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ALÒS TYPE DECOMPOSITION FORMULA FOR BARNDORFF-NIELSEN AND SHEPHARD MODEL

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Abstract. An Alòs type decomposition formula for call options is established for the Barndorff-Nielsen and Shephard model: an Ornstein-Uhlenbeck type stochastic volatility model driven by a subordinator without drift. Alòs [2] introduced a decomposition expression for the Heston model by using Ito's formula. In this paper, we extend it to the Barndorff-Nielsen and Shephard model. As far as we know, this is the first result on the Alòs type decomposition formula for models with infinite active jumps.

1. Introduction

Stochastic volatility models have drawn considerable attention in mathematical finance since they are very useful for capturing the volatility skew and smiles, but there is no closed-form option pricing formula for stochastic volatility models in general. Thus, some authors have presented decomposition expressions of option prices, which are useful to derive approximations of option prices and to analyze implied volatilities. Firstly, for continuous stochastic volatility models with no correlation between the asset price and the volatility processes, Hull and White [12] provided an option price expression with a conditional expectation of the Black-Scholes formula by substituting the future average volatility for the volatility in the Black-Scholes formula. Alòs [1] has extended it to correlated models by means of Malliavin calculus in order to deal with Ito's formula for anticipating processes, since the future average volatility is a non-adapted process. Besides, extensions to more general models have been done by [4], [5], [14] and so on. On the other hand, Alòs [2] obtained a new decomposition formula for the Heston model by using the average squared future volatility, instead of the future average volatility. Since the average squared future volatility is an adapted process, she made use of the classical Ito calculus, not the Malliavin calculus. The decomposition formula in [2] is given as the sum of the Black-Scholes formula and terms due to the volatility process. In addition, using the obtained decomposition expression, approximate option pricing formulas were also presented. This Alòs type decomposition formula has been extended to more general models by [16], [17] and so on. Among them, Merino et al. [15] has extended to stochastic volatility models with finite active

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jumps. Moreover, for the Heston model, Alòs et al. [3] suggested an approximation of the implied volatility and a calibration method by using the results of [2].

The objective of this paper is to obtain an Alòs type decomposition expression of call option prices for the Barndorff-Nielsen and Shephard (BNS) model by applying Ito's formula to the Black-Scholes formula. It is given as the sum of the Black-Scholes formula, a term due to the impact of the asset price jumps, and some residual terms due to the asset price jumps and changes of the volatility. Unlike [2], we use the current squared volatility value instead of the average squared future volatility, and substitute it to the volatility in the Black-Scholes formula. To our best knowledge, this is the first result of the Alòs type decomposition formula for models with infinite active jumps, but Jafari and Vives [14] derived a Hull-White type decomposition formula for models with infinite active jumps by means of Malliavin calculus. Now, the BNS model is a representative jump-type stochastic volatility model undertaken by [9], [10], and its volatility process is given by a non-Gaussian Ornstein-Uhlenbeck process. The BNS model is still being actively researched, e.g. Humayra and SenGupta [13] discussed an optimal hedging strategy for commodity markets for a refined BNS model, and Shantanu and SenGupta [20] analyzed the first-exit time for an approximate BNS model. For details on the BNS model, see also [18] and [19]. The BNS model has the following three features: First, the asset price process has jumps, but all jumps are negative. Second, there is no Brownian component in the volatility process. Third, the jump component is common between the asset price and the volatility processes. We note that the jumps might be infinite active. Our decomposition formula will be derived by making the most of these features of the BNS model.

The structure of this paper is as follows: We give some mathematical preliminaries and notations in the following section. Section 3 introduces our decomposition formula. Its proof is given in Section 4, and conclusions are summarized in Section 5.

2. Preliminaries

2.1. Model description. Consider throughout a financial market model in which only one risky asset and one riskless asset are tradable. Let $r \geq 0$ be the interest rate of our market, and $T > 0$ a finite time horizon. In the BNS model, the risky asset price at time $t \in [0, T]$ is described by

$$S_t := S_0 \exp\left\{\int_0^t \left(r + \mu - \frac{1}{2}\Sigma_u^2\right) du + \int_0^t \Sigma_u dW_u + \rho H_{\lambda t}\right\}, \quad t \in [0, T], \quad (2.1)$$

where $S_0 > 0$, $\rho \leq 0$, $\mu \in \mathbb{R}$, $\lambda > 0$, H is a subordinator without drift, and W is a 1-dimensional standard Brownian motion. Here Σ is the volatility process, of which squared process Σ^2 is given by an Ornstein-Uhlenbeck process driven by the subordinator H_λ, that is, the solution to the following stochastic differential equation:

$$d\Sigma_t^2 = -\lambda \Sigma_t^2 dt + dH_{\lambda t}, \quad t \in [0, T] \quad (2.2)$$

with $\Sigma_0^2 > 0$. Note that the asset price process S is defined on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$ with the usual condition, where $(\mathcal{F}_t)_{0 \leq t \leq T}$

is the filtration generated by W and H_λ. In addition, we denote by X the log price process $\log S$, that is,

$$X_t := \log S_t = \log S_0 + \int_0^t \left(r + \mu - \frac{1}{2}\Sigma_u^2\right) du + \int_0^t \Sigma_u dW_u + \rho H_{\lambda t}, \quad t \in [0,T]. \tag{2.3}$$

We note that the term $\rho H_{\lambda t}$ in (2.3) (or (2.1)) accounts for the leverage effect, which is a stylized fact such that the asset price declines at the moment when the volatility increases.

For later use, we enumerate some properties of Σ: Firstly, we have

$$\Sigma_t^2 = e^{-\lambda t}\Sigma_0^2 + \int_0^t e^{-\lambda(t-u)} dH_{\lambda u} \geq e^{-\lambda T}\Sigma_0^2 \tag{2.4}$$

for any $t \in [0,T]$, that is, Σ is bounded from below. Next, the integrated squared volatility is represented as

$$\int_t^T \Sigma_u^2 du = \epsilon(T-t)\Sigma_t^2 + \int_t^T \epsilon(T-u) dH_{\lambda u} \tag{2.5}$$

for any $t \in [0,T]$, where

$$\epsilon(t) := \frac{1 - e^{-\lambda t}}{\lambda}.$$

In addition, (2.5) implies

$$\int_0^T \Sigma_u^2 du \leq \frac{1}{\lambda}(H_{\lambda T} + \Sigma_0^2). \tag{2.6}$$

Now, we denote by N the Poisson random measure of H_λ. Hence, we have

$$H_{\lambda t} = \int_0^\infty z N([0,t], dz), \quad t \in [0,T].$$

Letting ν be the Lévy measure of H_λ, we find that

$$\widetilde{N}(dt, dz) := N(dt, dz) - \nu(dz)dt$$

is the compensated Poisson random measure. Note that ν is a σ-finite measure on $(0,\infty)$ satisfying

$$\int_0^\infty (z \wedge 1)\nu(dz) < \infty$$

by Proposition 3.10 of [11]. The asset price process S is also given as the solution to the following stochastic differential equation:

$$dS_t = S_{t-}\left\{\alpha dt + \Sigma_t dW_t + \int_0^\infty (e^{\rho z} - 1)\widetilde{N}(dt, dz)\right\}, \quad t \in [0,T],$$

where

$$\alpha := r + \mu + \int_0^\infty (e^{\rho z} - 1)\nu(dz).$$

Note that $S_t > 0$ holds for any $t \in [0,T]$.

Now, we introduce our standing assumption as follows:

Assumption 2.1. (1) $\mu = \int_0^\infty (1 - e^{\rho z})\nu(dz)$.

(2) $\int_1^\infty e^{2\epsilon(T)z}\nu(dz)<\infty$.

The above condition 1 implies that the discounted asset price process $\widehat{S}_t:=e^{-rt}S_t$ becomes a local martingale. On the other hand, the condition 2 ensures that

$$\int_0^\infty z^2\nu(dz)<\infty,$$

which yields $\mathbb{E}[H^2_{\lambda T}]<\infty$ by Proposition 3.13 of [11], and

$$\mathbb{E}\left[\sup_{t\in[0,T]}X_t^2\right]<\infty \tag{2.7}$$

by (2.6). In addition,

$$\mathbb{E}\left[\sup_{t\in[0,T]}S_t^2\right]<\infty \tag{2.8}$$

holds under the condition 2 from the view of Subsection 2.3 of [8]. Thus, $\widehat{S}$ is a square-integrable martingale under Assumption 2.1.

Example 2.2. We introduce two important examples of the squared volatility process Σ^2.

(1) The first one is the case where Σ^2 follows an IG-OU process. The corresponding Lévy measure ν is given by

$$\nu(dz)=\frac{\lambda a}{2\sqrt{2\pi}}z^{-\frac{3}{2}}(1+b^2z)\exp\left\{-\frac{1}{2}b^2z\right\}dz,\quad z\in(0,\infty),$$

where $a>0$ and $b>0$. Note that this is a representative example of the BNS model with infinite active jumps, that is, $\nu((0,\infty))=\infty$. In this case, the invariant distribution of Σ^2 follows an inverse-Gaussian distribution with parameters $a>0$ and $b>0$. Note that the condition 2 of Assumption 2.1 is satisfied whenever $\frac{b^2}{2}>2\epsilon(T)$

(2) The second example is the gamma-OU case. In this case, ν is described as

$$\nu(dz)=\lambda abe^{-bz}dz,\quad z\in(0,\infty),$$

and the invariant distribution of Σ^2 is given by a gamma distribution with parameters $a>0$ and $b>0$. If $b>2\epsilon(T)$, then the condition 2 of Assumption 2.1 is satisfied.

2.2. Black-Scholes formula. In this subsection, consider the so-called Black-Scholes model with volatility $\sigma>0$ and interest rate $r\geq 0$, and the call option with strike price $K>0$ and maturity $T>0$. We describe the call option price at time $t\in[0,T)$ with the log asset price $x\in\mathbb{R}$ by a function BS on not only t and x, but also squared volatility σ^2. Thus, the function $BS(t,x,\sigma^2)$, which is well-known as the Black-Scholes formula, is given as

$$BS(t,x,\sigma^2):=e^x\Phi(d^+)-Ke^{-r\tau_t}\Phi(d^-),\quad t\in[0,T),x\in\mathbb{R},\sigma>0, \tag{2.9}$$

where $\tau_t = T - t$, Φ is the cumulative distribution function of the standard normal distribution, and

$$d^{\pm} := \frac{x - \log K + r\tau_t}{\sigma\sqrt{\tau_t}} \pm \frac{\sigma\sqrt{\tau_t}}{2}. \tag{2.10}$$

For later use, we denote

$$x_z := x + \rho z,\ \sigma_z := \sqrt{\sigma^2 + z},\ \eta^{\pm} := r \pm \frac{\sigma^2}{2},\ \eta_z^{\pm} := r \pm \frac{\sigma_z^2}{2} = \eta^{\pm} \pm \frac{z}{2} \tag{2.11}$$

for $z > 0$, $x \in \mathbb{R}$ and $\sigma > 0$. Thus, $d^{\pm}$ is rewritten as

$$d^{\pm} = \frac{x - \log K + \eta^{\pm}\tau_t}{\sigma\sqrt{\tau_t}}.$$

Furthermore, we define

$$d_{\rho z}^{\pm} := \frac{x_z - \log K + \eta^{\pm}\tau_t}{\sigma\sqrt{\tau_t}} = d^{\pm} + \frac{\rho z}{\sigma\sqrt{\tau_t}}. \tag{2.12}$$

and

$$d_{\rho z, z}^{\pm} := \frac{x_z - \log K + \eta_z^{\pm}\tau_t}{\sigma_z\sqrt{\tau_t}} \tag{2.13}$$

for $z > 0$. We note that the time parameter t included in $d^{\pm}$, $d_{\rho z}^{\pm}$ and $d_{\rho z,z}^{\pm}$ might be replaced with u or s according to the situation. In addition, since we have

$$\lim_{t \to T} BS(t, x, \sigma^2) = (e^x - K)^+,$$

the domain of the function BS can be extended to $[0, T] \times \mathbb{R} \times (0, \infty)$, and we may define

$$BS(T, x, \sigma^2) := (e^x - K)^+.$$

For simplicity, substituting X_t and Σ_t^2 defined in (2.3) and (2.2) for x and σ^2 respectively in the function BS, we denote

$$BS_t := BS(t, X_t, \Sigma_t^2)$$

for $t \in [0, T]$.

More importantly, defining an operator $\mathcal{D}^{BS}$ as

$$\mathcal{D}^{BS} f(t, x, \sigma^2) := \left(\partial_t + \frac{\sigma^2}{2}\partial_x^2 + \eta^- \partial_x - r\right) f(t, x, \sigma^2)$$

for $\mathbb{R}$-valued function $f(t, x, \sigma^2)$, $t \in [0, T)$, $x \in \mathbb{R}$, $\sigma > 0$, we have

$$\mathcal{D}^{BS} BS(t, x, \sigma^2) = 0, \quad t \in [0, T), x \in \mathbb{R}, \sigma > 0. \tag{2.14}$$

We observe that partial derivatives of BS are given as

$$\partial_x BS(t, x, \sigma^2) = e^x \Phi(d^+), \tag{2.15}$$

$$\partial_x^2 BS(t, x, \sigma^2) = e^x \Phi(d^+) + \frac{e^x}{\sigma\sqrt{\tau_t}}\phi(d^+), \tag{2.16}$$

and

$$\partial_{\sigma^2} BS(t, x, \sigma^2) = \frac{\tau_t}{2}(\partial_x^2 - \partial_x) BS(t, x, \sigma^2) = \frac{\sqrt{\tau_t}}{2\sigma} e^x \phi(d^+), \tag{2.17}$$

where ϕ is the probability density function of the standard normal distribution. All of the above derivatives are positive functions. For later use, we define additionally the following operators for $\mathbb{R}$-valued function $f(t,x,\sigma^2)$, $t \in [0,T)$, $x \in \mathbb{R}$, $\sigma > 0$:

$$\Delta^{a,b} f(t,x,\sigma^2) := f(t,x+a,\sigma^2+b) - f(t,x,\sigma^2), \quad a,b \in \mathbb{R},$$
$$\mathcal{L}^z f(t,x,\sigma^2) := \Delta^{\rho z,0} f(t,x,\sigma^2) + \partial_x f(t,x,\sigma^2)(1-e^{\rho z}), \quad z > 0,$$

and

$$\overline{\mathcal{L}} f(t,x,\sigma^2) := \int_0^\infty \mathcal{L}^z f(t,x,\sigma^2)\nu(dz).$$

3. Main Results

In this section, we introduce our main result, that is, a decomposition formula for the BNS model introduced in Section 2. Recall that the discounted asset price process $\widehat{S}$ is a square-integrable martingale under Assumption 2.1. Thus, for the vanilla call option with strike price $K > 0$ and maturity $T > 0$, its price at time $t \in [0,T]$ is given as

$$V_t := e^{-r\tau_t}\mathbb{E}[BS_T | X_t, \Sigma_t^2].$$

In Theorem 3.1 below, we derive a decomposition expression of V_t by applying Ito's formula to the Black-Scholes function BS. Its proof is postponed until Section 4.

Theorem 3.1. *Under Assumption 2.1, we have, for $t \in [0,T]$,*

$$V_t = BS_t + \tau_t \overline{\mathcal{L}} BS_t + I_1 + I_2 + I_3 + I_4 + I_5. \tag{3.1}$$

Here, $I_1, \dots, I_5$ are defined as follows:

$$I_1 := \mathbb{E}\left[\int_t^T e^{-r(u-t)}\partial_{\sigma^2} BS_u(-\lambda\Sigma_u^2)du \middle| X_t, \Sigma_t^2\right],$$

$$I_2 := \mathbb{E}\left[\int_t^T e^{-r(u-t)}\int_0^\infty \left(\Delta^{\rho z,z} - \Delta^{\rho z,0}\right) BS_u \nu(dz)du \middle| X_t, \Sigma_t^2\right],$$

$$I_3 := \mathbb{E}\left[\int_t^T e^{-r(u-t)}\tau_u \partial_x \overline{\mathcal{L}} BS_u \mu du \middle| X_t, \Sigma_t^2\right],$$

$$I_4 := \mathbb{E}\left[\int_t^T e^{-r(u-t)}\tau_u \partial_{\sigma^2} \overline{\mathcal{L}} BS_u(-\lambda\Sigma_u^2)du \middle| X_t, \Sigma_t^2\right],$$

and

$$I_5 := \mathbb{E}\left[\int_t^T e^{-r(u-t)}\tau_u \int_0^\infty \Delta^{\rho z,z}\overline{\mathcal{L}} BS_u \nu(dz)du \middle| X_t, \Sigma_t^2\right],$$

where $\tau_u := T - u$.

Remark 3.2. In the decomposition formula (3.1), the first two terms in the right hand side are regarded as principal terms. In particular, the second term $\tau_t \overline{\mathcal{L}} BS_t$ represents the impact of the jumps of the asset price process. Indeed, it becomes 0 whenever $\rho = 0$. Note that this term converges to 0 with order 1 as the time to maturity τ_t tends to 0. Here we give interpretations of $I_1, \dots, I_5$ in turn. First of all, we can say that I_1 represents the influence of the continuous fluctuation of the squared volatility process Σ^2. Next, decomposing I_2 into the following two terms

$$\mathbb{E}\left[\int_t^T e^{-r(u-t)} \int_0^\infty \Delta^{0,z} BS_u \nu(dz) du \middle| X_t, \Sigma_t^2\right], \tag{3.2}$$

and

$$\mathbb{E}\left[\int_t^T e^{-r(u-t)} \int_0^\infty \left(\Delta^{\rho z,z} - \Delta^{\rho z,0} - \Delta^{0,z}\right) BS_u \nu(dz) du \middle| X_t, \Sigma_t^2\right], \tag{3.3}$$

we can say that (3.2) represents the impact of the jumps of the squared volatility process, but (3.3) is corresponding to the impact of that jumps occur simultaneously in the asset price process and the squared volatility process. As for the last three terms, the comparison between (3.1) and (4.4) below gives

$$I_3 + I_4 + I_5 = \mathbb{E}\left[\int_t^T e^{-r(u-t)} \overline{\mathcal{L}} BS_u du \middle| X_t, \Sigma_t^2\right] - \tau_t \overline{\mathcal{L}} BS_t.$$

Thus, the sum $I_3 + I_4 + I_5$ is corresponding to the residual part of the impact of the asset price jumps. Each I_3, I_4 and I_5 represents the interaction of the impact of the asset price jumps with the continuous fluctuation of the asset price process, the continuous fluctuation of the squared volatility process, and the fact that jumps occur simultaneously in the asset price and the squared volatility processes, respectively.

Remark 3.3. As mentioned in Section 1, the decomposition formula (3.1) is given as an extension of the result of [2] for Heston model, in which the average squared future volatility $\mathcal{V}_t^2$ has been substituted for the volatility in the Black-Scholes formula, where $\mathcal{V}_t^2$ is defined as

$$\mathcal{V}_t^2 := \frac{1}{\tau_t} \int_t^T \mathbb{E}[\Sigma_u^2 | \Sigma_t^2] du.$$

Note that $\mathcal{V}_t^2$ for the BNS model is given as

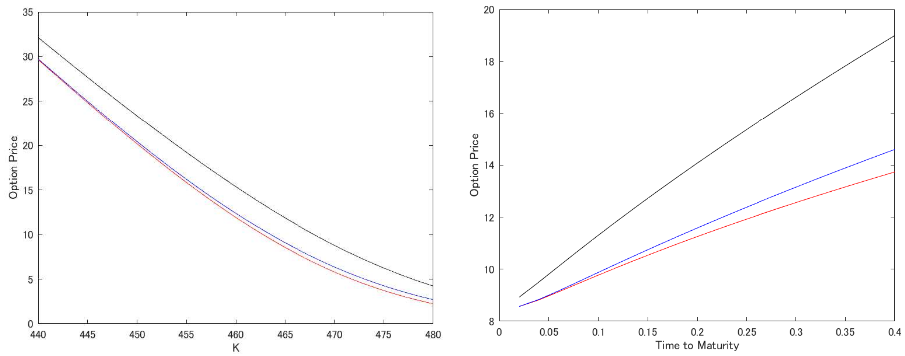
$$\mathcal{V}_t^2 = \frac{\epsilon(\tau_t)}{\tau_t} \Sigma_t^2 + \frac{1}{\lambda}\left(1 - \frac{\epsilon(\tau_t)}{\tau_t}\right) \int_0^\infty z \nu(dz)$$

by (2.4). In this paper, we use the current squared volatility value Σ_t^2, not $\mathcal{V}_t^2$, since the use of Σ_t^2 simplifies our calculations drastically. In addition, as indicated in Figure 1 below, the difference between $BS_t = BS(t, X_t, \Sigma_t^2)$ and $BS(t, X_t, \mathcal{V}_t^2)$ is sufficiently small. Thus, the choice of Σ_t^2 or $\mathcal{V}_t^2$ does not make a big impact.

4. Proof of Theorem 3.1

We shall show Theorem 3.1 by applying Ito's formula twice to the Black-Scholes function.

Step 1. Fix $s, t \in [0, T)$ with $s > t$ arbitrarily for the time being. Note that the function $e^{-ru} BS_u$, $u \in [s, t]$ is sufficiently smooth to apply Ito's formula. From



Panel (A) Panel (B)

Figure 1. We consider the IG-OU case of the BNS model introduced in Example 2.2. We fix t to 0 and set $\rho = -4.7039$, $\lambda = 2.4958$, $a = 0.0872$, $b = 11.98$, $r = 0.01$, $S_0 = 468.44$ and $\Sigma_0 = 0.064262$, where this parameter set comes from Table 5.1 of [18], who used S&P 500 index option price data on November 2, 1993. We note that the above parameter set meets Assumption 2.1. In this figure, we compute the values of V_0, $BS(0, X_0, \Sigma_0^2)$ and $BS(0, X_0, \mathcal{V}_0^2)$. Note that the values of V_0 are computed by the fast Fourier transform-based numerical scheme developed in Section 6 of [7] in order to compute the local risk-minimizing strategies for the BNS model as an extension of the so-called Carr-Madan method. Panel (A) shows the values of V_0, $BS(0, X_0, \Sigma_0^2)$ and $BS(0, X_0, \mathcal{V}_0^2)$ for the call options with strike price $K = 440, 440.1, \ldots, 480$ when the maturity T is fixed to 0.25. In Panel (B), fixing K to 460, and moving T instead from 0.02 to 0.40 at steps of 0.02, we compute the same values for the option with $K = 460$. The black, red and blue curves represent the values of V_0, $BS(0, X_0, \Sigma_0^2)$ and $BS(0, X_0, \mathcal{V}_0^2)$, respectively.

the view of Lemma 4.2 below, we have

$$
\begin{aligned}
e^{-rs} BS_s = e^{-rt} BS_t - r \int_t^s e^{-ru} BS_u du & \\
+ \int_t^s e^{-ru} \partial_t BS_u du + \int_t^s e^{-ru} \partial_x BS_u \left(r + \mu - \frac{\Sigma_u^2}{2} \right) du & \\
+ \int_t^s e^{-ru} (\partial_x BS_u) \Sigma_u dW_u + \frac{1}{2} \int_t^s e^{-ru} (\partial_x^2 BS_u) \Sigma_u^2 du & \\
+ \int_t^s e^{-ru} \partial_{\sigma^2} BS_u (-\lambda \Sigma_u^2) du & \\
+ \int_t^s e^{-ru} \int_0^\infty \Delta^{\rho z, z} BS_{u-} N(du, dz) &
\end{aligned}
$$

$$
\begin{aligned}
&= e^{-rt}BS_t + \int_t^s e^{-ru}\mathcal{D}^{BS}BS_u du + \int_t^s e^{-ru}\partial_x BS_u \mu du \\
&\quad + \int_t^s e^{-ru}(\partial_x BS_u)\Sigma_u dW_u + \int_t^s e^{-ru}\partial_{\sigma^2}BS_u(-\lambda\Sigma_u^2)du \\
&\quad + \int_t^s e^{-ru}\int_0^\infty \Delta^{\rho z, z}BS_{u-}N(du,dz). \qquad (4.1)
\end{aligned}
$$

Now, we take the conditional expectation given X_t and Σ_t^2 on both sides of (4.1). By (2.14) and Lemmas 4.1 and 4.2, we have

$$
\begin{aligned}
e^{-rs}\mathbb{E}[BS_s|X_t,\Sigma_t^2] &= e^{-rt}BS_t + \mathbb{E}\left[\int_t^s e^{-ru}\partial_x BS_u \mu du\Big|X_t,\Sigma_t^2\right] \\
&\quad + \mathbb{E}\left[\int_t^s e^{-ru}\partial_{\sigma^2}BS_u(-\lambda\Sigma_u^2)du\Big|X_t,\Sigma_t^2\right] \\
&\quad + \mathbb{E}\left[\int_t^s e^{-ru}\int_0^\infty \Delta^{\rho z,z}BS_u\nu(dz)du\Big|X_t,\Sigma_t^2\right]. \qquad (4.2)
\end{aligned}
$$

Taking the limitation on the left hand side as s tends to T, we have

$$
\lim_{s\to T}\mathbb{E}[BS_s|X_t,\Sigma_t^2] = \mathbb{E}[BS_T|X_t,\Sigma_t^2],
$$

since $|BS_s| \le \sup_{t\in[0,T]} S_t + K$, which is integrable. Next, the partial derivatives $\partial_x BS$ and $\partial_{\sigma^2}BS$ are positive by (2.15) and (2.17). Thus, the monotone convergence theorem provides that

$$
\lim_{s\to T}\mathbb{E}\left[\int_t^s e^{-ru}\partial_x BS_u du\Big|X_t,\Sigma_t^2\right] = \mathbb{E}\left[\int_t^T e^{-ru}\partial_x BS_u du\Big|X_t,\Sigma_t^2\right] \qquad (4.3)
$$

and

$$
\begin{aligned}
&\lim_{s\to T}\mathbb{E}\left[\int_t^s e^{-ru}\partial_{\sigma^2}BS_u(-\lambda\Sigma_u^2)du\Big|X_t,\Sigma_t^2\right] \\
&= \mathbb{E}\left[\int_t^T e^{-ru}\partial_{\sigma^2}BS_u(-\lambda\Sigma_u^2)du\Big|X_t,\Sigma_t^2\right].
\end{aligned}
$$

Moreover, from the view of the proof of Lemma 4.2, the dominated convergence theorem implies

$$
\begin{aligned}
&\lim_{s\to T}\mathbb{E}\left[\int_t^s e^{-ru}\int_0^\infty \Delta^{\rho z,z}BS_u\nu(dz)du\Big|X_t,\Sigma_t^2\right] \\
&= \mathbb{E}\left[\int_t^T e^{-ru}\int_0^\infty \Delta^{\rho z,z}BS_u\nu(dz)du\Big|X_t,\Sigma_t^2\right].
\end{aligned}
$$

To summarize the above, taking the limitation on both sides of (4.2) as s tends to T, and multiplying e^{rt} on both sides, we obtain

$$
V_t = BS_t + \mathbb{E}\left[\int_t^T e^{-r(u-t)}\partial_{\sigma^2}BS_u(-\lambda\Sigma_u^2)du\Big|X_t,\Sigma_t^2\right]
$$

$$+\mathbb{E}\left[\int_t^T e^{-r(u-t)}\int_0^\infty \{\Delta^{\rho z,z}BS_u+\partial_x BS_u(1-e^{\rho z})\}\nu(dz)du\middle|X_t,\Sigma_t^2\right]$$
$$=BS_t+I_1+I_2+\mathbb{E}\left[\int_t^T e^{-r(u-t)}\overline{\mathcal{L}}BS_u du\middle|X_t,\Sigma_t^2\right], \tag{4.4}$$

since $\mu=\int_0^\infty(1-e^{\rho z})\nu(dz)$.

Step 2. We shall calculate the last term of (4.4). First of all, we fix $t\in[0,T)$ arbitrarily, and define

$$F(u,x,\sigma^2):=e^{-r(u-t)}\tau_u\overline{\mathcal{L}}BS(u,x,\sigma^2),\quad u\in[t,T).$$

Lemma 4.3 ensures that, for any $s,t\in[0,T)$ with $t<s$, $\overline{\mathcal{L}}BS(u,x,\sigma^2)$ is a $C^{1,2,1}$-function on $[t,s]\times\mathbb{R}\times[e^{-\lambda T}\Sigma_0^2,\infty)$. We note that the domain of σ^2 is restricted to $[e^{-\lambda T}\Sigma_0^2,\infty)$ from the view of (2.4). Ito's formula, together with (4.11) in Lemma 4.3, implies

$$\begin{aligned}
F(s,X_s,\Sigma_s^2)&=F(t,X_t,\Sigma_t^2)-r\int_t^s e^{-r(u-t)}\tau_u\overline{\mathcal{L}}BS_u du\\
&\quad-\int_t^s e^{-r(u-t)}\overline{\mathcal{L}}BS_u du+\int_t^s e^{-r(u-t)}\tau_u\partial_t\overline{\mathcal{L}}BS_u du\\
&\quad+\int_t^s e^{-r(u-t)}\tau_u\partial_x\overline{\mathcal{L}}BS_u\left(r+\mu-\frac{\Sigma_u^2}{2}\right)du\\
&\quad+\int_t^s e^{-r(u-t)}\tau_u(\partial_x\overline{\mathcal{L}}BS_u)\Sigma_u dW_u\\
&\quad+\frac{1}{2}\int_t^T e^{-r(u-t)}\tau_u(\partial_x^2\overline{\mathcal{L}}BS_u)\Sigma_u^2 du\\
&\quad+\int_t^s e^{-r(u-t)}\tau_u\partial_{\sigma^2}\overline{\mathcal{L}}BS_u(-\lambda\Sigma_u^2)du\\
&\quad+\int_t^s e^{-r(u-t)}\tau_u\int_0^\infty \Delta^{\rho z,z}\overline{\mathcal{L}}BS_{u-}N(du,dz)\\
&=F(t,X_t,\Sigma_t^2)-\int_t^s e^{-r(u-t)}\overline{\mathcal{L}}BS_u du+\int_t^s e^{-r(u-t)}\tau_u\partial_x\overline{\mathcal{L}}BS_u\mu du\\
&\quad+\int_t^s e^{-r(u-t)}\tau_u(\partial_x\overline{\mathcal{L}}BS_u)\Sigma_u dW_u\\
&\quad+\int_t^s e^{-r(u-t)}\tau_u\partial_{\sigma^2}\overline{\mathcal{L}}BS_u(-\lambda\Sigma_u^2)du\\
&\quad+\int_t^s e^{-r(u-t)}\tau_u\int_0^\infty \Delta^{\rho z,z}\overline{\mathcal{L}}BS_{u-}N(du,dz).
\end{aligned}\tag{4.5}$$

We observe that the above integral with respect to $N(du,dz)$ is also well-defined by Lemma 4.5. Taking the conditional expectation on both sides of (4.5), we have

$$F(s,X_s,\Sigma_s^2)=F(t,X_t,\Sigma_t^2)-\mathbb{E}\left[\int_t^s e^{-r(u-t)}\overline{\mathcal{L}}BS_u du\middle|X_t,\Sigma_t^2\right]$$

$$
\begin{aligned}
&+\mathbb{E}\left[\int_t^s e^{-r(u-t)}\tau_u \partial_x \overline{\mathcal{L}}BS_u \mu du \Big| X_t, \Sigma_t^2\right] \\
&+\mathbb{E}\left[\int_t^s e^{-r(u-t)}\tau_u \partial_{\sigma^2} \overline{\mathcal{L}}BS_u (-\lambda\Sigma_u^2) du \Big| X_t, \Sigma_t^2\right] \\
&+\mathbb{E}\left[\int_t^s e^{-r(u-t)}\tau_u \int_0^\infty \Delta^{\rho z, z} \overline{\mathcal{L}}BS_u \nu(dz) du \Big| X_t, \Sigma_t^2\right] \qquad (4.6)
\end{aligned}
$$

by Lemmas 4.4 and 4.5.

Now, we take limits as s tends to T on both sides of (4.6). A similar argument with the proof of Lemma 4.2 yields

$$
\begin{aligned}
&\lim_{s\to T}\mathbb{E}\left[\int_t^s e^{-ru} \int_0^\infty \Delta^{\rho z, 0} BS_u \nu(dz) du \Big| X_t, \Sigma_t^2\right] \\
&= \mathbb{E}\left[\int_t^T e^{-ru} \int_0^\infty \Delta^{\rho z, 0} BS_u \nu(dz) du \Big| X_t, \Sigma_t^2\right],
\end{aligned}
$$

from which, together with (4.3),

$$
\lim_{s\to T}\mathbb{E}\left[\int_t^s e^{-r(u-t)} \overline{\mathcal{L}}BS_u du \Big| X_t, \Sigma_t^2\right] = \mathbb{E}\left[\int_t^T e^{-r(u-t)} \overline{\mathcal{L}}BS_u du \Big| X_t, \Sigma_t^2\right]
$$

holds. In addition, we have

$$
\begin{aligned}
&\lim_{s\to T}\mathbb{E}\left[\int_t^s e^{-r(u-t)}\tau_u \partial_x \overline{\mathcal{L}}BS_u du \Big| X_t, \Sigma_t^2\right] \\
&= \mathbb{E}\left[\int_t^T e^{-r(u-t)}\tau_u \partial_x \overline{\mathcal{L}}BS_u du \Big| X_t, \Sigma_t^2\right],
\end{aligned}
$$

and

$$
\begin{aligned}
&\lim_{s\to T}\mathbb{E}\left[\int_t^s e^{-r(u-t)}\tau_u \int_0^\infty \Delta^{\rho z, z} \overline{\mathcal{L}}BS_u \nu(dz) du \Big| X_t, \Sigma_t^2\right] \\
&= \mathbb{E}\left[\int_t^T e^{-r(u-t)}\tau_u \int_0^\infty \Delta^{\rho z, z} \overline{\mathcal{L}}BS_u \nu(dz) du \Big| X_t, \Sigma_t^2\right]
\end{aligned}
$$

from the views of the proofs of Lemmas 4.4 and 4.5, respectively. Summarizing the above with Lemmas 4.6 and 4.7, we obtain

$$
\begin{aligned}
&\mathbb{E}\left[\int_t^T e^{-r(u-t)} \overline{\mathcal{L}}BS_u du \Big| X_t, \Sigma_t^2\right] \\
&= F(t, X_t, \Sigma_t^2) + \mathbb{E}\left[\int_t^T e^{-r(u-t)}\tau_u \partial_x \overline{\mathcal{L}}BS_u \mu du \Big| X_t, \Sigma_t^2\right] \\
&\quad+\mathbb{E}\left[\int_t^T e^{-r(u-t)}\tau_u \partial_{\sigma^2} \overline{\mathcal{L}}BS_u (-\lambda\Sigma_u^2) du \Big| X_t, \Sigma_t^2\right] \\
&\quad+\mathbb{E}\left[\int_t^T e^{-r(u-t)}\tau_u \int_0^\infty \Delta^{\rho z, z} \overline{\mathcal{L}}BS_u \nu(dz) du \Big| X_t, \Sigma_t^2\right].
\end{aligned}
$$

This completes the proof of Theorem 3.1.

4.1. Lemmas.

Lemma 4.1.

$$\mathbb{E}\left[\int_t^s e^{-ru}(\partial_x BS_u)\Sigma_u dW_u \middle| X_t, \Sigma_t^2\right] = 0. \tag{4.7}$$

Proof. Since $\widehat{S}$ is a square integrable martingale, $\int_0^t \widehat{S}_u \Sigma_u dW_u$ is also a square integrable martingale. Thus, (2.15) yields that

$$\mathbb{E}\left[\int_0^T e^{-2ru}(\partial_x BS_u)^2 \Sigma_u^2 du\right] \le \mathbb{E}\left[\int_0^T \widehat{S}_u^2 \Sigma_u^2 du\right] < \infty,$$

which implies (4.7). □

Lemma 4.2. *The integral*

$$\int_t^s e^{-ru} \int_0^\infty \Delta^{\rho z, z} BS_{u-} N(du, dz)$$

is well-defined, and we have

$$\begin{aligned}
&\mathbb{E}\left[\int_t^s e^{-ru} \int_0^\infty \Delta^{\rho z, z} BS_{u-} N(du, dz) \middle| X_t, \Sigma_t\right] \\
&= \mathbb{E}\left[\int_t^s e^{-ru} \int_0^\infty \Delta^{\rho z, z} BS_u \nu(dz) du \middle| X_t, \Sigma_t\right]
\end{aligned}$$

for any $s, t \in [0, T)$ with $t < s$.

Proof. From the view of Subsection 4.3.2 (p.231) of Applebaum [6], it suffices to see

$$\int_0^T \int_0^\infty \mathbb{E}[|\Delta^{\rho z, z} BS_u|] \nu(dz) du < \infty.$$

Here, C denotes a positive constant, which may vary from line to line. For $d^\pm$ and $d^\pm_{\rho z, z}$ defined in (2.10) and (2.13) respectively, we have

$$\begin{aligned}
|d^\pm_{\rho z, z} - d^\pm| &\le \frac{|x - \log K + r\tau_t|}{\sqrt{\tau_t}} \left|\frac{1}{\sigma_z} - \frac{1}{\sigma}\right| + \frac{|\rho| z}{\sigma_z \sqrt{\tau_t}} + \frac{|\sigma_z - \sigma|\sqrt{\tau_t}}{2} \\
&\le \frac{|x - \log K + r\tau_t|}{\sqrt{\tau_t}} \frac{|\sigma - \sigma_z|}{\sigma \sigma_z} + \frac{|\rho| z}{\sigma \sqrt{\tau_t}} + \frac{z\sqrt{\tau_t}}{2(\sigma_z + \sigma)} \\
&\le \frac{|x - \log K + r\tau_t|}{\sqrt{\tau_t}} \frac{z}{2\sigma^3} + \frac{|\rho| z}{\sigma \sqrt{\tau_t}} + \frac{z\sqrt{\tau_t}}{4\sigma} \\
&\le C\left(\frac{|x| + 1}{\sqrt{\tau_t}} + \sqrt{\tau_t}\right) \frac{z}{\sigma \wedge \sigma^3},
\end{aligned} \tag{4.8}$$

where σ_z is defined in (2.11). Now, (4.8) implies

$$\begin{aligned}
&|\Delta^{\rho z, z} BS(t, x, \sigma^2)| \\
&= |e^{x_z}\Phi(d^+_{\rho z, z}) - K e^{-r\tau_t}\Phi(d^-_{\rho z, z}) - e^x \Phi(d^+) + K e^{-r\tau_t}\Phi(d^-)| \\
&\le e^{x_z}|\Phi(d^+_{\rho z, z}) - \Phi(d^+)| + e^x |e^{\rho z} - 1|\Phi(d^+) + K e^{-r\tau_t}|\Phi(d^-_{\rho z, z}) - \Phi(d^-)|
\end{aligned}$$

$$
\begin{aligned}
&\leq e^x \frac{1}{\sqrt{2\pi}}|d^+_{\rho z,z} - d^+| + e^x|\rho|z + Ke^{-r\tau_t}\frac{1}{\sqrt{2\pi}}|d^-_{\rho z,z} - d^-| \\
&< C(e^x+1)\left(\frac{|x|+1}{\sqrt{\tau_t}} + \sqrt{\tau_t}\right)\frac{z}{\sigma\wedge\sigma^3} + e^x|\rho|z \\
&< C(e^x+1)(|x|+1)\left(\frac{1}{\sqrt{\tau_t}} + \sqrt{\tau_t} + 1\right)\frac{z}{1\wedge\sigma\wedge\sigma^3}.
\end{aligned}
$$

Note that the second inequality is derived from

$$
|\Phi(d^+_{\rho z,z}) - \Phi(d^+)| = \left|\int_{d^+}^{d^+_{\rho z,z}} \phi(\vartheta)d\vartheta\right| \leq \frac{|d^+_{\rho z,z} - d^+|}{\sqrt{2\pi}},
$$

where ϕ is the probability density function of the standard normal distribution. Since the volatility process Σ is bounded from below by (2.4), we have

$$
\begin{aligned}
&\int_0^T\int_0^\infty \mathbb{E}[|\Delta^{\rho z,z}BS_u|]\nu(dz)du \\
&\leq C\int_0^T\left(\frac{1}{\sqrt{\tau_u}} + \sqrt{\tau_u} + 1\right)du\int_0^\infty z\nu(dz) \\
&\qquad \times\sqrt{\mathbb{E}\left[\left(\sup_{t\in[0,T]} S_t + 1\right)^2\right]\mathbb{E}\left[\left(\sup_{t\in[0,T]}|X_t| + 1\right)^2\right]} \\
&< \infty
\end{aligned}
\tag{4.9}
$$

by (2.8) and (2.7), from which Lemma 4.2 follows. □

Lemma 4.3. *For any t, $s \in [0,T)$ with $t < s$, and any partial derivative operator $\partial \in \{\partial_t, \partial_x, \partial_x^2, \partial_{\sigma^2}\}$, $\partial\overline{\mathcal{L}}BS(u,x,\sigma^2)$ exists for $(u,x,\sigma^2) \in [t,s]\times\mathbb{R}\times[e^{-\lambda T}\Sigma_0^2,\infty)$, and we have*

$$
\partial\overline{\mathcal{L}}BS(u,x,\sigma^2) = \overline{\mathcal{L}}\partial BS(u,x,\sigma^2). \tag{4.10}
$$

In particular,

$$
\mathcal{D}^{BS}\overline{\mathcal{L}}BS(u,x,\sigma^2) = 0 \tag{4.11}
$$

holds for $(u,x,\sigma^2) \in [t,s]\times\mathbb{R}\times[e^{-\lambda T}\Sigma_0^2,\infty)$.

Proof. First of all, we show (4.10) for ∂_x. By the definition of $\overline{\mathcal{L}}$, (2.9) and (2.15), we have

$$
\begin{aligned}
&\partial_x\overline{\mathcal{L}}BS(u,x,\sigma^2) \\
&= \partial_x\int_0^\infty \mathcal{L}^z BS(u,x,\sigma^2)\nu(dz) \\
&= \partial_x\int_0^\infty\Big\{e^{x_z}\Phi(d^+_{\rho z}) - Ke^{-r\tau_u}\Phi(d^-_{\rho z}) - e^x\Phi(d^+) + Ke^{-r\tau_u}\Phi(d^-) \\
&\qquad + e^x\Phi(d^+)(1-e^{\rho z})\Big\}\nu(dz) \\
&= \partial_x\int_0^\infty\left\{e^{x_z}(\Phi(d^+_{\rho z}) - \Phi(d^+)) - Ke^{-r\tau_u}(\Phi(d^-_{\rho z}) - \Phi(d^-))\right\}\nu(dz)
\end{aligned}
$$

$$
\begin{aligned}
&= e^x(1+\partial_x)\int_0^\infty e^{\rho z}(\Phi(d^+_{\rho z})-\Phi(d^+))\nu(dz) \\
&\quad - Ke^{-r\tau_u}\partial_x\int_0^\infty (\Phi(d^-_{\rho z})-\Phi(d^-))\nu(dz).
\end{aligned}
$$

We note that $d^\pm$ and $d^\pm_{\rho z}$ appeared in this proof are defined in (2.10) and (2.12) respectively, but time parameter t is replaced with u. Note that

$$
|\Phi(d^+_{\rho z})-\Phi(d^+)| \le \frac{|d^+_{\rho z}-d^+|}{\sqrt{2\pi}} = \frac{1}{\sqrt{2\pi}}\frac{|\rho|z}{\sigma\sqrt{\tau_u}}.
$$

Thus, $|\Phi(d^+_{\rho z})-\Phi(d^+)|$ is integrable with respect to $\nu(dz)$. Moreover, since ϕ' is bounded, that is, there is a constant $C_{\phi'}>0$ such that

$$
|\phi'(d)| < C_{\phi'} \tag{4.12}
$$

for any $d\in\mathbb{R}$, we have

$$
\begin{aligned}
|\partial_x(\Phi(d^+_{\rho z})-\Phi(d^+))| &= |(\partial_x d^+_{\rho z})\phi(d^+_{\rho z})-(\partial_x d^+)\phi(d^+)| \\
&= \frac{1}{\sigma\sqrt{\tau_u}}|\phi(d^+_{\rho z})-\phi(d^+)| \le \frac{1}{\sigma\sqrt{\tau_u}}\frac{C_{\phi'}|\rho|z}{\sigma\sqrt{\tau_u}},
\end{aligned}
$$

which is also integrable with respect to $\nu(dz)$. Similarly, we can see the integrability of $|\partial_x(\Phi(d^-_{\rho z})-\Phi(d^-))|$. Thus, (4.10) holds when $\partial=\partial_x$ from the view of the dominated convergence theorem.

As for ∂_x^2, we have

$$
\begin{aligned}
&\partial_x^2\overline{\mathcal{L}}BS(u,x,\sigma^2) \\
&= \partial_x\overline{\mathcal{L}}\partial_x BS(u,x,\sigma^2) \\
&= \partial_x\int_0^\infty\left\{\partial_x BS(u,x_z,\sigma^2)-\partial_x BS(u,x,\sigma^2)+\partial_x^2 BS(u,x,\sigma^2)(1-e^{\rho z})\right\}\nu(dz) \\
&= \partial_x\int_0^\infty\left\{e^{x_z}\Phi(d^+_{\rho z})-e^x\Phi(d^+)+\left(e^x\Phi(d^+)+\frac{e^x}{\sigma\sqrt{\tau_u}}\phi(d^+)\right)(1-e^{\rho z})\right\}\nu(dz) \\
&= \partial_x\int_0^\infty\left\{e^{x_z}(\Phi(d^+_{\rho z})-\Phi(d^+))+\frac{e^x}{\sigma\sqrt{\tau_u}}\phi(d^+)(1-e^{\rho z})\right\}\nu(dz) \\
&= e^x(1+\partial_x)\int_0^\infty\left\{e^{\rho z}(\Phi(d^+_{\rho z})-\Phi(d^+))+\frac{1}{\sigma\sqrt{\tau_u}}\phi(d^+)(1-e^{\rho z})\right\}\nu(dz)
\end{aligned}
$$

by (2.16). Thus, we can show (4.10) for ∂_x^2 by a similar argument with the case of ∂_x. Similarly, (4.10) holds for ∂_{σ^2}, since (2.17), together with (4.12), implies that

$$
\begin{aligned}
&\left|\partial_{\sigma^2}\mathcal{L}^z BS(u,x,\sigma^2)\right| \\
&= \left|\frac{\sqrt{\tau_u}}{2\sigma}e^x\left(e^{\rho z}\phi(d^+_{\rho z})-\phi(d^+)\right)+\frac{\sqrt{\tau_u}}{2\sigma}e^x\left(\phi(d^+)+\partial_x d^+\phi'(d^+)\right)(1-e^{\rho z})\right| \\
&\le \frac{\sqrt{\tau_u}}{2\sigma}e^x\left\{e^{\rho z}|\phi(d^+_{\rho z})-\phi(d^+)|+\frac{C_{\phi'}}{\sigma\sqrt{\tau_u}}(1-e^{\rho z})\right\} \\
&\le \frac{C_{\phi'}e^x}{2\sigma^2}(e^{\rho z}|\rho|z+1-e^{\rho z}) \le \frac{C_{\phi'}e^x}{2e^{-\lambda T}\Sigma_0^2}(e^{\rho z}|\rho|z+1-e^{\rho z}),
\end{aligned} \tag{4.13}
$$

which is integrable with respect to $\nu(dz)$. On the other hand, noting that

$$\partial_t d^{\pm} = \frac{x - \log K}{2\sigma \tau_u^{\frac{3}{2}}} - \frac{\eta^{\pm}}{2\sigma\sqrt{\tau_u}}$$

for $u \in [t,s] \subset [0,T)$, where $\eta^{\pm}$ is defined in (2.11), we can see (4.10) for ∂_t similarly.

Summarizing the above, together with (2.14), we have (4.11). □

Lemma 4.4.

$$\mathbb{E}\left[\int_t^s e^{-r(u-t)}\tau_u(\partial_x \overline{\mathcal{L}} BS_u)\Sigma_u dW_u \middle| X_t, \Sigma_t^2\right] = 0$$

for any $s,t \in [0,T)$ with $t < s$.

Proof. We show this lemma by the same way as the proof of Lemma 4.1. To this end, recall that

$$\begin{aligned}\partial_x \overline{\mathcal{L}} BS(u,x,\sigma^2) &= \overline{\mathcal{L}}\partial_x BS(u,x,\sigma^2)\\ &= e^x \int_0^\infty \left\{ e^{\rho z}\left(\Phi(d^+_{\rho z}) - \Phi(d^+)\right) + \frac{\phi(d^+)}{\sigma\sqrt{\tau_u}}(1 - e^{\rho z})\right\}\nu(dz).\end{aligned}$$

Thus, we have

$$\left|\partial_x \overline{\mathcal{L}} BS(u,x,\sigma^2)\right|^2 \le \frac{e^{2x}}{2\pi\sigma^2\tau_u}\left\{\int_0^\infty \left(e^{\rho z}|\rho| z + 1 - e^{\rho z}\right)\nu(dz)\right\}^2,$$

which implies

$$\begin{aligned}&\mathbb{E}\left[\int_t^s e^{-2r(u-t)}\tau_u^2(\partial_x \overline{\mathcal{L}} BS_u)^2\Sigma_u^2 du\right]\\ &\le Ce^{2rT}T\mathbb{E}\left[\int_t^s \widehat{S}_u^2 du\right] \le Ce^{2rT}T^2\mathbb{E}\left[\sup_{u\in[0,T]}|\widehat{S}_u|^2\right] < \infty\end{aligned}$$

for some $C > 0$. This completes the proof of Lemma 4.4. □

Lemma 4.5. *The integral*

$$\int_t^s e^{-r(u-t)}\tau_u \int_0^\infty \Delta^{\rho z, z}\overline{\mathcal{L}} BS_{u-} N(du,dz)$$

is well-defined, and we have

$$\begin{aligned}&\mathbb{E}\left[\int_t^s e^{-r(u-t)}\tau_u \int_0^\infty \Delta^{\rho z, z}\overline{\mathcal{L}} BS_{u-} N(du,dz)\middle| X_t, \Sigma_t^2\right]\\ &= \mathbb{E}\left[\int_t^s e^{-r(u-t)}\tau_u \int_0^\infty \Delta^{\rho z, z}\overline{\mathcal{L}} BS_u \nu(dz)du\middle| X_t, \Sigma_t^2\right]\end{aligned}$$

for any $s,t \in [0,T)$ with $t < s$.

Proof. By the same manner as Lemma 4.2, it suffices to see

$$\int_0^T \tau_u \int_0^\infty \mathbb{E}[|\Delta^{\rho z, z}\overline{\mathcal{L}} BS_u|]\nu(dz)du < \infty. \tag{4.14}$$

Recall that

$$\begin{aligned}\overline{\mathcal{L}}BS(t,x,\sigma^2) &= \int_0^\infty \Big\{ e^{x_z}\Phi(d^+_{\rho z}) - Ke^{-r\tau_u}\Phi(d^-_{\rho z}) - e^x\Phi(d^+) + Ke^{-r\tau_u}\Phi(d^-) \\ &\quad + e^x\Phi(d^+)(1-e^{\rho z})\Big\}\nu(dz) \\ &= \int_0^\infty \Big\{ e^{x_z}\left(\Phi(d^+_{\rho z}) - \Phi(d^+)\right) - Ke^{-r\tau_u}\left(\Phi(d^-_{\rho z}) - \Phi(d^-)\right)\Big\}\nu(dz).\end{aligned}$$

This implies

$$\begin{aligned}&\Delta^{\rho z,z}\overline{\mathcal{L}}BS(t,x,\sigma^2) \\ &= \int_0^\infty \Big\{ \mathcal{L}^w BS(t,x_z,\sigma_z^2) - \mathcal{L}^w BS(t,x,\sigma^2)\Big\}\nu(dw) \\ &= \int_0^\infty \Big\{ e^{x_{z+w}}\left(\Phi(d^+_{\rho z+\rho w,z}) - \Phi(d^+_{\rho z,z})\right) - Ke^{-r\tau_u}\left(\Phi(d^-_{\rho z+\rho w,z}) - \Phi(d^-_{\rho z,z})\right) \\ &\quad - e^{x_w}\left(\Phi(d^+_{\rho w}) - \Phi(d^+)\right) + Ke^{-r\tau_u}\left(\Phi(d^-_{\rho w}) - \Phi(d^-)\right)\Big\}\nu(dw) \\ &= \int_0^\infty \Big\{ e^{x_{z+w}}\int_{d^+_{\rho z,z}}^{d^+_{\rho z+\rho w,z}}\phi(\vartheta)d\vartheta - Ke^{-r\tau_u}\int_{d^-_{\rho z,z}}^{d^-_{\rho z+\rho w,z}}\phi(\vartheta)d\vartheta \\ &\quad - e^{x_w}\int_{d^+}^{d^+_{\rho w}}\phi(\vartheta)d\vartheta + Ke^{-r\tau_u}\int_{d^-}^{d^-_{\rho w}}\phi(\vartheta)d\vartheta\Big\}\nu(dw) \\ &= \frac{\rho}{\sigma_z\sqrt{\tau_t}}\int_0^\infty\int_0^w \Big\{ e^{x_{z+w}}\phi(d^+_{\rho z+\rho\zeta,z}) - Ke^{-r\tau_u}\phi(d^-_{\rho z+\rho\zeta,z})\Big\}d\zeta\nu(dw) \\ &\quad - \frac{\rho}{\sigma\sqrt{\tau_t}}\int_0^\infty\int_0^w \Big\{ e^{x_w}\phi(d^+_{\rho\zeta}) - Ke^{-r\tau_u}\phi(d^-_{\rho\zeta})\Big\}d\zeta\nu(dw) \\ &= \frac{\rho}{\sigma_z\sqrt{\tau_t}}\int_0^\infty\int_0^w \Big\{ e^{x_{z+w}}\phi(d^+_{\rho z+\rho\zeta,z}) - e^{x_{z+\zeta}}\phi(d^+_{\rho z+\rho\zeta,z})\Big\}d\zeta\nu(dw) \\ &\quad - \frac{\rho}{\sigma\sqrt{\tau_t}}\int_0^\infty\int_0^w \Big\{ e^{x_w}\phi(d^+_{\rho\zeta}) - e^{x_\zeta}\phi(d^+_{\rho\zeta})\Big\}d\zeta\nu(dw) \\ &= \frac{\rho e^x}{\sqrt{\tau_t}}\int_0^\infty\int_0^w (e^{\rho w} - e^{\rho\zeta})\Big\{\frac{e^{\rho z}}{\sigma_z}\phi(d^+_{\rho z+\rho\zeta,z}) - \frac{1}{\sigma}\phi(d^+_{\rho\zeta})\Big\}d\zeta\nu(dw). \end{aligned}\tag{4.15}$$

Note that the fifth equality of (4.15) comes from the following general fact:

$$e^x\phi(d^+) = Ke^{-r\tau_t}\phi(d^-)$$

for any $t\in[0,T)$, $x\in\mathbb{R}$ and $\sigma>0$. In addition, the following inequality holds:

$$\begin{aligned}&\left|\frac{e^{\rho z}}{\sigma_z}\phi(d^+_{\rho z+\rho\zeta,z}) - \frac{1}{\sigma}\phi(d^+_{\rho\zeta})\right| \\ &\le \phi(d^+_{\rho z+\rho\zeta,z})\left|\frac{e^{\rho z}}{\sigma_z} - \frac{1}{\sigma}\right| + \frac{1}{\sigma}|\phi(d^+_{\rho z+\rho\zeta,z}) - \phi(d^+_{\rho\zeta})| \\ &\le \frac{1}{\sqrt{2\pi}}\left|\frac{e^{\rho z}-1}{\sigma_z} + \frac{1}{\sigma_z} - \frac{1}{\sigma}\right| + \frac{C_{\phi'}}{\sigma}\Big\{|d^+_{\rho z,z} - d^+| + \frac{|\rho|\zeta}{\sqrt{\tau_t}}\left|\frac{1}{\sigma_z} - \frac{1}{\sigma}\right|\Big\}\end{aligned}$$

$$\leq \frac{1}{\sqrt{2\pi}}\left(\frac{|\rho|z}{\sigma}+\frac{z}{2\sigma^3}\right)+\frac{C_{\phi'}}{\sigma}\left\{C\left(\frac{|x|+1}{\sqrt{\tau_t}}+\sqrt{\tau_t}\right)\frac{z}{\sigma\wedge\sigma^3}+\frac{|\rho|\zeta}{\sqrt{\tau_t}}\frac{z}{2\sigma^3}\right\}$$

for some $C>0$. We observe that $C_{\phi'}$ is the positive constant defined in (4.12), and the last inequality is due to (4.8). Thus, (4.15) is less than

$$Ce^x(|x|+1)\left(\frac{1}{\tau_t}+\frac{1}{\sqrt{\tau_t}}+1\right)\frac{z}{\sigma\wedge\sigma^4}\int_0^\infty(w\wedge w^2)\nu(dw)$$

for some $C>0$. As a result, substituting u, X_u and Σ_u^2 for t, x and σ^2 respectively, we can see (4.14) by a similar way with (4.9). □

Lemma 4.6. $\lim_{s\to T}F(s,x,\sigma^2)=0$ *for any* $x\in\mathbb{R}$ *and* $\sigma>0$.

Proof. First of all, we have

$$\begin{aligned}&\tau_s\overline{\mathcal{L}}BS(s,x,\sigma^2)\\&=\tau_s\int_0^\infty\left\{e^{x_z}\left(\Phi(d_{\rho z}^+)-\Phi(d^+)\right)-Ke^{-r\tau_s}\left(\Phi(d_{\rho z}^-)-\Phi(d^-)\right)\right\}\nu(dz).\end{aligned}$$

Now, we evaluate the above integrand as follows:

$$\begin{aligned}&\tau_s\left|e^{x_z}\left(\Phi(d_{\rho z}^+)-\Phi(d^+)\right)-Ke^{-r\tau_s}\left(\Phi(d_{\rho z}^-)-\Phi(d^-)\right)\right|\\&\leq\tau_s\left\{e^{x_z}\frac{|\rho|z}{\sqrt{2\pi}\sigma\sqrt{\tau_s}}+K\frac{|\rho|z}{\sqrt{2\pi}\sigma\sqrt{\tau_s}}\right\}\leq\sqrt{T}\frac{|\rho|z}{\sqrt{2\pi}\sigma}(e^x+K),\end{aligned}$$

which is integrable with respect to $\nu(dz)$. Thus, the dominated convergence theorem implies

$$\lim_{s\to T}F(s,x,\sigma^2)=\int_0^\infty\lim_{s\to T}e^{-r(s-t)}\tau_s\mathcal{L}^zBS(s,x,\sigma^2)\nu(dz)=0.$$

□

Lemma 4.7.

$$\begin{aligned}&\lim_{s\to T}\mathbb{E}\left[\int_t^s e^{-r(u-t)}\tau_u\partial_{\sigma^2}\overline{\mathcal{L}}BS_u(-\lambda\Sigma_u^2)du\Big|X_t,\Sigma_t^2\right]\\&=\mathbb{E}\left[\int_t^T e^{-r(u-t)}\tau_u\partial_{\sigma^2}\overline{\mathcal{L}}BS_u(-\lambda\Sigma_u^2)du\Big|X_t,\Sigma_t^2\right].\end{aligned}$$

Proof. By (4.13), we have

$$|\partial_{\sigma^2}\overline{\mathcal{L}}BS(u,x,\sigma^2)|\leq C\frac{e^x}{2\sigma^2}$$

for some $C>0$. Thus, we can find a constant $C>0$ such that

$$\left|\int_t^s e^{-r(u-t)}\tau_u\partial_{\sigma^2}\overline{\mathcal{L}}BS_u(-\lambda\Sigma_u^2)du\right|\leq CT^2\sup_{u\in[0,T]}S_u,$$

which is integrable with respect to $\mathbb{P}$. Hence, Lemma 4.7 follows by the dominated convergence theorem. □

5. Conclusions

An Alòs type decomposition formula for the vanilla call option for the BNS model has been derived by using Ito's formula twice. Figure 1 shows that the values of V_0 are away from the values of $BS(0, X_0, \Sigma_0^2)$. This indicates that we need to develop an approximate option pricing formula by using our decomposition formula, but we leave it to future works. Besides, such an approximation would enable us to develop an approximation of implied volatilities and a calibration method for model parameters.

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