Communications on Stochastic Analysis

Volume 2 | Number 3 Article 7

12-1-2008

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Communications on Stochastic Analysis
Vol. 2, No. 3 (2008) 459-467

Serials Publications
www.serialspublications.com

A DECOMPOSITION OF MULTIPLE WIENER INTEGRALS BY THE LÉVY PROCESS AND LÉVY LAPLACIAN

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Abstract. In this paper, we consider the Lévy Laplacian acting on multiple Wiener integrals by the Lévy process, and give a necessary and sufficient condition for eigenfunctions of the Lévy Laplacian. Moreover we give a decomposition by eigenspaces consisting of multiple Wiener integrals by the Lévy process in terms of the Lévy Laplacian.

1. Introduction

An infinite dimensional Laplacian was introduced by P. Lévy [6]. This Laplacian was introduced into the framework of white noise analysis by T. Hida [1] and has been studied by many authors from various aspects. In the papers [8] and [9], the Laplacian acting on Gaussian and Poisson white noise functionals has eigenfunctions. The purpose of this paper is to extend this result and obtain a necessary and sufficient condition for eigenfunctions of the Lévy Laplacian acting on some class of Lévy white noise functionals. Moreover, we give a decomposition by eigenspaces consisting of multiple Wiener integrals by the Lévy process in terms of the Lévy Laplacian.

More precisely, in this paper, we consider the Lévy Laplacian acting on multiple Wiener integrals by the Lévy process $X = \{X(t)|t \in \mathbf{R}\}$ which the characteristic function is given by

$$E[e^{irX(t)}] = \exp\{tf_X(r)\}, \ r, t \in \mathbf{R},$$
$$f_X(r) = i\mu r - \frac{\sigma^2}{2}r^2 + \int_{|u|>0} \left(e^{iru} - 1 - \frac{iru}{1+u^2}\right)\frac{1+u^2}{u^2}d\beta(u),$$

where $\sigma \geq 0, \mu \in \mathbf{R}$ and β is a positive finite measure on $\mathbf{R}$ with $\beta(\{0\}) = \sigma^2$ and $\int_{\mathbf{R}} |u|^n d\beta(u) < +\infty$ for all $n \in \mathbf{N}$. Moreover, we give a necessary and sufficient condition for $\mathcal{U}$-transforms

$$\mathcal{U}[\langle\cdot, f_1\rangle \diamond \cdots \diamond \langle\cdot, f_n\rangle], f_1, \cdots, f_n \in S(\mathbf{R})$$

to be eigenfunctions of the Lévy Laplacian in the two following cases:

$$(T1) \quad \beta = \sigma^2\delta_0$$
$$(T2) \quad \sigma = 0, \beta = b\delta_a + (b - \mu a)\delta_{-a}$$
$$\text{for some } a > 0 \text{ and } b \geq 0 \text{ with } b \geq \mu a,$$

2000 *Mathematics Subject Classification.* 60H40.
Key words and phrases. Lévy Laplacian, Lévy process, decomposition, Wiener integrals.

where $\langle\cdot,\cdot\rangle$ is the canonical bilinear form on $S'(\mathbf{R})\times S(\mathbf{R})$ and $\diamond$ is the Wick product. In the last section, we have the following decomposition:

$$1)\quad (L^2)=\bigoplus_{n=0}^{\infty}W_n(0)\text{ holds if and only if (T1) holds,}$$

$$2)\quad (L^2)=\bigoplus_{n=0}^{\infty}W_n\Big(-\frac{n}{|T|}a^2\Big)\text{ holds if and only if }\sigma=0,\ \beta=\mu a\delta_a,$$

where (L^2) is the space of all quadratic integrable white noise functionals.

2. Preliminaries

Let $E=S(\mathbf{R})$ be the Schwartz space of rapidly decreasing $\mathbf{R}$-valued functions on $\mathbf{R}$ and let E^* be the dual space of E. The canonical bilinear form on $E^*\times E$ is denoted by $\langle\cdot,\cdot\rangle$.

Let $X=\{X(t)|t\in\mathbf{R}\}$ be a Lévy process on a probability space $(\Omega,\mathcal{F},P)$, which the characteristic function is given by

$$E[e^{irX(t)}]=\exp\{tf_X(r)\},\ r,t\in\mathbf{R},$$
$$f_X(r)=i\mu r-\frac{\sigma^2}{2}r^2+\int_{|u|>0}\Big(e^{iru}-1-\frac{iru}{1+u^2}\Big)\frac{1+u^2}{u^2}d\beta(u),$$

where $\sigma\geq 0$,$\mu\in\mathbf{R}$ and β is a positive finite measure on $\mathbf{R}$ with $\beta(\{0\})=\sigma^2$ and $\int_{\mathbf{R}}|u|^n d\beta(u)<+\infty$ for all $n\in\mathbf{N}$.

Set $C(\xi)=\exp\{\int_{\mathbf{R}}f_X(\xi(t))dt\}$, $\xi\in E$. Then by the Bochner-Minlos Theorem, there exists a probability measure Λ on E^* such that

$$\int_{E^*}e^{i\langle x,\xi\rangle}d\Lambda(x)=C(\xi),\ \xi\in E.$$

Let $(L^2)\equiv L^2(E^*,\Lambda)$ be the Hilbert space of $\mathbf{C}$-valued square-integrable functions on (E^*,Λ). We denote the (L^2)-norm by $||\cdot||_0$. The $\mathcal{U}$-transform $\mathcal{U}[\varphi]$ of $\varphi\in(L^2)$ is defined by

$$\mathcal{U}[\varphi](\xi)=C(\xi)^{-1}\int_{E^*}e^{i\langle x,\xi\rangle}\varphi(x)d\Lambda(x),\ \xi\in E$$

and the Wick product $\langle\cdot,f_1\rangle\diamond\cdots\diamond\langle\cdot,f_n\rangle$ of $\langle\cdot,f_j\rangle, j=1,\cdots,n$, is given by

$$\mathcal{U}[\langle\cdot,f_1\rangle\diamond\cdots\diamond\langle\cdot,f_n\rangle]=\mathcal{U}[\langle\cdot,f_1\rangle]\cdots\mathcal{U}[\langle\cdot,f_n\rangle]\ ,\ f_1,\cdots,f_n\in E.$$

Fixing a finite interval T on $\mathbf{R}$, we take an orthonormal basis $\{\zeta_n\}_{n=0}^{\infty}\subset E$ for $L^2(T)$ which is equally dense, uniformly bounded (see [4]) and satisfies the following condition

$$\sup_{N\geq 1}\frac{1}{N}\sum_{n=0}^{N-1}|\zeta_n|_0^2<\infty.$$

Let $\mathcal{D}_L$ denote the set of all $\varphi\in(L^2)$ such that the limit

$$\widetilde{\Delta}_L\mathcal{U}[\varphi](\xi)\equiv\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}(\mathcal{U}\varphi)''(\xi)(\zeta_n,\zeta_n),$$

exists for each $\xi \in E$ and the functional $\widetilde{\Delta}_L \mathcal{U}[\varphi]$ is in $\mathcal{U}[(L^2)]$. The Lévy Laplacian Δ_L on $\mathcal{D}_L$ is defined by

$$\Delta_L \varphi = \mathcal{U}^{-1} \widetilde{\Delta}_L \mathcal{U} \varphi, \ \varphi \in \mathcal{D}_L.$$

3. Conditions for Eigenfunctions of the Lévy Laplacian.

Let $F_{f_1,\cdots,f_n}(\xi) = \mathcal{U}[\langle \cdot, f_1\rangle \diamond \cdots \diamond \langle \cdot, f_n\rangle](\xi)$ with $\mathrm{supp} f_j \subset T, j = 1, \cdots, n$. Then we can calculate

$$\widetilde{\Delta}_L F_f(\xi) = -\frac{1}{|T|} \int_{\mathbf{R}} f(t) \int_{\mathbf{R}} u(1+u^2) e^{i\xi(t)u} d\beta(u) dt. \tag{3.1}$$

Proposition 3.1. *The functionals $F_{f_1,\cdots,f_n}$ for all $f_1, \cdots, f_n \in E$ are eigenfunctions of $\widetilde{\Delta}_L$ if and only if there exists $C \in \mathbf{R}$ such that the following equalities hold:*

$$\begin{aligned}
&(P1) \quad -\frac{n}{|T|} \int_{\mathbf{R}} u(1+u^2) d\beta(u) = C\Big\{\mu + \int_{\mathbf{R}} u d\beta(u)\Big\},\\
&(P2) \quad -\frac{n}{|T|} \int_{\mathbf{R}} u^2(1+u^2) d\beta(u) = C\Big\{\sigma^2 + \int_{\mathbf{R}} (1+u^2) d\beta(u)\Big\},\\
&(P3) \quad -\frac{n}{|T|} \int_{\mathbf{R}} u^{k+1}(1+u^2) d\beta(u) = C\Big\{\int_{\mathbf{R}} u^{k-1}(1+u^2) d\beta(u)\Big\}
\end{aligned}$$

for any $k \geq 2$.

Proof. Let $F_{f_1,\cdots,f_n}$ be an eigenfunction of $\widetilde{\Delta}_L$. Then there exists $C \in \mathbf{R}$ such that $\widetilde{\Delta}_L F_{f_1,\cdots,f_n} = C F_{f_1,\cdots,f_n}$. By (3.1), we can check that

$$\begin{aligned}
\widetilde{\Delta}_L F_{f_1,\cdots,f_n} &= \sum_{j=1}^n \mathrm{F}_{f_1}(\xi) \cdots \widetilde{\Delta}_L \mathrm{F}_{f_j}(\xi) \cdots \mathrm{F}_{f_n}(\xi)\\
&= -\frac{1}{|T|} \sum_{j=1}^n \prod_{1\leq k\leq n, k\neq j} \mathrm{F}_{f_k}(\xi) \int_{\mathbf{R}} f_j(t) \int_{\mathbf{R}} u(1+u^2) e^{i\xi(t)u} d\beta(u) dt\\
&= -\frac{1}{|T|} \sum_{j=1}^n \prod_{1\leq k\leq n, k\neq j} \mathrm{F}_{f_k}(\xi) \Big\{\sum_{l=0}^\infty \frac{i^l}{l!} \int_{\mathbf{R}} f_j(t) \xi(t)^l \int_{\mathbf{R}} u^{l+1}(1+u^2) d\beta(u) dt\Big\}
\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}
C F_{f_1,\cdots,f_n}(\xi) &= C F_{f_1}(\xi) \cdots F_{f_n}(\xi)\\
&= \frac{C}{n} \sum_{j=1}^n F_{f_j}(\xi) \prod_{1\leq k\leq n, k\neq j} \mathrm{F}_{f_k}(\xi).
\end{aligned} \tag{3.3}$$

Since

$$
\begin{aligned}
F_f(\xi) &= \mu \int_{\mathbf{R}} f(t)dt + i\sigma^2 \int_{\mathbf{R}} f(t)\xi(t)dt \\
&\quad + \int_{\mathbf{R}} f(t) \int_{|u|>0} \left\{ e^{i\xi(t)u} - \frac{1}{1+u^2} \right\} \frac{1+u^2}{u} d\beta(u)dt \\
&= \left\{ \mu + \int_{\mathbf{R}} ud\beta(u) \right\} \int_{\mathbf{R}} f(t)dt \\
&\quad + i\left\{ \sigma^2 + \int_{\mathbf{R}} (1+u^2)d\beta(u) \right\} \int_{\mathbf{R}} f(t)\xi(t)dt \\
&\quad + \sum_{l=2}^{\infty} \frac{i^l}{l!} \left\{ \int_{\mathbf{R}} u^{l-1}(1+u^2)d\beta(u) \right\} \int_{\mathbf{R}} f(t)\xi(t)^l dt, \qquad (3.4)
\end{aligned}
$$

where $f \in E$, by (3.2),(3.3) and (3.4), we have (P1), (P2) and (P3). Conversely, if there exists $C \in \mathbf{R}$ such that (P1), (P2) and (P3) hold, then we can check $\widetilde{\Delta}_L F_{f_1,\cdots,f_n} = CF_{f_1,\cdots,f_n}$ by the above calculations (3.2), (3.3) and (3.4). □

Theorem 3.2. *The functionals $F_{f_1,\cdots,f_n}$ for all $f_1,\cdots,f_n \in E$ are eigenfunctions of $\widetilde{\Delta}_L$ if and only if either of the following equality holds:*

$(T1)$ $\beta = \sigma^2\delta_0$

$(T2)$ $\sigma = 0, \beta = b\delta_a + (b-\mu a)\delta_{-a}$

for some $a > 0$ and $b \geq 0$ with $b \geq \mu a$

Proof. If the functionals $F_{f_1,\cdots,f_n}$ for all $f_1,\cdots,f_n \in E$ are eigenfunctions of $\widetilde{\Delta}_L$, then (P1), (P2) and (P3) hold. Since

$$
\int_{\mathbf{R}} u^{k-1}\left(u^2 + \frac{|T|}{n}C\right)\left(1+u^2\right)d\beta(u) = 0\ ,\ k \geq 2
$$

from (P3), we have

$$
\begin{aligned}
&\int_{\mathbf{R}} u^2\left(u^2 + \frac{|T|}{n}C\right)^2\left(1+u^2\right)d\beta(u) \\
&= \int_{\mathbf{R}} u^4\left(u^2 + \frac{|T|}{n}C\right)\left(1+u^2\right)d\beta(u) \\
&\quad + \frac{|T|}{n}C\int_{\mathbf{R}} u^2\left(u^2 + \frac{|T|}{n}C\right)\left(1+u^2\right)d\beta(u) \\
&= 0. \qquad (3.5)
\end{aligned}
$$

In the case of $C \geq 0$, by (3.5) we have

$$\beta(\mathbf{R} - \{0\}) = 0$$

and therefore β is expressed by

$$\beta = c\delta_0\ ,\ c \geq 0.$$

Since $\beta(\{0\}) = \sigma^2$, we get

$$c = \int_{\mathbf{R}} 1_{\{0\}}(u)d\beta(u) = \sigma^2.$$

Hence we have (T1). In the case of $C < 0$, by (3.5) we have

$$\beta(\mathbf{R} - \{0, a, -a\}) = 0,$$

where $a = \sqrt{-\frac{|T|C}{n}}$. This means that β is expressed by

$$\beta = c\delta_0 + b\delta_a + d\delta_{-a}\ ,\ \ b, c, d \geq 0.$$

Since $\beta(\{0\}) = \sigma^2$, we get

$$c = \int_{\mathbf{R}} 1_{\{0\}}(u) d\beta(u) = \sigma^2.$$

By (P2) we have

$$(b+d)a^2(1+a^2) = a^2\{2\sigma^2 + (b+d)(1+a^2)\}.$$

Hence

$$\sigma = 0.$$

Moreover, by (P1) we have

$$(b-d)a(1+a^2) = a^2\{\mu + (b-d)a\},$$

and hence

$$d = b - \mu a.$$

Then we have (T2). If β is given by the form (T1), we have

$$-\frac{n}{|T|}\int_{\mathbf{R}} u^k(1+u^2)d\beta(u) = 0,\ \ k \in \mathbf{N}.$$

Therefore (P1), (P2) and (P3) hold for arbitrary constant C if $\mu = 0$ and $\sigma = 0$, and hold by setting $C = 0$ if otherwise. By Proposition 3.1, the function $F_{f_1,\cdots,f_n}$ is an eigenfunction of $\widetilde{\Delta}_L$. If β is given by the form (T2), setting $C = -\frac{n}{|T|}a^2$, we have

$$\begin{aligned}
-\frac{n}{|T|}\int_{\mathbf{R}} u(1+u^2)d\beta(u) &= -\frac{n}{|T|}\mu a^2(1+a^2),\\
C\left\{\mu + \int_{\mathbf{R}} u d\beta(u)\right\} &= -\frac{n}{|T|}a^2(\mu + \mu a^2),\\
-\frac{n}{|T|}\int_{\mathbf{R}} u^2(1+u^2)d\beta(u) &= -\frac{n}{|T|}(2b - \mu a)a^2(1+a^2),\\
C\left\{\sigma^2 + \int_{\mathbf{R}} (1+u^2)d\beta(u)\right\} &= -\frac{n}{|T|}a^2(2b - \mu a)(1+a^2),\\
-\frac{n}{|T|}\int_{\mathbf{R}} u^{k+1}(1+u^2)d\beta(u) &= -\frac{n}{|T|}\{b + (-1)^{k+1}(b - \mu a)\}a^{k+1}(1+a^2),
\end{aligned}$$

and

$$C\left\{\int_{\mathbf{R}} u^{k-1}(1+u^2)d\beta(u)\right\} = -\frac{n}{|T|}a^2\{b + (-1)^{k-1}(b - \mu a)\}a^{k-1}(1+a^2),$$

where $k \geq 2$. Hence (P1), (P2) and (P3) hold. By Proposition 3.1, the function $F_{f_1,\cdots,f_n}$ is an eigenfunction of $\widetilde{\Delta}_L$. $\square$

Example 3.3. Standard Gaussian white noise measure

$$\mu = 0 \ , \ \beta = \sigma^2 \delta_0.$$

By Theorem 3.2, the function $F_{f_1,\cdots,f_n}$ is an eigenfunction of $\widetilde{\Delta}_L$ with $C = 0$.

Example 3.4. Poisson white noise measure

$$\mu = \frac{1}{2} \ , \ \beta = \frac{1}{2}\delta_1.$$

By Theorem 3.2, the function $F_{f_1,\cdots,f_n}$ is an eigenfunction of $\widetilde{\Delta}_L$ with $C = -\frac{n}{|T|}$.

Example 3.5. Gamma white noise measure

$$\mu = \int_0^\infty \frac{e^{-u}}{1+u^2} du \ , \ \beta(E) = \int_{E\cap[0,+\infty)} \frac{ue^{-u}}{1+u^2} du.$$

Since

$$\int_{\mathbf{R}} u^k (1+u^2) d\beta(u) = (2k)! \qquad (k \in \mathbf{N}),$$

by Proposition 3.1, there exist $f_1, \cdots, f_n \in E$ such that $F_{f_1,\cdots,f_n}$ is not an eigenfunction of $\widetilde{\Delta}_L$.

4. Decomposition by Eigenspaces of (L^2)

Let $K_0 = \mathbf{C}$, $K_n = LS\{\langle \cdot, f_1 \rangle \diamond \cdots \diamond \langle \cdot, f_n \rangle | f_1, \cdots, f_n \in E\}$ for each $n \in \mathbf{N}$. Define a space $\overline{K_n}$ by the completion of K_n in (L^2) with respect to $|| \cdot ||_0$. Set $\mathbf{K} = \bigoplus_{n=0}^{\infty} \overline{K_n}$.

Proposition 4.1. *(see [5]).* $\mathbf{K} = (L^2)$ *if and only if* $\beta = c\delta_a$ *for some* $c > 0$ *and* $a \in \mathbf{R}$.

Let $M = \{\varphi = \sum_{n=0}^{\infty} \varphi_n \in \mathbf{K} | \varphi_n \in K_n, \sum_{n=0}^{\infty} n^2 \|\varphi_n\|_0^2 < \infty\}$, and an operator $\overline{\Delta}_L$ on M is defined by

- $\overline{\Delta}_L \varphi = \Delta_L \varphi$ if $\varphi \in K_n$.
- $\overline{\Delta}_L \varphi = \sum_{n=0}^{\infty} \overline{\Delta}_L \varphi_n$ if $\varphi = \sum_{n=0}^{\infty} \varphi_n \in M$.

We define a norm $\| \cdot \|_M$ on M by

$$\|\varphi\|_M^2 = \sum_{n=0}^{\infty} n^2 \|\varphi_n\|_0^2, \ \varphi = \sum_{n=0}^{\infty} \varphi_n \in M.$$

Then the following Proposition holds.

Proposition 4.2. *If (T1) or (T2) holds, then* $\overline{\Delta}_L$ *is a continuous linear operator on* M *into* (L^2).

Proof. The linearity of $\overline{\Delta}_L$ is obvious. Let $\varphi, \phi \in M$. Then φ and ϕ are represented in the forms:

$$\varphi = \sum_{n=0}^{\infty} \varphi_n \ , \ \phi = \sum_{n=0}^{\infty} \phi_n \ , \ \varphi_n, \phi_n \in K_n.$$

Since (T1) or (T2) holds, there exists $C \in \mathbf{R}$ such that the following hold,

$$\begin{aligned} \|\overline{\Delta}_L \varphi - \overline{\Delta}_L \phi\|_0^2 &= \|\overline{\Delta}_L(\varphi - \phi)\|_0^2 \\ &= \|\sum_{n=0}^{\infty} \overline{\Delta}_L(\varphi_n - \phi_n)\|_0^2 \\ &= \|\sum_{n=0}^{\infty} nC(\varphi_n - \phi_n)\|_0^2 \\ &\leq C^2 \sum_{n=0}^{\infty} n^2 \|(\varphi_n - \phi_n)\|_0^2 \\ &= C^2 \|\varphi - \phi\|_M. \end{aligned}$$

Therefore, $\overline{\Delta}_L$ is a continuous linear operator on M into (L^2). □

Since the operator Δ_L is continuous on K_n, then $\overline{\Delta}_L$ can be extended to a continuous linear operator defined on $\overline{K_n}$. The extension is also denoted by $\overline{\Delta}_L$.

Proposition 4.3. *Let $n \in \mathbf{N}$. Then any φ in $\overline{K_n}$ is an eigenfunction of $\overline{\Delta}_L$ if and only if either of (T1) and (T2) holds.*

Proof. Let $n \in \mathbf{N}$. For each $m \in \mathbf{N}$ and $f_1, \cdots, f_m \in E$, take $\varphi_{f_1,\cdots,f_m}$ to be $\varphi_{f_1,\cdots,f_m} = \langle\cdot, f_1\rangle \diamond \cdots \diamond \langle\cdot, f_m\rangle$ in K_m, and put $\mathrm{F}_{f_1,\cdots,f_m} = \mathcal{U}[\varphi_{f_1,\cdots,f_m}]$. Since

$$\mathrm{F}_{f_1,\cdots,f_n}(\xi) = \langle\xi, f_1\rangle \cdots \langle\xi, f_n\rangle = \mathrm{F}_{f_1}(\xi) \cdots \mathrm{F}_{f_n}(\xi),$$

we have

$$\widetilde{\Delta}_L \mathrm{F}_{f_1,\cdots,f_n}(\xi) = \sum_{k=1}^{n} \mathrm{F}_{f_1}(\xi) \cdots \widetilde{\Delta}_L \mathrm{F}_{f_k}(\xi) \cdots \mathrm{F}_{f_n}(\xi).$$

Any element φ in K_n is expressed by

$$\varphi = \sum_{j=1}^{N} \varphi_{f_{j,1},\cdots,f_{j,n}}.$$

(1) Let $\mathrm{F} = \mathcal{U}[\varphi]$ and $C \in \mathbf{R}$. Then we have

$$\begin{aligned} & C\mathrm{F}(\xi) - \widetilde{\Delta}_L \mathrm{F}(\xi) \\ &= \sum_{j=1}^{N} \left\{ C\mathrm{F}_{f_{j,1},\cdots,f_{j,n}}(\xi) - \widetilde{\Delta}_L \mathrm{F}_{f_{j,1},\cdots,f_{j,n}}(\xi) \right\} \\ &= \sum_{j=1}^{N} \sum_{k=1}^{n} \left\{ \frac{C}{n} \mathrm{F}_{f_{j,k}}(\xi) - \widetilde{\Delta}_L \mathrm{F}_{f_{j,k}}(\xi) \right\} \prod_{1 \leq l \leq n, l \neq k} \mathrm{F}_{f_l}(\xi). \end{aligned}$$

Since $f_{j,k}$, $1 \leq j \leq N, 1 \leq k \leq n$ are arbitary, all F in $\mathcal{U}[K_n]$ are eigenfunctions of $\widetilde{\Delta}_L$ with same eigenvalue if and only if (P1),(P2) and (P3) hold. Hence by Theorem 3.2, (T1) or (T2) holds. By definition of $\overline{\Delta}_L$, all φ in K_n are eigenfunctions of $\overline{\Delta}_L$ with same eigenvalue if and only if (T1) or (T2) holds.

(2) In case that all φ in $\overline{K_n}$ are eigenfunctions of $\overline{\Delta}_L$ with same eigenvalue, since $K_n \subset \overline{K_n}$, (T1) or (T2) holds.

(3) In case that (T1) or (T2) holds, observe that

$$\overline{K_n} = K_n^b := LS\{\langle \cdot, \zeta_{k_1}\rangle \diamond \cdots \diamond \langle \cdot, \zeta_{k_n}\rangle | k_1, \cdots, k_n \in \mathbf{N}\}.$$

Then, for any $\varphi \in \overline{K_n}$, there exists $\{\varphi_j\} \subset K_n^b$ such that $\varphi_j \to \varphi$ as $j \to \infty$ in $\overline{K_n}$. Since φ_j's can be represented by

$$\varphi_j = \sum_{k_1, \cdots, k_n = 0}^{\infty} a^j_{k_1, \cdots, k_n} \langle \cdot, \zeta_{k_1}\rangle \diamond \cdots \diamond \langle \cdot, \zeta_{k_n}\rangle,$$

we have the following equalities:

$$\begin{aligned}
\overline{\Delta}_L \varphi &= \lim_{j\to\infty} \overline{\Delta}_L \varphi_j \\
&= \lim_{j\to\infty} \sum_{k_1, \cdots, k_n = 0}^{\infty} a^j_{k_1, \cdots, k_n} \overline{\Delta}_L \langle \cdot, \zeta_{k_1}\rangle \diamond \cdots \diamond \langle \cdot, \zeta_{k_n}\rangle \\
&= \lim_{j\to\infty} \sum_{k_1, \cdots, k_n = 0}^{\infty} a^j_{k_1, \cdots, k_n} \cdot C \langle \cdot, \zeta_{k_1}\rangle \diamond \cdots \diamond \langle \cdot, \zeta_{k_n}\rangle \\
&= C \lim_{j\to\infty} \sum_{k_1, \cdots, k_n = 0}^{\infty} a^j_{k_1, \cdots, k_n} \langle \cdot, \zeta_{k_1}\rangle \diamond \cdots \diamond \langle \cdot, \zeta_{k_n}\rangle \\
&= C\varphi
\end{aligned}$$

for some $C \in \mathbf{R}$. Therefore, all φ in $\overline{K_n}$ are eigenfunctions of $\overline{\Delta}_L$ with same eigenvalue if and only if (T1) or (T2) holds. $\square$

Let $W_n(\lambda) = \{\varphi \in \overline{K_n} | \overline{\Delta}_L \varphi = \lambda \varphi\}$ for any $n \in \mathbf{N} \cup \{0\}$ and $\lambda \in \mathbf{R}$. Then Proposition 4.1 and Proposition 4.3 imply the next theorem.

Theorem 4.4. *We have the following assertions:*

1) *The decomposition* $(L^2) = \bigoplus_{n=0}^{\infty} W_n(0)$ *holds if and only if (T1) holds.*
2) *The decomposition* $(L^2) = \bigoplus_{n=0}^{\infty} W_n\big(-\frac{n}{|T|}a^2\big)$ *holds if and only if* $\sigma = 0$ *and* $\beta = \mu a \delta_a$. *Here* $a \in \mathbf{R}$ *and* $\mu a > 0$.

Acknowledgment. The author would like to express his deep thanks to Professor K. Saitô of Meijo University for his guidance in writing this paper.

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