\ & \leq \frac{1}{4} (1 + \delta^{-1}) \mathbb{E} |\nabla \log J(\tilde{x}_s)|^2 + (1 + \delta) \left(\frac{d^2(x_0, y_0)}{T^2} + \frac{ns}{T(T-s)} \right). \end{aligned}$$

(3) *Set*

$$c_{p,J} = \sup_{0 \leq s \leq T} |\nabla \log J^{-\frac{1}{2}}(\tilde{x}_s)|_{L^p(\Omega)}.$$

$$F(p, r, t) := \frac{1}{t-r} \int_r^t \frac{s^{\frac{p}{2}}}{(T-s)^{\frac{p}{2}} T^{\frac{p}{2}}} ds. \quad (9.5)$$

There exist constants $c(p)$ and $c(p, n)$ such that

$$\begin{aligned} & \mathbb{E} \left(\frac{1}{t-r} \int_r^t |\nabla \log(k_{T-s}(\tilde{x}_s, y_0))|^p ds \right) \\ & \leq (2C_{p,J})^p + c(p) \frac{d^p(x_0, y_0)}{T^p} + c(p, n) F(p, r, t). \end{aligned}$$

(4) *Moreover for any $p > 0$,*

$$\begin{aligned} & \sup_{0 \leq t \leq T} \mathbb{E} \left(\frac{2}{t} \int_0^{\frac{t}{2}} |\nabla \log k_{T-s}(\tilde{x}_s, y_0)|^p ds \right) \\ & \leq (2C_{p,J})^p + c(p) \frac{d^p(x_0, y_0)}{T^p} + c(p, n) T^{-p/2}, \end{aligned} \quad (9.6)$$

(5) *and for $p < 2$,*

$$\begin{aligned} & \sup_{0 \leq t \leq T} \mathbb{E} \left(\frac{2}{t} \int_{\frac{t}{2}}^t |\nabla \log k_{T-s}(\tilde{x}_s, y_0)|^p ds \right) \\ & \leq (2C_{p,J})^p + c(p) \frac{d^p(x_0, y_0)}{T^p} + c(p, n) T^{-p/2}. \end{aligned}$$

Proof. The \mathbb{R}^n valued Brownian bridge is a Gaussian process, the exponential of its square distance function is integrable in L^a for $a < 1/2$ (Fernique's theorem). Since $d(\tilde{x}_s, y_0)$ is the n -dimensional Bessel bridge, $\sup_{s \leq t} \mathbb{E} e^{C d^2(\tilde{x}_s, y_0)}$ is finite for C sufficiently small, and any $p \geq 0$ and T , by choosing sufficiently small δ we see that $\Phi^h \left(e^{\int_0^t \Phi^h(\tilde{x}_s) ds}, t \leq T \right)$ is L^p bounded, concluding part (1).

Let $c(p, n)$ denote a constant depending only on p and n , varying from line to line. Using the explicit law of the n -dimensional Bessel bridge, we see that for some number $c(p, n)$,

$$\mathbb{E} d^p(\tilde{x}_s, y_0) \leq c(p) \left(\frac{(T-s)d(x_0, y_0)}{T} \right)^p + c(p, n) \left(\frac{s(T-s)}{T} \right)^{\frac{p}{2}}.$$

Furthermore $\mathbb{E} d^2(\tilde{x}_t, y_0) = \frac{(T-t)^2}{T^2} d^2(x_0, y_0) + \frac{nt(T-t)}{T}$, so the second part of statement (2) follows from applying standard inequalities to the following identity

$$|\nabla \log k_{T-t}(\tilde{x}_t, y_0)| = \left| \nabla \log J^{-\frac{1}{2}}(\tilde{x}_t) + \frac{(d\nabla d)(\tilde{x}_t, y_0)}{T-t} \right|.$$

We used the fact that $|\nabla d| = 1$. For any $p > 0$,

$$\begin{aligned} \mathbb{E} |\nabla \log k_{T-s}(\tilde{x}_s, y_0)|^p &\leq 2^p \mathbb{E} |\nabla \log J^{-\frac{1}{2}}(\tilde{x}_s)|^p + 2^p \frac{\mathbb{E} |(d\nabla d)(\tilde{x}_s, y_0)|^p}{(T-s)^p} \\ &\leq 2^p \mathbb{E} |\nabla \log J^{-\frac{1}{2}}(\tilde{x}_s)|^p + c(p) \frac{d^p(x_0, y_0)}{T^p} + c(p, n) \left(\frac{s}{T(T-s)} \right)^{\frac{p}{2}}, \end{aligned}$$

and so the first inequality in the statement of part (3) follows:

$$\begin{aligned} &\mathbb{E} \left(\frac{1}{t-r} \int_r^t |\nabla \log(k_{T-s}(\tilde{x}_s, y_0))|^p ds \right) \\ &\leq 2^p \sup_{r \leq s \leq t} \mathbb{E} |\nabla \log J^{-\frac{1}{2}}(\tilde{x}_s)|^p + c(p) \frac{d^p(x_0, y_0)}{T^p} + c(p, n) \frac{1}{t-r} \int_r^t \frac{s^{\frac{p}{2}}}{(T-s)^{\frac{p}{2}} T^{\frac{p}{2}}} ds. \end{aligned}$$

Replacing $r = \frac{t}{2}$ and t by $\frac{t}{2}$, and apply the following inequality from Lemma 9.2 to conclude the last two inequalities. \square

For an orthonormal basis $\{e_i\}_{i=1}^n$ of \mathbb{R}^n , we define as following a family of independent one dimensional Brownian motions on the probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{Q})$ where \mathbb{Q} is probability measure that is equivalent to \mathbb{P} on each \mathcal{F}_t , where $t < T$, with the density $\frac{d\mathbb{Q}}{d\mathbb{P}} = M_t$ where (M_t) is given in Lemma 9.1.

Definition 9.4. Set $\tilde{B}_t = (\tilde{B}_t^1, \dots, \tilde{B}_t^n)$, where for $1 \leq i \leq n$,

$$\tilde{B}_t^i := B_t^i + \int_0^t d \log(k_{T-s} e^{-h})(\tilde{u}_s e_i) ds.$$

For a number $p > 0$, the following notation will be used in the next lemma.

$$\begin{aligned} b_1(2p) &= \sup_{0 \leq s \leq T} \left| \nabla \log J^{-\frac{1}{2}}(\tilde{x}_s) \right|_{L^{2p}(\Omega)}, \\ b_2(2p) &= \sup_{0 \leq s \leq T} \left| \nabla h(\tilde{x}_s) \right|_{L^{2p}(\Omega)}, \\ b_3(2) &= \sup_{0 \leq s \leq T} \left| \tilde{W}_s^{(2)}(v_1, v_2) \right|_{L^2(\Omega)}, \\ b_4(2p) &= \sup_{0 \leq s \leq T} \left| \tilde{W}_s^{(2)}(v_1, v_2) \right|_{L^{2p}(\Omega)}, \\ A &= ((b_1(2p))^p + (b_2(2p))^p)^{\frac{1}{p}} b_4(2p). \end{aligned}$$

Lemma 9.5. *Assume C4a, C4b. Let $p \geq 1$. Then for a constant $c(p, n)$, depending only on p and n , the following estimate holds for any $t \in (0, T]$,*

$$\begin{aligned} & \left| \frac{2}{t} \int_0^{t/2} \langle d\{\tilde{x}_s\}, \tilde{W}_s^{(2)}(v_1, v_2) \rangle \right|_{L^p(\Omega)} \\ & \leq c(p, n) \left(b_4(2p) \frac{d(x_0, y_0)}{T} + b_4(2p) \frac{1}{\sqrt{T}} + b_3(2) \frac{1}{\sqrt{t}} + A \right). \end{aligned} \quad (9.7)$$

for all unit tangent vectors v_1, v_2 . Also the following holds for $0 < t \leq T$,

$$\begin{aligned} & \left| \int_0^t \frac{2}{t-r} \int_0^{(t-r)/2} \langle d\{\tilde{x}_s\}, \tilde{W}_s^{(2)}(v_1, v_2) \rangle dr \right|_{L^p(\Omega)} \\ & \leq c(p, n) \left(b_4(2p) d(x_0, y_0) + [b_4(2p) + b_3(2)] \sqrt{T} + AT \right). \end{aligned}$$

Furthermore, the following stochastic processes,

$$\begin{aligned} & \exp \left[\int_0^t \Phi^h(\tilde{x}_s) ds \right] \cdot \frac{2}{t} \int_0^{t/2} \langle d\{\tilde{x}_s\}, \tilde{W}_s^{(2)}(v_1, v_2) \rangle \\ & \exp \left[\int_0^t \Phi^h(\tilde{x}_s) ds \right] \int_0^t \frac{2}{t-r} \int_0^{(t-r)/2} \langle d\{\tilde{x}_s\}, \tilde{W}_s^{(2)}(v_1, v_2) \rangle dr \end{aligned}$$

are L^p bounded on $[\frac{T}{2}, T]$ and converge in L^1 as $t \uparrow T$.

Proof. Let us take the canonical representation of the process \tilde{x}_t so integration w.r.t. the martingale part of \tilde{x}_s is the same as w.r.t. $\tilde{u}_s d\tilde{B}_s$. Since $\|\Theta^h\| + \|\mathcal{R}\| \leq ce^{\delta_1 d(\cdot, y_0)}$, we apply Lemma 5.5 to conclude that $\sup_{s \leq t} \mathbb{E} \left(\left| \tilde{W}_s^{(2)}(v_1, v_2) \right|^q \right)$ is finite.

Let us denote $\tilde{W}_s^{(2)}(v_1, v_2)$ by $\tilde{W}_s^{(2)}$ to ease the notation. By Itô's isometry, for any $0 \leq r < t \leq T$,

$$\begin{aligned} \left| \frac{1}{t-r} \int_r^t \langle \tilde{u}_s dB_s, \tilde{W}_s^{(2)} \rangle \right|_{L^2(\Omega)} & \leq \frac{1}{t-r} \sqrt{\int_r^t \mathbb{E} |\tilde{W}_s^{(2)}|^2 ds} \\ & \leq \frac{1}{\sqrt{t-r}} \sup_{r \leq s \leq t} |\tilde{W}_s^{(2)}|_{L^2(\Omega)}. \end{aligned}$$

On the other hand, by the definition of \tilde{B}_t ,

$$\int_r^t \langle \tilde{u}_s d\tilde{B}_s, \tilde{W}_s^{(2)} \rangle = \int_r^t \langle \tilde{u}_s dB_s, \tilde{W}_s^{(2)} \rangle + \int_r^t d \log(k_{T-s} e^{-h}) \left(\tilde{W}_s^{(2)} \right) ds.$$

Below $c(p)$ stands for a constant depending on p . By multiple uses of Hölder's inequality and the elementary inequality $(a + b)^p \leq c(p)a^p + c(p)b^p$ we obtain the following for $p \geq 1$,

$$\begin{aligned} \mathbb{E} \left(\left| \frac{2}{t-r} \int_0^{\frac{t-r}{2}} \langle \tilde{u}_s d\tilde{B}_s, \tilde{W}_s^{(2)} \rangle \right|^p \right) &\leq c(p) \mathbb{E} \left(\left| \frac{2}{t-r} \int_0^{\frac{t-r}{2}} \langle \tilde{u}_s dB_s, \tilde{W}_s^{(2)} \rangle \right|^p \right) \\ &\quad + c(p) \mathbb{E} \left(\left| \frac{2}{t-r} \int_0^{\frac{t-r}{2}} d \log(k_{T-s} e^{-h}) (\tilde{W}_s^{(2)}) ds \right|^p \right). \\ \mathbb{E} \left(\left| \frac{2}{t-r} \int_0^{\frac{t-r}{2}} \langle \tilde{u}_s d\tilde{B}_s, \tilde{W}_s^{(2)} \rangle \right|^p \right) &\leq c(p) (t-r)^{-\frac{p}{2}} \sup_{0 \leq s \leq T} \left(\mathbb{E} |\tilde{W}_s^{(2)}|^2 \right)^{\frac{p}{2}} \\ &\quad + c(p) \frac{1}{t-r} \int_r^t \mathbb{E} \left(|\nabla \log(k_{T-s} e^{-h})(\tilde{x}_s)|^p \left| \tilde{W}_s^{(2)} \right|^p \right) ds. \end{aligned}$$

Since $\sup_{s \leq t} \mathbb{E} e^{\delta_2 d(\tilde{x}_s, y_0)} < \infty$, and since $|\nabla h|$ and $|\nabla \log J|$ are bounded by $\delta_2 d(\cdot, y_0)$, $|\nabla h(\tilde{x}_s)|$ and $|\nabla \log J(\tilde{x}_s)|$ are L^p bounded on $[0, T]$ for any $p \geq 1$. Since,

$$|\nabla \log(k_{T-s} e^{-h})(\tilde{x}_s)|^{2p} \leq c(p) |\nabla \log(k_{T-s})(\tilde{x}_s)|^{2p} + c(p) |\nabla h(\tilde{x}_s)|^{2p},$$

we obtain

$$\begin{aligned} &\frac{1}{t-r} \int_r^t \mathbb{E} \left(|\nabla \log(k_{T-s} e^{-h})(\tilde{x}_s)|^p \left| \tilde{W}_s^{(2)} \right|^p \right) ds \\ &\leq \left(\frac{2}{t-r} \int_0^{\frac{t-r}{2}} \mathbb{E} \left(|\nabla \log(k_{T-s})(\tilde{x}_s) - \nabla h(\tilde{x}_s)|^{2p} \right) \right)^{\frac{1}{2}} \sqrt{\frac{2}{t-r} \int_0^{\frac{t-r}{2}} \mathbb{E} \left(\left| \tilde{W}_s^{(2)} \right|^{2p} \right) ds} \\ &\leq (b_4(2p))^p \left(\frac{2}{t-r} \int_0^{\frac{t-r}{2}} \mathbb{E} \left(|\nabla \log(k_{T-s})(\tilde{x}_s)|^{2p} \right) + (b_2(2, p))^p \right)^{\frac{1}{2}}. \end{aligned}$$

On the other hand, by Lemmas 9.3 and 9.2 we have:

$$\begin{aligned} &\left(\frac{2}{t-r} \int_0^{\frac{t-r}{2}} \mathbb{E} |\nabla \log k_{T-s}(\tilde{x}_s)|^{2p} ds \right)^{\frac{1}{2}} \\ &\leq c(p) \left(\sup_{0 \leq s \leq T} \mathbb{E} |\nabla \log J^{-\frac{1}{2}}(\tilde{x}_s)|^{2p} + \frac{d^{2p}(x_0, y_0)}{T^{2p}} + \frac{2c(p, n)}{t-r} \int_0^{\frac{t-r}{2}} \frac{s^p}{(T-s)^p T^p} ds \right)^{\frac{1}{2}} \\ &\leq c(p) (b_1(2p))^p + c(p) \frac{d^p(x_0, y_0)}{T^p} + c(p, n) T^{-p/2}. \end{aligned}$$

Finally we obtain

$$\begin{aligned} &\mathbb{E} \left(\left| \frac{2}{t-r} \int_0^{\frac{t-r}{2}} \langle \tilde{u}_s d\tilde{B}_s, \tilde{W}_s^{(2)} \rangle \right|^p \right) \\ &\leq \frac{c(p)}{(t-r)^{\frac{p}{2}}} (b_3(2))^p + c(p) (b_4(2p))^2 [(b_1(2p))^p + (b_2(2p))^p] + \frac{d^p(x_0, y_0)}{T^p} + \frac{c(p, n)}{T^{\frac{p}{2}}} \\ &= c(p, n) \left((t-r)^{-\frac{p}{2}} (b_3(2))^p + A^p + (b_4(2p))^p \frac{d^p(x_0, y_0)}{T^p} + (b_4(2p))^p T^{-p/2} \right) \end{aligned}$$

For $r = 0$, this is (9.7), as required. For $p \geq 1$ integrating both sides of the above inequality from 0 to t with respect to r leads to

$$\begin{aligned} & \mathbb{E} \left| \int_0^t \frac{2}{t-r} \int_0^{(t-r)/2} \langle \tilde{u}_s d\tilde{B}_s, \tilde{W}_s^{(2)}(v_1, v_2) \rangle dr \right|^p \\ & \leq c(p, n) t^{\frac{p}{2}} (b_3(2))^p + t^p c(p, n) \left(A^p + (b_4(2p))^p \frac{d^p(x_0, y_0)}{T^p} + (b_4(2p))^p T^{-p/2} \right), \end{aligned}$$

giving the second required estimate.

From Lemma 9.3, $e^{\int_0^t \Phi^h(\tilde{x}_s) ds}$ is also L^p bounded for any $p \geq 1$, from which we see that $\exp \left[\int_0^t \Phi^h(\tilde{x}_s) ds \right] \cdot \frac{2}{t} \int_0^{t/2} \langle \tilde{u}_s d\tilde{B}_s, \tilde{W}_s^{(2)}(v_1, v_2) \rangle$ is L^p bounded. The L^p boundedness of the second stochastic process follows by the same argument, completing the proof of the lemma. \square

Definition 9.6. For $0 < t \leq T$ we set

$$\tilde{N}_t = \frac{4}{t^2} \int_{\frac{t}{2}}^t \langle d\{\tilde{x}_s\}, \tilde{W}_s(v_1) \rangle \int_0^{\frac{t}{2}} \langle d\{\tilde{x}_s\}, \tilde{W}_s(v_2) \rangle, \quad (9.8)$$

and $C_1(K, T) = \max_{0 \leq r \leq t \leq T} \frac{e^{Kt} - e^{Kr}}{K(t-r)}$. And for α, p positive we define

$$\begin{aligned} b_1(\alpha p) &= \sup_{0 \leq s \leq T} \left| \nabla \log J^{-\frac{1}{2}}(\tilde{x}_s) \right|_{L^{p\alpha}(\Omega)}, \\ b_2(\alpha p) &= \sup_{0 \leq s \leq T} |\nabla h(\tilde{x}_s)|_{L^{p\alpha}(\Omega)}, \\ a_1(\alpha, p, T) &= e^{|K|T} \left((b_1(\alpha'p))^2 + b_2(\alpha'p)^2 + (b_1(\alpha p))^2 + (b_2(\alpha p))^2 \right), \\ a_2(K, T) &= C_1(K, T) + e^{|K|T}. \end{aligned}$$

Lemma 9.7. Assume **C4a, C4b**. Let $\alpha, p \geq 1$ be real numbers such that $\alpha p < 2$. For a constant $c(p)$ depending only on p and n , the following holds for $t \in (0, T]$,

$$\left| \tilde{N}_t \right|_{L^p(\Omega)} \leq c(p, n) \left(a_1(\alpha, p, T) + e^{|K|T} \frac{d^2(x_0, y_0)}{T^2} + a_2(K, T) \frac{1}{T} \right).$$

Furthermore,

$$\begin{aligned} & \sup_{\frac{T}{2} \leq t \leq T} \left(\mathbb{E} \left| \int_0^t \tilde{N}_{t-r} dr \right|^p \right)^{\frac{1}{p}} \\ & \leq c(p, n) \left(a_1(\alpha, p, T) T + e^{|K|T} \frac{d^2(x_0, y_0)}{T} + a_2(K, T) \right). \end{aligned}$$

As a consequence, for any number $p \in [1, 2)$, (\tilde{N}_t) is L^p bounded on $[\frac{T}{2}, T]$; and so are the stochastic processes:

$$e^{\int_0^t \Phi^h(\tilde{x}_s) ds} \tilde{N}_t; \quad \text{and} \quad e^{\int_0^t \Phi^h(\tilde{x}_s) ds} \int_0^t |\tilde{N}_{t-r}| dr,$$

both converge in L^1 as $t \uparrow T$.

Proof. Let us take v_1, v_2 to be unit vectors and take the canonical representation of \tilde{x}_t so $d\{\tilde{x}_s\} = \tilde{u}_s d\tilde{B}_s$. We first compute $\mathbb{E}|\tilde{N}_{t-r}|^p$ for which it is sufficient to split the products. For $\alpha, \alpha' > 1$ conjugate, $\frac{1}{\alpha} + \frac{1}{\alpha'} = 1$, we have

$$\begin{aligned} & \mathbb{E}|\tilde{N}_{t-r}|^p \\ & \leq \left(\mathbb{E} \left(\left| \frac{2}{t-r} \int_{\frac{t-r}{2}}^{t-r} \langle \tilde{u}_s d\tilde{B}_s, \tilde{W}_s(v_1) \rangle \right|^{p\alpha} \right) \right)^{\frac{1}{\alpha}} \left(\mathbb{E} \left(\left| \frac{2}{t-r} \int_0^{\frac{t-r}{2}} \langle \tilde{u}_s d\tilde{B}_s, \tilde{W}_s(v_2) \rangle \right|^{p\alpha'} \right) \right)^{\frac{1}{\alpha'}}. \end{aligned}$$

Below c denotes a constant whose value may change from line to line. Using the lower bound $\rho^h \geq -K$ and Burkholder-Davis-Gundy inequality we see that, for any $0 \leq r < t \leq T$,

$$\mathbb{E} \left| \frac{1}{t-r} \int_r^t \langle \tilde{u}_s dB_s, \tilde{W}_s(v_i) \rangle \right|^p \leq \frac{c(p)}{(t-r)^p} \mathbb{E} \left(\int_r^t |\tilde{W}_s(v_i)|^2 ds \right)^{\frac{p}{2}} \leq \frac{c(p)C_1(K, T)^{\frac{p}{2}}}{(t-r)^{\frac{p}{2}}},$$

where $C_1(K, T) = \max_{0 \leq r \leq t \leq T} \frac{-e^{Kr} + e^{Kt}}{K(t-r)}$ if $K \neq 0$, and $C_1(K, T) = 1$ for $K = 0$. By the definition of \tilde{B} , the estimate above, and repeated use of the elementary inequality $(a+b)^p \leq c(p)a^p + c(p)b^p$ we obtain the following estimates:

$$\begin{aligned} & \mathbb{E} \left(\left| \frac{1}{t-r} \int_r^t \langle \tilde{u}_s d\tilde{B}_s, \tilde{W}_s(v_1) \rangle \right|^{\alpha p} \right) \\ & \leq c(p, \alpha) \mathbb{E} \left(\left| \frac{1}{t-r} \int_r^t \langle \tilde{u}_s dB_s, \tilde{W}_s(v_1) \rangle \right|^{\alpha p} \right) \\ & \quad + c(p, \alpha) \mathbb{E} \left(\left| \frac{1}{t-r} \int_r^t \langle \nabla \log(k_{T-s} e^{-h}), \tilde{W}_s(v_1) \rangle \right|^{\alpha p} \right) \\ & \leq c(p, \alpha) \frac{(C_1(K, T))^{\frac{\alpha p}{2}}}{(t-r)^{\frac{\alpha p}{2}}} + c(p, \alpha) \mathbb{E} \left(\frac{1}{t-r} \int_r^t |\nabla \log(k_{T-s} e^{-h})|^{\alpha p} |\tilde{W}_s(v_1)|^{\alpha p} ds \right) \\ & \leq c(p, \alpha) \frac{(C_1(K, T))^{\frac{\alpha p}{2}}}{(t-r)^{\frac{\alpha p}{2}}} \\ & \quad + c(p, \alpha) e^{\frac{1}{2}|K|p\alpha T} \mathbb{E} \left(\frac{1}{t-r} \int_r^t |\nabla \log k_{T-s}(\tilde{x}_s)|^{\alpha p} ds + \frac{1}{t-r} \int_r^t |\nabla h(\tilde{x}_s)|^{\alpha p} ds \right). \end{aligned}$$

We apply Lemma 9.3 to obtain that,

$$\begin{aligned} & \mathbb{E} \left(\left| \frac{2}{t-r} \int_{\frac{t-r}{2}}^{t-r} \langle \tilde{u}_s d\tilde{B}_s, \tilde{W}_s(v_1) \rangle \right|^{\alpha p} \right) \\ & \leq c(\alpha, p) \frac{(C_1(K, T))^{\frac{\alpha p}{2}}}{(t-r)^{\frac{\alpha p}{2}}} \\ & \quad + c(\alpha, p) e^{\frac{1}{2}|K|\alpha p T} \left((b_1(\alpha, p))^{\alpha p} + (b_2(\alpha, p))^{\alpha p} + \frac{d^{\alpha p}(x_0, y_0)}{T^{\alpha p}} \right) \\ & \quad + c(\alpha, p) e^{\frac{1}{2}|K|\alpha p T} F(\alpha p, (t-r)/2, t-r), \end{aligned} \tag{9.9}$$

where $F(\alpha p, r, t) = \frac{1}{t-r} \int_r^t \left(\frac{s}{(T-s)T} \right)^{\frac{1}{2}\alpha p} ds$. Consequently, the p^{th} moment of $|\tilde{N}_{t-r}|^p$ has an upper bound of the same form:

$$\begin{aligned} & \mathbb{E}|\tilde{N}_{t-r}|^p \\ & \leq \left| \frac{2}{t-r} \int_{\frac{t-r}{2}}^{t-r} \langle \tilde{u}_s d\tilde{B}_s, \tilde{W}_s(v_1) \rangle \right|_{L^{p\alpha}(\Omega)}^p \left| \frac{2}{t-r} \int_0^{\frac{t-r}{2}} \langle \tilde{u}_s d\tilde{B}_s, \tilde{W}_s(v_2) \rangle \right|_{L^{p\alpha'}(\Omega)}^p \\ & \leq c(\alpha, p) \left(D_1 + e^{\frac{1}{2}|K|pT} \left((b_1(\alpha p))^p + (b_2(\alpha p))^p + \frac{d^p(x_0, y_0)}{T^p} + (F(\alpha p, \frac{t-r}{2}, t-r))^{\frac{1}{\alpha}} \right) \right) \\ & \quad \times \left(D_1 + e^{\frac{1}{2}|K|pT} \left((b_1(\alpha' p))^p + (b_2(\alpha' p))^p + \frac{d^p(x_0, y_0)}{T^p} + (F(\alpha' p, 0, t-r))^{\frac{1}{\alpha'}} \right) \right). \end{aligned}$$

where $D_1 = \frac{(C_1(K, T))^{\frac{p}{2}}}{(t-r)^{\frac{p}{2}}}$. If $\alpha p < 2$, we may apply Lemma 9.2,

$$\begin{aligned} & \sup_{0 \leq t \leq T} (F(\alpha p, (t-r)/2, t-r))^{\frac{1}{\alpha}} \leq c(p, \alpha) T^{-p/2}, \\ & \sup_{0 \leq t \leq T} (F(\alpha' p, 0, t-r))^{\frac{1}{\alpha'}} \leq c(p, \alpha) T^{-p/2}. \end{aligned}$$

Thus, for $b_3 := (b_1(\alpha' p))^2 + b_2(\alpha' p))^2 + (b_1(\alpha p))^2 + (b_2(\alpha p))^2$,

$$\left(\mathbb{E}|\tilde{N}_{t-r}|^p \right)^{\frac{1}{p}} \leq c(\alpha, p) \left(C_1(K, T) \frac{1}{t-r} + e^{|K|T} \left(b_3 + \frac{d^2(x_0, y_0)}{T^2} + \frac{1}{T} \right) \right).$$

Set $b_4 := (b_1(\alpha' p))^{2p} + b_2(\alpha' p))^{2p} + (b_1(\alpha p))^{2p} + (b_2(\alpha p))^{2p}$. Since L^p norms in a probability space increases with p we may assume that $p > 1$ and apply Hölder's inequality to obtain:

$$\begin{aligned} & \mathbb{E} \left| \int_0^t \tilde{N}_{t-r} dr \right|^p \\ & \leq t^{p-1} \mathbb{E} \left(\int_0^t \left| \frac{1}{t-r} \int_r^t \langle \tilde{u}_s d\tilde{B}_s, \tilde{W}_s(v_1) \rangle \cdot \frac{1}{r} \int_0^r \langle \tilde{u}_s d\tilde{B}_s, \tilde{W}_s(v_2) \rangle \right|^p dr \right) \\ & \leq t^{p-1} \int_0^t \left| \frac{2}{t-r} \int_{\frac{t-r}{2}}^{t-r} \langle \tilde{u}_s d\tilde{B}_s, \tilde{W}_s(v_1) \rangle \right|_{L^{p\alpha}(\Omega)}^p \cdot \left| \frac{2}{t-r} \int_0^{\frac{t-r}{2}} \langle \tilde{u}_s d\tilde{B}_s, \tilde{W}_s(v_2) \rangle \right|_{L^{p\alpha'}(\Omega)}^p dr \\ & \leq c(\alpha, p) t^{p-1} \int_0^t \left(\frac{(C_1(K, T))^p}{(t-r)^p} + e^{|K|pT} \left(b_4 + \frac{d^{2p}(x_0, y_0)}{T^{2p}} + T^{-p} \right) \right) dr \\ & \leq c(\alpha, p) \left((C_1(K, T))^p + T^p e^{|K|pT} \left(b_4 + \frac{d^{2p}(x_0, y_0)}{T^{2p}} + T^{-p} \right) \right). \end{aligned}$$

This completes the proof. \square

9.2. Hessian formula in terms of the semi-classical bridge. We are ready to prove a formula for $\nabla d\rho_T^{h, V}$ in terms of the semi-classical bridge. The idea is to use Grisanov transform to transform our Hessian formula in terms of the h -Brownian motion to the

semi-classical bridge with terminal time T , which is valid on any interval $[0, t]$ where $t < T$. The previous estimates allow us to take $t \rightarrow T$ in the formula. Set

$$\tilde{V}_{a,t} = (V(\tilde{x}_a) - V(x_0))e^{-\int_a^t (V(\tilde{x}_s) - V(x_0))ds}.$$

Theorem 9.8. *Assume **C4**. Let V be a bounded Hölder continuous function. Then the following formula holds where $T > 0$:*

$$\begin{aligned} & \frac{\nabla dp_T^{h,V}(v_1, v_2)}{k_T(x_0, y_0)} e^{h(x_0) - h(y_0) + V(x_0)T} \\ &= \mathbb{E} \left[e^{\int_0^T \Phi^h(\tilde{x}_s) ds} \tilde{N}_T \right] + \frac{2}{T} \mathbb{E} \left[e^{\int_0^T \Phi^h(\tilde{x}_s) ds} \int_0^{T/2} \langle d\{\tilde{x}_s\}, \tilde{W}_s^{(2)}(v_1, v_2) \rangle \right] \\ & \quad + \int_0^T \mathbb{E} \left[\tilde{V}_{T-r,T} e^{\int_0^T \Phi^h(\tilde{x}_s) ds} \frac{2}{T-r} \int_0^{(T-r)/2} \langle d\{\tilde{x}_s\}, \tilde{W}_s^{(2)}(v_1, v_2) \rangle \right] dr \\ & \quad + \int_0^T \mathbb{E} \left[\tilde{V}_{T-r,T} e^{\int_0^T \Phi^h(\tilde{x}_s) ds} \tilde{N}_{T-r} \right] dr. \end{aligned}$$

Proof. We may assume $V(x_0) = 0$, and work with $V - V(x_0)$ otherwise. We may also take $x_t = \pi(u_t)$ and $\tilde{x}_t = \pi(\tilde{u}_t)$ so $d\{x_s\} = u_s dB_s$. In the formula of Theorem 7.6, we take $f = k_{T-\epsilon}(\cdot, y_0)\phi$ where ϵ is a positive number smaller than T , ϕ is a smooth function with compact support such that $\phi(y_0) = 1$, thus

$$\begin{aligned} & \nabla dP_t^{h,V}(k_{T-\epsilon}(\cdot, y_0)\phi)(v_1, v_2) \\ &= \mathbb{E} [k_{T-\epsilon}(x_t, y_0)\phi(x_t)N_t] + \mathbb{E} \left[k_{T-\epsilon}(x_t, y_0)\phi(x_t) \frac{2}{t} \int_0^{t/2} \langle u_s dB_s, W_s^{(2)}(v_1, v_2) \rangle \right] \\ & \quad + \int_0^t \mathbb{E} \left[V(x_{t-r})k_{T-\epsilon}(x_t, y_0)\phi(x_t) e^{-\int_{t-r}^t V(x_s) ds} \frac{2}{t-r} \int_0^{(t-r)/2} \langle u_s dB_s, W_s^{(2)}(v_1, v_2) \rangle dr \right] \\ & \quad + \int_0^t \mathbb{E} \left[V(x_{t-r})k_{T-\epsilon}(x_t, y_0)\phi(x_t) e^{-\int_{t-r}^t V(x_s) ds} N_{t-r} dr \right]. \end{aligned}$$

Take $\epsilon \uparrow t$ in the above identity, followed by taking $t \uparrow T$. The left hand side is:

$$\begin{aligned} & \lim_{t \uparrow T} \lim_{\epsilon \uparrow t} \nabla dP_t^{h,V}(k_{T-\epsilon}(\cdot, y_0)\phi)(v_1, v_2) \\ &= \lim_{t \uparrow T} \lim_{\epsilon \uparrow t} \nabla d \left(\int_M p_t^{h,V}(\cdot, z) k_{T-\epsilon}(z, y_0)\phi(z) dz \right) (v_1, v_2) \\ &= \lim_{t \uparrow T} \lim_{\epsilon \uparrow t} \int_M \nabla dp_t^{h,V}(\cdot, z)(v_1, v_2) k_{T-\epsilon}(z, y_0)\phi(z) dz \\ &= \nabla dp_T^{h,V}(\cdot, y_0)(v_1, v_2). \end{aligned}$$

In the last step we use the fact that k_{T-t} is an approximation of the delta measure at y_0 , $J(y_0) = 1$, and ϕ has compact support, so we may exchange the order of taking limits and integration.

On the right hand side, we note that $k_{T-\epsilon}(\cdot, y_0)\phi$ is bounded uniformly in ϵ for ϵ small, V is bounded. Also, by Lemma 5.5, the stochastic integrals on the right hand side are L^p

integrable for any $p \geq 1$. We may therefor interchange the order of taking limits and taking expectations. The right hand side, after taking $\epsilon \uparrow t$, is:

$$\begin{aligned} & \mathbb{E} [k_{T-t}(x_t, y_0)\phi(x_t)N_t] + \mathbb{E} \left[k_{T-t}(x_t, y_0)\phi(x_t) \frac{2}{t} \int_0^{t/2} \langle u_s dB_s, W_s^{(2)}(v_1, v_2) \rangle \right] \\ & + \int_0^t \mathbb{E} \left[V(x_{t-r})k_{T-t}(x_t, y_0)\phi(x_t)e^{-\int_{t-r}^t V(x_s)ds} \frac{2}{t-r} \int_0^{(t-r)/2} \langle u_s dB_s, W_s^{(2)}(v_1, v_2) \rangle \right] dr \\ & + \int_0^t \mathbb{E} \left[V(x_{t-r})k_{T-t}(x_t, y_0)\phi(x_t)e^{-\int_{t-r}^t V(x_s)ds} N_{t-r} \right] dr. \end{aligned}$$

To the above, we make a Girsanov transform and apply Lemma 9.1 to transfer the x_t process to the \tilde{x}_t process and obtain:

$$\begin{aligned} & \mathbb{E} [M_t k_{T-t}(\tilde{x}_t, y_0)\phi(\tilde{x}_t)\tilde{N}_t] + \mathbb{E} \left[M_t k_{T-t}(\tilde{x}_t, y_0)\phi(\tilde{x}_t) \frac{2}{t} \int_0^{t/2} \langle \tilde{u}_s d\tilde{B}_s, \tilde{W}_s^{(2)}(v_1, v_2) \rangle \right] \\ & + \int_0^t \mathbb{E} \left[M_t V(\tilde{x}_{t-r})k_{T-t}(\tilde{x}_t, y_0)\phi(\tilde{x}_t)e^{-\int_{t-r}^t V(\tilde{x}_s)ds} \frac{2}{t-r} \int_0^{(t-r)/2} \langle \tilde{u}_s d\tilde{B}_s, \tilde{W}_s^{(2)}(v_1, v_2) \rangle \right] dr \\ & + \int_0^t \mathbb{E} \left[M_t V(\tilde{x}_{t-r})k_{T-t}(\tilde{x}_t, y_0)\phi(\tilde{x}_t)e^{-\int_{t-r}^t V(\tilde{x}_s)ds} \tilde{N}_{t-r} \right] dr. \end{aligned}$$

The $k_{T-t}(\tilde{x}_t, y_0)$ term in the above integrals cancels with that in M_t , where

$$M_t = e^{h(\tilde{x}_t) - h(x_0)} \frac{k_T(x_0, y_0)}{k_{T-t}(\tilde{x}_t, y_0)} \exp \left[\int_0^t \Phi^h(\tilde{x}_s) ds \right].$$

Set $A_t = e^{h(\tilde{x}_t) - h(x_0)} k_T(x_0, y_0) \exp \left[\int_0^t \Phi^h(\tilde{x}_s) ds \right]$, the right hand side simplifies to the following expression:

$$\begin{aligned} & \mathbb{E} [A_t \phi(\tilde{x}_t)\tilde{N}_t] + \mathbb{E} \left[A_t \phi(\tilde{x}_t) \frac{2}{t} \int_0^{t/2} \langle \tilde{u}_s d\tilde{B}_s, \tilde{W}_s^{(2)}(v_1, v_2) \rangle \right] \\ & + \int_0^t \mathbb{E} \left[A_t V(\tilde{x}_{t-r})\phi(\tilde{x}_t)e^{-\int_{t-r}^t V(\tilde{x}_s)ds} \frac{2}{t-r} \int_0^{(t-r)/2} \langle \tilde{u}_s d\tilde{B}_s, \tilde{W}_s^{(2)}(v_1, v_2) \rangle \right] dr \\ & + \int_0^t \mathbb{E} [A_t V(\tilde{x}_{t-r})\phi(\tilde{x}_t)e^{-\int_{t-r}^t V(\tilde{x}_s)ds} \tilde{N}_{t-r}] dr. \end{aligned}$$

We are now free to take the limit $t \uparrow T$. Since $\phi(\tilde{x}_t) \rightarrow 1$ and Φ^h does not depend on time, $A_t \phi(\tilde{x}_t)$ converges almost surely to

$$A_T \phi(\tilde{x}_T) = e^{h(y_0) - h(x_0)} k_T(x_0, y_0) \exp \left[\int_0^T \Phi^h(\tilde{x}_s) ds \right].$$

Since ϕ has compact support and $\int_0^T \Phi^h(\tilde{x}_s)$ is L^p bounded L^p for all p , the above convergence holds in L^p for any p . By Lemma 9.7, we may exchange the order of taking the limit $t \rightarrow T$ and taking expectation the convergence of the earlier long expression to the following :

$$\nabla dp_T^{h,V}(\cdot, y_0)(v_1, v_2)$$

$$\begin{aligned}
&= \mathbb{E} \left[A_T \tilde{N}_T \right] + \mathbb{E} \left[A_T \frac{2}{T} \int_0^{T/2} \langle \tilde{u}_s d\tilde{B}_s, \tilde{W}_s^{(2)}(v_1, v_2) \rangle \right] \\
&+ \int_0^T \mathbb{E} \left[A_T V(\tilde{x}_{T-r}) e^{-\int_{T-r}^T V(\tilde{x}_s) ds} \frac{2}{T-r} \int_0^{(T-r)/2} \langle \tilde{u}_s d\tilde{B}_s, \tilde{W}_s^{(2)}(v_1, v_2) \rangle \right] dr \\
&+ \int_0^T \mathbb{E} \left[A_T V(\tilde{x}_{T-r}) e^{-\int_{T-r}^T V(\tilde{x}_s) ds} \tilde{N}_{T-r} \right] dr.
\end{aligned}$$

To summarise this is exactly $\nabla dp_T^{h,V}(\cdot, y_0)(v_1, v_2)$ and the proof is complete. \square

10. Exact Gaussian Bounds for Weighted Laplacian

Let $T > 0$ and $x_0 \in M$. Set $\beta_T^h = e^{\int_0^T \Phi^h(\tilde{x}_s) ds}$ and $Z_T = \frac{e^{\int_0^T \Phi^h(\tilde{x}_s) ds}}{\mathbb{E} \left[e^{\int_0^T \Phi^h(\tilde{x}_s) ds} \right]}$. Let $ST_{x_0}M$ denote the unit sphere in the tangent space $T_{x_0}M$.

Corollary 10.1. *Assume **C4**. Let $q \geq 1$. Then there exists a constant c , explicitly given in the proof, s.t. for any $v_1, v_2 \in ST_{x_0}M$,*

$$|\nabla dp_T^h(v_1, v_2)| \leq c T^{-\frac{n}{2}} e^{-\frac{d^2(x_0, y_0)}{2T}} J_{y_0}^{-\frac{1}{2}}(x_0) e^{h(y_0) - h(x_0)} |\beta_T^h|_{L^q(\Omega)} \left(\frac{d^2}{T^2} + \frac{1}{T} + 1 \right)$$

Also, the following formula holds:

$$\frac{\nabla dp_T^h(v_1, v_2)}{p_T^h(x_0, y_0)} = \mathbb{E} \left[Z_T \tilde{N}_T \right] + \mathbb{E} \left[Z_T \frac{2}{T} \int_0^{T/2} \langle d\{\tilde{x}_s\}, \tilde{W}_s^{(2)}(v_1, v_2) \rangle \right]. \quad (10.1)$$

Furthermore, there exists a function C locally bounded in t and explicitly given in the proof, such that

$$\begin{aligned}
|\nabla d \log p_T^h| &\leq C(T, K, q, \delta_1, \delta_2) |Z_T|_{L^q(\Omega)} \left(\frac{d^2}{T^2} + \frac{1}{T} + 1 \right) \\
&+ |Z_T|_{L^2(\Omega)}^2 \frac{1}{T} \left(\frac{e^{KT} - 1}{KT} \right).
\end{aligned}$$

Proof. Let $v_1, v_2 \in T_{x_0}M$. In Theorem 9.8 where we take $V = 0$

$$\begin{aligned}
&\frac{\nabla dp_T^h(v_1, v_2)}{k_T(x_0, y_0)} e^{h(x_0) - h(y_0)} \\
&= \mathbb{E} \left[e^{\int_0^T \Phi^h(\tilde{x}_s) ds} \tilde{N}_T \right] + \frac{2}{T} \mathbb{E} \left[e^{\int_0^T \Phi^h(\tilde{x}_s) ds} \int_0^{T/2} \langle d\{\tilde{x}_s\}, \tilde{W}_s^{(2)}(v_1, v_2) \rangle \right],
\end{aligned}$$

to see that for constant $p \geq 1$ and p' satisfying $\frac{1}{p} + \frac{1}{p'} = 1$,

$$\begin{aligned}
&\frac{|\nabla dp_T^h(v_1, v_2)|}{k_T(x_0, y_0)} e^{h(x_0) - h(y_0)} \\
&\leq \left| e^{\int_0^T \Phi^h(\tilde{x}_s) ds} \right|_{L^{p'}(\Omega)} \left(|N_T|_{L^p(\Omega)} + \left| \frac{2}{T} \int_0^{T/2} \langle d\{\tilde{x}_s\}, \tilde{W}_s^{(2)}(v_1, v_2) \rangle \right|_{L^p(\Omega)} \right).
\end{aligned}$$

The first required estimate follows from the following estimates in Lemma 9.7 and Lemma 9.5:

$$\begin{aligned} \left| \tilde{N}_T \right|_{L^p(\Omega)} &\leq c(p, n) \left(a_1(\alpha, p, T) + e^{|K|T} \frac{d^2(x_0, y_0)}{T^2} + a_2(K, T) \frac{1}{T} \right), \\ \left| \frac{2}{T} \int_0^{T/2} \langle d\{\tilde{x}_s\}, \tilde{W}_s^{(2)}(v_1, v_2) \rangle \right|_{L^p(\Omega)} \\ &\leq c(p, n) \left(b_4(2p) \frac{d(x_0, y_0)}{T} + (b_4(2p) + b_3(2)) \frac{1}{\sqrt{T}} + A \right). \end{aligned}$$

We take $q = p'$ and set

$$S_T = a_1(\alpha, p, T) + A + e^{|K|T} \frac{d^2(x_0, y_0)}{T^2} + b_4(2p) \frac{d(x_0, y_0)}{T} + \frac{a_2(K, T)}{T} + \frac{b_4(2p) + b_3(2)}{\sqrt{T}}.$$

Then there exists a universal constant $c(p, n)$ s.t.

$$\left| \nabla dp_T^h(v_1, v_2) \right| \leq c(q, n) T^{-\frac{n}{2}} e^{-\frac{d^2(x_0, y_0)}{2T}} J_{y_0}^{-\frac{1}{2}}(x_0) e^{h(y_0) - h(x_0)} |\beta_T^h|_{L^q(\Omega)} S_T.$$

Next by the Girsanov transform in Lemma 9.1, we see that

$$p_T^h(x_0, y_0) = k_T(x_0, y_0) e^{h(y_0) - h(x_0)} \mathbb{E} \left[e^{\int_0^T \Phi^h(\tilde{x}_s) ds} \right],$$

formula (10.1) follows immediately. To formula (10.1) we apply Hölder's inequality to see

$$\begin{aligned} &\left| \frac{\nabla dp_T^h(v_1, v_2)}{p_T(x_0, y_0)} \right| \\ &\leq |Z_T|_{L^{p'}(\Omega)} c(p, n) \left(b_4(2p) \frac{d(x_0, y_0)}{T} + (b_4(2p) + b_3(2)) \frac{1}{\sqrt{T}} + A \right) \\ &\quad + |Z_T|_{L^{p'}(\Omega)} c(p, n) \left(a_1(\alpha, p, T) + e^{|K|T} \frac{d^2(x_0, y_0)}{T^2} + a_2(K, T) \frac{1}{T} \right). \end{aligned}$$

Finally, the following formula holds,

$$dp_T^h(\cdot, y_0) = \frac{1}{T} e^{h(y_0) - h(x_0)} k_T(x_0, y_0) \mathbb{E} \left[e^{\int_0^T \Phi^h(\tilde{x}_s) ds} \left(\int_0^T \langle \tilde{W}_r(\cdot), \tilde{u}_r d\tilde{B}_r \rangle \right) \right],$$

under our assumptions. This was proved in [71] under the assumption that Φ^h is assumed to be bounded from below. It can be seen from the following formula

$$dP_T^{h, V} f(v) = \frac{1}{T} \mathbb{E} \left[f(x_T) \int_0^T \langle W_s(v), u_s dB_s \rangle \right],$$

holds for f bounded under the conditions that ρ^h bounded from below. The estimates in this section and the proof of Theorem 9.8 allow us to conclude this under **C4(b)** on the growth of Φ^h , and use the exponential integrability of the distance function from y_0 , c.f. Lemma 4.2. In particular we obtain

$$\left| d \log p_T^h(\cdot, y_0) \right|^2 = \left| \mathbb{E} \left[Z_T \left(\frac{1}{T} \int_0^T \langle \tilde{W}_r(\cdot), \tilde{u}_r d\tilde{B}_r \rangle \right) \right] \right|^2 \leq \mathbb{E}(|Z_T|^2) \frac{1}{T} \left(\frac{e^{KT} - 1}{KT} \right).$$

Putting them together using $\nabla d \log p_t = \frac{\nabla dp_t}{p_t} - \nabla \log p_t \otimes \nabla \log p_t$,

$$\begin{aligned} & |\nabla d \log p_T^h(v_1, v_2)| \\ & \leq |Z_T|_{L^{p'}(\Omega)} c(p, n) \left(b_4(2p) \frac{d(x_0, y_0)}{T} + (b_4(2p) + b_3(2)) \frac{1}{\sqrt{T}} + A \right) \\ & \quad + |Z_T|_{L^{p'}(\Omega)} c(p, n) \left(a_1(\alpha, p, T) + e^{|K|T} \frac{d^2(x_0, y_0)}{T^2} + a_2(K, T) \frac{1}{T} \right) \\ & \quad + |Z_T|_{L^2(\Omega)}^2 \frac{1}{T} \left(\frac{e^{KT} - 1}{KT} \right). \end{aligned}$$

This completes the proof of the corollary. \square

Summary To conclude, we obtained exact Gaussian estimates for ∇dp_t^h on manifolds with a pole under condition **C4**; and formulas for $\text{Hess } P_t^{h,V} f$ and for $p_t^{h,V}$ on general manifolds, from which estimates are given under condition **C2**.

Remark 10.2. Noted added: Since this article appeared in our arxiv version, there have been beautiful advancements. See [87, 90]. In a more recent work, an intrinsic formula featuring a more ‘local’ nature, is obtained in [21] for the heat semigroup on complete Riemannian manifolds without any curvature restrictions. See also [48].

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