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A CLASS OF EXTREME X-HARMONIC FUNCTIONS

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ABSTRACT. Salisbury and Verzani introduced a class of martingales for the Brownian superprocess related to conditionings of the process to exit the boundary of a bounded domain in $\mathbb{R}^d$ in a particular way. The corresponding class of functions, denoted $H_{g,h_1,\dots,h_N}$, was generalized by Dynkin to more general superprocesses and shown to be X-harmonic. Salisbury and Verzani conjectured that a certain choice of g and h's would yield minimal functions in the Brownian case. This paper shows that this conjecture is true.

1. Introduction

The measure-valued processes called superprocesses have been well-studied in recent years (e.g. [2], [7], [3], [5]). However, certain areas are still relatively unknown. The Martin Boundary theory of X-harmonic functions, as summarized in [4], is an example. The aim of this paper is to extend the class of known extreme X-harmonic functions for super Brownian motion to include a family of harmonic functions first identified in [12]. This family arises when one conditions the exit measure for the superprocess to behave in certain ways.

Before stating the theorem, we need a number of definitions.

1.1. Superprocesses. For our purposes, a superprocess will be defined as in [4]. That is, let $E \subset \mathbb{R}^d$, $\mathcal{M}(E)$ be the set of finite measures on E and $\mathcal{B}_+(E)$ be the class of all positive Borel functions on E. Let (ξ, Π) be a diffusion in E with generator L and Green and Poisson operators G_D and K_D in D. Suppose that for every open set $D \subset E$ and every $\mu \in \mathcal{M}(E)$ there is a random measure $(X_D, \mathbb{P}_\mu)$ on $\mathbb{R}^d$ such that for every $f \in \mathcal{B}_+$ we have

$$\mathbb{P}_\mu(\exp -\langle f, X_D\rangle) = \exp -\langle V_D(f), \mu\rangle, \tag{1.1}$$

where $\langle f, \nu\rangle$ is the integral of f against ν and the function $u = V_D(f)$ satisfies

$$u + G_D\psi(u) = K_D(f).$$

The family $(X_D, \mathbb{P}_\mu)$ is called an (L, ψ)-superprocess, V_D is the transition operator. Existence is known for a wide class of ψ, in particular $\psi(x) = x^2/2$ which is the Brownian case considered in this paper.

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1.2. ***X*-harmonic functions.** Recall, a function, h, is called L-harmonic if for all $x \in E$ one has $h(x) = \Pi_x(h(\zeta_{\tau_D}))$ for all $D \Subset E$ (D is a bounded domain with closure in E), τ_D, as usual, standing for the exit time from D. In analogy, a function $H : \mathcal{M}(E) \to \mathbb{R}$ is called X-harmonic if for all finite μ supported on a compact set and D a subset of E, $\mathbb{P}_\mu(H(X_D)) = H(\mu)$. In particular, if h is L-harmonic, then $H(\mu) := \langle h, \mu\rangle$ is X-harmonic, as in general one has by the properties of the mean operator that

$$\mathbb{P}_\mu(\langle v, X_D\rangle) = \int \Pi_x(v(\zeta_{\tau_D}))\mu(dx).$$

1.3. The *X*-harmonic functions $H_{g,h_1,\ldots,h_N}$. In the papers [12] and [11] a class of X-harmonic functions was introduced that arose in the study of conditioning the superprocess to behave in a certain way as it exited E. To describe the process we need to review killing and transformations of processes.

1.3.1. *Killed Processes.* Let $g \in \mathcal{B}_+$, the process ξ killed at rate g is a strong Markov process with generator denoted L^g, lifetime denoted by ζ that satisfies

$$\Pi^g_x(\xi_t \in A, \zeta > t) = \Pi_x((\exp - \int_0^t g(\xi_s)ds), \xi_t \in A, \zeta > t), \tag{1.2}$$

where, the ζ on the right hand is for the process under L.

We define the Green and Poisson operators for the killed process as usual

$$\begin{aligned} G^g_D(f)(x) &= \Pi^g_x(\int_0^{\tau_D} f(\xi_t)dt) \qquad (1.3)\\ &= \Pi_x \int_0^{\tau_D} (\exp - \int_0^t g(\xi_s)ds) f(\xi_t)dt,\\ K^g_D(f)(x) &= \Pi^g_x((\exp - \int_0^{\tau_D} g(\xi_s)ds) f(\xi_{\tau_D})). \qquad (1.4)\end{aligned}$$

1.3.2. *Conditioned Processes.* The u- transform is defined for positive, L- harmonic functions as follows. Let τ be a stopping time, then

$$\Pi^u_x(\Phi_\tau(\xi)1_{\zeta>\tau}) = \frac{1}{u(x)}\Pi_x(\Phi_\tau(\xi)u(\xi_\tau)1_{\zeta>\tau}) \tag{1.5}$$

for $\Phi_\tau(\xi) \in \sigma\{\xi_s : s \le \tau\}$. If $0 < u < \infty$ in E then this new process is a diffusion. If u is L-harmonic, it dies on the boundary of D, if u is a potential then the process dies in the interior of D and will satisfy

$$\Pi^u_x[\Phi(\xi)] = \frac{1}{u(x)} \int_0^\infty \Pi_x(\Phi(\xi(\cdot \wedge t)) f(\xi_t)1_{\zeta>t}dt. \tag{1.6}$$

For X-harmonic functions H we can define the H-transform[1] of X_n accordingly. In particular we have

$$\mathbb{P}^H_\mu(F(X_D)) = \frac{1}{H(\mu)}\mathbb{P}_\mu(F(X_D)H(X_D)). \tag{1.7}$$

[1][4] uses exit laws and term this the F-transform.

1.3.3. *The functions* $H_{u,f_1,\dots,f_n}$. The equation $Lu = \psi(u)$ in E is intimately connected to the theory of superprocess. Let $\mathcal{U}$ denote the class of all positive $\mathcal{C}^2$ functions which solve the equation. Let $u \in \mathcal{U}$ be chosen, and assume $f_1, f_2, \dots, f_N$ are positive functions. For a finite set C let $\mathcal{P}_r(C)$ denote the set of all permutations of C with r factors and $\mathcal{P}(C)$ the set of all permutations of C. Then for each subset C of $\{1, 2, \dots, N\}$ define functions

$$v^C := \begin{cases} K_D^{\tilde{u}}(f_i) & C = \{i\}, \\ G_D^{\tilde{u}}(\sum_{r\geq 2} q_r \sum_{\mathcal{P}_r(C)}(v^{C_1} \cdots v^{C_r})) & |C| > 1, \end{cases} \tag{1.8}$$

where $\tilde{u} = \psi^{'}(V_D(u))$ and q_r is defined by

$$q_1 = 1, \quad q_r(x) = (-1)^r \psi_r(V_D(u)) \text{ for } r \geq 2. \tag{1.9}$$

Using $\prod_C$ to indicate the product over all factors of the permuation C, set

$$H_{u,f_1,\cdots,f_N}(\mu) := \exp -\langle u, \mu\rangle \sum_{C\in\mathcal{P}(\{1,2,\dots,N\})} \prod_C \langle v^{C_i}, \mu\rangle := \exp -\langle u, \mu\rangle \tilde{H}(\mu). \tag{1.10}$$

It is shown in [4] that if the f_i are positive solutions to the equation

$$Lv = \psi^{'}(u)v$$

in E then $H_{u,f_1,\cdots,f_N}$ is a X-harmonic for a wide class of ψ. As well, the f_i are harmonic for the process ξ killed at rate $\tilde{u}$ and $v^{\{i\}} = K_D^{\tilde{u}}(f_i) = f_i$.

In [12] the above was shown for $\psi = x^2/2$. In this case, (1.10) simplifies quite a bit. The function u solves $V_D(u) = u$ as it is in $\mathcal{U}$ and $\psi^{'} = x$ so $\tilde{u} = u$. The terms ψ_r in (1.9) involve derivatives, and in this case, $\psi_r = 0$ for $r \geq 2$. Thus, the term for v^C in (1.8) when $|C| > 1$ simplifies to

$$v^C(x) = G_D^u(\sum_{\mathcal{P}_2(C)} v^{C_1}v^{C_2}).$$

In the case when u is the maximal solution in $\mathcal{U}$ and f_i are minimal harmonic for L^g it was conjectured in section 6 of [12] that the functions $H_{u,f_1,\dots,f_N}$ should be minimal.

1.4. Extreme X-harmonic functions. In [4] the notion of extreme X-harmonic functions is developed. In particular, extreme X-harmonic functions may be defined as minimal X-harmonic functions on E. That is, fix a reference point $x_0 \in E$ and let $H(X, x_0)$ denote all non-negative X-harmonic functions in E normalized so that $H(\delta_{x_0}, x_0) = 1$. Let $H_e(X)$ be all non-zero elements of $H(X, x_0)$ with the property that if for any $\tilde{H} \in H(X, x_0)$ that if $\tilde{H} \leq H$ then $\tilde{H} = cH$ for some c. These are the extreme X-harmonic functions. Furthermore, there is a unique representation (due in this context to Evans and Perkins) in terms of these extreme functions

$$H(\mu) = \int_{H_e(X,x_0)} K(\mu, \gamma)\nu^H(d\gamma),$$

where ν is a finite measure supported on the extreme elements.

Now we state the main theorem of this paper:

Theorem 1.1. *Let E be a bounded domain in $\mathbb{R}^d$, L be the generator for Brownian motion in E and $\psi(x) = x^2/2$. Assume*

(1) *The function $g(x)$ is the unique, maximal solution to $L(u) = \psi(u)$ in E with infinite boundary condition.*
(2) *The functions $\{h_i, i = 1, \dots, N\}$ are extreme L^g-harmonic functions.*

Then $H_{g,h_1,\dots,h_N}$ is an extreme X-harmonic function.

Remark 1.2. Clearly from the definition of v^C in (1.8) if the h_i are not minimal, then H can not be minimal. In the case $N = 0$ it is clear that g must be maximal, this is not the case in the theorem though where $N \geq 1$ is assumed. In the course of the proof, it becomes clear why g needs to be the maximal solution.

2. Preliminaries

2.1. Quick sketch of proof. The proof will follow several steps analogous to those used in a proof by Evans ([6]) as outlined in [4]. The goal is to construct a process related to the H-transform of the superprocess. A process in [12] was shown to have an equivalent Laplace transform. This process is extended here to show that and its "backbone" process (similar to the immortal partical of Evans) is related to the H-transform. Then it is shown that only the backbone process contributes to the tail events. Finally, the backbone process is shown to have a trivial tail.

The proof requires the following ingredients.

2.2. An associated Markov chain. Fix $\{D_n\}$ to be any sequence of domains with $D_n \Subset E$, $D_n \subset \bar{D_n} \subset D_{n+1}$ and $\cup D_n = E$. We define transient Markov chains by $\xi_n := \xi_{\tau_{D_n}}$ and $X_n := X_{D_n}$. Let $\mathcal{F}_{\subset D}$ be the σ-algebra generated by $X_{D'}$ when $D^{'} \subset D$, $\mathcal{F}_n := \mathcal{F}_{\subset D_n}$ and $\mathcal{F}_{\supset D}$ be generated by $X_{D''}$ when $D^{''} \supset D$. The latter chain is Markov due to the Markov property of the superprocess, namely if $A \in \mathcal{F}_{\subset D}$ and $B \in \mathcal{F}_{\supset D}$ then

$$\mathbb{P}_\mu(AB) = \mathbb{P}_\mu(A\mathbb{P}_{X_D}(B)). \tag{2.1}$$

2.3. The extended moment formula. The fact that $H_{g,h_1,\dots,h_N}$ is harmonic follows from a moment formula (Theorem 4.1 of [4]) which takes the following form with $C = \{1, 2, \dots, n\}$.

$$\mathbb{P}_\mu[\exp -\langle u, X_D\rangle \prod_{i\in C} \langle f_i, X_D\rangle] = \mathbb{P}_\mu[\exp -\langle u, X_D\rangle] \sum_{\gamma\in\mathcal{P}(C)} \prod_{|\gamma|} \langle v^C \gamma_i, \mu\rangle. \tag{2.2}$$

3. The Superprocess Conditioned to not Charge the Boundary

The class $\mathcal{U}$ is defined as all functions which are solutions to $L(u) = \psi(u)$. If g is in $\mathcal{U}$ then the function $H_g(\mu) = \exp -\langle g, \mu\rangle$ is X-harmonic. Let the H-transform defined by H_g be referenced with a tilde. and the resulting process be denoted $\tilde{X}$.

Then for $D \Subset E$ we have

$$\begin{aligned}\tilde{\mathbb{P}}_\mu(\exp -\langle u, \tilde{X}_D\rangle) &= \frac{1}{H_g(\mu)}\mathbb{P}_\mu(\exp -\langle u, X_D\rangle \exp -\langle g, X_D\rangle) \\ &= \frac{1}{\exp\langle g,\mu\rangle} \exp -\langle V_D(u+g),\mu\rangle, \quad \text{by (1.1)} \\ &= \exp -\langle \tilde{V}_D(u),\mu\rangle,\end{aligned} \tag{3.1}$$

where

$$\tilde{V}_D(u)(x) := (V_D(u+g) - g)(x). \tag{3.2}$$

Lemma 3.1. *If g is maximal and $H_g(\mu) > 0$, then under $\tilde{\mathbb{P}}$ the process $\tilde{X}_n$ is a super process conditioned to not charge the boundary of E.*

Proof. The H_g-transform of X is a strong Markov, measure-valued process. We show that for all μ with $H_g(\mu) > 0$ one has for $\epsilon > 0$ $\lim_n \tilde{\mathbb{P}}_\mu(\langle 1, \tilde{X}_n\rangle > \epsilon) = 0$.

By Chebyshev's inequality we have for a fixed $\epsilon > 0$

$$\begin{aligned}0 &\le \tilde{\mathbb{P}}_\mu[\langle 1, \tilde{X}_n\rangle > \epsilon] \\ &\le \frac{1}{\epsilon}\tilde{\mathbb{P}}_\mu[\langle 1, \tilde{X}_n\rangle] \\ &= \frac{1}{\epsilon H_g(\mu)}\mathbb{P}_\mu[\langle 1, X_n\rangle H_g(X_n)] \\ &= \frac{1}{\epsilon H_g(\mu)} \exp -\langle V_n(g),\mu\rangle\langle K_n^g(1),\mu\rangle \qquad \text{by (2.2)} \\ &\le \frac{\langle K_n^g(1),\mu\rangle}{H_g(\mu)}.\end{aligned}$$

The proof will follow by showing that $K_n^g(1)$ goes to 0 as n goes to ∞. As $g \in \mathcal{U}$, $V_n(g) = g$, it is enough to show

$$\begin{aligned}K_n^g(1)(x) &:= \Pi_x[\exp -\int_0^{\tau_n} \Psi'(V_n(g)(\xi_s))ds] \\ &= \Pi_x[\exp -\int_0^{\tau_n} \Psi'(g(\xi_s))ds] \to 0.\end{aligned} \tag{3.3}$$

By assumption, the function g is the maximal solution in E and the unique solution with infinite boundary condition. From [3] chapter 11, there exists a function with fine trace representation $(\partial^M E, 0)$ which is the minimal σ-moderate solution in $\mathcal{U}$ which blows up on the boundary of E. Because we assume such a solution is unique it is our g. Thus the singular set $\mathrm{SG}(g)$ is $\partial^M E$ which says that for all y in $\partial^M E$ we have

$$\Pi_x^y[\int_0^\zeta \psi'(g)(\xi_s)ds = \infty] = 1. \tag{3.4}$$

In particular, $\Pi_x^y - a.s$ one has

$$\int_0^{\tau_n} \psi'(g(\xi_s))ds \to \infty. \tag{3.5}$$

We need to show this is true for the unconditioned process. We use the characterization of the h-transform as a conditioned process. (That is, as in proposition 2.7 of [1] one has $\Pi_{x_0}[A] = \int_{\partial E} \Pi^z_{x_0}[A] P_{x_0}[X_{\tau_D} \in dz]$ for $\mathcal{F}_{\tau_D}$-measurable A.) As equation (3.5) is true for all $y \in \partial E$ we have with $A_{n,M}$ denoting the event $\{\int_0^{\tau_n} \psi^{'}(g(\xi_s))ds > M\}$ that

$$\begin{aligned}\Pi_x(A_{n,M}) &= \int \Pi_x(A_{n,M} \mid \xi_{\tau_E} = y)\Pi_x(\xi_{\tau_E} \in dy) \\ &= \int \Pi^y_x(A_{n,M})\Pi_x(\xi_{\tau_E} \in dy) \\ &\to \int 1\, \Pi_x(\xi_{\tau_E} \in dy) = 1, \quad \text{ as } n \to \infty.\end{aligned}$$

Thus, for all M we have

$$\lim_n \Pi_x(\int_0^{\tau_n} \Psi'(g(\xi_s))ds > M) = 1,$$

which yields

$$\lim_n \Pi_x(\int_0^{\tau_n} \Psi'(g(\xi_s))ds = \infty) = 1.$$

Consequently equation (3.3) holds. □

4. Labeled Trees

The backbone process will be a labeled tree – a branching diffusion where the particles carry along a certain label that makes the process Markov. This section fixes some notation for labeled trees and then defines the Markov process that will be the backbone process for the conditioned superprocess

4.1. Definition of a labeled tree. We define a labeled tree in terms of an index set which governs the branching, a collection of paths which are the segments of the trees and a consistent collection of labels.

An index set: Let $\mathcal{T} = \{\delta\} \otimes \cup_n \mathbb{N}^n$. For an element $a = (\delta, a_1, a_2, \ldots, a_n)$, define the length $|a| = n$ ($|\delta| = 0$), the truncation operators for $j \le n$ by $t_j(a) = (\delta, a_1, \ldots, a_{n-j})$ if $|a| \ge 1$ and $t(\delta) = \delta$. Set $t(a) := t_1(a)$. A set of tree-indices is a subset $\mathcal{I} \in \mathcal{T}$ subject to the tree-consistency requirement that

$$a \in \mathcal{I} \implies t(a) \in \mathcal{I}. \tag{4.1}$$

For a set of tree indices $\mathcal{I}$ define sisters$(a) := \{b \in \mathcal{I} : t(b) = t(a)\}$.

A set of labels: Let $C \subset \mathbb{N}$ be a set of labels for the tree. A **tree-adapted labeling** is a map $\psi : a \mapsto C^a$ from $\mathcal{I}$ into subsets of C such that

$$C^{t(a)} = \cup_{b \in \text{sisters}(a)} C^b \tag{4.2}$$

and the C^a,C^b are disjoint if a, b are sisters. Denote $\mathcal{T}(C)$ to be the class of all *binary* trees with distinct tree-adapted labelings. such that $\psi(\delta) = C$. Such trees have a recursive decomposition as follows

$$\mathcal{T}(C) = \{\delta, C\} \cup \cup_{\mathcal{P}_2(C)} \mathcal{T}(C_1) \otimes \mathcal{T}(C_2). \tag{4.3}$$

The first term corresponds to the tree which does not branch, the other terms decompose a branching tree into the the trees that made from the first two branches.

A tree based on $\mathcal{I}$: . A continuous tree with a tree-index set $\mathcal{I}$ is defined as a mapping from $\mathcal{I}$ into the space of continuous paths in E with finite lifetime ζ. That is, each path is a pair (w, ζ) with the understanding that $w(\cdot)$ is continuous on $[0, \zeta)$ with a left limit at ζ. We denote the mapping with superscripts: $\phi : a \mapsto (w^a, \zeta^a)$.

We impose a **continuity condition** on the paths:

$$w^a(0) = w^{t(a)}(\zeta^{t(a)}-). \tag{4.4}$$

For a finite index set $\mathcal{I}$, define the **boundary** of $\mathcal{I}$, $\partial\mathcal{I}$, to be those a which are not the image of $t(b)$ for some $b \in \mathcal{I}$ ($\partial\mathcal{I} := \mathcal{I} \setminus \{t(a) : a \in \mathcal{I}\}$).

Define the **history** of a path $y^a(s)$ by

$$y^a(s) = \begin{cases} w^\delta(s) & s < \zeta^\delta \\ w^{t_j(a)}(s - \zeta^{t_{j+1}(a)}) & \end{cases} \tag{4.5}$$

where j is chosen so that

$$\sum_{i=j+1}^{n} \zeta^{t_i(a)} < s \le \sum_{i=j}^{n} \zeta^{t_i(a)}.$$

This is just the branch from the root to the path associated with a.

Isomorphisms: We are really concerned about trees labeled by subsets of C and not the index set. Suppose we have two index sets $\mathcal{I}$ and $\mathcal{I}^{'}$ with maps ϕ and ψ (and $\phi^{'}, \psi^{'}$), and a map $\theta : (I) \mapsto \mathcal{I}^{'}$ such that

$$\phi(a) = \phi^{'}(\theta(a)), \quad \psi(a) = \psi^{'}(\theta(a)),$$

then we say the two labeled trees are isomorphic.

A Forest: A forest will be defined as a finite collection of labeled trees. The labeling sets are assumed to be disjoint. We use the same notation for a tree and a forest, letting context be the guide. We define $|\Upsilon|$ as the number of trees in a forest. If we wish to refer to individual trees within a forest we will use use a superscript. That is, despite the awkwardness, we enumerate a forest by $\Upsilon = \{\Upsilon^1, \ldots, \Upsilon^{|\Upsilon|}\}$.

4.1.1. *Tree functionals.* We define several functionals on a labeled tree.

Tree integrals: Let f be a real-valued real function. Define the tree integral of f as an integral along each segment of the tree:

$$\langle\langle f, \Upsilon \rangle\rangle = \sum_{a \in \mathcal{I}} \int_0^{\zeta^a} f(w^a(s))ds. \tag{4.6}$$

We remark that the tree integral is obviously linear, and by Fubini's theorem satisfies the following for positive functions:

$$\langle\langle \int f(\cdot, t)dt, \Upsilon \rangle\rangle = \int \langle\langle f(\cdot, t), \Upsilon \rangle\rangle dt. \tag{4.7}$$

A Forest Integral: The forest integral uses the same notation and is simply the tree integral over all the trees in the forest:

$$\langle\langle u, \Upsilon\rangle\rangle = \sum_{i=1}^{|\Upsilon|} \langle\langle u, \Upsilon^i\rangle\rangle.$$

The pre-D tree.: Let $D \Subset E$ and define the tree stopped on first exiting D by setting $\mathcal{I}_D = \{a \in \mathcal{I} : \zeta^{t_n(a)} + \cdots + \zeta^{t_1(a)} \leq \tau_D(y^a)\}$ (the paths have not exited D before their last branch), $\bar{w}^a(\cdot)$ for $a \in \mathcal{I}_D$ by $\bar{w}^a = w^a$ if $\tau_D(y^a) = \infty$ and $(\bar{w}^a(\cdot), \tau_D(\bar{w}^a))$ if $\tau_D(y^a) < \infty$. Thus, the paths are always inside D and the boundary paths just exit D. The labels stay the same only restricted to $\mathcal{I}_D$.

We refer to the stopped tree as Υ_D. A stopped forest has a similar definition.

The post-D forest: For a tree Υ and a domain $D \Subset E$ we refer to the post-D forest as the paths after first exiting D. In particular, For each $a \in \partial\mathcal{I}_D$ we have a tree started at $\bar{w}^a(\bar{\zeta}^a-)$ (the $\bar{w}$ are for the stopped tree, the w reference the original tree). The initial segment is $\tilde{w}^\delta(t) = w^a(t+\tau_D(w^a)), 0 \leq t \leq \zeta^a - \tau_D(w^a) := \tilde{\zeta}^\delta$, the other segments are the same as before only reindexed. The new index set for the tree corresponding to $a \in \partial\mathcal{I}$ is $\mathcal{I} \circ \Theta_D := \{t_{|a|}(b) : b \in \mathcal{I}, t_j(b) = a \text{ for some } j \geq 0\}$.

We refer to this post-D forest by the notation $\Upsilon \circ \Theta_D$.

The exit tree: We will be interested in how the tree exits D. For boundary paths in Υ_D we have the exit points $\{x^a := w^a(\zeta^a-) : a \in \partial\mathcal{I}_D\}$ and their corresponding labels. Let $\Upsilon^D = \otimes(x^a, C^a)$ (we use $\Upsilon^{i,D}$ if we wish to refer to tree i in the forest Υ). Clearly, C^a partitions the original label set C. We will refer to the state space $\mathcal{E} = \mathcal{E}_{D,C}$ in several ways as is convenient. In particular,one of these ways:

- as a product of points and disjoint subsets

$$\cup_{r=1}^{|C|} \otimes_{i=1}^r (x_i \times C_i); x_i \in E, C = (C_1, \ldots, C_r),$$

- as a vector of points and a partition of C

$$\cup_{r=1}^{|C|} E^r \times \mathcal{P}_r(C),$$

- or, as an atomic measure on E with mass r and a partition

$$\cup_{r=1}^{|C|} \mathcal{M}_{A,r}(E) \times \mathcal{P}_r(C).$$

We will refer to the pieces of an exit tree as follows: $m \,\raisebox{0pt}{\textbf{.}}\, \Upsilon^D$ will denote the atomic measure $\sum \delta_{x_i}$ formed by the exit points $x_1, x_2, \ldots x_r$ which are denoted by $x \,\textbf{.}\, \Upsilon^D$ and $p \,\textbf{.}\, \Upsilon^D$ refers to the partition of the set C given by Υ^D.

Remark 4.1. Equipped with this notation we remark for future usage the relationships for measurable u, v

$$\langle\langle u, \Upsilon\rangle\rangle = \langle\langle u, \Upsilon_D\rangle\rangle + \langle\langle u, \Upsilon \circ \Theta_D\rangle\rangle \tag{4.8}$$

and if $D \Subset D'$ then

$$\langle v, \Upsilon^{D'} \rangle := \langle v, m \centerdot \Upsilon_{D'} \rangle = \sum_{i=1}^{|p \centerdot \Upsilon^D|} \langle v, m \centerdot (\Upsilon \circ \Theta_D)^{i,D'} \rangle. \tag{4.9}$$

That is, the tree integral breaks up into two pieces depending if the paths have exited D yet, and the exit points are unchanged when the shift is applied.

4.2. Definition of Υ_D, Υ^D. We now construct a random labeled tree that gives the backbone of the conditioned process. This is the same process identified in [12].

4.2.1. *A non-homogeneous branching diffusion.* Let C be a subset of $\{1, 2, \dots, N\}$ and functions $h_i : i \in C$ be $L^{\tilde{g}}$-harmonic. Recalling the definition of v^C based on h in (1.8), we inductively define a branching diffusion under a measure Π_x^C as follows:

The process starts at x and evolves as a $L^{\tilde{g}, v^C}$ particle until its lifetime ζ.

If $|C| = 1$ then this occurs on the boundary of E and our tree is just $\mathcal{I} = \{\delta\}$, w^δ the path up to its lifetime ζ^δ and labeling $C^\delta = C$.

If $|C| > 1$ then the function v^C is a potential as it satisfies

$$v^C = G^g(\sum_{\mathcal{P}_2} (v^{C_1} v^{C_2})(\cdot)).$$

Thus the $L^{\tilde{g}, v^C}$ process will die in the interior of E. Pick a random partitioning of C according to the probability

$$P((C_1, C_2)) = (\frac{v^{C_1} v^{C_2}}{\sum_{\mathcal{P}_2(C)} v^{\tilde{C}_1} v^{\tilde{C}_2}})(\xi_{\zeta-}), \quad (C_1, C_2) \in \mathcal{P}_2(C). \tag{4.10}$$

Then for $i = \{1, 2\}$, let $a_i = (\delta, i)$ and the a_i-particle evolve as a $L^{g, v^{C_i}}$ particle started at $\xi_{\zeta-}$ stopped at its lifetime and labeled by C_i. Again if $|C_i| > 1$ these particles will die in the interior.

Repeat this until eventually there are $|C|$ particles each evolving as a $L^{\tilde{g}, h_i}$ particle which die on the boundary of E.

This process will be continuous Π_x^C a.s. and so will be a labeled tree.

4.2.2. *The branching forest.* We extend the previous definition to a forest. Let C be an initial index set, and $\gamma \in \mathcal{P}(C)$ and points $x_1, x_2, \dots x_{|\gamma|}$ be given. The forest Υ evolving from $x \times \gamma$ is given by evolving $|\gamma|$ independent trees started from the x's. Let $\Pi_{\otimes(x_i \times \gamma_i)}$ be the measure under which Υ evolves. Then one has for all $D \Subset E$ by independence

$$\begin{aligned} &\Pi_{\otimes(x_i \times \gamma_i)}(\exp - \langle\langle u, \Upsilon_D \rangle\rangle \exp - \langle v, \Upsilon^D \rangle) \\ &= \prod_{i=1}^{|\gamma|} \Pi_{x_i}^{\gamma_i}(\exp - \langle\langle u, \Upsilon_D^i \rangle\rangle \exp - \langle v, \Upsilon^{i,D} \rangle). \end{aligned} \tag{4.11}$$

We now show that the branching forest is a Markov process in the following sense.

Lemma 4.2. *Let $D \Subset D' \Subset E$. Then for any initial planting of the forest $x \times \gamma$ we have*

$$\begin{aligned}&\mathit{\Pi}_{\otimes(x_i\times\gamma_i)}[\exp - \langle\langle u, \Upsilon'_D\rangle\rangle \exp - \langle v, \Upsilon^{D'}\rangle] \\ &\quad = \mathit{\Pi}_{\otimes(x_i\times\gamma_i)}(\exp - \langle\langle u, \Upsilon_D\rangle\rangle \mathit{\Pi}_{\Upsilon^D}[\exp - \langle\langle u, \Upsilon'_D\rangle\rangle \exp - \langle v, \Upsilon^{D'}\rangle]).\end{aligned}$$

Proof. We note that it is clear from the construction of Υ under $\mathit{\Pi}$ that given the information contained in Υ^D (the exit points and labels) that the pre- and post-D forests are independent. Thus, from the decompositions in (4.8) and (4.9) we have

$$\begin{aligned}&\mathit{\Pi}_{\otimes(x_i\times\gamma_i)}[\exp - \langle\langle u, \Upsilon_{D'}\rangle\rangle \exp - \langle v, \Upsilon^{D'}\rangle] \\ &\quad = \mathit{\Pi}_{\otimes(x_i\times\gamma_i)}[\exp - \langle\langle u, \Upsilon_D\rangle\rangle \exp - \langle\langle u, (\Upsilon \circ \Theta_D)_{D'}\rangle\rangle \exp - \langle v, (\Upsilon \circ \Theta_D)^{D'}\rangle] \\ &\quad = \mathit{\Pi}_{\otimes(x_i\times\gamma_i)}[\exp - \langle\langle u, \Upsilon_D\rangle\rangle \mathit{\Pi}[\exp - \langle\langle u, (\Upsilon \circ \Theta_D)_{D'}\rangle\rangle \exp - \langle v, (\Upsilon \circ \Theta_D)^{D'}\rangle \\ &\qquad \mid \Upsilon^D]] \\ &\quad = \mathit{\Pi}_{\otimes(x_i\times\gamma_i)}[\exp - \langle\langle u, \Upsilon_D\rangle\rangle \mathit{\Pi}_{\Upsilon^D}[\exp - \langle\langle u, \Upsilon_{D'}\rangle\rangle \exp - \langle v, \Upsilon^{D'}\rangle]].\end{aligned}$$

□

5. Proof of Theorem 1.1

The goal of the proof is to give a tractable representation of the H-transform of X_n. This representation is in terms of a measure, Z_n, and a "backbone", Υ (below). One then shows that the tail information is determined by only the backbone process. This relates the tail of the superprocess to that of a diffusion which allows us to say the tail σ-field is trivial.

5.1. The random points $(Y_D \times \Upsilon^D, Q_x^\gamma)$**.** The following lemma is used to construct a piece of the conditioned process. In particular, it describes a measure Y_D and a backbone process Υ^D.

Lemma 5.1. *Let $D \Subset E$ and μ be a finite measure on D. For each $x \in D$ and $\gamma \subset \{1, 2, \dots, N\}$ there exists a random point $(Y_D \times \Upsilon^D) \in \mathcal{M}(D) \times \mathcal{E}_{D,\gamma}$ and measure Q_x^γ for which*

$$Q_x^\gamma(\exp - \langle u, Y_D\rangle \exp - \langle v, \Upsilon^D\rangle) = \mathit{\Pi}_x^\gamma[\exp - \langle\langle \tilde{V}_D(u), \Upsilon_D\rangle\rangle \exp - \langle v, \Upsilon^D\rangle],$$

for all measurable $u, v \geq 0$.

Proof. Using the results of section 5 of [4] applied to the infinitely divisible measure $\tilde{X}_D$ we have that there exists a canonical measure $\tilde{R} = \tilde{R}_D$ which satisfies

$$\tilde{\mathbb{P}}_\mu(\exp - \langle u, \tilde{X}_D\rangle) = \exp - \int_{\mathcal{M}} (1 - e^{-\langle u, \nu\rangle}) \tilde{R}(\mu, d\nu).$$

Fix a labeled tree $\Upsilon = \Upsilon_D$ stopped on exiting D and define a Poisson random measure on $\mathcal{M}$ (Λ, P^Υ) with intensity

$$\lambda(C) = \langle\langle \tilde{R}_D(\cdot, C), \Upsilon_D\rangle\rangle. \tag{5.1}$$

Then, we can define Y_D by its action on measurable functions:

$$\langle u, Y_D\rangle = \int_{\mathcal{M}} \langle u, \nu\rangle \Lambda(d\nu). \tag{5.2}$$

Now by the definition of a Poisson random measure with intensity λ we have

$$P^{\Upsilon}[\exp - \int_{\mathcal{M}} F(\nu)\Lambda(d\nu)] = \exp - \int_{\mathcal{M}} (1 - e^{-F(\nu)})\lambda(d\nu).$$

In particular, when $F(\nu) = \langle u, \nu\rangle$ this becomes

$$P^{\Upsilon}[\exp - \int_{\mathcal{M}} \langle u, \nu\rangle\Lambda(d\nu)] = \exp - \int_{\mathcal{M}} (1 - e^{-\langle u,\nu\rangle})\lambda(d\nu). \tag{5.3}$$

However, by (3.1) we have

$$\tilde{\mathbb{P}}[\exp - \langle u, \tilde{X}_D\rangle] = \exp - \langle \tilde{V}_D(u), \mu\rangle = \exp - \int_{\mathcal{M}} (1 - e^{-\langle u,\nu\rangle})\tilde{R}(\mu, d\nu)$$

or, for all x

$$\tilde{V}_D(u)(x) = \int_{\mathcal{M}} (1 - e^{-\langle u,\nu\rangle})\tilde{R}(x, d\nu), \tag{5.4}$$

which leads to

$$\begin{aligned} P^{\Upsilon}[e^{-\langle u, Y_D\rangle}] &= P^{\Upsilon}(\exp - \int_{\mathcal{M}} \langle u, \nu\rangle\Lambda(d\nu)) && \text{(by (5.2))}\\ &= \exp - \int_{\mathcal{M}} (1 - e^{-\langle u,\nu\rangle})\lambda(d\nu)) && \text{(by (5.3))}\\ &= \exp - \int_{\mathcal{M}} (1 - e^{-\langle u,\nu\rangle})\langle\langle \tilde{R}(\cdot, d\nu), \Upsilon\rangle\rangle && \text{(by (5.1))}\\ &= \exp - \langle\langle \int_{\mathcal{M}} (1 - e^{-\langle u,\nu\rangle})\tilde{R}(\cdot, d\nu), \Upsilon\rangle\rangle && \text{(by (4.7))}\\ &= \exp - \langle\langle \tilde{V}_D(u), \Upsilon\rangle\rangle. && \text{(by (5.4))} \end{aligned}$$

Finally, define a measure Q_x^{γ} on $\mathcal{M} \times \mathcal{E}$ by randomizing Υ according to Π_x^{γ}:

$$Q_x^{\gamma}(C) = \int \Pi_x^{\gamma}(d\Upsilon)P_x^{\Upsilon}((Y_D, \Upsilon^D) \in C).$$

Then the pair (Y_D, Υ^D) satisfies

$$\begin{aligned} Q_x^{\gamma}(\exp - \langle u, Y_D\rangle \exp - \langle v, \Upsilon^D\rangle) &= \int \Pi_x^{\gamma}(d\Upsilon)P_x^{\Upsilon}(\exp - \langle u, Y_D\rangle \exp - \langle v, \Upsilon^D\rangle) \\ &= \Pi_x^{\gamma}[\exp - \langle\langle \tilde{V}_D(u), \Upsilon_D\rangle\rangle \exp - \langle v, \Upsilon^D\rangle]. \end{aligned} \tag{5.5}$$

□

5.2. The process $(Z_n \times \Upsilon^n, Q_{\mu,\otimes(x_i\times\gamma_i)})$. Fix a "forest" $\gamma \in \mathcal{P}(\{1,2,\dots,N\})$, starting points $x_1, x_2, \dots, x_{|\gamma|}$ and initial measure μ supported in D_r. Then define the process $(Z_n \times \Upsilon^n, Q_{\mu,\otimes(x_i\times\gamma_i)})$ as follows. For each $i = 1,2,\dots,|\gamma|$ define independent $Y_n^i, \Upsilon^{i,n}$ under $Q_{x_i}^{\gamma_i}$ as above. Then set for $n \geq r$

$$\begin{aligned} Z_n(\omega) &= \tilde{X}_n(\omega_0) + \sum_{i=1}^{|\gamma|} Y_n^i(\omega_i), \\ \Upsilon^n(\omega) &= \otimes \Upsilon^{i,n}(\omega_i), \\ Q_{\mu,\otimes(x_i,\gamma_i)(d\omega)} &= \tilde{\mathbb{P}}_\mu(d\omega_0) Q_{x_1}^{\gamma_1}(d\omega_1)\cdots Q_{x_{|\gamma|}}^{\gamma_{|\gamma|}}(d\omega_{|\gamma|}). \end{aligned} \tag{5.6}$$

Lemma 5.2. *For all $\gamma \in \mathcal{P}$, and $x_1,\dots,x_{|\gamma|} \in E$, The process $(Z_n \times \Upsilon^n)$ is a Markov process under $Q_{\mu,\otimes(x_i\times\gamma_i)}$. Furthermore, the process Υ^n under Q evolves like the branching diffusion based on $h_1,\dots,h_n$ described in section 4.2 stopped on leaving D_n, that is $(\Upsilon^{D_n}, \mathit{\Pi})$.*

Remark 5.3. The process is *almost* an example of Branching Exit Markov system on E ([3]). The difference being the need to carry the extra information contained in the labels to make a Markov process.

Proof. The last claim follows from the first by projecting onto the second coordinate after remarking that the law is correct by (5.5)

$$Q_{\mu,\otimes(x_i\times\gamma_i)}[\exp -\langle v, \Upsilon^n\rangle] = \mathit{\Pi}_{\otimes(x_i\times\gamma_i)}[\exp -\langle v, \Upsilon^n\rangle].$$

To show $Z_n \times \Upsilon^n$ is Markov, it suffices to show if $\mu \in \mathcal{M}(D_r)$, $x_i \in D_R$ then for $r < m < n$ that for u, v measurable functions that

$$\begin{aligned} &Q_{\mu,\otimes(x_i\times\gamma_i)}[\exp -\langle u, Z_n\rangle \exp -\langle v, \Upsilon^n\rangle] \\ &\quad = Q_{\mu,\otimes(x_i\times\gamma_i)}[Q_{Z_m\times\Upsilon^m}[\exp -\langle u, Z_n\rangle \exp -\langle v, \Upsilon^n\rangle]]. \end{aligned} \tag{5.7}$$

Working from the inside out, right to left we have

$$\begin{aligned} &Q_{Z_m\times\Upsilon^m}[\exp -\langle u, Z_n\rangle \exp -\langle v, \Upsilon^n\rangle] \\ &\quad = Q_{Z_m\times\Upsilon^m}[\exp -\langle u, X_n\rangle \exp - \sum_{i=1}^{|p.\Upsilon^m|} \langle u, Y_n^i\rangle \exp - \sum_{i=1}^{|p.\Upsilon^m|} \langle v, \Upsilon^{i,n}\rangle \\ &\quad = \tilde{\mathbb{P}}_{Z_m}[\exp -\langle u, \tilde{X}_n\rangle \prod_{i=1}^{|p.\Upsilon^m|} Q_{x^{i,m}}^{\Upsilon^{i,m}}[\exp -\langle u, Y_n^i\rangle \exp -\langle v, \Upsilon^{i,n}\rangle]] \\ &\quad = \exp -\langle \tilde{V}_n(u), \tilde{X}_m\rangle \prod_{j=1}^{|\gamma|} \exp -\langle u, Y_m^j\rangle \\ &\qquad \prod_{i=1}^{|p.\Upsilon^m|} Q_{x^{i,m}}^{\Upsilon^{i,m}}[\exp -\langle u, Y_n^i\rangle \exp -\langle v, \Upsilon^{i,n}\rangle]. \end{aligned}$$

Then the right hand side of (5.7) becomes by (5.6)

$$\begin{aligned}
&\tilde{\mathbb{P}}_\mu[\exp -\langle \tilde{V}_n(u), \tilde{X}_m\rangle] \prod_{j=1}^{|\gamma|} \exp -\langle u, Y_m^j\rangle Q_{x_j}^{\gamma^j}[\exp -\langle u, Y_m^j\rangle \\
&\qquad \prod_{i=1}^{|p_\bullet \Upsilon^m|} Q_{x^{i,m}}^{\Upsilon^{i,m}}[\exp -\langle u, Y_n^i\rangle \exp -\langle v, \Upsilon^{i,n}\rangle]] \\
&\quad = \exp -\langle \tilde{V}_m(\tilde{V}_n(u)), \mu\rangle \prod_{j=1}^{|\gamma|} \exp -\langle u, Y_m^j\rangle \mathit{\Pi}_{x_j}^{\gamma^j}[\\
&\qquad \exp -\langle\langle \tilde{V}_m(u), \Upsilon_m^j\rangle\rangle \mathit{\Pi}_{x^{i,m}}^{\Upsilon^{i,m}}[\exp -\langle\langle \tilde{V}_n(u), \Upsilon_n^i\rangle\rangle \exp -\langle v, \Upsilon^{i,n}\rangle]].
\end{aligned}$$

The Markov property of $\tilde{X}_n$ ensures us that $\tilde{V}_m(\tilde{V}_n(u)) = \tilde{V}_n(u)$ (from simplifying $\tilde{\mathbb{P}}_{\delta_x}[\exp -\langle u, \tilde{X}_n\rangle] = \mathbb{P}_{\delta_x}(\tilde{\mathbb{P}}_{\tilde{X}_m}[\exp -\langle u, \tilde{X}_n\rangle]))$. We have using (5.2)

$$\begin{aligned}
&\exp -\langle \tilde{V}_n(u), \mu\rangle \mathit{\Pi}_{\otimes(x_j,\gamma^k)}[\exp -\langle\langle \tilde{V}_m(u), \Upsilon_m^j\rangle\rangle \\
&\quad \times \mathit{\Pi}_{\Upsilon^m}[\exp -\langle\langle \tilde{V}_n(u), \Upsilon_n^i\rangle\rangle \exp -\langle v, \Upsilon^{i,n}\rangle]] \\
&\qquad = \exp -\langle \tilde{V}_n(u), \mu\rangle \mathit{\Pi}_{\otimes(x_j,\gamma^k)}[\exp -\langle\langle \tilde{V}_n(u), \Upsilon_n^i\rangle\rangle \exp -\langle v, \Upsilon^{i,n}\rangle] \\
&\quad = \tilde{\mathbb{P}}_\mu[\exp -\langle u, \tilde{X}_n\rangle] \prod_{j=1}^{|\gamma|} Q_{(X_j,\gamma^j)}[\exp -\langle u, Y^n\rangle \exp -\langle v, \Upsilon^n\rangle] \\
&\quad = Q_{\mu,\otimes(X_j,\gamma^j)}[\exp -\langle u, Z_n\rangle \exp -\langle v, \Psi^n\rangle].
\end{aligned}$$

□

Finally define a measure $\hat{\mathbb{P}}_\mu$ by randomizing the choice of γ and the x_i as follows. Let $P_\mu(d\gamma)$ be the probability of selecting a random element of $\mathcal{P}(\{1,\ldots,N\})$ with probability

$$P_\mu(d\gamma) = \frac{\prod_\gamma \langle v^{\gamma_i}\mu, \rangle}{\sum_{\mathcal{P}(\{1,\ldots,N\})} \prod_{\tilde{\gamma}} \langle v^{\tilde{\gamma}_i}, \mu\rangle}. \tag{5.8}$$

Then given a forest, we plant it by selecting starting points $x_i, x_2, \ldots x_{|\gamma|}$ according to the probability measure P_μ^γ as follows

$$P_\mu^\gamma(dx_1 \cdots dx_{|\gamma|}) = \frac{1}{\prod_j \langle v^{\gamma_j}, \mu\rangle} \prod_i v^{\gamma_i}(x_i)\mu(dx_i). \tag{5.9}$$

Then we can define the measure $\hat{\mathbb{P}}_\mu$ on $\mathcal{M}$ as

$$\hat{\mathbb{P}}_\mu(d\nu) = \int P_\mu(d\gamma) P_\mu^\gamma(dx_1 \cdots dx_{|\gamma|}) Q_{\mu,\otimes(x_i,\gamma_i)}(d\nu). \tag{5.10}$$

Where we abuse notation for the measure Q by restricting it to just Z_n.

5.2.1. *Tree picture of $v^\gamma Q_x^\gamma$.* We next rewrite $Q_x^\gamma[\exp -\langle u, Y_D\rangle \cdot \exp -\langle v, \Upsilon^D\rangle]$ in terms of integrals over labeled trees.

Lemma 5.4. *Let $D \Subset E$, $x \in D$ and $\gamma \subset C$. Set*

$$\phi^\gamma(x) = v^\gamma(x) Q_x^\gamma[\exp -\langle u, Y_D\rangle \exp -\langle v, \Upsilon^D\rangle].$$

Then we have the recursive formula

$$\phi^\gamma(x) = K_D^{u+g}[v^\gamma(\cdot)e^{-v(\cdot)}](x) + G_D^{u+g}[\sum_{\mathcal{P}_2(\gamma)} (\phi^{\gamma_1}(\cdot)\phi^{\gamma_2}(\cdot)](x). \tag{5.11}$$

Remark 5.5. This is essentially contained in Lemma 4.4 of [12]. As such, we skip the proof but note that here is where we use the fact that $\psi(x) = x^2/2$ so that $\psi^{'}(x) = x$ when we identify

$$\begin{aligned} K_D^{u+g}(f)(x) &:= \Pi_x[(\exp - \int_0^{\tau_D} \psi^{'}(V_D(u+g))(\xi_s)ds) f(\xi_{\tau_d})] \\ &= \Pi_x[(\exp - \int_0^{\tau_D} V_D(u+g)(\xi_s)ds) f(\xi_{\tau_d})], \end{aligned}$$

and similarly for G_D^{u+g}.

Remark 5.6. A closer inspection of (5.11) reveals an underlying tree picture for binary branching. Recall the decomposition of tree-adapted, labeled trees in (4.3). If we define an operator $\mathbb{L} : \mathcal{T}(\gamma) \mapsto \mathbb{R}_+$ by integrating with K_D^{u+g} on the leaves and G_D^{u+g} on the interior branches, then (5.11) becomes the following by (4.3)

$$v^\gamma(x) Q_x^\gamma[\exp -\langle u, Y_D\rangle \exp -\langle v, \Upsilon^D\rangle] = \sum_{T \in \mathcal{T}(\gamma)} \mathbb{L}(T). \tag{5.12}$$

5.3. The Relationship between Z_n and X_n. We have constructed a Markov process $(Z_n, \hat{\mathbb{P}}_\mu)$, we now show how this process is related to the H-transform of the superprocess.

Lemma 5.7. *The Markov processes $(X_n, \mathbb{P}_\mu^H)$ and $(Z_n, \hat{\mathbb{P}}_\mu)$ have the same law.*

Note: We will use the fact that the tail σ-field for X_n and Z_n are identical.

Proof. We follow the proof outlined in [4]. In particular, we use lemma 8.1 due to Rogers and Pitman reproduced here for ease of reference.

Lemma 5.8 ((Rogers, Pitman) from Lemma 8.1 of [4])**.** *Suppose that, for every $n = 0, 1, \dots,$ a measurable mapping ϕ_n from a measurable space S_n to a measurable space $S_n^{'}$ is given. Let $\Lambda_n(y, \cdot)$ be a Markov kernel from $S_n^{'}$ to S_n such that, for every $y \in S_n^{'}$ the measure $\Lambda_n(y, \cdot)$ is concentrated on $\phi_n^{-1}(y)$. Let a Markov transition function p in $\{S_n\}$ is related to a Markov transition function q in $\{S_n\prime\}$ by a formula*

$$q(r, y; n, B) = \int_{S_n \times S_n} \Lambda_r(y, dx) p(r, x; n, d\tilde{x}) 1_B[\phi_n(\tilde{x})]. \tag{5.13}$$

If

$$\int_{S_n} \Lambda_r(y, dx) p(r, x; n, B) = \int_{S_n^{'}} q(r, y; n, d\tilde{y}) \Lambda_n(\tilde{y}, B), \tag{5.14}$$

then the Markov processes $(X_n, \mathbb{P}_{r,x})$ *and* $(Y_n, Q_{r,y})$ *corresponding to* p *and* q *are related by the formula*

$$Q_{r,y} f_r(Y_r) \cdots f_n(Y_n) = \mathbb{P}^*_{r,y} f_r(\phi_r(X_r)) \cdots f_n(\phi_n(X_n))$$

where $0 \leq r < n$, f_i *is a positive measurable function on* $S_i^{'}$ *and*

$$\mathbb{P}^*_{r,y} = \int \Lambda_r(y, dx) \mathbb{P}_{r,x}.$$

We use the above lemma with the following

$$\begin{aligned} S_n &= \mathcal{M}(E_n, \mathcal{E}), \\ S_n^{'} &= \mathcal{M}(E_n), \\ \phi(\mu \times \cdot) &= \mu, \\ \hat{\mu} &= \int P_\mu(d\gamma) P_\mu^\gamma(dx_1 \cdots dx_{|\gamma|}), \\ \Lambda_n(\mu, \cdot) &= \delta_\mu \times \hat{\mu}. \end{aligned}$$

By construction, Λ is concentrated on ϕ^{-1} and Λ is a Markov kernel. The trouble is verifying equations (5.13) and (5.14).

For a measurable f we have

$$\begin{aligned} \Lambda_n(\mu, f) &= \int_{\mathcal{M}(E_n \times \mathcal{E})} \Lambda_n(\mu, d\nu \times d\Upsilon) f(\nu \times \Upsilon) \\ &= \int \hat{\mu}(d\Upsilon) f(\mu \times \Upsilon). \end{aligned}$$

To verify (5.13), apply the above to the function $\exp -\langle u, \mu \rangle$ to get that it is enough to show

$$\begin{aligned} \mathbb{P}^H_\mu[\exp -\langle u, X_n \rangle] &= \int \hat{\mu}(d\Upsilon) Q_{\mu,\Upsilon}[\exp -\langle u, \phi(Z_n \times \Upsilon_n) \rangle] \qquad (5.15) \\ &= \int P_\mu(d\gamma) P_\mu^\gamma(dx_1 \cdots dx_{|\gamma|}) Q_{\mu, \otimes(x_i \times \gamma_i)}[\exp -\langle u, Z_n \rangle]. \end{aligned}$$

The formula (5.15) is previously shown in remark 5.9 of [12]. It is not repeated here although the notation is different as the basic ideas are presented in the verification of (5.16).

To verify (5.14) it is enough to show it for functions of the form $\exp -\langle u, \mu \rangle \cdot \exp -\langle v, \Upsilon \rangle$ which means we need to verify

$$\begin{aligned} &\int \hat{\mu}(d\Upsilon) Q_{\mu,\Upsilon}[\exp -\langle u, Z_n \rangle \exp -\langle v, \Upsilon^n \rangle] \qquad (5.16) \\ &\quad = \mathbb{P}^H_\mu(\Lambda_n(X_n, \exp -\langle u, \cdot \rangle \exp -\langle v, \cdot \rangle)) \\ &\quad = \mathbb{P}^H_\mu[\exp -\langle u, X_n \rangle \int P_{X_n}(d\gamma) P_{X_n}^\gamma(dx_1 \cdots dx_{|\gamma|}) \exp -\langle v, \sum \delta_{x_i} \rangle. \end{aligned}$$

We begin with the H-transformed side. Using (5.8), $H(\mu) = \exp -\langle g, \mu\rangle \cdot \tilde{H}(\mu)$ and (5.9) we have

$$\begin{aligned}
&\mathbb{P}^H_\mu[\exp -\langle u, X_n\rangle \int P_{X_n}(d\gamma) P^\gamma_{X_n}(dx_1\cdots dx_{|\gamma|}) \exp -\langle v, \sum \delta_{x_i}\rangle] \\
&= \mathbb{P}^H_\mu[\exp -\langle u, X_n\rangle \frac{\sum_{\gamma\in\mathcal{P}(\{1,2,\dots,N\})}\prod_\gamma \langle v^{\gamma_i}, X_n\rangle}{\sum_{\tilde\gamma\in\mathcal{P}(\{1,2,\dots,N\})}\prod_{\tilde\gamma}\langle v^{\tilde\gamma}_i, X_n\rangle} \\
&\quad\cdot \frac{\prod_\gamma \int v^{\gamma_i}(x_i) e^{v(x_i)} X_n(dx_i)}{\prod_\gamma \langle v^{\gamma_i}, X_n\rangle}] \\
&= \mathbb{P}^H_\mu[\exp -\langle u, X_n\rangle \frac{1}{\sum_{\tilde\gamma\in\mathcal{P}(\{1,2,\dots,N\})}\prod_{\tilde\gamma}\langle v^{\tilde\gamma_i}, X_n\rangle} \prod_\gamma \langle v^{\gamma_i}(\cdot)e^{-v(\cdot)}, X_n\rangle] \\
&= \frac{1}{H(\mu)}\mathbb{P}_\mu[\exp -\langle u, X_n\rangle \frac{1}{\tilde H(X_n)} H(X_n) \sum_{\gamma\in\mathcal{P}(\{1,2,\dots,N\})} \prod_\gamma \langle v^{\gamma_i}(\cdot)e^{-v(\cdot)}, X_n\rangle] \\
&= \frac{1}{\tilde H(\mu)\exp -\langle g, \mu\rangle} \sum_{\gamma\in\mathcal{P}(\{1,2,\dots,N\})} \mathbb{P}_\mu[\exp -\langle u+g, X_n\rangle] \prod_\gamma \langle v^{\gamma_i}(\cdot)e^{-v(\cdot)}, X_n\rangle].
\end{aligned}$$

Which by Theorem 4.2 of [4] becomes:

$$\frac{1}{\tilde H(\mu)\exp -\langle g, \mu\rangle} \exp -\langle V_D(u+g), \mu\rangle \sum_{\gamma\in\mathcal{P}(\{1,2,\dots,N\})} \sum_{\Lambda(\gamma)} \langle \mathbb{L}_\Lambda, \mu\rangle. \tag{5.17}$$

The new notations Λ and $\mathbb{L}_\Lambda$ are from [4] and correspond to integrals over a certain type of forest and will be explained further below.

We can simplify the "Q" side of (5.16) to get

$$\begin{aligned}
&\int P_\mu(d\gamma) P^\gamma_\mu(dx_1\cdots x_{|\gamma|}) Q_{\mu,\otimes(x_i\times\gamma_i)}[\exp -\langle u, Z_n\rangle \exp -\langle v, \Upsilon^n\rangle] \\
&= \frac{\sum_{\gamma\in\mathcal{P}(\{1,2,\dots,N\})}\prod_\gamma \langle v^{\gamma_i}, \mu\rangle}{\sum_{\tilde\gamma\in\mathcal{P}(\{1,2,\dots,N\})}\prod_{\tilde\gamma}\langle v^{\tilde\gamma_i}, \mu\rangle} \frac{1}{\prod_\gamma \langle v^{\tilde\gamma_i}, \mu\rangle} \tilde{\mathbb{P}}_\mu[\exp -\langle u, \tilde X_n\rangle] \\
&\quad \cdot \prod_{i=1}^{\gamma} \int v^{\gamma_i}(x_i)\mu(dx_i) Q^{\gamma_i}_{x_i}[\exp -\langle u, Y^i_n\rangle \exp -\langle v, \Upsilon^{i,n}\rangle]) \\
&= \frac{1}{\tilde H(\mu)} \tilde{\mathbb{P}}_\mu[\exp -\langle u, \tilde X_n\rangle] \\
&= \sum_{\gamma\in\mathcal{P}(\{1,2,\dots,N\})} \prod_\gamma \int v^{\gamma_i}(x)\mu(dx) Q^{\gamma_i}_x[\exp -\langle u, Y^i_n\rangle \exp -\langle v, \Upsilon^{i,n}\rangle] \\
&= \frac{1}{\tilde H(\mu)} \tilde{\mathbb{P}}_\mu[\exp -\langle u, \tilde X_n\rangle] \\
&= \sum_{\gamma\in\mathcal{P}(\{1,2,\dots,N\})} \prod_\gamma \langle v^{\gamma_i}(\cdot) Q^{\gamma_i}_\cdot[\exp -\langle u, Y^i_n\rangle \exp -\langle v, \Upsilon^{i,n}\rangle], \mu\rangle
\end{aligned}$$

$$= \frac{1}{\tilde{H}(\mu)} \tilde{\mathbb{P}}_\mu[\exp -\langle u, \tilde{X}_n\rangle] \sum_{\gamma\in\mathcal{P}(\{1,2,\ldots,N\})} \prod_{\gamma} \langle \phi^{\gamma_i}, \mu\rangle$$

$$= \frac{1}{\tilde{H}(\mu)} \tilde{\mathbb{P}}_\mu[\exp -\langle u, \tilde{X}_n\rangle] \sum_{\gamma\in\mathcal{P}(\{1,2,\ldots,N\})} \prod_{\gamma} \sum_{T\in\mathcal{T}(\gamma_i)} \mathbb{L}(T). \quad \text{(by (5.12))} \tag{5.18}$$

But (5.17) and (5.18) show the two sides of (5.16) are equal provided

$$\sum_{\gamma\in\mathcal{P}(\{1,2,\ldots,N\})} \prod_{\gamma} \sum_{T\in\mathcal{T}(\gamma_i)} \mathbb{L}(T) = \sum_{\tilde{\gamma}\in\mathcal{P}(\{1,2,\ldots,N\})} \sum_{\Lambda(\tilde{\gamma})} \langle \mathbb{L}_\Lambda, \mu\rangle. \tag{5.19}$$

The left hand side becomes the sum of all forests with initial planting given by γ which consist of tree-adapted labelings. That is

$$\sum_{\gamma\in\mathcal{P}(\{1,2,\ldots,N\})} \sum_{\mathcal{T}(\gamma)} \prod_{i=1}^{|\gamma|} \mathbb{L}(T_i). \tag{5.20}$$

The right hand side is described in section 4 of [4] as a sum over frames, which are rooted trees described by their exit sets or leaves. The functions $\mathbb{L}$ are identical, where the exit leaves are labeled by functions $v^{\tilde{\gamma}}e^{-v}$ for some $\tilde{\gamma}$. The equivalence comes as one side describes the trees by their initial planting, the other by their final leave structure. As both describe forests, and each tree in the forest has a 1-1 relationship with its root labeling (γ_i) and it's final leaf structure $(\cup_{i\in C}\tilde{\gamma}_i, C \in \mathcal{P}(1,2,\ldots,|\tilde{\gamma}|))$ the two sides represent the same value. □

5.4. Tail events for $(Z_n \times \Upsilon^n)$**.** The measure Z_n has two contributions – the initial mass started at μ and evolving under $\tilde{\mathbb{P}}$ and the measures Y_n correspond to mass "immigrated" along the backbone process. In all cases, this mass is conditioned not to charge the boundary of E by the function g. As such, there should be no contribution to the tail events from this mass. The following quantifies this. At first glance it indicates that the initial mass μ is not important, but using the Markov property, we will see that it applies to the measure Z_n as well, that is all the mass that accumulates as the process leaves D_n.

Lemma 5.9. *Let C be a tail event of $Z_n \times \Upsilon^n$. Then*

$$Q_{\mu,\otimes(x_i\times\gamma_i)}[C] = Q_{0,\otimes(x_i\times\gamma_i)}[C].$$

Proof. The proof of this is nearly identical to that in step 4 of Theorem 8.1 in [4]. It is repeated here for ease of reference.

We have by (5.6) that for functions $g(Z_n \times \Upsilon^n) = \exp -\langle u, Z_n\rangle \exp -\langle v, \Upsilon^n\rangle$ that

$$Q_{\mu,(x\times\gamma)}(g(Z_n \times \Upsilon^n)) \\ = \int \tilde{\mathbb{P}}_\mu(d\omega_0) Q_{0,(x\times\gamma)}(d\omega_1) g[(\tilde{X}(\omega_0) + Z_n(\omega_1)) \times \Upsilon^n(\omega_1)].$$

By the multiplicative systems theorem, this holds true for all measurable g.

In particular, let C be a tail event of $Z_n \times \Upsilon_n$. That is, C is in $\cap_n \mathcal{F}_{\supset D_n}$. Then by the Markov property of $Z_n \times \Upsilon^n$ we have

$$\begin{aligned} Q_{\mu,(x\times\gamma)}[C] &= Q_{\mu,(x\times\gamma)}[Q_{Z_n\times\Upsilon^n}[C]] \\ &= \int \tilde{\mathbb{P}}_\mu(d\omega_0) Q_{0\times\Upsilon^n}(d\omega_1)[g(\tilde{X}_n(\omega_0) + Z_n(\omega_1)\times\Upsilon^n)], \end{aligned}$$

where $g(\cdot) = Q_\cdot[C]$. By considering the two cases $\tilde{X}_n = 0$ or not we have the above decomposes into two terms $I_n + J_n$ with

$$\begin{aligned} I_n &= \int \tilde{\mathbb{P}}_\mu(d\omega_0) 1(\tilde{X}_n(\omega_0) = 0) Q_{0\times\Upsilon^n}(d\omega_1)[g(\tilde{X}_n(\omega_0) + Z_n(\omega_1)\times\Upsilon^n)] \\ &= \tilde{\mathbb{P}}_\mu(\tilde{X}_n = 0) Q_{0,(x\times\gamma)}[C] \to Q_{0,(x\times\gamma)}[C], \end{aligned}$$

and

$$\begin{aligned} J_n &= \int \tilde{\mathbb{P}}_\mu(d\omega_0) 1(\tilde{X}_n(\omega_0) \neq 0) Q_{0\times\Upsilon^n}(d\omega_1)[g(\tilde{X}_n(\omega_0) + Z_n(\omega_1)\times\Upsilon^n)] \\ &\le \tilde{\mathbb{P}}_\mu(\tilde{X}_n \neq 0) \to 0. \end{aligned}$$

Thus, letting $n \to \infty$ we get $Q_{\mu,(x\times\gamma)}[C] = Q_{0,(x\times\gamma)}[C]$. □

We now define a process on the N-fold Cartesian product of E by combining N independent copies of ξ. Let $E^N = \otimes_{i=1}^N E$, and define a process

$$(\Xi_n, \Pi^{h^N}), \quad \Xi_n = (\xi_n^1, \xi_n^2, \dots \xi_n^N)$$

with generator $L^{N,g} = \otimes_{I=1}^N L^g$. Finally, let

$$h^N(x_1, \dots, x_N) = \prod_{i=1}^N h_i(x_i).$$

For short we write $h = h^N$.

Lemma 5.10. *The function $h(x)$ is $L^{N,g}$ harmonic. The h-transform of Ξ has the following form*

$$\Pi^h_{(x_1,\dots,x_N)}(A_1 \times \dots \times A_N) = \prod_{i=1}^N \Pi^{h_i}_{x_i}(A_i).$$

Proof. Clearly h is harmonic as it is in each term. Further, by the definition of the h-transform, we have with $x = (x_1, \dots, x_N)$

$$\begin{aligned} \Pi^h_x(A_1 \times \dots \times A_N) &= \frac{1}{h(x)} \Pi_x(A_1 \times \dots \times A_N; h(\Xi_n)) \\ &= \frac{1}{h(x)} \Pi_x(A_1 \times \dots \times A_N; h_1(\xi_1) \cdots h_N(\xi_N)) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{h(x)} \Pi_x(A_1 h_1(\xi_n) \times \cdots \times A_N h_N(\xi_n^1)) \\
&= \frac{1}{h(x)} \Pi_{x_1}(A_1 h_1(\xi_n^1)) \cdots \Pi_{x_N}(A_N h_N(\xi_n^N)) \\
&= \frac{h_1(x_1)\cdots h_N(x_N)}{h(x)} \prod_{i=1}^{N} \Pi_{x_i}^{h_i}(A_i)) \\
&= \prod_{i=1}^{N} \Pi_{x_i}^{h_i}(A_i).
\end{aligned}
$$

□

To finish the proof we will show the tail σ-field for Υ^n is the same as that of Ξ and that this process has a trivial tail field under our assumptions on H and g.

Lemma 5.11. *The following implications hold*

$$\Pi^{h^N} \textit{ is trivial } \implies \mathit{\Pi}_{\cdot} \textit{ is trivial } \implies Q_{\mu,\cdot} \textit{ is trivial } \implies \hat{\mathbb{P}}_\mu \textit{ is trivial.}$$

Proof. Work from the right-hand side back. Recall, a tail σ field is trivial if the probability of any event is either 0 or 1. Let A be a tail event of $Z_n \times \Upsilon^n$ under $Q_{\mu\times\cdot}$. Then we have

$$\hat{\mathbb{P}}_\mu[A] = \langle Q_{\mu,\cdot}[A], \mu\rangle = \langle 1_\Gamma(A), \mu\rangle = 1_\Gamma(A).$$

Hence it is trivial.

Next, suppose A is a tail event of $Z_n \times \Upsilon^n$ under Q. Set $\phi(\cdot) = Q_{0,\cdot}[A]$. Then by lemma 5.9 and the Markov property of $Z_n \times \Upsilon^n$ we have with Υ^r denoting any starting forest and μ concentrated on D_r

$$
\begin{aligned}
\phi(\Upsilon^r) &= Q_{0,\Upsilon^r}[A] = Q_{\mu,\Upsilon^r}[A] = Q_{\mu,\Upsilon^r}[Q_{Z_n\times\Upsilon^n}[A]] \\
&= Q_{\mu,\Upsilon^r}[Q_{0\times\Upsilon^n}[A]] = Q_{0,\Upsilon^r}[Q_{0\times\Upsilon^n}[A]] = Q_{0,\Upsilon^r}[\phi(\Upsilon^n)].
\end{aligned}
$$

So $\phi(\cdot)$ is harmonic and $\phi(\Upsilon^n)$ is a bounded martingale. It converges to $\bar{\phi}$ say. We have Υ^n under Q_{0,Υ^r} evolves as Υ^n under $\mathit{\Pi}_{\Upsilon^r}$ and so by lemma 5.2, $\hat{\phi}$ is in the tail σ-field of $(\Upsilon^n, \mathit{\Pi})$ hence is trivial by assumption. Thus,

$$Q_{\mu,\Upsilon^r}[A] = Q_{0,\Upsilon^r}[Q_{0\times\Upsilon^n}[A]] = \mathit{\Pi}_{\Upsilon^r}(\phi(\Upsilon^n)) \to \mathit{\Pi}_{\Upsilon^r}(\bar{\phi}) = 1_\Gamma(A).$$

That, is the σ-field is trivial.

Finally, we show the first implication. Let A be in the tail field of $(\Upsilon^n, \mathit{\Pi})$. Then set $\phi(\cdot) = \mathit{\Pi}_{\cdot}[A]$ as before. We have, again,

$$\phi(\Upsilon^r) = \mathit{\Pi}_{\Upsilon^r}[A] = \mathit{\Pi}_{\Upsilon^r}[\mathit{\Pi}_{\Upsilon^n}[A]] = \mathit{\Pi}_{\Upsilon^r}[\phi(\Upsilon^n)].$$

So $\phi(\cdot)$ is harmonic, and $\phi(\Upsilon^n)$ is a martingale which is bounded, hence convergent to $\bar{\phi}$ say.

We have the following decomposition for all $n > r$

$$
\begin{aligned}
\phi(\Upsilon^r) &= \mathit{\Pi}_{\Upsilon^r}[\mathit{\Pi}_{\Upsilon_n}[A]; |p \centerdot \Upsilon^n| = N] + \mathit{\Pi}_{\Upsilon^r}[\mathit{\Pi}_{\Upsilon_n}[A]; |p \centerdot \Upsilon^n| < N] \\
&= I_n + J_n.
\end{aligned}
$$

We have by the construction of Υ^n under $\mathit{\Pi}$ $J_n \to 0$. Thus, if I_n converges to an indicator independent of Υ^r the result will follow.

Recall, that once the branching tree Υ_n is labeled by a set $C = \{i\}$ the particle evolves like a h_i transform, independently of the other particles given their starting points. Thus we have for all $m > n$

$$\begin{aligned} I_n &= \mathit{\Pi}_{\Upsilon^r}[\mathit{\Pi}_{\Upsilon_n}[\mathit{\Pi}_{\Upsilon^m}[A]]; |p \centerdot \Upsilon^n| = N] \\ &= \mathit{\Pi}_{\Upsilon^r}[\Pi^{h^N}_{x \centerdot \Upsilon^n}[\mathit{\Pi}_{\Xi_m \times \{1,2,\ldots,N\}}[A]]] \\ &\to \mathit{\Pi}_{\Upsilon^r}[\Pi^{h^N}_{x \centerdot \Upsilon^n}[\bar{\phi}]] \\ &= 0, \end{aligned}$$

as $\bar{\phi}$ is in the tail of Ξ under Π^{h^N}. □

Theorem(1.1) is proven if we can establish the following lemma which states that the tail σ-field process composed of N-independent h-processes is trivial when each of the processes has a trivial tail field. By the last lemma and the fact that the tail fields of $(Z_n, \hat{P}\mu)$ and $(X_n, \mathbb{P}^H_\mu)$ are identical.

Lemma 5.12. *The tail σ-field of Ξ_n is trivial.*

Remark 5.13. There are two different ways to prove this – do it for the continuous process and discretize, or do it for the discrete processes directly. To do the former, we would generalize the *bitransform* approach in [10] to *multitransforms* applied in the rectangular setting of E^N. Then use the fact due to Molchanov [8] (see Taylor [13] as well) that the extreme harmonic functions for products are products of extreme harmonic functions for each coordinate. Finally, one could discretize by considering the appropriate lower-levels in the language of [10].

However, the transient nature of the discretized process Ξ_n makes a direct proof possible and so it is shown.

We let $p(r, x; n, x^{'})$ be the transition probability for Ξ_n to go from $x \in \partial D_r$ to $x^{'} \in \partial D_n$. As usual, define the Green operator by

$$G_r(f)(x) = \Pi^N_{r,x}(\sum_{n \geq r} f(\Xi_n)) = \int \sum_{n \geq r} p(r, x; n, x^{'}) f(x^{'}) = \int g_r(x, x^{'}) f(x^{'}).$$

Note, by the transient nature of Ξ_n that if $x^{'} \in \partial D_n$ then $g_r(x, x^{'}) = p(r, x; n, x^{'})$.

Define the Martin kernel by

$$K_r(x, y) = \frac{g_r(x, y)}{g_r(x^0, y)}, (x, y) \in E^N \times E^N$$

and the Martin boundary to be the minimal topology for which $K_r(x, y)$ extends continuously to $E^N \times \bar{E}^N$ and which separates points. Such a compactification of E^N exists and furthermore, there is a representation of Ξ-harmonic functions over the extreme Ξ-harmonic functions:

$$h(x) = \int K_r(x, y) \nu^h(dy)$$

Now, due to the simple form of $g_r(x, y)$ as a single term, it is clear that $g_r(x, y)$ factors as follows

$$g_r(x, y) = \prod_{i=1}^{N} p^{h_i}(e, x^i; n, y^i),$$

where p^{h_i} is the transition function for the conditioned process ξ^i_n with $x = (x^1, x^2, \ldots, x^N)$. As such, the martin kernel also factors on $E^N \times E^N$. As each individual factor converges, this extends continuously to $\bar{E}$ and we have that the Martin kernel factors on the whole space $E^N \times \bar{E^N}$.

By the uniqueness of the representation of Ξ-harmonic functions in terms of extreme elements, we get that h^N is extreme and hence minimal.

To see that it has trivial tail field then follows by a standard argument (e.g. §5.5 of [4]). Namely, if C is a tail event, and one sets $\phi(x) = \Pi^{h^N}_x(C)$ then as

$$\phi(x) = \Pi^{h^N}_x(\Pi^{h^N}_{\Xi_n}[C])$$

so that $(\phi h)(x) = \Pi_x((\phi h)(\Xi_n))$ and so ϕh is Ξ-harmonic, which by minimality implies ϕ is a constant. That ϕ is 0 or 1 comes from the fact that $\phi(X_n)$ is a bounded martingale hence convergent, and by the Markov property converges to 1_C.

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