[Journal of Stochastic Analysis](https://digitalcommons.lsu.edu/josa)

Volume 2 | Number 2 Article 8

June 2021

# Berry-Esseen Bounds for Approximate Maximum Likelihood Estimators in the α-Brownian Bridge

Khalifa Es-Sebaiy
*Kuwait University, Kuwait City, Kuwait*, khalifa.essebaiy@ku.edu.kw

Jabrane Moustaaid
*Cadi Ayyad University, Marrakesh, Morocco*, jabrane.mst@gmail.com

Idir Ouassou
*Cadi Ayyad University, Marrakesh, Morocco*, i.ouassou@uca.ma

Journal of Stochastic Analysis
Vol. 2, No. 2 (2021) Article 8 (19 pages)
DOI: 10.31390/josa.2.2.08



# BERRY-ESSEEN BOUNDS FOR APPROXIMATE MAXIMUM LIKELIHOOD ESTIMATORS IN THE $\alpha$-BROWNIAN BRIDGE

KHALIFA ES-SEBAIY, JABRANE MOUSTAAID*, AND IDIR OUASSOU

ABSTRACT. Let $T > 0,\ \alpha > \frac{1}{2}$. In this work we consider the problem of estimating the drift parameter of the $\alpha$-Brownian bridge defined as $dX_t = -\alpha\frac{X_t}{T-t}dt + dW_t,\ 0 \le t < T$, where $W$ is a standard Brownian motion. Assume that the process X is observed equidistantly in time with the step size $\Delta_n := \frac{T}{n+1},\ t_i = i\Delta_n,\ i = 0, ..., n$. We will propose two approximate maximum likelihood estimators $\hat{\alpha}_n$ and $\bar{\alpha}_n$ for the drift parameter $\alpha$ based on the discrete observations $X_{t_i}, i = 0, ..., n$. The consistency of those estimators is studied. Explicit bounds for the Kolmogorov distance in the central limit theorem for the estimators $\hat{\alpha}_n$ and $\bar{\alpha}_n$ are obtained.

## 1. Introduction

Let $T \in (0,\infty)$ be fixed. We consider the $\alpha$-Brownian bridge process $X := \{X_t,\ t \in [0,T)\}$, defined as the solution to the stochastic differential equation

$$X_0 = 0;\ dX_t = -\alpha\frac{X_t}{T-t}dt + dW_t,\ 0 \le t < T, \tag{1.1}$$

where $W$ is a standard Brownian motion, and $\alpha > 0$ is unknown parameter to be estimated.
Because (1.1) is linear, it is immediate to solve it explicitly; one then gets the following formula:

$$X_t = (T-t)^{\alpha}\int_0^t (T-s)^{-\alpha}\,dW_s,\ 0 \le t < T. \tag{1.2}$$

An important problem related to the $\alpha$-Brownian bridge (1.1) is to estimate the parameter $\alpha$ when one observes the whole trajectory of $X$. For more information and further references concerning the subject, we refer the reader to [2], as well as [9] and [8].

The maximum likelihood estimator (MLE) of the unknown parameter $\alpha$ based on continuous observations, is given by

$$\widetilde{\alpha}_t = -\left(\int_0^t \frac{X_u}{T-u}\,dX_u\right) \Big/ \left(\int_0^t \frac{X_u^2}{(T-u)^2}\,du\right), \quad t < T. \tag{1.3}$$

Received 2021-4-25; Accepted 2021-6-14; Communicated by the editors.
2010 *Mathematics Subject Classification.* Primary 60F05; Secondary 60H07.
*Key words and phrases.* $\alpha$-Brownian bridge, discrete-time observation, approximate maximum likelihood estimator, Kolmogorov distance, rate of convergence, Malliavin calculus.
* Corresponding author.

In (1.3), the integral with respect to $X$ must of course be understood in the Itô sense.

From a practical point of view, in parametric inference, it is more realistic and interesting to consider asymptotic estimation for the diffusion processes based on discrete observations. We assume that the process $X$ is observed equidistantly in time with the step size $\Delta_n := \frac{T}{n+1}$, $t_i = i\Delta_n$, $i = 0, ..., n$ and $T_n$ denotes the length of the observation window. Then we will consider the following approximate maximum likelihood estimators

$$\hat{\alpha}_n := -\frac{\sum_{i=1}^{n} \frac{X_{t_{i-1}}}{(T-t_{i-1})} \left(X_{t_i} - X_{t_{i-1}}\right)}{\Delta_n \sum_{i=1}^{n} \frac{X_{t_{i-1}}^2}{(T-t_{i-1})^2}}, \tag{1.4}$$

and

$$\bar{\alpha}_n := -\frac{\frac{1}{2}\left(\frac{X_{T_n}^2}{\Delta_n} - \Delta_n \sum_{i=1}^{n} \frac{X_{t_{i-1}}^2}{(T-t_{i-1})^2} - \log(n+1)\right)}{\Delta_n \sum_{i=1}^{n} \frac{X_{t_{i-1}}^2}{(T-t_{i-1})^2}}. \tag{1.5}$$

To our knowledge there is no study of the asymptotic behavior of the estimators $\hat{\alpha}_n$ and $\bar{\alpha}_n$. Our goal in the present paper is to investigate the consistency and the rate of convergence to normality of the estimators $\hat{\alpha}_n$ and $\bar{\alpha}_n$.

In the case of continuous observations, the asymptotic behavior of the MLE $\widetilde{\alpha}_t$, gevin by (1.3), of $\alpha$ based on the observation $\{X_s, 0 \leq s \leq t\}$ as $t \uparrow T$ has been studied in [2]. They proved that the MLE $\widetilde{\alpha}_t$ of $\alpha$ is strongly consistent and asymptotically normal in the case $\alpha > \frac{1}{2}$ . Recently, Es-Sebaiy and Moustaaid [6] obtained, when $\alpha > \frac{1}{2}$, an optimal rate of Kolmogorov distance for central limit theorem of the MLE $\widetilde{\alpha}_t$ in the following sense: there exist constants $0 < c < C < \infty$, depending only on $\alpha$ and $T$, such that for all $t$ sufficiently near $T$,

$$\frac{c}{\sqrt{|\log(T-t)|}} \leq \sup_{z\in\mathbb{R}} \left| \mathbb{P}\left(\sqrt{\frac{|\log(T-t)|}{2\alpha-1}}\,(\alpha - \widetilde{\alpha}_t) \leq z\right) - \mathbb{P}(Z \leq z) \right| \leq \frac{C}{\sqrt{|\log(T-t)|}},$$

where $Z$ denotes a standard normal random variable.

On the other hand, there exists a rich literature on the parametric estimation problems based on discrete observations. In the case of the Ornstein-Uhlenbeck process defined as solution to the equation $dX_t = -\theta X_t dt + dW_t,\ t \geq 0,\ X_0 = 0$, with $\theta > 0$, Bishwal and Bose [3] obtained an upper bound in Kolmogorov distance for normal approximation of the approximate maximum likelihood estimators for the drift parameter $\theta$ on the basis of discrete observations of the process $X$. We mention that Es-Sebaiy [5] studied the least squares estimator of $\theta$ based on the sampling data $X_i, i = 1, ..., n$ when the standard Brownian motion $W$ is replaced by a fractional Brownian motion, see also [14, 15].

The rest of the paper is structured as follows. In Section 2 we give the basic tools of Malliavin calculus needed throughout the paper. In Section 3 we prove the consistency and obtain the rate of convergence to normality of the estimators $\hat{\alpha}_n$ and $\bar{\alpha}_n$ .

## 2. Preliminaries

In this section, we recall some elements from stochastic analysis that we will need in the paper. See [12], and [13] for details. Any real, separable Hilbert space $\mathfrak{H}$ gives rise to an isonormal Gaussian process: a centered Gaussian family $(G(\varphi), \varphi \in \mathfrak{H})$ of random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathbf{E}(G(\varphi)G(\psi)) = \langle \varphi, \psi \rangle_{\mathfrak{H}}$. In this paper, it is enough to use the classical Wiener space, where $\mathfrak{H} = L^2([0,T])$, though any $\mathfrak{H}$ will also work. In the case $\mathfrak{H} = L^2([0,T])$, $G$ can be identified with the stochastic differential of a Wiener process $\{W_t, t \in [0,T]\}$ and one interprets $G(\varphi) := \int_0^T \varphi(s)\, dW(s)$.

The Wiener chaos of order $p$, denoted by $\mathfrak{H}_p$, is defined as the closure in $L^2(\Omega)$ of the linear span of the random variables $H_p(G(\varphi))$, where $\varphi \in \mathfrak{H}$, $\|\varphi\|_{\mathfrak{H}} = 1$ and $H_p$ is the Hermite polynomial of degree $p$. The multiple Wiener stochastic integral $I_p$ with respect to $G \equiv W$, of order $p$ is an isometry between the Hilbert space $\mathfrak{H}^{\odot p} = L^2_{sym}([0,T]^p)$ (symmetric tensor product) equipped with the scaled norm $\sqrt{p!}\|\cdot\|_{\mathfrak{H}^{\otimes p}}$ and the Wiener chaos of order $p$ under $L^2(\Omega)$'s norm, that is, the multiple Wiener stochastic integral of order $p$:

$$I_p : \left(\mathfrak{H}^{\odot p}, \sqrt{p!}\|\cdot\|_{\mathfrak{H}^{\otimes p}}\right) \longrightarrow \left(\mathfrak{H}_p, L^2(\Omega)\right)$$

is a linear isometry defined by $I_p(f^{\otimes p}) = H_p(G(f))$.

• **Multiple Wiener-Itô integral.** If $f \in L^2([0,T]^p)$ is symmetric, we can also rewrite $I_p(f)$ as the following iterated adapted Itô stochastic integral:

$$\begin{aligned} I_p(f) &= \int_{[0,T]^p} f(t_1,\ldots,t_p)dW_{t_1}\ldots dW_{t_p} \\ &= p! \int_0^T dW_{t_1} \int_0^{t_1} dW_{t_2} \ldots \int_0^{t_{p-1}} dW_{t_p} f(t_1,\ldots,t_p). \end{aligned} \tag{2.1}$$

• **The Wiener chaos expansion.** For any $F \in L^2(\Omega)$, there exists a unique sequence of functions $f_p \in \mathfrak{H}^{\odot p}$ such that

$$F = \mathbf{E}[F] + \sum_{p=1}^{\infty} I_p(f_p),$$

where the terms are all mutually orthogonal in $L^2(\Omega)$ and

$$\mathbf{E}\left[I_p(f_p)^2\right] = p!\|f_p\|^2_{\mathfrak{H}^{\otimes p}}.$$

• **Product formula and contractions.** For any integers $p, q \geq 1$ and symmetric integrands $f \in \mathfrak{H}^{\odot p}$ and $g \in \mathfrak{H}^{\odot q}$,

$$I_p(f)I_q(g) = \sum_{r=0}^{p\wedge q} r! \binom{p}{r}\binom{q}{r} I_{p+q-2r}(f\widetilde{\otimes}_r g); \tag{2.2}$$

where $f \otimes_r g$ is the contraction of order $r$ of $f$ and $g$ which is an element of $\mathfrak{H}^{\otimes(p+q-2r)}$ defined by

$$\begin{aligned}(f \otimes_r g)(s_1, ..., s_{p-r}, t_1, ..., t_{q-r}) \quad := \quad & \int_{[0,T]^{p+q-2r}} f(s_1, ..., s_{p-r}, u_1, ..., u_r) \\ & g(t_1, ..., t_{q-r}, u_1, ..., u_r)\, du_1 ... du_r,\end{aligned}$$

while $f \widetilde{\otimes}_r g$ denotes its symmetrization. More generally the symmetrization $\tilde{f}$ of a function $f$ is defined by $\tilde{f}(x_1, \dots, x_p) = \frac{1}{p!} \sum_{\sigma} f(x_{\sigma(1)}, ..., x_{\sigma(p)})$ where the sum runs over all permutations $\sigma$ of $\{1, ..., p\}$.

• **Kolmogorov distance between random variables**. If $X, Y$ are two real-valued random variables, then the Kolmogorov distance between the law of $X$ and the law of $Y$ is given by

$$d_{Kol}(X, Y) := \sup_{z \in \mathbb{R}} \left| \mathbb{P}(X \leq z) - \mathbb{P}(Y \leq z) \right|.$$

• **The derivative operator**. We denote by $\mathcal{S}$ the class of smooth cylindrical functionals of the form

$$F = f\left(W(h_1), W(h_2), ..., W(h_n)\right), \tag{2.3}$$

where $n \geq 1$ $f \in C_b^{\infty}(\mathbb{R}^n, \mathbb{R})$ and $h_1, h_2, ..., h_n \in \mathfrak{H}$.
The derivative operator $D$ of a smooth cylindrical random variable $F$ of the form (2.3) is defined as the $\mathfrak{H}$-valued random variable

$$DF = \sum_{i=1}^{n} \partial_i f\left(W(h_1), W(h_2), ..., W(h_n)\right) h_i.$$

In this way the derivative $DF$ is an element of $L^2(\Omega; \mathfrak{H})$. We denote by $D^{1,2}$ the closure of $\mathcal{S}$ with respect to the norm defined by

$$\|F\|_{1,2} = \mathbf{E}(F^2) + \mathbf{E}(\|DF\|_{\mathfrak{H}}^2).$$

**Theorem 2.1** ([11])**.** *Let $F = I_q(g)$ with $q \geq 2$ and $g \in L^2([0,T]^q)$. Then,*

$$d_{kol}(F, N) \leq \sqrt{\mathbf{E}\left[\left(1 - \frac{1}{q}\|DF\|_{\mathfrak{H}}^2\right)^2\right]}, \tag{2.4}$$

*where $N \sim \mathcal{N}(0,1)$.*

**Proposition 2.2** ([11])**.** *Let $F = I_q(g)$ with $q \geq 2$ and $g \in L^2([0,T]^q)$. Then,*

$$\mathbf{E}\left[\left(1 - \frac{1}{q}\|DF\|_{\mathfrak{H}}^2\right)^2\right] \tag{2.5}$$

$$\leq \left(1 - q!\|g\|_{\mathfrak{H}^{\otimes q}}^2\right)^2 \tag{2.6}$$

$$+q^2 \sum_{r=1}^{q-1} (2q-2r)!\,(r-1)!^2 \binom{q-1}{r-1} \times \|g \otimes_r g\|_{\mathfrak{H}^{\otimes 2(q-r)}}^2$$

**Lemma 2.3** (Wick's lemma). *Let $(Z_1, Z_2, Z_3, Z_4)$ be a Gaussian random vector with zero mean . Then*

$$\mathbf{E}\left[Z_1Z_2Z_3Z_4\right] = \mathbf{E}\left[Z_1Z_2\right]\mathbf{E}\left[Z_3Z_4\right] + \mathbf{E}\left[Z_1Z_3\right]\mathbf{E}\left[Z_2Z_4\right] + \mathbf{E}\left[Z_1Z_4\right]\mathbf{E}\left[Z_2Z_3\right].$$

Throughout the paper $Z$ denotes a standard normal random variable. Also, $C$ denotes a generic positive constant (perhaps depending on $\alpha$ and $T$, but not on anything else), which may change from line to line.

## 3. Asymptotic Behavior of the Approximate Maximum Likelihood Estimators

This section is devoted to study the consistency and the asymptotic distribution of the approximate maximum likelihood estimators $\hat{\alpha}_n$ and $\bar{\alpha}_n$ given by (1.4) and (1.5) respectively.

Let us fix the notations needed in what follows. Define for every $t \in [0, T)$,

$$\begin{aligned}
Y_t &:= \frac{X_t}{T-t},\ B_{T_n} := \int_0^{T_n} Y_t dW_t, \\
I_n &:= \Delta_n \sum_{i=1}^n Y_{t_{i-1}}^2,\ A_n := \sum_{i=1}^n \int_{t_{i-1}}^{t_i} Y_{t_{i-1}}\left[Y_{t_{i-1}} - Y_t\right] dt, \\
I_{T_n} &:= \int_0^{T_n} Y_t^2 dt,\ B_n := \sum_{i=1}^n \int_{t_{i-1}}^{t_i} Y_{t_{i-1}} dW_t.
\end{aligned}$$

Notice that for every $t, s \in [0, T)$, the covariance of $X_t$ and $X_s$ is

$$Cov\left(X_t, X_s\right) = \begin{cases} \frac{(T-s)^\alpha (T-t)^\alpha}{1-2\alpha}\left(T^{1-2\alpha} - \left(T - (s \wedge t)\right)^{1-2\alpha}\right) & \text{if } \alpha \neq \frac{1}{2}, \\ \\ (T-s)^\alpha (T-t)^\alpha \log\left(\frac{T}{T-s\wedge t}\right) & \alpha = \frac{1}{2}. \end{cases} \tag{3.1}$$

In particular $X_t$ is a normally distributed random variable with mean $\mathbf{E}X_t = 0$ and with variance

$$\mathbf{E}\left[X_t^2\right] = \begin{cases} \frac{T}{1-2\alpha}\left(\frac{T-t}{T}\right)^{2\alpha} - \frac{T-t}{1-2\alpha} & \text{if } \alpha \neq \frac{1}{2}, \\ \\ (T-t)\log\left(\frac{T}{T-t}\right) & \alpha = \frac{1}{2}. \end{cases} \tag{3.2}$$

In order to prove our main results we will make use of the following lemmas.

**Lemma 3.1.** *Suppose that $\alpha > \frac{1}{2}$. Let $0 < \varepsilon < 1$. Then*

$$\mathbb{P}\left(\left|\lambda_n I_{T_n} - 1\right| > \varepsilon\right) \leq \frac{C}{\varepsilon^2 \log(n+1)}, \tag{3.3}$$

*and*

$$\mathbb{P}\left(\left|\lambda_n\left(I_n - I_{T_n}\right)\right| > \varepsilon\right) \leq \frac{C}{\varepsilon^2 \log^2(n+1)}, \tag{3.4}$$

*where $\lambda_n = \frac{2\alpha+1}{\log(n+1)}$.*

*Proof.* Applying the Itô formula, see [7], we have

$$\frac{2\alpha-1}{\log(n+1)}I_{T_n}-1=\frac{2}{\log(n+1)}\int_0^{T_n}\frac{X_t}{(T-t)}dW_t-\frac{X_{T_n}^2}{\Delta_n\log(n+1)}. \tag{3.5}$$

On the other hand, by combining the Lemma 2.3 and (3.2), we get that

$$\mathbf{E}\left[X_{T_n}^4\right]=3\left(\mathbf{E}\left[X_{T_n}^2\right]\right)^2=\frac{3\left(\Delta_n-T^{1-2\alpha}\Delta_n^{2\alpha}\right)^2}{(2\alpha-1)^2},$$

and

$$\begin{aligned}\mathbf{E}\left[I_{T_n}\right] &= \int_0^{T_n}\mathbf{E}\left[\frac{X_t^2}{(T-t)^2}\right]dt\\ &= \int_0^{T_n}\frac{(T-t)-T^{1-2\alpha}(T-t)^{2\alpha}}{(2\alpha-1)(T-t)^2}dt\\ &= \frac{(2\alpha-1)\log(n+1)+T^{1-2\alpha}\Delta_n^{2\alpha-1}-1}{(2\alpha-1)^2}. \end{aligned} \tag{3.6}$$

Hence, by Chebyshev inequality we have

$$\begin{aligned}&\mathbb{P}\left(|\lambda_n I_{T_n}-1|>\varepsilon\right)\\ &\leq\frac{1}{\varepsilon^2}\mathbf{E}\left[|\lambda_n I_{T_n}-1|^2\right]\\ &=\frac{1}{\varepsilon^2}\mathbf{E}\left[\left|\frac{2}{\log(n+1)}\int_0^{T_n}\frac{X_t}{(T-t)}dW_t-\frac{X_{T_n}^2}{\Delta_n\log(n+1)}\right|^2\right]\\ &\leq\frac{1}{\varepsilon^2}\left\{\frac{8}{\log^2(n+1)}\mathbf{E}\left[\left(\int_0^{T_n}\frac{X_t}{(T-t)}dW_t\right)^2\right]+2\mathbf{E}\left[\frac{X_{T_n}^4}{\Delta_n^2\log^2(n+1)}\right]\right\}\\ &=\frac{2}{\varepsilon^2}\left\{\frac{4}{\log^2(n+1)}\mathbf{E}\left[I_{T_n}\right]+\frac{\mathbf{E}\left[X_{T_n}^4\right]}{\Delta_n^2\log^2(n+1)}\right\}\\ &=\frac{2}{\varepsilon^2}\left\{\frac{4\left((2\alpha-1)\log(n+1)+T^{1-2\alpha}\Delta_n^{2\alpha-1}-1\right)}{(2\alpha-1)^2\log^2(n+1)}\right.\\ &\qquad\left.+\frac{3\left(\Delta_n-T^{1-2\alpha}\Delta_n^{2\alpha}\right)^2}{(2\alpha-1)^2\Delta_n^2\log^2(n+1)}\right\}\\ &\leq\frac{C}{\varepsilon^2\log(n+1)}.\end{aligned}$$

Again, by Chebyshev inequality we have

$$\mathbb{P}\left(|\lambda_n\left(I_n-I_{T_n}\right)|>\varepsilon\right)\leq\frac{\lambda_n^2}{\varepsilon^2}\mathbf{E}\left[|(I_n-I_{T_n})|^2\right]. \tag{3.7}$$

On the other hand

$$\mathbf{E}\left[|(I_n-I_{T_n})|^2\right] = \mathbf{E}\left[\left|\sum_{i=1}^n\int_{t_{i-1}}^{t_i}\left(Y_{i-1}^2-Y_t^2\right)dt\right|^2\right]$$

$$
\begin{aligned}
&= \mathbf{E}\left[\sum_{i=1}^{n}\left(\int_{t_{i-1}}^{t_i}\left(Y_{t_{i-1}}+Y_t\right)\left(Y_{t_{i-1}}-Y_t\right)dt\right)^2\right] \\
&\quad +2\sum_{i<j}\int_{t_{i-1}}^{t_i}\int_{t_{j-1}}^{t_j}\left\{\mathbf{E}\left[\left(Y_{t_{i-1}}-Y_t\right)\left(Y_{t_{j-1}}+Y_s\right)\right.\right. \\
&\quad \left.\left.\left(Y_{t_{i-1}}+Y_t\right)\left(Y_{t_{j-1}}-Y_s\right)\right]\right\} dsdt \\
&\quad =: b_{1,n}+b_{2,n}.
\end{aligned}
$$

We begin with studying the term $b_{1,n}$. Using Lemma 2.3 together with Cauchy Schwarz inequality, we have

$$
\mathbf{E}\left[\left(Y_{t_{i-1}}+Y_t\right)^2\left(Y_{t_{i-1}}-Y_t\right)^2\right] \le 3\mathbf{E}\left[\left(Y_{t_{i-1}}+Y_t\right)^2\right]\mathbf{E}\left[\left(Y_{t_{i-1}}-Y_t\right)^2\right].
$$

Moreover, by (3.1), we have

$$
\begin{aligned}
&(2\alpha-1)\mathbf{E}\left[\left(Y_{t_{i-1}}-Y_t\right)^2\right] \\
&= \left(T-t_{i-1}\right)^{-1} - T^{1-2\alpha}\left(T-t_{i-1}\right)^{2\alpha-2} + \left(T-t\right)^{-1} - T^{1-2\alpha}\left(T-t\right)^{2\alpha-2} \\
&\quad -2\left(T-t_{i-1}\right)^{-\alpha}\left(T-t\right)^{\alpha-1} + 2T^{1-2\alpha}\left(T-t_{i-1}\right)^{\alpha-1}\left(T-t\right)^{\alpha-1} \\
&= \left(T-t_{i-1}\right)^{-1} + \left(T-t\right)^{-1} - 2\left(T-t_{i-1}\right)^{-\alpha}\left(T-t\right)^{\alpha-1} \\
&\quad -T^{1-2\alpha}\left(\left(T-t_{i-1}\right)^{\alpha-1} - \left(T-t\right)^{\alpha-1}\right)^2 \\
&\le \left(T-t_{i-1}\right)^{-1} + \left(T-t\right)^{-1}.
\end{aligned}
$$

Similarly, we can obtain

$$
(2\alpha-1)\mathbf{E}\left[\left(Y_{t_{i-1}}+Y_t\right)^2\right] \le 4\left(T-t\right)^{-1}.
$$

This implies that,

$$
\begin{aligned}
b_{1,n} &\le \mathbf{E}\left[\sum_{i=1}^{n}\Delta_n\int_{t_{i-1}}^{t_i}\left(Y_{t_{i-1}}+Y_t\right)^2\left(Y_{t_{i-1}}-Y_t\right)^2 dt\right] \\
&\le C\Delta_n\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\left(T-t\right)^{-1}\left(\left(T-t_{i-1}\right)^{-1}+\left(T-t\right)^{-1}\right)dsdt \\
&= C\Delta_n\left\{\sum_{i=1}^{n}\left(\left(T-t_i\right)^{-1}-\left(T-t_{i-1}\right)^{-1}\right)\right. \\
&\qquad \left. +\sum_{i=1}^{n}\left(T-t_{i-1}\right)^{-1}\left(\log(T-t_{i-1})-\log(T-t_i)\right)\right\} \\
&= C\Delta_n\left\{\frac{1}{\Delta_n}+\frac{1}{T}+\frac{1}{\Delta_n}\sum_{k=1}^{n}\frac{\log(k+1)-\log(k)}{k+1}\right\} \\
&\le C\Delta_n\left\{\frac{1}{\Delta_n}+\frac{1}{T}+\frac{1}{\Delta_n}\sum_{k=1}^{n}\frac{1}{k^2}\right\} \le C'. \qquad (3.8)
\end{aligned}
$$

We now study $b_{2,n}$. Applying Lemma 2.3, we can write

$$\begin{aligned}
&\mathbf{E}\left[\left(Y_{t_{i-1}}+Y_t\right)\left(Y_{t_{j-1}}+Y_s\right)\left(Y_{t_{i-1}}-Y_t\right)\left(Y_{t_{j-1}}-Y_s\right)\right]\\
&=\mathbf{E}\left[\left(Y_{t_{i-1}}+Y_t\right)\left(Y_{t_{j-1}}+Y_s\right)\right]\mathbf{E}\left[\left(Y_{t_{i-1}}-Y_t\right)\left(Y_{t_{j-1}}-Y_s\right)\right]\\
&+\mathbf{E}\left[\left(Y_{t_{i-1}}+Y_t\right)\left(Y_{t_{i-1}}-Y_t\right)\right]\mathbf{E}\left[\left(Y_{t_{j-1}}+Y_s\right)\left(Y_{t_{j-1}}-Y_s\right)\right]\\
&+\mathbf{E}\left[\left(Y_{t_{i-1}}+Y_t\right)\left(Y_{t_{j-1}}-Y_s\right)\right]\mathbf{E}\left[\left(Y_{t_{j-1}}+Y_s\right)\left(Y_{t_{i-1}}-Y_t\right)\right]\\
&=: U_{1,n}+U_{2,n}+U_{3,n}.
\end{aligned}$$

Hence,

$$\begin{aligned}
b_{2,n} \quad = \quad & 2\sum_{i<j}\int_{t_{i-1}}^{t_i}\int_{t_{j-1}}^{t_j}\left\{\mathbf{E}\left[\left(Y_{t_{i-1}}+Y_t\right)\left(Y_{t_{j-1}}+Y_s\right)\right]\right.\\
&\left.\mathbf{E}\left[\left(Y_{t_{i-1}}-Y_t\right)\left(Y_{t_{j-1}}-Y_s\right)\right]\right\}dsdt\\
&+2\sum_{i<j}\int_{t_{i-1}}^{t_i}\int_{t_{j-1}}^{t_j}\left\{\mathbf{E}\left[\left(Y_{t_{i-1}}+Y_t\right)\left(Y_{t_{i-1}}-Y_t\right)\right]\right.\\
&\left.\mathbf{E}\left[\left(Y_{t_{j-1}}+Y_s\right)\left(Y_{t_{j-1}}-Y_s\right)\right]\right\}dsdt\\
&+2\sum_{i<j}\int_{t_{i-1}}^{t_i}\int_{t_{j-1}}^{t_j}\left\{\mathbf{E}\left[\left(Y_{t_{i-1}}+Y_t\right)\left(Y_{t_{j-1}}-Y_s\right)\right]\right.\\
&\left.\mathbf{E}\left[\left(Y_{t_{j-1}}+Y_s\right)\left(Y_{t_{i-1}}-Y_t\right)\right]\right\}dsdt\\
=: \quad & V_1(n)+V_2(n)+V_3(n).
\end{aligned}$$

First we study the term $V_1(n)$. Let us consider two cases. Assume first that $\alpha \geq 1$, so

$$\begin{aligned}
&(2\alpha-1)\mathbf{E}\left[\left(Y_{t_{i-1}}-Y_t\right)\left(Y_{t_{j-1}}-Y_s\right)\right]\\
&=\Big((T-t_{j-1})^{\alpha-1}-(T-s)^{\alpha-1}\Big)\Big((T-t_{i-1})^{-\alpha}-(T-t)^{-\alpha}\\
&+T^{1-2\alpha}\Big((T-t)^{\alpha-1}-(T-t_{i-1})^{\alpha-1}\Big)\Big)\\
&\leq 0,
\end{aligned}$$

where we used $t_{i-1} \leq t < t_i$, $t_{j-1} \leq s < t_j$ and $\alpha \geq 1$ in the last inequality. Moreover

$$\begin{aligned}
&(2\alpha-1)\mathbf{E}\left[\left(Y_{t_{i-1}}+Y_t\right)\left(Y_{t_{j-1}}+Y_s\right)\right]\\
&=\Big((T-t_{j-1})^{\alpha-1}+(T-s)^{\alpha-1}\Big)\\
&\Big((T-t_{i-1})^{-\alpha}-T^{1-2\alpha}(T-t_{i-1})^{\alpha-1}+(T-t)^{-\alpha}-T^{1-2\alpha}(T-t)^{\alpha-1}\Big)\\
&\geq 0.
\end{aligned}$$

This gives

$$V_1(n) \leq 0. \tag{3.9}$$

In the second case, we suppose $\frac{1}{2} < \alpha \leq 1$, so

$$
\begin{aligned}
0 \le (2\alpha-1)\mathbf{E}\left[\left(Y_{t_{i-1}}-Y_t\right)\left(Y_{t_{j-1}}-Y_s\right)\right] &\le \Big((T-t_{j-1})^{\alpha-1}-(T-s)^{\alpha-1}\Big) \\
&\quad \Big((T-t_{i-1})^{-\alpha}-(T-t)^{-\alpha}\Big) \\
&\le C\,(s-t_{j-1})\,(t-t_{j-1}) \\
&\quad (T-s)^{\alpha-2}\,(T-t)^{-\alpha-1}\,,
\end{aligned}
$$

and

$$
0 \le (2\alpha-1)\mathbf{E}\left[\left(Y_{t_{i-1}}+Y_t\right)\left(Y_{t_{j-1}}+Y_s\right)\right] \le C\,(T-s)^{\alpha-1}\,(T-t)^{-\alpha}\,.
$$

Thus

$$
\begin{aligned}
|V_1(n)| &\le C\sum_{i<j}\int_{t_{i-1}}^{t_i}\int_{t_{j-1}}^{t_j}(s-t_{j-1})\,(t-t_{j-1})\,(T-t)^{-2\alpha-1}\,(T-s)^{2\alpha-3}\,dtds \\
&\le C\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(t-t_{j-1})\,(T-t)^{-2\alpha-1}\,dt\,\times \\
&\qquad \sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}(s-t_{j-1})\,(T-s)^{2\alpha-3}\,ds \\
&\le C\left(\Delta_n\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(T-t)^{-2\alpha-1}\,dt\right)\left(\Delta_n\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}(T-s)^{2\alpha-3}\,ds\right) \\
&\le C\left(1-T^{2\alpha-2}\Delta_n^{2-2\alpha}-T^{-2\alpha}\Delta_n^{2\alpha}+T^{-2}\Delta_n^{2}\right) \\
&\le C. \qquad (3.10)
\end{aligned}
$$

For the term $V_2(n)$, when $\alpha \ge 1$, we have

$$
\begin{aligned}
&(2\alpha-1)^2\,|U_{2,n}| \\
&= \Big|\Big((T-t_{i-1})^{-1}-(T-t)^{-1}-T^{1-2\alpha}\,(T-t_{i-1})^{2\alpha-2}+T^{1-2\alpha}\,(T-t)^{2\alpha-2}\Big) \\
&\Big((T-t_{j-1})^{-1}-(T-s)^{-1}-T^{1-2\alpha}\,(T-t_{j-1})^{2\alpha-2}+T^{1-2\alpha}\,(T-s)^{2\alpha-2}\Big)\Big| \\
&\le \Big((T-t)^{-1}-(T-t_{i-1})^{-1}+C\Big)\Big((T-s)^{-1}-(T-t_{j-1})^{-1}+C\Big)\,.
\end{aligned}
$$

This implies

$$
\begin{aligned}
|V_2(n)| &\le C\sum_{i<j}\int_{t_{i-1}}^{t_i}\int_{t_{j-1}}^{t_j}\Big((T-t)^{-1}-(T-t_{i-1})^{-1}+C\Big) \qquad (3.11) \\
&\quad \Big((T-s)^{-1}-(T-t_{j-1})^{-1}+C\Big)\,dsdt \\
&\le C\sum_{0\le i,j\le n}\int_{t_{i-1}}^{t_i}\int_{t_{j-1}}^{t_j}\Big((T-t)^{-1}-(T-t_{i-1})^{-1}+C\Big) \qquad (3.12) \\
&\quad \Big((T-s)^{-1}-(T-t_{j-1})^{-1}+C\Big)\,dsdt
\end{aligned}
$$

$$
\begin{aligned}
&= C\left(\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\Big((T-t)^{-1}-(T-t_{i-1})^{-1}+C\Big)\,dt\right)^2\\
&= C\left(\sum_{i=1}^{n}\Big(\log(T-t_{i-1})-\log(T-t_i)-\Delta_n(T-t_{i-1})^{-1}+\Delta_n\Big)\right)^2\\
&= C\left\{\log(n+1)-\sum_{k=2}^{n+1}\frac{1}{k}+n\Delta_n\right\}^2\\
&\leq C\left\{\frac{n}{n+1}+n\Delta_n\right\}^2\\
&\leq C. \qquad (3.13)
\end{aligned}
$$

When $\frac{1}{2}<\alpha\leq 1$, we have

$$
\begin{aligned}
(2\alpha-1)^2\left|U_{2,n}\right| \leq & \Big((T-t)^{-1}-(T-t_{i-1})^{-1}+T^{1-2\alpha}(t-t_{i-1})(T-t)^{2\alpha-3}\Big)\\
& \Big((T-s)^{-1}-(T-t_{j-1})^{-1}+T^{1-2\alpha}(t-t_{j-1})(T-t)^{2\alpha-3}\Big),
\end{aligned}
$$

which leads to

$$
\begin{aligned}
|V_2(n)| &\leq C\left(\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\Big((T-t)^{-1}-(T-t_{i-1})^{-1}\right.\\
&\qquad\left. +T^{1-2\alpha}(t-t_{i-1})(T-t)^{2\alpha-3}\Big)\,dt\right)^2\\
&= C\left\{\log(n+1)-\sum_{k=2}^{n+1}\frac{1}{k}+T^{1-2\alpha}\Delta_n\left(\Delta_n^{2\alpha-2}-T^{2\alpha-2}\right)\right\}^2\\
&\leq C\left\{\frac{n}{n+1}+T^{2\alpha-2}\Delta_n^{2\alpha-1}-\Delta_n\right\}^2\\
&\leq C. \qquad (3.14)
\end{aligned}
$$

For $V_3(n)$, we can write

$$
\begin{aligned}
V_3(n) &= 2\sum_{i<j}\int_{t_{i-1}}^{t_i}\int_{t_{j-1}}^{t_j}\Big(\mathbf{E}\left[Y_{t_{i-1}}\left(Y_{t_{j-1}}-Y_s\right)\right]\mathbf{E}\left[Y_{t_{j-1}}\left(Y_{t_{i-1}}-Y_t\right)\right]\\
&\quad+\mathbf{E}\left[Y_{t_{i-1}}\left(Y_{t_{j-1}}-Y_s\right)\right]\mathbf{E}\left[Y_s\left(Y_{t_{i-1}}-Y_t\right)\right]\\
&\quad+\mathbf{E}\left[Y_t\left(Y_{t_{j-1}}-Y_s\right)\right]\mathbf{E}\left[Y_{t_{j-1}}\left(Y_{t_{i-1}}-Y_t\right)\right]\\
&\quad+\mathbf{E}\left[Y_t\left(Y_{t_{j-1}}-Y_s\right)\right]\mathbf{E}\left[Y_s\left(Y_{t_{i-1}}-Y_t\right)\right]\Big)\\
&=: V_{3,1}(n)+V_{3,2}(n)+V_{3,3}(n)+V_{3,4}(n)
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
&(2\alpha-1)\left|\mathbf{E}\left[Y_{t_{j-1}}\left(Y_{t_{i-1}}-Y_t\right)\right]\right|\\
&\qquad=\Big|(T-t_{j-1})^{\alpha-1}\Big((T-t_{i-1})^{-\alpha}-(T-t)^{-\alpha}\\
&\qquad\qquad+T^{1-2\alpha}(T-t)^{\alpha-1}-T^{1-2\alpha}(T-t_{i-1})^{\alpha-1}\Big)\Big|,
\end{aligned}
$$

and

$$\begin{aligned}
&(2\alpha-1)\left|\mathbf{E}\left[Y_{t_{i-1}}\left(Y_{t_{j-1}}-Y_s\right)\right]\right| \\
&=\left|(T-t_{j-1})^{\alpha-1}-(T-s)^{\alpha-1}\right|\left((T-t_{i-1})^{-\alpha}-T^{1-2\alpha}\left(T-t_{i-1}\right)^{\alpha-1}\right) \\
&\leq\left|(T-t_{j-1})^{\alpha-1}-(T-s)^{\alpha-1}\right|(T-t_{i-1})^{-\alpha}.
\end{aligned}$$

Hence, when $\frac{1}{2}<\alpha<2$, we have

$$\begin{aligned}
&(2\alpha-1)\left|\mathbf{E}\left[Y_{t_{j-1}}\left(Y_{t_{i-1}}-Y_t\right)\right]\right| \\
&\leq C\left(T-t_{j-1}\right)^{\alpha-1}\Big(\left(t-t_{i-1}\right)\left(T-t\right)^{-\alpha-1}+\left(t-t_{i-1}\right)\left(T-t\right)^{\alpha-2}\Big),
\end{aligned}$$

and

$$(2\alpha-1)\left|\mathbf{E}\left[Y_{t_{i-1}}\left(Y_{t_{j-1}}-Y_s\right)\right]\right|\leq C(s-t_{j-1})\left(T-t_{i-1}\right)^{-\alpha}\left(T-s\right)^{\alpha-2}.$$

When $\alpha\geq 2$, we have

$$\begin{aligned}
&(2\alpha-1)\left|\mathbf{E}\left[Y_{t_{j-1}}\left(Y_{t_{i-1}}-Y_t\right)\right]\right| \\
&\leq C\left(T-t_{j-1}\right)^{\alpha-1}\Big(\left(t-t_{i-1}\right)\left(T-t\right)^{-\alpha-1}+\left(t-t_{i-1}\right)\left(T-t_{i-1}\right)^{\alpha-2}\Big),
\end{aligned}$$

and

$$(2\alpha-1)\left|\mathbf{E}\left[Y_{t_{i-1}}\left(Y_{t_{j-1}}-Y_s\right)\right]\right|\leq C(s-t_{j-1})\left(T-t_{i-1}\right)^{-\alpha}\left(T-t_{j-1}\right)^{\alpha-2}.$$

As a consequence, when $\alpha\geq 2$, we have

$$\begin{aligned}
&\left|V_{3,1}(n)\right| \\
\leq\ & C\sum_{i<j}\int_{t_{i-1}}^{t_i}\int_{t_{j-1}}^{t_j}\left(s-t_{j-1}\right)\left(T-t_{i-1}\right)^{-\alpha}\left(T-t_{j-1}\right)^{\alpha-2}\left(T-t_{j-1}\right)^{\alpha-1} \\
&\Big(\left(t-t_{i-1}\right)\left(T-t\right)^{-\alpha-1}+\left(t-t_{i-1}\right)\left(T-t_{i-1}\right)^{\alpha-2}\Big)\,dsdt \\
\leq\ & C\sum_{i<j}\int_{t_{i-1}}^{t_i}\int_{t_{j-1}}^{t_j}\left(s-t_{j-1}\right)\left(T-t_{i-1}\right)^{-\alpha}\left(T-t_{j-1}\right)^{\alpha-2} \\
&\left(T-t_{j-1}\right)^{\alpha-1}\left(t-t_{i-1}\right)\left(T-t\right)^{-\alpha-1}dsdt \\
&+C\sum_{i<j}\int_{t_{i-1}}^{t_i}\int_{t_{j-1}}^{t_j}\left(s-t_{j-1}\right)\left(T-t_{i-1}\right)^{-\alpha}\left(T-t_{j-1}\right)^{\alpha-2} \\
&\left(T-t_{j-1}\right)^{\alpha-1}\left(t-t_{i-1}\right)\left(T-t_{i-1}\right)^{\alpha-2}dsdt \\
\leq\ & C\sum_{i<j}\int_{t_{i-1}}^{t_i}\int_{t_{j-1}}^{t_j}\left(s-t_{j-1}\right)\left(T-t_{i-1}\right)^{-2}\left(T-t_{j-1}\right)^{-2}\left(t-t_{i-1}\right)dsdt \\
&+C\sum_{i<j}\int_{t_{i-1}}^{t_i}\int_{t_{j-1}}^{t_j}\left(s-t_{j-1}\right)\left(T-t_{i-1}\right)^{-2}T^{2\alpha-3}(t-t_{i-1})dsdt \\
\leq\ & C\left(\sum_{k=2}^{n+1}\frac{1}{k^2}\right)^2+Cn\Delta_n^2\sum_{k=2}^{n+1}\frac{1}{k^2} \\
\leq\ & C.
\end{aligned}$$

When $1 \leq \alpha \leq 2$, we have

$$\begin{aligned}
& |V_{3,1}(n)| \\
\leq\ & \sum_{i<j} \int_{t_{i-1}}^{t_i} \int_{t_{j-1}}^{t_j} (s - t_{j-1}) (T - t_{i-1})^{-\alpha} (T - s)^{\alpha-2} (T - t_{j-1})^{\alpha-1} \\
& \left( (t - t_{i-1}) (T - t)^{-\alpha-1} + (t - t_{i-1}) (T - t)^{\alpha-2} \right) dsdt \\
\leq\ & C \sum_{i<j} \int_{t_{i-1}}^{t_i} \int_{t_{j-1}}^{t_j} (s - t_{j-1}) (T - t_{i-1})^{-\alpha} (T - s)^{\alpha-2} \\
& (T - t_{j-1})^{\alpha-1} (t - t_{i-1}) (T - t)^{-\alpha-1} dsdt \\
& + C \sum_{i<j} \int_{t_{i-1}}^{t_i} \int_{t_{j-1}}^{t_j} (s - t_{j-1}) (T - t_{i-1})^{-\alpha} (T - s)^{\alpha-2} \\
& (T - t_{j-1})^{\alpha-1} (t - t_{i-1}) (T - t)^{\alpha-2} dsdt \\
\leq\ & C \sum_{i<j} \int_{t_{i-1}}^{t_i} \int_{t_{j-1}}^{t_j} (s - t_{j-1}) (T - s)^{-2} (t - t_{i-1}) (T - t)^{-2} dsdt \\
& + C \sum_{i<j} \int_{t_{i-1}}^{t_i} \int_{t_{j-1}}^{t_j} (s - t_{j-1}) (T - s)^{\alpha-2} (t - t_{i-1}) (T - t)^{-2} dsdt \\
\leq\ & C \left( \sum_{k=1}^{n} \frac{1}{k} - \log(n+1) \right)^2 + C \left( \Delta_n^{\alpha} - T^{\alpha-1} \Delta_n \right) \left( \sum_{k=1}^{n} \frac{1}{k} - \log(n+1) \right) \\
\leq\ & C \left( \frac{n}{n+1} \right)^2 + C \left( \Delta_n^{\alpha} - T^{\alpha-1} \Delta_n \right) \left( \frac{n}{n+1} \right) \leq C'.
\end{aligned}$$

Finally, when $\frac{1}{2} < \alpha \leq 1$, we have

$$\begin{aligned}
& |V_{3,1}(n)| \\
& \leq C \sum_{i<j} \int_{t_{i-1}}^{t_i} \int_{t_{j-1}}^{t_j} (s - t_{j-1}) (T - s)^{2\alpha-3} (t - t_{i-1}) \times \\
& \qquad\qquad (T - t_{i-1})^{-\alpha} (T - t)^{-\alpha-1} dsdt \\
& \quad + C \sum_{i<j} \int_{t_{i-1}}^{t_i} \int_{t_{j-1}}^{t_j} (s - t_{j-1}) (T - s)^{2\alpha-3} (t - t_{i-1}) (T - t)^{-2} dsdt \\
& \leq C \Delta_n^{-\alpha-1} \sum_{i<j} \int_{t_{i-1}}^{t_i} \int_{t_{j-1}}^{t_j} (s - t_{j-1}) (T - s)^{2\alpha-3} (t - t_{i-1}) (T - t_{i-1})^{-\alpha} dsdt \\
& \quad + C \sum_{i<j} \int_{t_{i-1}}^{t_i} \int_{t_{j-1}}^{t_j} (s - t_{j-1}) (T - s)^{2\alpha-3} (t - t_{i-1}) (T - t)^{-2} dsdt \\
& \leq C \left( 1 - T^{-\alpha} \Delta_n^{\alpha} - T^{2\alpha-2} \Delta_n^{2-2\alpha} + T^{\alpha-2} \Delta_n^{2-\alpha} \right) \\
& \qquad\qquad + C \left( \Delta_n^{2\alpha-1} - T^{2\alpha-2} \Delta_n \right) \left( \frac{n}{n+1} \right)
\end{aligned}$$

$\leq C.$

Similarly, we can use the same way as $V_{3,1}(n)$, we get that

$$|V_{3,2}(n)| \leq C, \ |V_{3,3}(n)| \leq C, \ |V_{3,4}(n)| \leq C.$$

As a consequence,

$$|V_3(n)| \leq C. \tag{3.15}$$

Combining (3.9), (3.10), (3.13), (3.14) and (3.15), we deduce that

$$|b_{2,n}| \leq C. \tag{3.16}$$

Therefore, the facts (3.8) and (3.16) achieve the proof of the desired result. □

**Lemma 3.2.** *Suppose that $\alpha > \frac{1}{2}$. Let $0 < \varepsilon < 1$. Then*

$$\mathbb{P}\left(\lambda_n \left|B_{T_n}\right| > \varepsilon\right) \leq \frac{C}{\varepsilon^2 \log(n+1)}, \tag{3.17}$$

*and*

$$\mathbb{P}\left(\lambda_n \left|B_n - B_{T_n}\right| > \varepsilon\right) \leq \frac{C}{\varepsilon^2 \log^2(n+1)}, \tag{3.18}$$

*where $\lambda_n = \frac{2\alpha+1}{\log(n+1)}$.*

*Proof.* By using (3.6), we have

$$\begin{aligned}
\mathbf{E}\left|B_{T_n}\right|^2 &= \mathbf{E}\left|\int_0^{T_n} \frac{X_t}{(T-t)} dW_t\right|^2 \\
&= \mathbf{E}\left[I_{T_n}\right] \\
&= \frac{(2\alpha-1)\log(n+1) + T^{1-2\alpha}\Delta_n^{2\alpha-1} - 1}{(2\alpha-1)^2}.
\end{aligned}$$

On the other hand by Chebyshev inequality we obtain

$$\begin{aligned}
\mathbb{P}\left(\lambda_n \left|B_{T_n}\right| > \varepsilon\right) &\leq \frac{\lambda_n^2}{\varepsilon^2} \mathbf{E}\left|B_{T_n}\right|^2 \\
&\leq \frac{C}{\varepsilon^2 \log(n+1)}.
\end{aligned}$$

Let us now prove (3.18). By Chebyshev inequality we have

$$\mathbb{P}\left(\lambda_n \left|B_n - B_{T_n}\right| > \frac{\varepsilon(1-\varepsilon)}{3}\right) \leq \frac{\lambda_n^2}{\varepsilon^2} \mathbf{E}\left[\left|B_n - B_{T_n}\right|^2\right].$$

Moreover,

$$\begin{aligned}
\mathbf{E}\left[\left|B_n - B_{T_n}\right|^2\right] &= \mathbf{E}\left(\left|\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{X_{t_{i-1}}}{(T-t_{i-1})} dW_t - \int_0^{T_n} \frac{X_t}{(T-t)} dW_t\right|^2\right) \\
&= \mathbf{E}\left(\left|\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \left(\frac{X_{t_{i-1}}}{(T-t_{i-1})} - \frac{X_t}{(T-t)}\right) dW_t\right|^2\right)
\end{aligned}$$

$$= \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \mathbf{E}\left[\left(\frac{X_{t_{i-1}}}{(T-t_{i-1})} - \frac{X_t}{(T-t)}\right)^2\right] dt.$$

On the other hand,

$$\begin{aligned}
&(2\alpha-1)\mathbf{E}\left[\left(\frac{X_{t_{i-1}}}{(T-t_{i-1})} - \frac{X_t}{(T-t)}\right)^2\right] \\
&= (T-t_{i-1})^{-1} - T^{1-2\alpha}(T-t_{i-1})^{2\alpha-2} + (T-t)^{-1} - T^{1-2\alpha}(T-t)^{2\alpha-2} \\
&-2(T-t_{i-1})^{-\alpha}(T-t)^{\alpha-1} + 2T^{1-2\alpha}(T-t_{i-1})^{\alpha-1}(T-t)^{\alpha-1} \\
&\le (T-t_{i-1})^{-1} + (T-t)^{-1} - 2(T-t_{i-1})^{-\alpha}(T-t)^{\alpha-1} \\
&= (T-t_{i-1})^{-1} - (T-t)^{-1} + 2(T-t)^{-1} - 2(T-t_{i-1})^{-\alpha}(T-t)^{\alpha-1} \\
&= (T-t_{i-1})^{-1} - (T-t)^{-1} + 2(T-t)^{\alpha-1}\left((T-t)^{-\alpha} - (T-t_{i-1})^{-\alpha}\right) \\
&\le (t-t_{i-1})(T-t)^{-2} + 2(T-t)^{\alpha-1}\left(\alpha(t-t_{i-1})(T-t)^{-\alpha-1}\right) \\
&= C(t-t_{i-1})(T-t)^{-2}.
\end{aligned}$$

Thus

$$\begin{aligned}
\mathbf{E}\left[|B_n - B_{T_n}|^2\right] &\le C\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(t-t_{i-1})(T-t)^{-2}\,dt \\
&\le C\left(\frac{n}{n+1}\right).
\end{aligned}$$

Then the desired result is obtained. □

**3.1. Consistency of the approximate maximum likelihood estimators.** The following result establishes the consistency of the approximate maximum likelihood estimators $\hat{\alpha}_n$ and $\bar{\alpha}_n$.

**Theorem 3.3.** *Suppose that $\alpha > \frac{1}{2}$. Then, we have*

$$\hat{\alpha}_n \longrightarrow \alpha \text{ and } \bar{\alpha}_n \longrightarrow \alpha,$$

*in probability as $n \longrightarrow \infty$.*

*Proof.* By using (1.1), we have

$$\begin{aligned}
\alpha - \hat{\alpha}_n &= \frac{\alpha I_n + \sum_{i=1}^n Y_{t_{i-1}}\left(X_{t_i} - X_{t_{i-1}}\right)}{I_n} \\
&= \frac{\alpha I_n + \sum_{i=1}^n Y_{t_{i-1}}\int_{t_{i-1}}^{t_i} dX_t}{I_n} \\
&= \frac{\alpha I_n + \sum_{i=1}^n Y_{t_{i-1}}\int_{t_{i-1}}^{t_i}(-\alpha Y_t dt + dW_t)}{I_n} \\
&= \frac{\alpha\left(I_n - \sum_{i=1}^n \int_{t_{i-1}}^{t_i} Y_{t_{i-1}} Y_t dt\right) + \sum_{i=1}^n Y_{t_{i-1}}\int_{t_{i-1}}^{t_i} dW_t}{I_n} \\
&= \alpha\frac{A_n}{I_n} + \frac{B_n}{I_n}, \qquad (3.19)
\end{aligned}$$

and again by (1.1) together with (3.5) we can write

$$\alpha - \bar{\alpha}_n = \frac{2\alpha - 1}{2}\frac{(I_n - I_{T_n})}{I_n} + \frac{B_{T_n}}{I_n}. \tag{3.20}$$

Let $0 < \varepsilon < 1$ and $\lambda_n = \frac{2\alpha-1}{\log(n+1)}$. We have

$$\begin{aligned}
\mathbb{P}\left(|\alpha - \hat{\alpha}_n| > \varepsilon\right) &= \mathbb{P}\left(\left|\frac{\alpha A_n + B_n}{I_n}\right| > \varepsilon\right) \\
&= \mathbb{P}\left(\left|\frac{\lambda_n\left(\alpha A_n + B_n\right)}{\lambda_n I_n}\right|\right) \\
&\leq \mathbb{P}\left(|\lambda_n\left(\alpha A_n + B_n\right)| > \varepsilon(1-\varepsilon)\right) + \mathbb{P}\left(|\lambda_n I_n - 1| > \varepsilon\right) \\
&:= J_n + K_n.
\end{aligned}$$

We begin by studying the term $J_n$. We have,

$$\begin{aligned}
J_n &= \mathbb{P}\left(|\lambda_n\left(\alpha A_n + B_n\right)| > \varepsilon(1-\varepsilon)\right) \\
&= \mathbb{P}\left(|\lambda_n\left(\alpha A_n + (B_n - B_{T_n}) + B_{T_n}\right)| > \varepsilon(1-\varepsilon)\right) \\
&\leq \mathbb{P}\left(\alpha\lambda_n\left|A_n\right| > \frac{\varepsilon(1-\varepsilon)}{3}\right) + \mathbb{P}\left(\lambda_n\left|B_n - B_{T_n}\right| > \frac{\varepsilon(1-\varepsilon)}{3}\right) \\
&\quad + \mathbb{P}\left(\lambda_n\left|B_{T_n}\right| > \frac{\varepsilon(1-\varepsilon)}{3}\right).
\end{aligned}$$

Proceeding similar to the estimation of $\mathbf{E}\left[|(I_n - I_{T_n})|^2\right]$, we deduce that

$$\mathbb{P}\left(\alpha\lambda_n\left|A_n\right| > \frac{\varepsilon(1-\varepsilon)}{3}\right) \leq \frac{C}{\log^2(n+1)\left(\varepsilon(1-\varepsilon)\right)^2}. \tag{3.21}$$

Therefore, from Lemma 3.2, we obtain

$$J_n \leq \frac{C}{\left(\varepsilon(1-\varepsilon)\right)^2\log(n+1)}. \tag{3.22}$$

For the term $K_n$, it follows from Lemma 3.1 that

$$\begin{aligned}
K_n &= \mathbb{P}\left(|\lambda_n\left(I_n - I_{T_n}\right) + \lambda_n I_{T_n} - 1| > \varepsilon\right) \\
&\leq \mathbb{P}\left(|\lambda_n\left(I_n - I_{T_n}\right)| > \frac{\varepsilon}{2}\right) + \mathbb{P}\left(|\lambda_n I_{T_n} - 1| > \frac{\varepsilon}{2}\right) \\
&\leq \frac{C}{\varepsilon^2\log(n+1)}. \qquad (3.23)
\end{aligned}$$

Combining (3.22) and (3.23), we deduce that $\hat{\alpha}_n$ converges in probability to $\alpha$ as $n \to \infty$. The proof of the consistency of $\bar{\alpha}_n$ is quite similar to the proof above. Thus the desired result is obtained. □

**3.2. Rate of convergence of the approximate maximum likelihood estimators.** In order to investigate the Berry-Esseen-type bounds in the Kolmogorov distance for the approximate maximum likelihood estimators $\hat{\alpha}_n$ and $\bar{\alpha}_n$, we will need the following lemmas.

**Lemma 3.4** ([10], page 78)**.** *Let $f$ and $g$ be two real-valued random variables with $g \neq 0$ $\mathbb{P}-a.s.$ Then, for any $\epsilon > 0$,*

$$d_{kol}\left(\frac{f}{g}, Z\right) \leq d_{kol}(f, Z) + \mathbb{P}(|g-1| > \epsilon) + \epsilon,$$

*where $Z \sim \mathcal{N}(0,1)$.*

**Lemma 3.5** ([1], page 280)**.** *Let $f$ and $g$ be two real-valued random variables. Then, for any $\epsilon > 0$,*

$$d_{kol}(f+g, Z) \leq d_{kol}(f, Z) + \mathbb{P}(|g| > \epsilon) + \frac{\epsilon}{\sqrt{2\pi}},$$

*where $Z \sim \mathcal{N}(0,1)$.*

In the next Theorem we give an explicit bounds for the Kolmogorov distance in the central limit theorem for the estimators $\hat{\alpha}_n$ and $\bar{\alpha}_n$.

**Theorem 3.6.** *Let $0 < \varepsilon < 1$ and $\alpha > \frac{1}{2}$. Then there exist a positive constant $C$, depending only on $\alpha$ and $T$, such that*

$$\sup_{z\in\mathbb{R}} |\mathbb{P}(\beta_n(\alpha - \hat{\alpha}_n) \leq z) - \mathbb{P}(Z \leq z)| \leq \frac{C}{\sqrt{\log(n+1)}} + \frac{C}{\varepsilon^2 \log(n+1)} + 3\varepsilon, \tag{3.24}$$

*and*

$$\sup_{z\in\mathbb{R}} |\mathbb{P}(\beta_n(\alpha - \bar{\alpha}_n) \leq z) - \mathbb{P}(Z \leq z)| \leq \frac{C}{\sqrt{\log(n+1)}} + \frac{C}{\varepsilon^2 \log(n+1)} + 2\varepsilon, \tag{3.25}$$

*where $\beta_n = \sqrt{\frac{\log(n+1)}{2\alpha-1}}$.*

*Proof.* Using (3.19) and Lemma 3.4, we obtain

$$\begin{aligned}
&\sup_{z\in\mathbb{R}}\left|\mathbb{P}\left(\sqrt{\frac{\log(n+1)}{2\alpha-1}}(\alpha - \hat{\alpha}_n) \leq z\right) - \mathbb{P}(Z \leq z)\right| \\
&= \sup_{z\in\mathbb{R}}\left|\mathbb{P}\left(\sqrt{\frac{\log(n+1)}{2\alpha-1}}\left(\frac{\alpha A_n + B_n}{I_n}\right) \leq z\right) - \mathbb{P}(Z \leq z)\right| \\
&= \sup_{z\in\mathbb{R}}\left|\mathbb{P}\left(\sqrt{\frac{2\alpha-1}{\log(n+1)}}(\alpha A_n + B_n) \leq z\right) - \mathbb{P}(Z \leq z)\right| \\
&+ \mathbb{P}\left(\left|\frac{2\alpha-1}{\log(n+1)} I_n - 1\right| > \varepsilon\right) + \varepsilon \\
&:= K_1 + K_2 + \varepsilon.
\end{aligned}$$

On the other hand, from Lemma 3.5, we have

$$K_1 = \sup_{z\in\mathbb{R}}\left|\mathbb{P}\left(\sqrt{\frac{2\alpha-1}{\log(n+1)}} B_n \leq z\right) - \mathbb{P}(Z \leq z)\right|$$

$$
\begin{aligned}
& +\mathbb{P}\left(\left|\alpha\sqrt{\frac{2\alpha-1}{\log(n+1)}}A_n\right|>\varepsilon\right)+\varepsilon \\
:= \quad & L_1+L_2+\varepsilon.
\end{aligned}
$$

Again using Lemma 3.5, we obtain

$$
\begin{aligned}
L_1 &= \sup_{z\in\mathbb{R}}\left|\mathbb{P}\left(\sqrt{\frac{2\alpha-1}{\log(n+1)}}\left(B_n-B_{T_n}+B_{T_n}\right)\le z\right)-\mathbb{P}\left(Z\le z\right)\right| \\
&= \sup_{z\in\mathbb{R}}\left|\mathbb{P}\left(\sqrt{\frac{2\alpha-1}{\log(n+1)}}B_{T_n}\le z\right)-\mathbb{P}\left(Z\le z\right)\right|+ \\
&\quad \mathbb{P}\left(\sqrt{\frac{2\alpha-1}{\log(n+1)}}|B_n-B_{T_n}|>\varepsilon\right)+\varepsilon.
\end{aligned}
$$

From Lemma 3.2, we deduce that

$$
\mathbb{P}\left(\sqrt{\frac{2\alpha-1}{\log(n+1)}}|B_n-B_{T_n}|>\varepsilon\right)\le\frac{C}{\varepsilon^2\log(n+1)}. \tag{3.26}
$$

Moreover, it follows from (1.2) and (2.1) that,

$$
\begin{aligned}
&\sqrt{\frac{2\alpha-1}{\log\left(n+1\right)}}B_{T_n} \\
&=\sqrt{\frac{2\alpha-1}{\log\left(n+1\right)}}\int_0^{T_n}\frac{X_s}{T-s}dW_s \\
&=\sqrt{\frac{2\alpha-1}{\log\left(n+1\right)}}\int_0^{T_n}\int_0^{s}\left(T-s\right)^{\alpha-1}\left(T-r\right)^{-\alpha}dW_rdW_s \\
&=\frac{1}{2}\sqrt{\frac{2\alpha-1}{\log\left(n+1\right)}}\int_0^{T_n}\int_0^{T_n}\left(T-s\vee r\right)^{\alpha-1}\left(T-s\wedge r\right)^{-\alpha}dW_rdW_s \\
&=: I_2\left(f_{T_n}\right),
\end{aligned}
$$

where $f_{T_n}$ is a symmetric function defined by

$$
f_{T_n}(u,v)=\frac{1}{2}\sqrt{\frac{2\alpha-1}{\log\left(n+1\right)}}\left(T-u\vee v\right)^{\alpha-1}\left(T-u\wedge v\right)^{-\alpha}\mathbb{1}_{[0,T_n]^2}(u,v).
$$

Since the function $f_{T_n}$ is symmetric, we have

$$
\begin{aligned}
\|f_{T_n}\|^2_{\mathfrak{H}^{\otimes 2}} &= \frac{2\alpha-1}{4\log\left(n+1\right)}\int_{[0,T_n]^2}\left(T-x\vee y\right)^{2\alpha-2}\left(T-x\wedge y\right)^{-2\alpha}dxdy \\
&= \frac{2\alpha-1}{2\log\left(n+1\right)}\int_0^{T_n}dy\int_0^{y}\left(T-y\right)^{2\alpha-2}\left(T-x\right)^{-2\alpha}dx \\
&= \frac{1}{2\log\left(n+1\right)}\int_0^{T_n}\left(\left(T-y\right)^{-1}-T^{-2\alpha+1}\left(T-y\right)^{2\alpha-2}\right)dy
\end{aligned}
$$

$$= \frac{1}{2\log(n+1)}\left(\log(n+1) + \frac{(\Delta_n/T)^{2\alpha-1}}{(2\alpha-1)} - \frac{1}{(2\alpha-1)}\right).$$

Thus,

$$2\|f_{T_n}\|^2_{\mathfrak{H}^{\otimes 2}} - 1 = \frac{(\Delta_n/T)^{2\alpha-1} - 1}{(2\alpha-1)\log(n+1)},$$

hence

$$\left|2\|f_{T_n}\|^2_{\mathfrak{H}^{\otimes 2}} - 1\right| \leq \frac{C}{\log(n+1)}.$$

Moreover, from [6], we have

$$\|f_{T_n} \otimes_1 f_{T_n}\|^2_{\mathfrak{H}^{\otimes 2}} \leq \frac{C}{\log(n+1)}.$$

Consequently, by combining (2.4) and (2.7), we obtain

$$\sup_{z\in\mathbb{R}}\left|\mathbb{P}\left(\sqrt{\frac{2\alpha-1}{\log(n+1)}}B_{T_n} \leq z\right) - \mathbb{P}(Z \leq z)\right| \leq \frac{C}{\sqrt{\log(n+1)}}. \tag{3.27}$$

For the term $L_2$, by using (3.21), we deduce that

$$\mathbb{P}\left(\left|\alpha\sqrt{\frac{2\alpha-1}{\log(n+1)}}A_n\right| > \varepsilon\right) \leq \frac{C}{\varepsilon^2\log(n+1)}. \tag{3.28}$$

Finally, from Lemma 3.1, we have

$$K_2 \leq \frac{C}{\varepsilon^2\log(n+1)}. \tag{3.29}$$

Combining (3.26), (3.27), (3.28) and (3.29) we deduce (3.24). The proof of (3.25) is quite similar to the proof above. Thus the desired result is obtained.

□

**Corollary 3.7.** *Let $\frac{1}{6} \leq \beta < \frac{1}{2}$ and $\alpha > \frac{1}{2}$. Then there exist a positive constant $C$, depending only on $\alpha$ and $T$, such that*

$$\sup_{z\in\mathbb{R}}\left|\mathbb{P}\left(\sqrt{\frac{\log(n+1)}{2\alpha-1}}(\alpha - \hat{\alpha}_n) \leq z\right) - \mathbb{P}(Z \leq z)\right| \leq C\log^{\beta-\frac{1}{2}}(n+1),$$

*and*

$$\sup_{z\in\mathbb{R}}\left|\mathbb{P}\left(\sqrt{\frac{\log(n+1)}{2\alpha-1}}(\alpha - \bar{\alpha}_n) \leq z\right) - \mathbb{P}(Z \leq z)\right| \leq C\log^{\beta-\frac{1}{2}}(n+1).$$

**Example 3.8.** Upon choosing $\beta = \frac{1}{6}$ we obtain

$$\sup_{z\in\mathbb{R}}\left|\mathbb{P}\left(\sqrt{\frac{\log(n+1)}{2\alpha-1}}(\alpha - \hat{\alpha}_n) \leq z\right) - \mathbb{P}(Z \leq z)\right| \leq \frac{C}{\log^{\frac{1}{3}}(n+1)},$$

and

$$\sup_{z\in\mathbb{R}}\left|\mathbb{P}\left(\sqrt{\frac{\log(n+1)}{2\alpha-1}}(\alpha - \bar{\alpha}_n) \leq z\right) - \mathbb{P}(Z \leq z)\right| \leq \frac{C}{\log^{\frac{1}{3}}(n+1)}.$$

## References


1. Babu, G. J.; Ghosh, M. and Singh, K: On rates of convergence to normality for $\phi$-mixing process,*Sankhya Ser.* A 40 (1978), 278–293.
2. Barczy M. and Pap G.: Asymptotic behavior of maximum likelihood estimator for time inhomogeneous diffusion processes, *J. Statist. Plan. Infer.* **140** (2010) , no. 6, 1576–1593.
3. Bishwal J. P. N and Pap A.: Rates of convergence of approximate naximum likelihood estimators in the Ornstein-Uhlenbeck process, *Computers and Mathematics with Applications* **42** (2001) , 23–38.
4. Brennan, M. J. and Schwartz, E. S.: Arbitrage in stock index futures, *The Journal of Business* **63(1)** (1990), S7–S31.
5. Es-Sebaiy, K.: Berry-Esséen bounds for the least squares estimator for discretely observed fractional Ornstein-Uhlenbeck processes, *Statistics and Probability Letters* **83** (2013), 2372–2385.
6. Es-Sebaiy, K., Moustaaid, J.: Optimal Berry-Esséen bound for maximum likelihood estimation of the drift parameter in $\alpha$-Brownian bridge, *J. Korean Stat. Soc* (2020).
7. Friedman, A.: *Stochastic Differential Equations*, Vol. I, Academic Press, New York, 1975.
8. Görgens, M. and Thulin, M.: Bias-correction of the maximum likelihood estimator for the $\alpha$-Brownian bridge, *Statistics and Probability Letters* **93** (2014) 78–86.
9. Mansuy, R.: On a one-parameter generalization of the Brownian bridge and associated quadratic functionals, *J. Theoret. Probab* **17** (2004) , no. 4, 1021–1029.
10. Michael, R. and Pfanzagl, J.: The accuracy of the normal approximation for minimum contrast estimate, *Z. Warh. Theor. verw Gebiete* **18** (1971) , 37–84.
11. Nourdin, I. and Peccati, G.: Stein's method on Wiener chaos, *Probability Theory and Related Fields* **145 (1-2)** (2007) 75–118.
12. Nourdin, I. and Peccati, G.: *Normal Approximations with Malliavin Calculus : From Stein's Method to Universality*, Cambridge Tracts in Mathematics 192. Cambridge University Press, Cambridge, 2012.
13. Nualart, D.: *The Malliavin Calculus and Related Topics*, Springer-Verlag, Berlin, 2006.
14. Prakasa Rao, B. L. S.: Statistical Inference from sampled data for stochastic processes, *Contemporary Mathematics* **80** (1988), 249–284.
15. Prakasa Rao, B. L. S.: *Statistical inference for fractional diffusion processes*, Wiley Series in Probability and Statistics, Chichester, John Wiley & Sons, 2010.



Khalifa Es-Sebaiy: Department of Mathematics, Faculty of Science, Kuwait University, Kuwait City, Kuwait
*E-mail address*: khalifa.essebaiy@ku.edu.kw

Jabrane Moustaaid: National School of Applied Sciences, Cadi Ayyad University, Marrakesh, Morocco
*E-mail address*: jabrane.mst@gmail.com

Idir Ouassou: National School of Applied Sciences, Cadi Ayyad University, Marrakesh, Morocco
*E-mail address*: i.ouassou@uca.ma