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## OPTIMAL HEDGING OF PATH-DEPENDENT OPTIONS IN DISCRETE TIME INCOMPLETE MARKET

NORMAN JOSEPHY, LUCY KIMBALL, AND VICTORIA STEBLOVSKAYA\*

**ABSTRACT.** We consider hedging of a path-dependent European style option with convex continuous payoff in a discrete time incomplete market, where underlying stock price jumps are distributed over a bounded interval. The incompleteness of the market produces an interval of no-arbitrage option prices for the path-dependent option. Upper and lower bounds for the no-arbitrage price interval are developed. Explicit formulas for a no-arbitrage option price and a non-self-financing hedging strategy are given. Each non-self-financing hedging strategy produces an accumulated residual amount. Theoretical results are applied to the case of an arithmetic Asian option. A numerical algorithm for constructing the non-self-financing hedging strategy that maximizes the accumulated residual amount is developed. The algorithm is tested on various underlying stocks and the Standard & Poor 500 Index.

### 1. Introduction

Pricing and hedging of path-dependent options has interested both mathematicians and practitioners in the last few decades. An Asian option is a typical path-dependent derivative where the pay-off function depends on the average of quoted stock prices over a prescribed period of time. The integral average is addressed in Geman and Yor [8]. Under the lognormal assumption on the underlying asset price distribution, they obtain an analytical expression for the Laplace transform of the Asian option price. They also calculate moments of the distribution for the integral average of the underlying asset prices.

Pricing and hedging of Asian options with the pay-off depending on the arithmetic average of the underlying asset prices has been the subject of many studies, but even in the simplest case of the Black-Scholes market no closed-form analytical solution has been obtained. An approach based on approximating the distribution of the arithmetic average has been developed by several authors (see e.g. Thurnbull and Wakeman [23], Levy [12], Vorst [26] and references therein). Namely, Thurnbull and Wakeman consider a lognormal approximation for the distribution of the arithmetic average and obtain a closed-form formula for the option price. They also derive recursive formulas for the moments of the distribution of the average. M. Jacques [10], building on the results of Thurnbull and Wakeman (as well as using the alternative inverse Gaussian approximation), derives explicit formulas

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for a self-financing hedging strategy that approximately replicates the arithmetic Asian option. We would like to emphasize here that the above methods have been developed for a complete Black-Scholes market environment.

Further pricing and hedging methods for Asian options are based on Monte-Carlo simulations (see e.g. Broadie and Glasserman [2], Broadie, Glasserman and Kou [3] and references therein) as well as partial differential equations (see e.g. Dewynne and Wilmott [5], Vecer [24], Vecer [25] and references therein). These methods do not typically result in reliable hedging strategies.

Another approach is based on upper and lower bounds for the Asian option prices. The first attempts to derive such bounds are attributed to Curran [4], Roger and Shi [18], and Thompson [22]. An approach based on comonotonic theory has been developed by Dhaene et al in [6] and [7]. Building on these results, Albrecher et al [1] suggested a static super-hedge consisting of purchasing and holding a portfolio of European call options with strikes and times to expiration chosen in an optimal way. This study has been developed for a more realistic market environment, namely for an incomplete market where an underlying asset price is assumed to follow a Lévy process.

Our present work assumes an incomplete market environment, but we consider a discrete time market model. Our market consists of a stock, a bond, and a path-dependent option with convex continuous pay-off function. Incompleteness of our market results from the fact that stock price jumps are assumed to be distributed over a bounded interval (unlike in a complete binomial Cox-Ross-Rubinstein model where the stock price jumps take only two possible values). Our model extends on earlier incomplete multi-nomial market models (see e.g. Tessitore and Zabczyk [21], Wolczynska [27] and references therein). Earlier studies of the present model belong to A. Nagaev et al (see [14]-[17]) as well as to L. Ruschendorf [19]. In [19], the upper and lower bounds on the no-arbitrage prices of European style options with convex payoff functions are derived. In [15]-[17], non-self-financing super-hedges for vanilla call and put options are built, and asymptotic properties of their residuals are studied. An alternative approach to non-self-financing hedging in incomplete markets can be found in [20], although the theoretical foundation and goals of their research differ from the research presented here.

Building on theoretical results of A. Nagaev et al and L. Ruschendorf, we have previously developed a two-stage algorithm for optimal hedging of European path-independent options (e.g. European call options) with short times to expiration (see [13] and [11]). In the present paper we extend both theoretical and numerical results of our previous work to the case of path-dependent European style options. The necessary theoretical background is developed in sections 2 through 4. We describe a market model in section 2. Here we also give explicit formulas for the lower and upper bounds on the no-arbitrage prices of a path-dependent option with convex pay-off function. In section 3 we discuss the extreme case of a non-self-financing minimum cost super-hedge and give formulas for its residuals. We proceed to a more realistic case in section 4. Here we take a position of a seller of a path-dependent option with convex continuous pay-off function who sells the option for a no-arbitrage market price and wishes to hedge his/her short position in the option. We derive the formula for a no-arbitrage option price in

our model (Proposition 4.1) which builds theoretical foundation for the first stage of our algorithm. We also discuss the set of possible non-self-financing hedging strategies that may be constructed in this case. The residuals of such strategies can take non-positive values. This gives rise to our optimization criterion (maximize the accumulated residual from a non-self-financing hedging strategy) and builds theoretical foundation for the second stage of our numerical algorithm.

The description of our numerical algorithm for hedging an arithmetic average Asian option is described in section 5. Although we use the same two-stage algorithm structure developed in our previous studies, the implementation for path-dependent options differs significantly. In the first stage, we determine a market calibrated set of model parameters by matching model parameters with the stock price historical volatility (see section 5.1 for details). In the second stage, an optimal parameter pair is chosen from the market calibrated set of parameters by applying the optimization criterion to residual values computed over a large number of simulated asset value paths. The large amount of computation required for evaluating the residuals of path-dependent options is reduced by using a table of pre-computed prices of the Asian option over a finely spaced grid of days to expiration and stock values (see section 5.2 for details). For this study we chose to utilize historical asset price data in a bootstrap simulation to simulate the asset value paths, leaving alternative approaches to our future work. We assume that the underlying asset price process has independent, identically distributed jumps with bounded support, but place no additional assumptions on the jump process distribution.

The numerical results of applying our algorithm to the hedging of arithmetic average Asian options with three underlying stocks (H.J. Heinz, ExxonMobil, and Cisco Systems) are documented in section 6. Section 7 contains the results of applying our algorithm to the hedging of an arithmetic average Asian option on the S&P 500 Index. The paper finishes with an Appendix containing the proofs of two key propositions stated in the paper.

## 2. Market Model and No-arbitrage Option Price Interval

Let us consider a discrete time model for a risky asset (stock)  $s_k$  evolving in time as follows:

$$s_k = s_0 \xi_1 \xi_2 \cdots \xi_k, 1 \leq k \leq n,$$

where the stock price jumps  $\xi_k = \frac{s_k}{s_{k-1}}$  are assumed to be random variables distributed over a bounded interval  $[D, U]$ ,  $D < U$  at every time step  $k$ . No further assumptions are made on a distribution for  $\xi_k$ .

Let  $b_k$  denote a time  $t = k$  value of a risk-free bond with fixed interest rate  $r \geq 0$ , evolving from an initial value  $b_0 > 0$  at time  $t = 0$  as follows:

$$b_k = b_0(1 + r)^k.$$

Within the above market model we consider a European type path-dependent option with the convex payoff function  $f(s_0, s_1, \dots, s_n) \geq 0$  (here  $n$  is a number of time steps to expiration).

We assume the usual no-arbitrage condition

$$D < 1 + r < U \quad (2.1)$$

holds in this market. This discrete time market model is incomplete. This fact has been established in the case of the multi-nomial model (see e.g. [21]). Even in the simplest trinomial model, there is an infinite set of equivalent martingale measures. As a result, there is no unique no-arbitrage price for a contingent claim. Our model generalizes the multi-nomial model allowing the stock price jumps  $\xi_k$  at every time step  $k$  to be distributed over a bounded interval. As a result of model incompleteness, for each contingent claim there is an open interval of no-arbitrage prices. Upper and lower bounds for no-arbitrage prices of claims with convex pay-off functions are considered in [19] (see also references to earlier works therein).

For fixed number of steps to expiration  $n$  we denote by  $\overline{X}_k(f, D, U)$  (respectively  $\underline{x}_k(f, D, U)$ ) the upper bound (lower bound) of the no-arbitrage price interval at time  $k$  ( $k = 0, \dots, n - 1$ ) for a path-dependent option with the convex payoff function  $f$ , assuming that our option is evaluated in the framework of the above market model. Let  $CRR_k(f, D, U)$  stand for a (unique) price at time  $k$  of the same path-dependent option evaluated in the framework of the Cox-Ross-Rubinstein binomial model. We have the following Proposition.

**Proposition 2.1.** *For any  $k = 0, \dots, n - 1$ , we have*

$$\overline{X}_k(f, D, U) = CRR_k(f, D, U), \quad (2.2)$$

$$\underline{x}_k(f) = (1 + r)^{-(n-k)} f(s_0, \dots, s_{k-1}, s_k, s_k(1 + r), \dots, s_k(1 + r)^{n-k}). \quad (2.3)$$

*Proof.* This proposition follows immediately from Theorem 3 in [19]. We remark that the lower bound of the no-arbitrage option price interval does not depend on the parameters  $D, U$ .  $\square$

For the remainder of the paper we will use the following simplified notation:

$$\begin{aligned} \overline{X}_k(f, D, U) &\equiv \overline{X}_k(D, U), \\ CRR_k(f, D, U) &\equiv CRR_k(D, U), \\ \underline{x}_k(f) &\equiv \underline{x}_k, \quad k = 0, \dots, n - 1. \end{aligned}$$

As is well-known, by means of the risk-neutral valuation, the CRR option price  $CRR_k(D, U)$  equals the expected payoff from the option at maturity (in the risk-neutral world) discounted at the risk-free interest rate  $r$ . An explicit formula for  $CRR_k(D, U)$  is rather complicated. Additionally, this formula is usually replaced by approximating formulas when it comes to numerical evaluation of path-dependent options. Nevertheless, for clarity of presentation, we present this formula here (see Proposition 2.2 below), and we give a sketch of a proof in Appendix. Proposition 2.2 also gives explicit formulas for a unique replicating hedging strategy which exists in the framework of the CRR model due to its completeness. Let  $\gamma_k(D, U)$  be the number of stocks in the hedging portfolio kept during the time period  $[k, k + 1)$  ( $k = 0, \dots, n - 1$ ), and let  $\beta_k(D, U)$  be the number of bonds in that portfolio. Then the sequence  $(\gamma_k, \beta_k)$  ( $k = 0, \dots, n - 1$ ) defines a unique

self-financing hedging strategy replicating the option, in other words, the perfect hedge.

**Proposition 2.2.** *In the framework of the CRR binomial model with parameters  $D, U$  the price at time  $k$  of a path-dependent option with a payoff function  $f$  is given by*

$$CRR_k(D, U) = g_k^{s_0, \dots, s_{k-1}}(D, U, s_k), \quad k = 1, \dots, n-1; \quad (2.4)$$

$$CRR_0(D, U) = g_0(D, U, s_0), \quad (2.5)$$

where  $s_i$  is the stock price at time  $i$ , and the functions  $g_k$  and  $g_0$  are defined as follows:

$$\begin{aligned} g_k^{a_0, \dots, a_{k-1}}(D, U, x) &= (1+r)^{-(n-k)} \sum_{j=0}^{n-k} [p(D, U)]^j [1-p(D, U)]^{n-k-j} \\ &\times F_j^{a_0, \dots, a_{k-1}}(D, U, x), \quad k = 1, \dots, n-1, \end{aligned} \quad (2.6)$$

and

$$g_0(D, U, x) = (1+r)^{-n} \sum_{j=0}^n [p(D, U)]^j [1-p(D, U)]^{n-j} F_j(D, U, x). \quad (2.7)$$

Here we have

$$p(D, U) = \frac{(1+r) - D}{U - D} \quad (2.8)$$

and the functions  $F_j$  are defined in the following way:

$$\begin{aligned} F_j^{a_0, \dots, a_{k-1}}(D, U, x) &= \sum_{(i_1, \dots, i_{n-k}) \in I_{kj}} \\ &f(a_0, \dots, a_{k-1}, x, xU^{i_1}D^{1-i_1}, \dots, xU^{i_1+\dots+i_{n-k}}D^{n-k-(i_1+\dots+i_{n-k})}), \end{aligned} \quad (2.9)$$

where  $f$  is a payoff function of the option, and  $I_{kj}$  is defined as follows:

$$I_{kj} = \{(i_1, \dots, i_{n-k}) : i_1 + \dots + i_{n-k} = j, i_m = 0, 1, 1 \leq m \leq n-k\}, \quad (2.10)$$

and

$$\begin{aligned} F_j(D, U, x) &= \sum_{(i_1, \dots, i_{n-k}) \in I_j} \\ &f(x, xU^{i_1}D^{1-i_1}, \dots, xU^{i_1+\dots+i_{n-k}}D^{n-(i_1+\dots+i_{n-k})}), \end{aligned} \quad (2.11)$$

where

$$I_j = \{(i_1, \dots, i_n) : i_1 + \dots + i_n = j, i_m = 0, 1, 1 \leq m \leq n\}. \quad (2.12)$$

The unique self-financing replicating hedging strategy is given by:

$$\gamma_k(D, U) = \frac{g_{k+1}^{s_0, \dots, s_k}(D, U, s_k U) - g_{k+1}^{s_0, \dots, s_k}(D, U, s_k D)}{s_k(U - D)}, \quad (2.13)$$

$$\beta_k(D, U) = \frac{U g_{k+1}^{s_0, \dots, s_k}(D, U, s_k D) - D g_{k+1}^{s_0, \dots, s_k}(D, U, s_k U)}{(1+r)b_k(U - D)}. \quad (2.14)$$

Let us give a more intuitive explanation of the formulas (2.9) through (2.12). The functions  $F_j^{a_0, \dots, a_{k-1}}(D, U, x)$  can be represented as follows:

$$F_j^{a_0, \dots, a_{k-1}}(D, U, x) = \sum_{i=1}^{\binom{n-k}{j}} f(a_0, \dots, a_{k-1}, x, \text{path}_i^{n-k}, xU^j D^{n-k-j}), \quad (2.15)$$

where  $f$  is a pay-off function of the option,  $\binom{n-k}{j}$  is a binomial coefficient, and  $\text{path}_i^{n-k}$  stands for the  $i$ -th binary path consisting of  $n-k$  steps with the starting point  $x$  and terminal point  $xU^j D^{n-k-j}$ .

Similarly, the functions  $F_j(D, U, x)$  can be represented as follows:

$$F_j(D, U, x) = \sum_{i=1}^{\binom{n}{j}} f(x, \text{path}_i^n, xU^j D^{n-j}), \quad (2.16)$$

where  $\text{path}_i^n$  stands for the  $i$ -th binary path consisting of  $n$  steps with the starting point  $x$  and terminal point  $xU^j D^{n-j}$ .

### 3. Non-self-financing Minimum Cost Super Hedge

Let us consider a hypothetical extreme case: suppose that a path-dependent option with a convex pay-off function  $f$  has been sold at time  $k=0$  for the upper bound price  $\bar{X}_0(D, U)$ . The option seller creates a portfolio consisting of  $\tilde{\gamma}_0$  stocks and  $\tilde{\beta}_0$  bonds with the intention of hedging the short position in the option. The seller re-balances the portfolio at each time instant  $t=k$  ( $k=1, \dots, n-1$ ) creating a dynamic trading strategy  $(\tilde{\gamma}_k, \tilde{\beta}_k)$  ( $k=0, \dots, n-1$ ).

Suppose that for each  $k=0, \dots, n-1$  the quantities  $\tilde{\gamma}_k$  and  $\tilde{\beta}_k$  are chosen as follows:

$$\begin{aligned} \tilde{\gamma}_k &= \gamma_k(D, U), \\ \tilde{\beta}_k &= \beta_k(D, U), \end{aligned}$$

where  $\gamma_k(D, U)$  and  $\beta_k(D, U)$  define the self-financing hedging strategy in the framework of the CRR model (see (2.13) respectively (2.14)).

**Proposition 3.1.** *The dynamic trading strategy  $(\tilde{\gamma}_k, \tilde{\beta}_k)$  ( $k=0, \dots, n-1$ ) represents a non-self-financing minimum cost super-hedging strategy whose associated portfolio value at every time instant  $t=k$  is greater than or equal to the value of the option.*

*Proof.* Suppose at each time instant  $k$ , the option seller liquidates the portfolio constructed in the prior period  $[k-1, k)$  and uses the proceeds to construct a new portfolio for the current period  $[k, k+1)$ . Using (2.13) and (2.14), one can write

the liquidation value of the prior period portfolio as follows:

$$\begin{aligned}
v_k(D, U) &= \gamma_{k-1}(D, U)s_k + \beta_{k-1}(D, U)b_k \\
&= \gamma_{k-1}(D, U)s_{k-1}\xi_k + \beta_{k-1}(D, U)b_{k-1}(1+r) \\
&= \frac{U - \xi_k}{U - D}g_k^{s_0, \dots, s_{k-1}}(D, U, s_{k-1}D) \\
&\quad + \frac{\xi_k - D}{U - D}g_k^{s_0, \dots, s_{k-1}}(D, U, s_{k-1}U). \tag{3.1}
\end{aligned}$$

On the other hand, the funds required to construct the new period portfolio, or set-up cost, is given by the time  $k$  option value (which equals the upper bound of the no-arbitrage price interval at time  $t = k$ ):

$$\bar{X}_k(D, U) = g_k^{s_0, \dots, s_{k-1}}(D, U, s_{k-1}\xi_k). \tag{3.2}$$

The difference between the liquidation value (3.1) and the set-up cost (3.2) is a residual amount  $\delta_k$

$$\begin{aligned}
\delta_k(D, U) &= \frac{U - \xi_k}{U - D}g_k^{s_0, \dots, s_{k-1}}(D, U, s_{k-1}D) \\
&\quad + \frac{\xi_k - D}{U - D}g_k^{s_0, \dots, s_{k-1}}(D, U, s_{k-1}U) - g_k^{s_0, \dots, s_{k-1}}(D, U, s_{k-1}\xi_k). \tag{3.3}
\end{aligned}$$

We need to prove that

$$\delta_k(D, U) \geq 0, \quad k = 1, \dots, n. \tag{3.4}$$

Without loss of generality, let us consider the single-step model:  $n = 1$ . In this case the residual amount (3.3) takes the following form:

$$\delta_1(D, U) = \frac{U - \xi_1}{U - D}f(s_0, s_0D) + \frac{\xi_1 - D}{U - D}f(s_0, s_0U) - f(s_0, s_0\xi_1). \tag{3.5}$$

Since every  $\xi_1 \in [D, U]$  can be represented as

$$\xi_1 = U \frac{\xi_1 - D}{U - D} + D \frac{U - \xi_1}{U - D},$$

and since

$$\frac{\xi_1 - D}{U - D} + \frac{U - \xi_1}{U - D} = 1,$$

it follows immediately from the convexity of  $f$  that  $\delta_1(D, U) \geq 0$ . Moreover,  $\delta_1(D, U) = 0$  only if  $\xi_1 = D$  or  $\xi_1 = U$ . The case of  $n > 1$  can be handled by induction.  $\square$

On the basis of Proposition 3.1, the strategy  $(\gamma_k(D, U), \beta_k(D, U))$  (which represents a self-financing replicating hedge in the framework of the CRR model) may be used as a non-self-financing minimum cost super hedge in the framework of our extended model. Indeed, at every time step, after each portfolio liquidation prior to the construction of the next time period portfolio, the option seller withdraws the non-negative residual amount  $\delta_k(D, U)$  and invests it at the risk-free interest



rate  $r$ . At option maturity, the withdrawn residuals will accumulate to the value  $\Delta_n(D, U)$  where

$$\Delta_n(D, U) = \delta_1(D, U)(1+r)^{n-1} + \delta_2(D, U)(1+r)^{n-2} + \dots + \delta_n(D, U). \tag{3.6}$$

We will refer to the quantity  $\Delta_n(D, U)$  as the minimum cost super hedge residual.

#### 4. Non-self-financing Hedging Strategies and Their Residuals

Let us consider a more realistic situation where a path-dependent option with a convex pay-off function has been sold at time zero for a price that is lower than the upper bound  $\overline{X}_0(U, D)$ , but still falls within the open interval of no-arbitrage option prices  $(\underline{x}_0(U, D), \overline{X}_0(U, D))$ . In contrast to the extreme case of the minimum cost super-hedge that produces a non-negative accumulated residual, a non-self-financing hedging strategy constructed in this case will produce a possibly negative accumulated residual. None the less, our goal is to choose model parameters to maximize the produced expected accumulated residual.

In order to explain how such a trading strategy can be constructed, consider the quantity  $\overline{x}_k(d, u)$  given as follows:

$$\overline{x}_k(d, u) = g_k^{s_0, \dots, s_{k-1}}(d, u, s_k), \quad k = 1, \dots, n-1, \tag{4.1}$$

$$\overline{x}_0(d, u) = g_0(d, u, s_0), \tag{4.2}$$

where  $s_k$  is the stock price at time  $k$  and  $g_k$  is defined in (2.6), (2.7), where the boundary parameters  $D, U$  are replaced with a pair of numbers  $(d, u)$  such that

$$D \leq d \leq 1+r \leq u \leq U. \tag{4.3}$$

**Proposition 4.1.** *Let  $f : R^{n+1} \rightarrow R$  be a continuous convex function. The function  $\overline{x}_k$  maps the set of  $(d, u)$  pairs satisfying (4.3) onto the price interval  $[\underline{x}_k, \overline{X}_k(D, U)]$  defined in (2.2), (2.3), (2.4), (2.5). When  $(d, u) = (D, U)$  then  $\overline{x}_k(d, u) = \overline{X}_k(D, U)$  and when  $(d, u) = (1+r, 1+r)$  then  $\overline{x}_k(d, u) = \underline{x}_k$ .*

The above proposition infers that for any choice of  $d$  and  $u$  satisfying

$$D < d < 1+r < u < U, \tag{4.4}$$

the quantity  $\overline{x}_k(d, u)$  given by (4.1),(4.2) falls within the no-arbitrage option price interval

$$\underline{x}_k < \overline{x}_k(d, u) < \overline{X}_k(D, U), \quad k = 0, \dots, n-1. \tag{4.5}$$

Now suppose we know a time zero market price  $x_0$  of the path-dependent option with the convex continuous pay-off function  $f$ . If we assume that there is no arbitrage in the market, then the amount  $x_0$  can be identified with a point in the no-arbitrage option price open interval  $(\underline{x}_0, \overline{X}_0(D, U))$ . Moreover, there exists at least one pair  $(d, u)$  satisfying (4.4) such that

$$x_0 = \overline{x}_0(d, u), \tag{4.6}$$

where  $\overline{x}_0(d, u)$  is given by (4.2). We will refer to  $\overline{x}_k(d, u)$  ( $k = 0, \dots, n-1$ ) as a no-arbitrage price of the option at time  $t = k$ . For a given  $x_0$ , there exists an infinite set of  $(d, u)$  pairs such that (4.6) holds. This set forms a level curve of the

function  $\bar{x}_0$ . For any pair  $(d, u)$  satisfying (4.6), the option seller uses the amount  $x_0$  to initiate the creation of the dynamic hedging strategy

$$(\gamma_k(d, u), \beta_k(d, u)), \quad k = 0, \dots, n-1, \quad (4.7)$$

where  $\gamma_k(d, u)$  and  $\beta_k(d, u)$  are defined in (2.13) and (2.14) respectively, with the boundary parameters  $D, U$  replaced with the values  $d, u$ :

$$\gamma_k(d, u) = \frac{g_{k+1}^{s_0, \dots, s_k}(d, u, s_k u) - g_{k+1}^{s_0, \dots, s_k}(d, u, s_k d)}{s_k(u-d)}, \quad (4.8)$$

$$\beta_k(d, u) = \frac{u g_{k+1}^{s_0, \dots, s_k}(d, u, s_k d) - d g_{k+1}^{s_0, \dots, s_k}(d, u, s_k u)}{(1+r)b_k(u-d)}. \quad (4.9)$$

Note that there are an infinite number of dynamic portfolio strategies defined by the formulas (4.8) and (4.9). These strategies are distinguished by the values of the parameters  $(d, u)$ .

At every time step  $k = 1, \dots, n-1$ , the option seller re-balances his/her portfolio as described in the previous section. Namely, the investor liquidates the portfolio constructed in the prior period  $[k-1, k)$  and uses the proceeds to set up a new portfolio for the current period  $[k, k+1)$ . The difference between the liquidation value of the prior period portfolio and the set-up cost of the current portfolio produces a residual amount

$$\begin{aligned} \delta_k(d, u) &= \frac{u - \xi_k}{u - d} g_k^{s_0, \dots, s_{k-1}}(d, u, s_{k-1} d) \\ &+ \frac{\xi_k - d}{u - d} g_k^{s_0, \dots, s_{k-1}}(d, u, s_{k-1} u) - g_k^{s_0, \dots, s_{k-1}}(d, u, s_{k-1} \xi_k) \end{aligned} \quad (4.10)$$

This formula is an analog of the formula (3.3) for the residual amount produced at time  $k$  by the minimum cost super hedge. In the more realistic setting of this section, the residual amount can take both positive and negative values depending on the value of the stock price jump  $\xi_k$  at time  $k$ .

**Proposition 4.2.** *For a convex pay-off function  $f$ , the residual amount (4.10) possesses the following properties*

- (i)  $\delta_k(d, u) > 0$  if  $d < \xi_k < u$ ,
- (ii)  $\delta_k(d, u) = 0$  if  $\xi_k = d$  or  $\xi_k = u$ ,
- (iii)  $\delta_k(d, u) < 0$  if  $D < \xi_k < d$  or  $u < \xi_k < U$ .

*Proof.* Property (ii) is straightforward. Properties (i) and (iii) follow from the convexity arguments similar to those used in the proof of Proposition 3.1.  $\square$

The hedging strategy  $(\gamma_k(d, u), \beta_k(d, u))$  defined by (4.8), (4.9) is in general non-self-financing. Indeed, at each time step  $k = 1, \dots, n$  if the residual  $\delta_k(d, u)$  is positive, the investor will withdraw the residual amount from the liquidated proceeds; if  $\delta_k(d, u)$  is negative, he/she will add the residual amount in order to cover the set-up cost of the current portfolio. The local residuals  $\delta_k(d, u)$  produce an accumulated residual

$$\Delta_n(d, u) = \delta_1(d, u)(1+r)^{n-1} + \delta_2(d, u)(1+r)^{n-2} + \dots + \delta_n(d, u). \quad (4.11)$$

In contrast to the minimum cost super hedge accumulated residual  $\Delta_n(D, U)$  defined in (3.6), the accumulated residual (4.11) can take negative values, in which

case it is interpreted as a loss to the investor. A positive accumulated residual (4.11) is interpreted as a gain. The main purpose of the numerical algorithm presented in the following sections is to maximize the expected value of the accumulated residual (4.11).

*Remark 4.3.* The boundary parameters  $D, U$  (as well as the exact upper bound of the no-arbitrage option price interval  $\bar{X}_k(D, U)$ ) play a purely theoretical role in our setting. As long as one imposes a no-arbitrage assumption, one can successfully set up and maintain a non-self-financing hedging strategy without knowing the parameters  $D, U$ .

For the remainder of this paper we will assume an arithmetic Asian option pay-off function  $f$ ,

$$f(s_0, \dots, s_n) = \left( \frac{1}{n+1} \sum_{i=0}^n s_i - K \right)_+, \quad (4.12)$$

where  $K$  is the option strike price.

## 5. Algorithm Design and Implementation

Based on the theory presented in previous sections, the incompleteness of our market model results in an infinite number of possible no-arbitrage Asian option prices located within the open interval  $(\underline{x}_k(D, U), \bar{X}_k(D, U))$ ,  $k = 0, \dots, n-1$ . For a given time zero no-arbitrage Asian option price  $x_0$  in  $(\underline{x}_0(D, U), \bar{X}_0(D, U))$ , one has an infinite choice of  $(d, u)$  pairs such that (4.6) holds. The set of  $(d, u)$  pairs such that (4.6) holds forms a level curve of the function  $\bar{x}_0$  which we will denote by  $\Omega$ . Each  $(d, u)$  pair gives rise to a non-self financing hedging strategy defined by (4.8) and (4.9) and each hedging strategy provides an accumulated residual defined by (4.11). Our algorithm design is based on determining one  $(d, u)$  pair and associated hedging strategy that is most beneficial to the seller of the option.

**5.1. Algorithm design.** We begin by determining a set of  $(d, u)$  pairs that reflect the current market environment. We will always assume that there is no arbitrage in the market. Then, if a market time zero Asian option price  $x_0$  is available, it can be associated with the no-arbitrage option price  $\bar{x}_0(d, u)$  by (4.6), and the  $(d, u)$  pairs satisfying (4.6) provide a market calibrated curve  $\Omega$ . In the case of Asian options (or other so-called over the counter options) this approach is not feasible due to the limited public availability of the market option prices. An additional constraint is the computational difficulty involved in solving (4.6) for  $(d, u)$ .

As an alternative, we follow an approach based on using an underlying asset historical volatility. We recall here that in the framework of the CRR model when constructing a binomial tree to represent movements of the underlying asset price, the parameters  $d$  and  $u$  (characterizing the downward (respectively upward) movements of the underlying asset price) are chosen to match the volatility of the asset price (see Hull [9] for details). The market model presented in this paper extends the CRR model (see section 2), and the parameters  $(d, u)$  play a different role in our setting. Nevertheless, the no-arbitrage Asian option price  $\bar{x}_0(d, u)$  associated with our model for fixed  $(d, u)$  (see (4.6), (4.2)) can be interpreted as

a unique Asian option price evaluated using the CRR model with the parameters  $(d, u)$ . Based on this observation, we follow the standard approach of matching the parameters  $(d, u)$  with the historical asset volatility as detailed below.

It is shown in Hull [9] that matching the first two moments of the return on the underlying asset with the parameters  $(d, u)$ , one obtains the equation

$$\rho(u + d) - ud - \rho^2 = \sigma^2 \Delta t, \quad (5.1)$$

where  $\Delta t$  is the time step in years,  $\rho = 1 + r\Delta t$  is the accumulation factor associated with the risk-free interest rate  $r$  and  $\sigma$  is the underlying asset annual volatility. We solve equation (5.1) numerically by choosing a realistic finite set of  $d$  values and solving for the corresponding  $u$  values. This produces a finite set of  $(d, u)$  pairs  $\Sigma$ . The set  $\Sigma$  represents a market calibrated parameter set that numerically approximates the level curve  $\Omega$ .

The final stage of the algorithm determines a unique optimal parameter pair (denoted by  $(d^*, u^*)$ ) from the market calibrated parameter set  $\Sigma$  that is most beneficial to the seller of the option. Theoretically, the choice of optimal parameter pair  $(d^*, u^*)$  and corresponding hedging strategy  $(\gamma^*, \beta^*)$  is based on the following optimization criterion

$$\max_{(d,u) \in \Omega} E(\Delta(d, u)), \quad (5.2)$$

where  $\Delta(d, u) \equiv \Delta_n(d, u)$  is the accumulated residual value defined in (4.11) and  $\Omega$  is the market calibrated level curve described earlier in this section. Numerically, this is achieved by the following steps.

Step 1. Let  $n$  be the number of time steps to option expiration and let  $s_0$  be the current underlying asset price. We use the historical underlying asset price path of length  $n$  to create  $m$  alternative asset price paths of length  $n$ , each of which may be viewed as a potential future asset price path. In order to do so, we sample with replacement from the set of historical asset price jumps and compute price paths from the cumulative product of the sampled price jumps, initialized with the current asset value  $s_0$ .

Step 2. For each  $(d, u)$  in the market calibrated parameter set  $\Sigma$ , we evaluate the associated accumulated residual  $\Delta(d, u)$  on each of the  $m$  alternative asset price paths created in Step 1; the accumulated residuals are then averaged over  $m$  alternative paths. The averaged accumulated residual numerically approximates the expected accumulated residual  $E(\Delta(d, u))$ .

Step 3. The largest averaged accumulated residual identifies the optimal parameter pair  $(d^*, u^*)$  and corresponding hedging strategy  $(\gamma^*, \beta^*)$ .

**5.2. Algorithm implementation.** Implementing the algorithm as described requires simulation of  $m$  alternative asset price paths. Creating alternative asset price paths can be achieved in a number of different ways. For example, one can use time series methods, Monte Carlo simulation, etc. For this initial study, we have assumed independent, identically distributed jumps and chosen the method of sampling asset price jumps with replacement. More sophisticated simulation techniques will be explored in the future.

Another implementation issue involves the evaluation of each single period residual defined in (4.10) for each  $(d, u)$  in the market calibrated parameter set  $\Sigma$  and

each alternative asset price path. While in theory computing the single period residuals exactly using equation (4.10) is possible, it is computationally infeasible due to the number of terms  $g_k^{s_0, \dots, s_{k-1}}(u, d, s)$  for all  $k, s_0, \dots, s_{k-1}$  and current asset value  $s$ . We thus replace the computation of each individual  $g_k$  for each  $d, u$  and  $s$  values appearing in our path with a table lookup of pre-computed values of  $g_k$ .

Let us recall that each quantity  $g_k^{s_0, \dots, s_{k-1}}(u, d, s)$  can be interpreted as the (unique) Asian option price evaluated within the framework of the CRR model with parameters  $(d, u)$  and  $n - k$  time steps to expiration. There are a number of well established computational techniques for approximating Asian option prices (see e.g. [12], [24], [28]). After a careful study and comparison we have chosen the method described in Thompson [22] for computation of the  $g_k^{s_0, \dots, s_{k-1}}(u, d, s)$  values based on accuracy and rapid execution time. We apply the Thompson algorithm to construct the table of  $g_k^{s_0, \dots, s_{k-1}}(u, d, s)$  values over a finely spaced grid of days to expiration and asset values  $s$  for fixed strike, interest rate and asset volatility.

## 6. Numerical Results for Asian Stock Options

We test our algorithm on six Asian stock options with thirty days to expiration described in Table 1. The underlying stock choice represents low volatility (H.J. Heinz Co), moderate volatility (ExxonMobil) and high volatility (Cisco Systems) stocks for the time period December 14, 2006 to January 25, 2007. For each underlying stock, the algorithm is tested for two Asian option strike prices  $K$ , one satisfying  $s_0 < K$ , and one satisfying  $s_0 > K$ . The risk-free interest rate is fixed at 4% per year for all computations.

Stock prices are collected from December 14, 2006 to March 9, 2007. The collected data is divided into two data sets: the set of values from December 14, 2006 to January 25, 2007 for model fitting (referred to as model fitting data) and the set of values from January 26, 2007 through March 9, 2007 for testing the model (referred to as testing data).

**6.1. Comparison of market calibrated hedging strategies for model fitting data.** In this section we compare the expected accumulated residual value produced by the optimal parameter pair  $(u^*, d^*)$  and associated hedging strategy  $(\gamma^*, \beta^*)$  with the expected accumulated residual values produced by other hedging strategies associated with other parameter pairs in the market calibrated parameter set  $\Sigma$ . The accumulated residual value is averaged over 100 bootstrap sample stock paths. The range of values for the expected accumulated residuals for the options described in Table 1 are presented in Table 2.

For the Heinz data with strike price  $K = 45$ , the expected accumulated residual values for the market calibrated parameters range from a low of -0.40 to a maximum of 0.10 (the value associated with the optimal market calibrated parameter pair). Thus, our chosen hedging strategy provides up to 500% improvement in the gain for the option seller. In the case of the Heinz option with strike  $K = 48$ , the expected accumulated residual values range from 0.10 to 0.70. Our chosen hedging

| Company name   | Strike price | $s_0$ | Annual Volatility |
|----------------|--------------|-------|-------------------|
| H.J. Heinz Co. | 45           | 46.4  | 0.1758            |
| H.J. Heinz Co. | 48           | 46.4  | 0.1758            |
| ExxonMobil     | 72           | 73.6  | 0.2247            |
| ExxonMobil     | 75           | 73.6  | 0.2247            |
| Cisco Systems  | 24           | 26.35 | 0.475             |
| Cisco Systems  | 28           | 26.35 | 0.475             |

TABLE 1. Asian stock options used in data analysis

| Company name            | Strike price | Min $E(\Delta)$ | Max $E(\Delta)$ |
|-------------------------|--------------|-----------------|-----------------|
| H.J. Heinz Co. (HNZ-45) | 45           | -0.40           | 0.10            |
| H.J. Heinz Co. (HNZ-48) | 48           | 0.10            | 0.70            |
| ExxonMobil (XOM-72)     | 72           | -0.15           | 0.25            |
| ExxonMobil (XOM-75)     | 75           | 0.45            | 0.80            |
| Cisco Systems (CSCO-24) | 24           | 0.28            | 0.65            |
| Cisco Systems (CSCO-28) | 28           | 0.52            | 1.20            |

TABLE 2. Range of expected accumulated residual values for single asset options

strategy produces an expected accumulated residual up to seven times as large as other strategies.

Expected accumulated residual values range from -0.15 to 0.25 for the Exxon-Mobil data with strike  $K = 72$  and from 0.45 to 0.80 for strike  $K = 75$ . The optimal value is approximately four times as large as other possible values in both cases.

The high volatility Cisco stock price data produce expected accumulated residual values in the interval  $[0.28, 0.65]$  when  $K = 24$  and in  $[0.52, 1.20]$  when  $K = 28$ . It is interesting to note that the expected residual values for the lower strike prices are consistently above the values for the higher strike prices for all data sets. The lowest volatility data set (H.J. Heinz) produces the widest range of values for expected accumulated residuals and the highest volatility data (Cisco Systems) produces the smallest range of values.

**6.2. Comparison of market calibrated hedging strategies for testing data.** The results given in section 6.1 document the advantages of using the optimal hedging strategy based on the optimal parameter pair in comparison to choosing from the range of other market calibrated hedging strategies. We now evaluate our hedging strategy by applying it to an asset price time series that was not used in determining the optimal parameter pair. Let us recall that we used stock price data from December 14, 2006 to January 25, 2007 to compute our optimal parameter pair  $(d^*, u^*)$  and corresponding hedging strategy  $(\gamma^*, \beta^*)$ . Now we will apply that strategy in the financial environment of the stock price data from January 26, 2007 to March 9, 2007. We compare the accumulated residual of our optimal hedging strategy  $(\gamma^*, \beta^*)$  (denoted by  $\Delta^*$ ) over this time period to

those of the strategies based on other parameter pairs in the market calibrated parameter set. The results of the comparison are shown in Figure 1. Each dot in Figure 1 represents the value of the accumulated residual associated with a hedging strategy in the market calibrated set. The large dot indicates the value  $\Delta^*$  of the accumulated residual associated with the optimal parameter pair. The accumulated residual values  $\Delta^*$  for all six tested options fall at the high end of the realized accumulated residual values.

A numerical evaluation of these results is provided by examining how close the accumulated residual  $\Delta^*$  is to the the maximum realized accumulated residual as measured by the percentage difference from the maximum as a fraction of the range of realized values. For five of the six test cases the percent difference ranges from 3.3% to 14.3%. More specifically, the accumulated residual value  $\Delta^*$  for the Cisco data with strike  $K = 24$  (CSCO-24) is within the top 8.5% of the possible values and the Cisco data with strike  $K = 28$  (CSCO-28) produces  $\Delta^*$  within the top 14.3% of the possible values. For the Heinz data and strike  $K = 45$  (HNZ-45) the accumulated residual value  $\Delta^*$  is in the top 10.2%, but with strike  $K = 48$  (HNZ-48)  $\Delta^*$  falls in the bottom 25%. The  $\Delta^*$  value for the ExxonMobil data with  $K = 72$  (XOM-72) is in the top 13% and for  $K = 75$  (XOM-75) the value is in the top 3.3%.

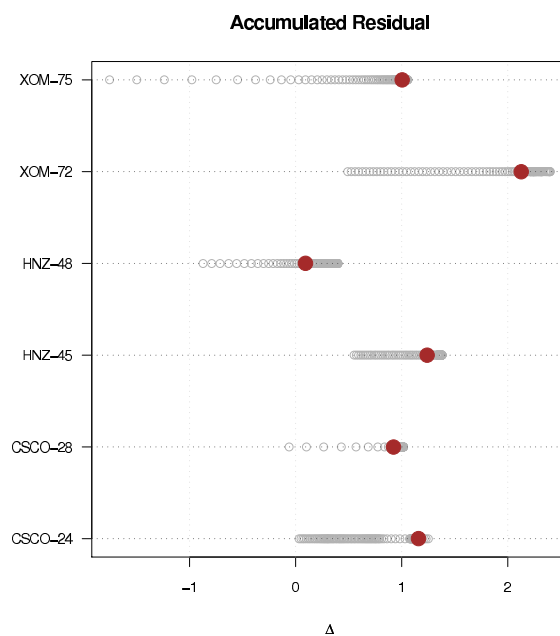


FIGURE 1. Accumulated residual values for stock options using testing data

| Strike price | Min $\Delta$ | Max $\Delta$ | $\Delta^*$ |
|--------------|--------------|--------------|------------|
| 900          | -57          | 45.52        | 45.50      |
| 1012         | -134         | 57.27        | 57.30      |
| 1124         | -199         | 34.37        | 6.86       |
| 1237         | -225         | 12.24        | -16.6      |
| 1349         | -218         | 3.03         | -20.9      |

TABLE 3. Range of accumulated residual values for S&amp;P Index options

## 7. Numerical Results for Asian S&P Index Options

As a final test, we apply our algorithm to the S&P Index data that is utilized in a different approach to Asian option pricing and hedging in [1]. We follow the parameter choice of [1] in using a risk-free interest rate of  $r = .007$  and an annual volatility of  $\sigma = 0.2008$ . As in our previous tests, we divide the collected data into a set of values for model fitting and a set of values for testing. For this set of tests, the fitting period is April, 2001 to April, 2002 and the testing period is April, 2002 to April, 2003. The S&P Index paths consist of 12 monthly values over a one year period. In order to create a reasonable sample of jumps, we identify the dates of the first, second, third and fourth Monday to Friday for each month from April, 2001 to April, 2002. The data is used to create sample paths consisting of the first Mondays of each month, the first Tuesdays of each month, up to the fourth Fridays of each month. This creates 20 sample paths spanning the April, 2001 to April, 2002 year. The S&P Index value jumps are used to create bootstrap paths for evaluating the expected accumulated residual.

**7.1. Comparison of market calibrated hedging strategies for testing data.** Table 3 presents the range of accumulated residual values for the S&P 500 Index testing data with five strike prices using all pairs in the market calibrated parameter set  $\Delta$ . The  $\Delta^*$  value is the accumulated residual value associated with the hedging strategy produced by the algorithm. For the options with strikes of  $K = 900$  and  $K = 1012$ , the accumulated residual associated with our chosen strategy practically coincides with the maximum realized accumulated residual value. For the option with strike  $K = 1124$ , the  $\Delta^*$  value is within 20% of the maximum possible value, where the percent difference from the maximum value is measured as the difference from the maximum as a fraction of the range of realized values. For the two options with strikes  $K = 1237$  and  $K = 1349$  the  $\Delta^*$  value is within 12.2% and 10.8%, respectively, of the maximum value. The results are depicted in Figure 2 where each dot represents an accumulated residual associated with a parameter pair in the market calibrated parameter set and the large dot indicates the residual  $\Delta^*$  corresponding to the optimal parameter pair.

**7.2. Comparison of accumulated residual value with naive super-hedge.** We compare our accumulated residual value in hedging an Asian option with monthly re-balancing to a naive super-hedge. The naive super-hedge consists of a purchased European call option with the same strike price and time to maturity as the sold Asian option. The set-up cost for constructing the naive super-hedge



| Strike price | Algorithmic $\Delta^*$ | Naive $\Delta_{SH}$ |
|--------------|------------------------|---------------------|
| 900          | 45.5                   | -50.02              |
| 1012         | 57.30                  | -31.27              |
| 1124         | 6.86                   | -40.16              |
| 1237         | -16.6                  | -34.33              |
| 1349         | -20.9                  | -21.85              |

TABLE 4. Comparison of algorithmic accumulated residual and naive super-hedge

is the difference between the premium collected for the sold Asian option and the premium paid for the purchased European option. The residual value  $\Delta_{SH}$  for the naive super-hedge can then be determined by computing the difference between the set up cost for the naive super-hedge and the net value of the options at maturity  $t = n$ ,

$$\Delta_{SH} = (\text{Hedge set-up cost}) e^{rn} + (s_n - K)_+ - \left( \frac{1}{n+1} \sum_{i=0}^n s_i - K \right)_+ \quad (7.1)$$

where  $s_i$  are the S&P 500 Index values at time  $i = 0, 1, \dots, n$  and  $r$  is the risk-free interest rate. Table 4 compares the  $\Delta^*$  produced by our algorithm to the  $\Delta_{SH}$  produced by the naive super-hedge. Our algorithm outperforms the naive

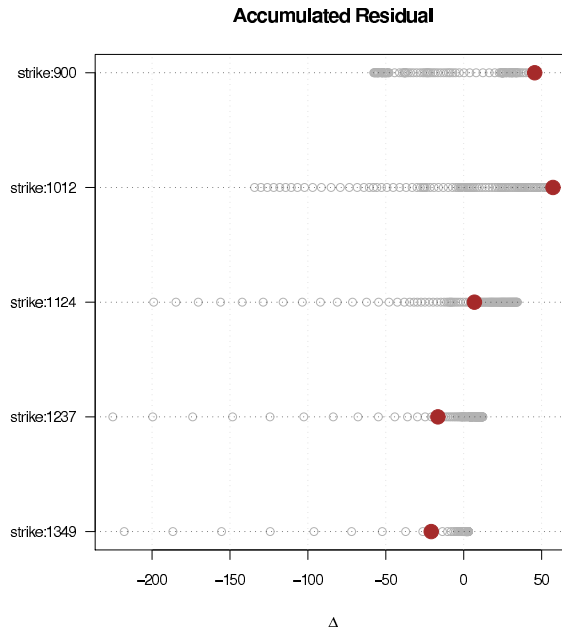


FIGURE 2. Accumulated residual values for S&P 500 Index options using testing data

super-hedge in all cases. For  $K = 900$  there is nearly a 200% gain in following the

algorithmic hedging strategy as compared to the naive super-hedge. The difference between the algorithmic  $\Delta^*$  and the naive  $\Delta_{SH}$  decreases as  $K$  becomes larger than  $s_0$ . The results suggest that the algorithm may be more effective when  $s_0 > K$ .

## 8. Appendix

**Proof of Proposition 2.2.** Consider the CRR model with parameters  $(D, U)$  at one time step to expiration ( $t = n - 1$ ). Our goal is to find an explicit expression for the CRR option price and trading strategy.

We take the position of an option seller who wishes to hedge the potential liability of the sold option being exercised. Suppose the option with the convex pay-off function  $f$  is sold at time  $t = n - 1$  for the price  $CRR_{n-1}$ . The option seller uses the amount  $CRR_{n-1}$  to cover the set-up cost of a portfolio consisting of  $\gamma_{n-1}$  stocks and  $\beta_{n-1}$  bonds with the intention of hedging the short position in the option:

$$CRR_{n-1} = \gamma_{n-1}s_{n-1} + \beta_{n-1}b_{n-1}. \quad (8.1)$$

At time  $t = n$  the value of the portfolio changes to

$$CRR_n = \gamma_{n-1}s_n + \beta_{n-1}b_n = \gamma_{n-1}s_{n-1}\xi_n + \beta_{n-1}b_{n-1}(1+r).$$

In order to obtain a perfect hedging strategy (which is possible due to the completeness of the CRR model) we need to set up the equation:

$$CRR_n = f(s_0, s_1, \dots, s_{n-1}, s_n). \quad (8.2)$$

We stress here that at time  $t = n - 1$  the values  $s_0, s_1, \dots, s_{n-1}$  are known while the value of  $s_n$  is unknown. Equation (8.2) is equivalent to the system of equations:

$$\begin{aligned} \gamma_{n-1}s_{n-1}D + \beta_{n-1}b_{n-1}(1+r) &= f(s_0, s_1, \dots, s_{n-1}, s_{n-1}D), \\ \gamma_{n-1}s_{n-1}U + \beta_{n-1}b_{n-1}(1+r) &= f(s_0, s_1, \dots, s_{n-1}, s_{n-1}U). \end{aligned} \quad (8.3)$$

Solving the system (8.3) one obtains:

$$\gamma_{n-1} = \frac{f(s_0, \dots, s_{n-1}, s_{n-1}U) - f(s_0, \dots, s_{n-1}, s_{n-1}D)}{s_{n-1}(U - D)}, \quad (8.4)$$

$$\beta_{n-1} = \frac{Uf(s_0, \dots, s_{n-1}, s_{n-1}D) - Df(s_0, \dots, s_{n-1}, s_{n-1}U)}{(1+r)b_{n-1}(U - D)}. \quad (8.5)$$

Plugging (8.4) and (8.5) into (8.1) we obtain a unique CRR option price at time  $t = n - 1$ :

$$\begin{aligned} CRR_{n-1} &= (1+r)^{-1}(p(D, U)f(s_0, \dots, s_{n-1}, s_{n-1}U) \\ &\quad + [1 - p(D, U)]f(s_0, \dots, s_{n-1}, s_{n-1}D)) \end{aligned} \quad (8.6)$$

The right hand side of (8.6) equals  $g_{n-1}^{s_0, \dots, s_{n-2}}(D, U, s_{n-1})$  (see (2.6)).

Proceeding backward in time in a similar manner produces the formulas (2.4) through (2.14).

**Proof of Proposition 4.1.** It is sufficient to consider a single-step model with an option that expires at time  $t = 1$ . Let the pair  $(d, u)$  satisfy (4.3). We will compare two quantities:

$$\bar{X}_0(D, U) = (1 + r)^{-1}(p(D, U)f(s_0, s_0U) + [1 - p(D, U)]f(s_0, s_0D)) \quad (8.7)$$

and

$$\bar{x}_0(d, u) = (1 + r)^{-1}(p(d, u)f(s_0, s_0u) + [1 - p(d, u)]f(s_0, s_0d)), \quad (8.8)$$

where  $p$  is given by (2.8). Let us define a real valued function  $h$  on  $[D, d] \times [u, U]$  as follows:

$$h(x, y) = \frac{(1 + r) - x}{y - x} f(s_0, s_0y) + \frac{y - (1 + r)}{y - x} f(s_0, s_0x). \quad (8.9)$$

It follows that

$$\begin{aligned} h(D, U) &= (1 + r)\bar{X}_0(D, U) \\ h(d, u) &= (1 + r)\bar{x}_0(d, u). \end{aligned} \quad (8.10)$$

In order to prove the proposition, we need to show that  $h$  is non-decreasing along the direction of the vector  $l = \langle D - d, U - u \rangle$ .

First suppose that the pay-off function  $f$  is continuously differentiable in  $R^2$ . Then it follows that for every  $x \in R^1$  the function  $\phi_x(y) = f(x, s_0y)$  is continuously differentiable in  $R^1$ . Moreover, it follows from the convexity of  $f$  that  $\phi_x(y)$  is convex in  $y \in R^1$ .

Let us evaluate the directional derivative  $(h'(x, y), l)$ . We have:

$$\begin{aligned} (h'(x, y), l) &= \frac{(1 + r - y)(D - d)}{(y - x)^2} (\phi_{s_0}(y) - \phi_{s_0}(x) - \phi'_{s_0}(x)(y - x)) \\ &+ \frac{(1 + r - x)(U - u)}{(y - x)^2} (\phi_{s_0}(x) - \phi_{s_0}(y) - \phi'_{s_0}(y)(x - y)) \end{aligned} \quad (8.11)$$

It follows from the convexity of  $\phi_x(y)$  that the latter quantity is non-negative for all  $(x, y) \in [D, d] \times [u, U]$ .

Now we will drop the assumption of continuous differentiability of  $f$  in  $R^2$ . We will assume though that  $f$  is continuous in  $R^2$ . Then for every fixed  $x$ ,  $\phi_x(y)$  is a convex continuous function on  $[D, d] \cup [u, U]$ . Therefore, at each point  $z_0$  of the set  $(D, d) \cup (u, U)$  there exist left-sided and right-sided continuous derivatives  $D_- \phi_x(z_0)$  and  $D_+ \phi_x(z_0)$ , moreover:

$$D_- \phi_x(z_0) \leq D_+ \phi_x(z_0). \quad (8.12)$$

Suppose the function  $\phi$  is continuously differentiable everywhere but at the point  $z_0 \in [D, d]$ . We will split the interval  $[D, d]$  into two parts:  $[D, d] = [D, z_0] \cup [z_0, d]$  and accordingly we will split  $[u, U]$  as follows:  $[u, U] = [u, y_0] \cup [y_0, U]$ , where  $y_0$  is the second coordinate of the point  $(z_0, y_0)$  lying on the vector  $l$ . Note that for every point  $(x, y) \in (D, z_0) \times (y_0, U) \cup (z_0, d) \times (u, y_0)$  we have  $(h'(x, y), l) \geq 0$ . Therefore it suffices to show that

$$\lim_{x \rightarrow z_0^+, y \rightarrow y_0} (h'(x, y), l) \leq \lim_{x \rightarrow z_0^-, y \rightarrow y_0} (h'(x, y), l). \quad (8.13)$$

We have,

$$\begin{aligned} & \lim_{x \rightarrow z_0^+, y \rightarrow y_0} (h'(x, y), l) \\ &= \frac{(1+r-y_0)(D-d)}{(y_0-z_0)^2} (\phi_{s_0}(y_0) - \phi_{s_0}(z_0) - D_+ \phi_{s_0}(z_0)(y_0-z_0)) \\ &+ \frac{(1+r-z_0)(U-u)}{(y_0-z_0)^2} (\phi_{s_0}(z_0) - \phi_{s_0}(y_0) - \phi'_{s_0}(y_0)(z_0-y_0)) \quad (8.14) \end{aligned}$$

Further,

$$\begin{aligned} & \lim_{x \rightarrow z_0^-, y \rightarrow y_0} (h'(x, y), l) \\ &= \frac{(1+r-y_0)(D-d)}{(y_0-z_0)^2} (\phi_{s_0}(y_0) - \phi_{s_0}(z_0) - D_- \phi_{s_0}(z_0)(y_0-z_0)) \\ &+ \frac{(1+r-z_0)(U-u)}{(y_0-z_0)^2} (\phi_{s_0}(z_0) - \phi_{s_0}(y_0) - \phi'_{s_0}(y_0)(z_0-y_0)) \quad (8.15) \end{aligned}$$

Comparing (8.14) and (8.15), we have by means of (8.12) that (8.13) holds.

The multi-step case is proven in a similar manner.

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