2015

# Excluding Two Minors of the Petersen Graph

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# EXCLUDING TWO MINORS OF THE PETERSEN GRAPH

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Department of Mathematics

by
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May 2015

# Acknowledgments

First and foremost I would like to dedicate this to my wife Shoshanah and my beautiful baby girl Kaylee Belle. You are my reasons, my motivations, everything.

I would also like to thank Professor Guoli Ding. He has been invaluable for this project, and is partially responsible for stoking my borderline obsession with the Petersen Graph. Thanks as well to Dr. Oporowski and Dr. Oxley for all of my preliminary graph and matroid theory education.

This would not have been possible without each of you.

# Table of Contents

# Abstract

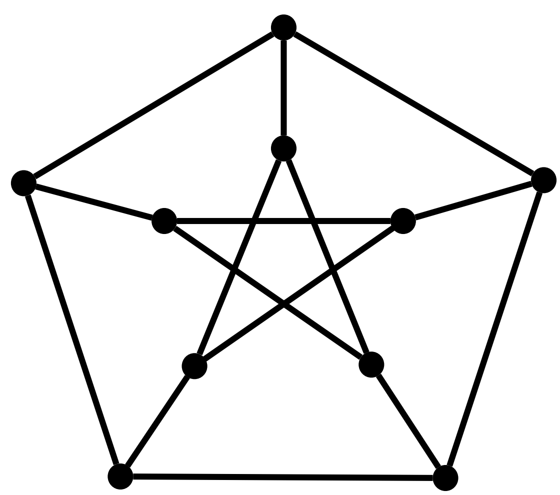
In this dissertation, we begin with a brief survey of the Petersen graph and its role in graph theory. We will then develop an alternative decomposition to clique sums for 3-connected graphs, called T-sums. This decomposition will be used in Chapter 2 to completely characterize those graphs which have no $P_3$ minor, where $P_3$ is a graph with 7 vertices, 12 edges, and is isomorphic to the graph created by contracting three edges of a perfect matching of the Petersen Graph. In Chapter 3, we determine the structure of any large internally 4-connected graph which has no $P_2$ minor, where $P_2$ is a graph on 8 vertices, 13 edges, and is isomorphic to the graph created by contracting two edges of a perfect matching of the Petersen Graph.

# Chapter 1
# Introduction

## 1.1 Background

In 1880, Tait famously proposed a simplification of the 4-color theorem involving polyhedron edges. Notably, he discovered that a 3-edge coloring of the dual of a planar graph could be extended to a 4-coloring of the original graph. Believing this to be the best way to approach the 4-color problem, he set about attempting to prove every cubic graph is 3-edge colorable.

To Tait's dismay, he immediately found that any cubic graph with a bridge could not be 3-edge colored. Tait restated his conjecture in terms of edges of polyhedra. In 1898, Petersen would restate this conjecture as "every bridgeless cubic graph is 1-factorable." To which, he also provided a very elegant counterexample, a counterexample published by Kempe [8] in 1886. This graph would from then on be called the Petersen Graph, which we will denote $P_0$.



$P_0$

If this had been the only appearance of $P_0$, it probably would not have gained the infamy it has today. But, as pervasive as the proverbial bad penny, it kept creeping into other problems. $P_0$ is the smallest snark. It is the smallest hypohamiltonian graph. It is the unique (3,5)-cage. Jaeger has conjectured that every bridgeless graph has a cycle continuous mapping to $P_0$. The Petersen Family of graphs forms the forbidden minor family for linklessly embeddable graphs. In perhaps the most famous Petersen-related conjecture, Tutte [20] proposed that "every bridgeless graph without a $P_0$ minor admits a nowhere-zero 4-flow." There have been recent results towards this. Robertson, Sanders, Seymour, and Thomas have settled this conjecture for cubic graphs in [16], [17], and [4].

**Theorem 1.1.** *Every 2-connected cubic graph with no Petersen minor admits a nowhere-zero 4-flow.*

Recently Wang, Zhang, and Zhang [22] have made progress towards removing the 'cubic' requirement in the previous theorem. Their theorem uses $P_3$, a minor of $P_0$ constructed by contracting two non-adjacent edges.

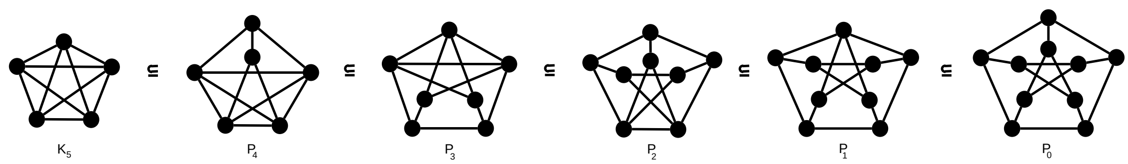
**Theorem 1.2.** *Let $G$ be a bridgeless graph. If $G$ contains no $P_3$ minor, then $G$ admits a nowhere-zero 4-flow.*

Tutte's conjecture for all Petersen-free graphs remains open. This brings us to the crux of the problem. $P_0$ is a counterexample to many a conjecture, and logically, if $P_0$ is a counterexample, then any graph that contains $P_0$ as a minor is also a counterexample. An important question then arises, "Which graphs contain $P_0$? Which do not?"

### 1.2 Related Exclusion Results

As it turns out, this question is a bit complex due to the size of $P_0$. Ten vertices and 15 edges do not make the examination of possible graph expansions from $P_0$

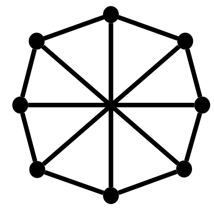
an easy task. Even narrowing the question to 3-connected graphs does little to lessen the difficulty. However, the problem becomes a bit more manageable from the other direction, specifically, when we consider the graphs that are minors of $P_0$. Naturally, if a graph $G$ were a minor of $P_0$, and a third graph $H$ did not contain $G$ as a minor, we could safely say that $H$ does not contain $P_0$ either. Using the Splitter Theorem [18], we may grow $K_5$ to $P_0$ creating the following sequence of graphs.




It should be noted that the intervening graphs are unique. This can be easily seen considering $K_5$ has 5 vertices and 10 edges, while $P_0$ has 10 vertices and 15 edges. The Splitter Theorem allows for only two operations: splitting a vertex and adding an edge. The first operation increases both the number of vertices and edges by 1, whereas the second only increases the number of edges by 1. To achieve the requisite number of vertices for $P_0$, 5 uncontractions are necessary, bringing the total number of edges to 15. As each vertex of $K_5$ has degree 4, it may only be split once. It only remains to see by inspection, that the order and manner in which each vertex is uncontracted is irrelevant.

Continuing our thought exercise, we consider that $K_5$ is a minor of $P_0$. Hence, any graph without a $K_5$ minor is also without a $P_0$ minor. Wagner has given a classification of $K_5$-free graphs already [21].

**Theorem 1.3.** *A graph is $K_5$-free if and only if it can be constructed from $V_8$ and planar graphs by 0-, 1-, 2-, and 3-sums.*

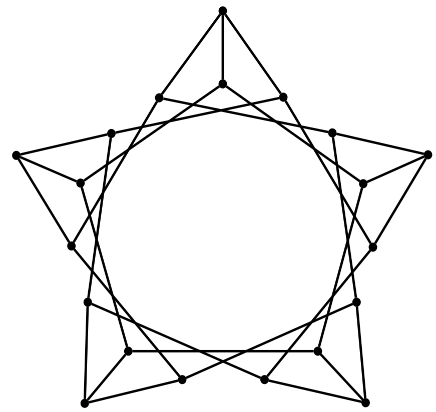



Tutte was able to characterize the family of 3-connected graphs that do not contain $P_4$. Specifically, the graphs that do not contain $P_4$ are exactly those graphs that do not contain $K_5$ along with $K_5$ itself. We can then say that any graph of this type is also $P_0$-free, as clearly $P_4$-free graphs do not contain $P_0$. In fact, observing the sequence of graphs shown previously, we can say that the collections of minor excluded graphs for each graph in the sequence form an ascending chain of inclusion:

$$\{K_5\text{-free }\} \subseteq \{P_4\text{-free }\} \subseteq \{P_3\text{-free }\} \subseteq \{P_2\text{-free }\} \subseteq \{P_1\text{-free }\} \subseteq \{P_0\text{-free }\}$$

Put simply, as we exclude the larger graphs in the sequence, we will obtain collections that are closer to the exclusion of $P_0$. While there are currently no structure theorems for $P_0$-free graphs, if we restrict the types of graphs we consider, there is a result by Robertson, Seymour, and Thomas [14] about $P_0$-free graphs. We call a graph $G$ *apex*, if $G \backslash v$ is planar for some $v \in V(G)$. A graph $G$ is *doublecross* if $G$ can be drawn on the plane with two crossings in the same region.

**Theorem 1.4.** *Let $G$ be a cyclically 5-connected cubic graph with no $P_0$ minor, and assume for every set $A \subseteq V(G)$ with $|A|$, $|V(G) - A| \geq 6$, there are at least 6 edges of $G$ incident with both $A$ and $V(G) - A$. Then $G$ is apex, doublecross, or it is isomorphic to Starfish (shown below).*



Additionally, Robertson, Seymour, and Thomas gave a classification of graphs that do not contain any of the Petersen Family as minors [15]. Petersen family graphs are those seven graphs which are obtained from $K_6$ through $Y - \Delta$ and $\Delta - Y$ exchanges, of which $P_0$ is a member. A graph $G$ has a linkless embedding if it may be embedded in 3-space such that any two disjoint cycles of $G$ have a zero linking number.

**Theorem 1.5.** *The following are equivalent:*

1. *$G$ has a linkless embedding.*
2. *$G$ has no minor isomorphic to a member of the Petersen Family.*

Even though there are currently no explicit structure theorems for Petersen-free graphs, or for that matter any graph which is similar in size to Petersen, there have been several results on the exclusion of smaller graphs, some of which we have already mentioned. Towards the larger end, the families that are Cube-free, Octahedron-free, and $V_8$-free have been characterized by Maharry [9], Ding [3], and Robertson, respectively. Each of these graphs has 12 edges, which makes them rather large for current results. Our goal in Chapter 4 is to give a complete characterization of $P_3$-free graphs, also a 12 edge graph.

### 1.3 Results for $P_3$ and $P_2$

In Chapter 4 of this dissertation, we prove the following structure theorem for $P_3$-free graphs. Details for definitions are given in Chapter 4.

**Theorem 1.6.** *A 3-connected graph $G$ has no $P_3$ minor if and only if $G$ is one of $\{V_8, K_{3,3}^{2,2}, K_{3,3}^{3,2}, K_6, K_5\}$ or $G$ is constructible in the following manner: Let $H$ be constructed from 3-connected planar graphs by repeated 3-sums, then 3-sum copies of $K_5$ to $H$.*

Additionally, in Chapter 5, while we were not able to give a complete characterization of $P_2$-free graphs, we were able to classify those graphs which are quasi-4-connected, sufficiently large, and $P_2$-free. Again details for definitions are given in chapter 5.

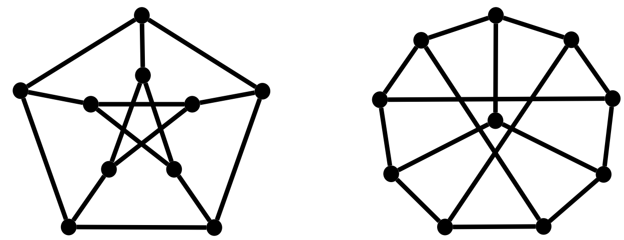
**Theorem 1.7.** *For every integer $n \geq 6$, there exists a number $N$ such that every quasi-4-connected graph $G$ of order at least $N$ contains a $P_2$ minor, unless $G$ is a member of $\boldsymbol{K}_{4,n}$, $\boldsymbol{M}_n$, or $\boldsymbol{SM}_n$.*

# Chapter 2
# Notation and Terminology

## 2.1 Basics

While any *graph* $G$ is a finite collection of elements called *vertices* $V(G)$ together with a collection $E(G)$ of two-element subsets of $V(G)$ called *edges*, it is easier when dealing with graphs to represent them pictorially. We will represent any vertex of $G$ as a dot and any edge of $G$ as a line segment connecting the two relevant vertices. It is possible that an edge of $G$ could connect a vertex to itself, a *loop*, or that an edge of $G$ connects two vertices that are already connected with an edge, a *parallel* edge. Graphs that have parallel edges are called *multigraphs*, graphs without parallel edges and loops are called *simple* graphs. A graph may have any number of vertices and any number of edges. The *order* of $G$ is $|V(G)|$, or the number of vertices in $G$. We will assume that all graphs in this paper are of finite order. Since loops and parallel edges do not change the results of our dissertation, we will assume that all graphs herein are simple graphs.

Note that when representing graphs pictorially, the actual orientation of the vertices and edges is largely irrelevant. We will place vertices in different places for purposes of clarity and aesthetics. The way in which a graph is drawn however, does not change the graph. We consider a graph drawn two different ways to be the same graph. The following two graphs are both Petersen though the representations differ.



As a convention, we label the vertices of a graph with numbers or letters. These serve no purpose save to facilitate tracking the movement of each vertex during graph operations. We will refer to edges by their endpoints, i.e. the edge between vertex $x$ and vertex $y$ will be called $xy$. We say then that vertex $x$ is *adjacent* to vertex $y$, and that vertex $x$ is *incident* with edge $xy$. The collection of vertices that are adjacent to $x$ are called the *neighbors* of $x$.

We will use $G \backslash e$ to represent the deletion of an edge $e$ from a graph $G$. Then $G \backslash e$ is the graph with vertex set $V(G \backslash e) = V(G)$ and edge set $E(G \backslash e) = E(G) - \{e\}$. The deletion of a vertex $v$ from $G$ is similar, though we also by necessity need to remove any edges which were incident with $v$. $G \backslash v$ would then be the graph with vertex set $V(G \backslash v) = V(G) - \{v\}$ and edge set $E(G \backslash v) = \{e \mid e \in E(G)$ and $e$ is not incident with $v\}$.

For a graph $G$, any graph $H$ such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ is called a *subgraph* of $G$. More generally, a subgraph of $G$ is any graph that can be obtained through the deletion of edges and vertices of $G$. We call a graph $H$ an *induced* subgraph of $G$ if $S = V(H) \subset V(G)$ and any two vertices $x$ and $y$ are adjacent in $H$ if and only if they are adjacent in $G$. We can think of an induced subgraph of $G$ as a graph that can be obtained from $G$ by deletion of vertices only. It is natural to think of any subset $S$ of the vertex set $V(G)$ as inducing a particular subgraph of $G$, namely the induced subgraph of $G$ with vertex set $S$, notated as $G[S]$.

## 2.2 Connectivity

The idea of connectivity will be very important in this dissertation. Define $P(v_0, v_n)$ to be a sequence of vertices $v_0, v_1, ..., v_n$ in a graph $G$ such that $v_i \neq v_j$ for $i \neq j$, and $v_{i-1}v_i \in E(G)$ for all $i \in \{1, 2, ..., n\}$. We call $P(u, v)$ a *u,v-path* in $G$. Two vertices $x, y$ of a graph $G$ are said to be connected in $G$ if there is a $P(x, y)$ path in $G$. Again, this is much more natural to think about pictorially. Vertices $x$ and $y$ are connected if we can start at $x$ and "traverse" edges of $G$ to eventually arrive at $y$. We can define then, the length of a path $P(v_0, v_n)$ to be $n$. A graph $G$ is *connected* if there is a non-zero length path between any two vertices of $G$. We call a maximal connected subgraph of a graph $G$ a *component* of $G$, and should $G$ have only a single component, we say $G$ is a connected graph.

This does not mean however, that all of the graphs we will consider here are equally connected. We will need to consider cases of higher connectivity, as higher connectivity gives us many powerful tools to use (the Splitter Theorem chiefly among them).

We will define a *vertex cut*, or *n-cut*, of a connected graph $G$ to be a minimal set of vertices $\{v_1, v_2, ..., v_n\}$ such that $G\backslash\{v_1, v_2, ..., v_n\}$ is disconnected. That is, the deletion of the vertices in the cut will result in $G$ having more than one component. We can use the size of the vertex cuts of $G$ to classify the connectivity of $G$. Suppose $G$ has a minimum vertex cut of size $n$. Then we say $G$ is *k-connected* for all $k \leq n$. As an example, the three neighbors of any vertex in $P_0$ are a vertex cut separating that vertex from the rest of the graph. As $P_0$ does not have a vertex cut of smaller size, we can say $P_0$ is 3-connected. In general, complete graphs $K_n$ can not be disconnected by the removal of any number of vertices. By convention, we consider $K_n$ to have connectivity $n - 1$.

This idea of $k$-connectedness can be linked to paths between vertices. Menger [10] has a very well known theorem relating the two ideas. This theorem takes many forms, but the one used by this dissertation is the following [1].

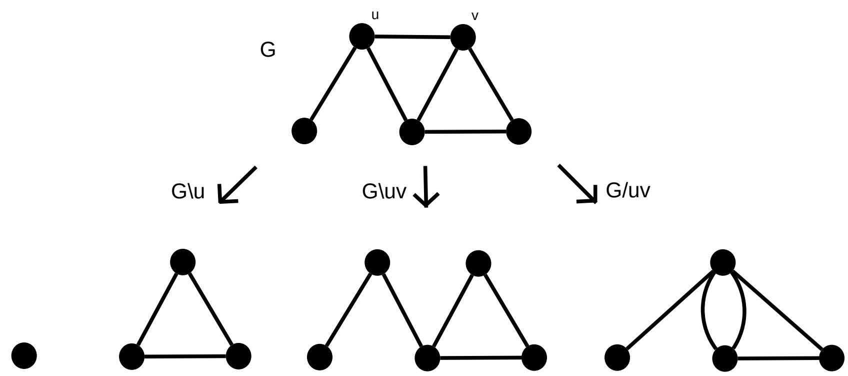
**Theorem 2.1.** *A graph is $k$-connected if and only if it contains $k$ internally disjoint paths between any two vertices.*

We define a *separation* of $G$ to be an ordered pair $(A, B)$ such that $A, B \subset V(G)$, $A \cup B = V(G)$ and there are no edges between $A \backslash B$ and $B \backslash A$. A separation is considered *proper* if $A \backslash B$ and $B \backslash A$ are both non-empty. If $|A \cap B| = k$ then $(A, B)$ is a *$k$-separation*.

Suppose $G$ is a 3-connected graph. If for every proper 3-separation $(A, B)$ of $G$, one of $A$ or $B$ contains exactly 4 vertices, then we say that $G$ is *quasi 4-connected* (q-4-c). We define $G$ to be *weakly 4-connected* (w-4-c) if $G$ is q-4-c and for every proper 3-separation $(A, B)$ where $A \cap B = \{x, y, z\}$, there is at most one edge between the cut vertices $x, y$, and $z$. Lastly we define $G$ to be *internally 4-connected* (i-4-c), if the order of $G$ is at least 5, $G$ is w-4-c, and there are no edges among $x, y$, and $z$. We can see that although $P_0$ is not 4-connected, it is i-4-c. We note that, in particular, $K_4$ is not i-4-c.

## 2.3 Minors

In addition to deleting edges, we may also *contract* an edge $xy$. This is done by deleting the edge $xy$ and identifying the two vertices $x$ and $y$ as a single vertex, and is represented $G/xy$. It should be noted that even if $G$ is a simple graph, this could result in parallel edges. Again it is useful to think of these operations pictorially, as shown below.




Any graph $H$ obtained from $G$ through a series of deletions and contractions of edges and vertices is called a *minor* of $G$, or symbolically $H \subseteq G$.

It should be noted that the operation of vertex deletion is somewhat superfluous if $G$ is connected, as the deletion of a vertex $v$ from $G$ can be simulated by deleting all but a single edge incident with $v$, and then contracting the final edge incident with $v$. If this is the case, we may consider that any minor $H$ of a connected graph $G$ can be obtained through edge contractions and deletions only. It should be noted that in general, the order in which these contractions and deletions are performed is irrelevant. Or equivalently, for any $e, f \in E(G)$, $G\backslash e/f = G/f\backslash e$. A brief proof of this can be found in [13].

We can then think of creating an $H$ minor by first contracting a subset of edges, and then deleting a subset of the remaining edges. It is natural to think of this as contracting subsets of the vertex set to single vertices which will be the vertices of our $H$ minor, and then deleting any unnecessary edges to obtain $H$. We can then partition the vertex set of $G$ into $n$ sets $V_1, V_2, ...V_n$, where each $G[V_i]$ may be contracted to, and hence directly corresponds to $v_i \in V(H)$. It should be noted here, that each $G[V_i]$ is connected. We will call each of these $V_i$ *blocks* of the $H$ minor in $G$.

Two graphs $G$ and $H$ are considered to be isomorphic if there exists an isomorphism $i : V(G) \longrightarrow V(H)$ such that $uv$ is an edge of $G$ if and only if $i(u)i(v)$ is an edge of $H$. Isomorphic graphs will be used interchangeably, and in these cases we will usually just say "$G$ is $H$." As such, when we talk about minors we will typically employ a shortcut in nomenclature. If $G$ contains a minor that is isomorphic to a graph $H$, we will typically say $G$ *contains* $H$ instead of $G$ *contains a minor that is isomorphic to* $H$. If a graph $G$ does not contain $H$ as a minor, we say that $G$ is $H$-free.

We will use a similar expression to refer to graphs without a certain structure. A $\Delta$-*less* graph $G$ is a graph with no triangles as subgraphs. Equivalently, $G$ is $\Delta$-less if and only if there does not exist $\{x, y, z\} \subseteq V(G)$ such that each $x, y, z$ are pairwise adjacent.

Some of the theorems used in this dissertation reference subdivisions of graphs. A graph $H$ is a *subdivision* of $G$, if $H$ can be obtained from $G$ by replacing each edge $xy$ of $G$ with a non-zero length path with ends $x$ and $y$. These paths are called the *branches* of $H$. Clearly then, if $H$ is a subdivision of $G$, $G \subseteq H$.

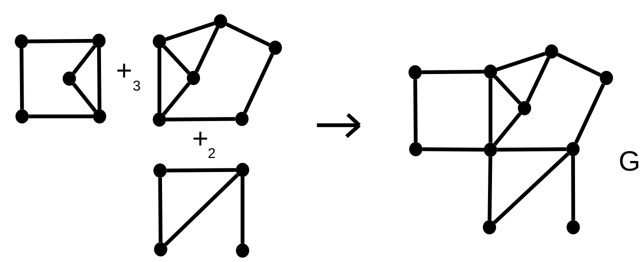
# Chapter 3
# Graphs Sums

## 3.1 Clique Sums

In chapter 4, we provide a method for building all of those graphs which are $P_3$-free by pasting together much smaller graphs which are also $P_3$-free. In order to do this, we need to define a process for summing graphs.

A *clique* in $G$ is a collection of vertices $X \subseteq V(G)$ such that every pair of vertices in $X$ is adjacent. Suppose graphs $G$ and $H$ both contain cliques of size $k$. The *$k$-clique sum*, or equivalently *$k$-sum*, of $G$ and $H$ is formed by identifying pairs of vertices of the two cliques to form a single clique, and then possibly deleting some of the edges between the clique vertices.

It is natural then, to think of creating larger graphs by summing smaller graphs to each other repeatedly. Below is shown a graph $G$, which is the clique sum of several graphs.




Because we *may* delete some of the clique edges each time we $k$-sum, there is an associativity to the $k$-sum operation. If $G$ is the 3-sum of $G_1$, $G_2$, and then $G_3$, then we may obtain $G$ by first summing $G_2$ to $G_3$, and then summing with $G_1$. Consider an edge $e$ within a clique to be summed. If $e$ only participates in a single sum, then there is no issue of order. If $e$ participates in multiple sums, and for instance, is deleted after the sum of $G_2$ and $G_3$, then instead we may just retain edge $e$ after the $G_2$ and $G_3$ sum, and subsequently delete $e$ after summing with $G_1$. In this manner, we can think of larger graphs as being able to be *decomposed* into a sequence of clique sums of smaller graphs.

With regard to a $k$-connected graph $H$, we may use such decompositions to characterize those graphs which do not contain $H$ as a minor. This is a commonly used result, as found in [3].

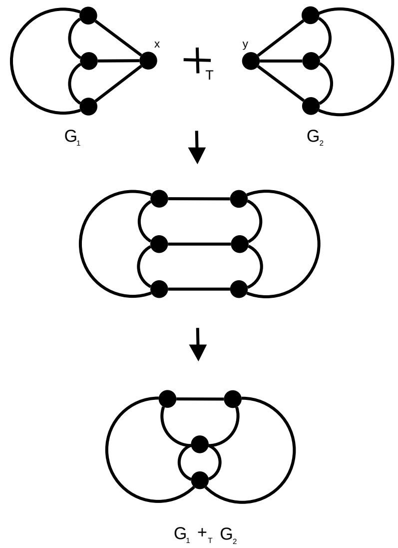
**Theorem 3.1.** *If $H$ is $k$-connected ($k = 1, 2, 3$), then $H$-free graphs are precisely 0-, ..., $(k-1)$-sums of $K_1$, $K_2$, ...,$K_k$ and $k$-connected $H$-free graphs.*

### 3.2 T-sums

In order to give a characterization of $P_3$-free graphs, however, we need to introduce a new graph operation called a *Triad-sum* (from here on will be referred to as a T-sum).

**Definition 3.2.** *Let $G_1, G_2$ be two 3-connected graphs, each containing a cubic vertex $x$ and $y$, respectively. $G$ is then a T(x,y)-sum of $G_1$ and $G_2$ if the order of $G$ is at least the order of $G_1$ and $G_2$, and $G$ is obtained from $G_1$ and $G_2$ by the following process:*

1. *Delete $x$ and $y$.*
2. *Add a matching between the neighbors of $x$ and $y$.*
3. *Contract some, all, or none of the edges in the matching.*
4. *Simplify any parallel edges that may have been created.*




It should be noted that the T-sum of two graphs is not unique due to the construction process. This, however presents no problems with decompositions. Our intent is to utilize these T-sums as an alternative decomposition to clique sums for the purposes of examining graphs that are $P_3$-free. To explore the relationship between T-sums and minor-inclusion, we need the following results.

**Lemma 3.3.** *Let $G$ be a T-sum of two 3-connected graphs $G_1$ and $G_2$. Then $G$ is also 3-connected.*

*Proof.* Let $G_1$ and $G_2$ be 3-connected graphs with cubic vertices $v_1$ and $v_2$, respectively, to be summed. Let $\{x_i, y_i, z_i\}$ be the 3 neighbors of $v_i$ in $G_i$ for $i = 1, 2$. Finally, let $G$ be a T$(v_1, v_2)$-sum of $G_1$ and $G_2$ by adding a $x_1x_2, y_1y_2, z_1z_2$ matching. Note that to facilitate notation, we will first consider the case where none of the matching edges are contracted.

By Menger's Theorem, we know $G$ to be 3-connected if there are 3 internally disjoint $u_1, u_2$ paths in $G$ for any $u_1, u_2 \in V(G)$. Define $A = V(G_1) - \{v_1\}$ and $B = V(G_2) - \{v_2\}$. By symmetry then, there are only two possibilities for the vertices $u_1$ and $u_2$. Either $u_1 \in A$ and $u_2 \in B$, or $u_1, u_2 \in A$.

Suppose $u_1 \in A$ and $u_2 \in B$. Then since $G_1$ is 3-connected, there exist in $G_1$ three internally disjoint paths from $u_1$ to $v_1$. Since $v_1$ is cubic, the existence of these paths also guarantees the existence of internally disjoint paths $P(u_1, x_1), P(u_1, y_1)$, and $P(u_1, z_1)$ in $G_1$. By this same reasoning, $G_2$ contains internally disjoint paths $P(x_2, u_2), P(y_2, u_2)$, and $P(z_2, u_2)$. We see then, that the concatenation of these paths $P(u_1, x_1) \cup P(x_2, u_2), P(u_1, y_1) \cup P(y_2, u_2)$, and $P(u_1, z_1) \cup P(z_2, u_2)$ gives three internally disjoint $u_1, u_2$ paths in $G$.

Suppose instead then, that $u_1, u_2 \in A$. Then there exist three internally disjoint $u_1, u_2$ paths $P_1, P_2$, and $P_3$ in $G_1$ as $G_1$ is 3-connected. Because $v_1$ is cubic, at most one of these paths contains vertex $v_1$. If $v_1$ is contained in none of $P_1, P_2$, or $P_3$, then $P_1, P_2$, and $P_3$ are internally disjoint $u_1, u_2$ paths in $G$. We assume then that $v_1$ is contained in, say $P_1$. We know that $P_1$ must also contain two neighbors of $v_1$, and hence by symmetry, must be of the form $P_1 = P(u_1, x_1) \cup \{v_1\} \cup P(y_1, u_2)$. Now, since $G_2$ is 3-connected, it must contain at least one $x_2, y_2$ path that does not contain the vertex $v_2$. Let $P(x_2, y_2)$ be such a path. If we let $P_1' = P(u_1, x_1) \cup P(x_2, y_2) \cup P(y_1, u_2)$, then we see that $P_1', P_2$, and $P_3$ are three internally disjoint $u_1, u_2$ paths in $G$.

We consider then, the contraction of one of the matching edges in the T-sum, say $e = x_1x_2$. We aim to construct 3 disjoint $u_1, u_2$ paths in $G/e$ for every $u_1, u_2$. First suppose $u_1, u_2 \neq x$ and $u_1 \in A$, $u_2 \in B$. Then we know $G$ contains three disjoint $u_1, u_2$ paths $P_1, P_2, P_3$. At most one of these contains $..., x_1, x_2, ....$ Replacing this sequence with $..., x, ...$ will give the third disjoint path in $G/e$. The case where $u_1 = x$, $u_2 \in A$ is similar. We know there to be 3 disjoint $x_2, u_2$ paths $P_1, P_2, P_3$ in $G$. At most one of these begins $x_2, x_1, ....$ Replacing this sequence with $x, ...$ gives the third disjoint path in $G/e$.

It remains to consider then, that $u_1, u_2 \neq x$ and $u_1, u_2 \in A$. In this case, we know there to be 3 disjoint $u_1, u_2$ paths $P_1, P_2, P_3$ in $G$. If one of these paths, say $P_1$, is of the form $P(u_1, w_1) \cup P(x_1, x_2) \cup P(w_2, u_2)$, then the path $P_1' = P(u_1, w_1) \cup \{x\} \cup P(w_2, u_2)$ along with $P_2, P_3$ will give 3 disjoint $u_1, u_2$ paths in $G/e$. Clearly, if only one of $x_1$ or $x_2$ is contained within $P_1, P_2, P_3$, then the replacement of the included $x_i$ with $x$ gives 3 disjoint paths.

By symmetry, the only remaining possibility is that $x_1 \in P_1$ and $x_2 \in P_2$. We know then that $P_2 = P(u_1, y_1) \cup P(y_2, x_2) \cup [P(x_2, z_2) - \{x_2\}] \cup P(z_1, u_2)$. Since $G_2$ is 3-connected, there are three internally disjoint $y_2, z_2$ paths $P_1', P_2', P_3'$ in $G_2$. At most one of these contains $v_2$ and at most one of these contains $x_2$. There exists at least one then, $P_2'$, which does not contain $v_2$ or $x_2$. We modify $P_2$ by letting $P_2'' = P(u_1, y_1) \cup P_2' \cup P(z_1, u_2)$, and then $P_1, P_2'', P_3$ give 3 internally disjoint $u_1, u_2$ paths in $G/e$.

□

**Lemma 3.4.** *Let $xy$ be an edge of a 3-connected graph $G$, and let $\{x, y, z\}$ be a 3-cut of $G$. Then $G \backslash xy$ is 3-connected unless one of $x$ or $y$ has degree 3.*

*Proof.* Suppose $G$ is 3-connected and $x, y, z$ are vertices as described. Suppose either $x$ or $y$ has degree 3. Then $G \backslash xy$ has a degree 2 vertex and is not 3 connected.

Suppose $G$ is 3-connected, and $x$ and $y$ both have degree greater than 3. We suppose then that $G \backslash xy$ is not 3-connected. It must then be disconnected or contain a 1- or 2- cut. However, should $G \backslash xy$ be disconnected, then $G$ would be either disconnected, or $x$ and $y$ would be cut vertices of $G$. Similar reasoning leads to the conclusion that if $G \backslash xy$ has a cut vertex $v$, then either $v$ is a cut vertex of $G$ or $\{x, v\}$ and $\{y, v\}$ are 2-cuts of $G$. So, we may assume then that $G \backslash xy$ has a 2-cut.

Suppose $G\backslash xy$ contains a 2-cut, $\{u, v\}$. There exists then a separation $(A, B)$ of $G\backslash xy$ such that $A \cap B = \{u, v\}$. Consider then, that for $G$ to have been 3-connected, $x \in A\backslash B$ and $y \in B\backslash A$. Otherwise $\{u, v\}$ would be a 2-cut of $G$ as well. This leaves two possibilities: $z = u$ or $z \neq u$.

In the first case $z = u$, we consider that $G$ is 3-connected and has a separation $(X, Y)$ such that $X \cap Y = \{x, y, z\}$. By symmetry, $v \in X\backslash Y$. We know that $Y\backslash X$ is non-empty, and there are at least 3 edges, each with one end in $\{x, y, z\}$ and the other end in $Y\backslash X$. This, however, contradicts the assumption that $\{z, v\}$ is a 2-cut of $G$.

We may assume then, that $z \neq u$. By symmetry, we may also assume that $z \in A\backslash B$. We note that the scenario where $u, v \in X$ derives a contradiction by the same argument as the $z = u$ case. We assume then, that $u \in X$ and $v \in Y$. By assumption $d(y) \geq 3$ in $G\backslash xy$, and as $\{u, v\}$ is a 2-cut of $G\backslash xy$, we know there are no neighbors of $y$ in $A\backslash B$. This implies that there must be another vertex $a \in B\backslash A$ that is adjacent to $y$. Since $\{x, y, z\}$ is a 3-cut of $G$ we know that there does not exist a $u, v$ path in $B$ disjoint from $y$ containing $a$. This however, implies that $\{y, v\}$ is a 2-cut of $G$, which is a contradiction.

□

**Lemma 3.5.** *Let $G$ be a 3-connected graph, and $\{x, y, z\}$ be a 3-cut of $G$. Let $(A, B)$ be a separation of $G$ such that $A \cap B = \{x, y, z\}$. We create a new graph $G'_A$ by adding one additional vertex $v_A$ to $G[A]$ where $v_A$ is adjacent to $x, y, z$. Then $G'_A$ is 3-connected unless some $u \in \{x, y, z\}$ has degree 2 in $G'_A$.*

*Proof.* Suppose one of $\{x, y, z\}$ has degree 2 in $G'_A$. Clearly, then it is not 3-connected. Now, we assume the opposite, that each of $\{x, y, z\}$ have at least degree 3 in $G'_A$.

Suppose $G'_A$ is not 3-connected. Then there are several possibilities, namely that $G'_A$ is disconnected, only 1-connected, or only 2-connected. However, $G'_A$ being disconnected would imply that $G$ is disconnected, as $x, y, z$ would all belong to a single component of $G'_A$. As $G$ is assumed to be connected, this is a contradiction. We focus on the two remaining possibilities.

Suppose $G'_A$ is only 1-connected, and that $v$ is a cut vertex. Then $G'_A$ has a proper separation $(A', B')$ such that $A' \cap B' = v$. Then neither $A'$ nor $B'$ contains all of $x, y, z$. We may assume by symmetry that $x, y \in A'$ and $z \in B'$. Now, for $\{x, y, z\}$ to be a 3-cut of $G$, there exists $w \in A \backslash B$. Then either $w \in A' \backslash B'$, which would imply that $\{x, y\}$ is a 2-cut of $G$, or $w \in B' \backslash A'$, which would imply $z$ is a cut vertex of $G$. Either case derives a contradiction, since $G$ is 3-connected. Hence $v$ is not a cut vertex.

Suppose, by symmetry that $x$ is a cut vertex of $G'_A$. Then $G'_A$ has a proper separation $(A', B')$ such that $A' \cap B' = x$. As $v$ is not a cut vertex, we may assume that $v, y, z$ are in $A' \backslash B'$ and $x$ is adjacent to another vertex in $A'$. We know then, that $B' \backslash A'$ is not empty, and hence $x$ is a cut vertex of $G$. As $G$ is 3-connected, we know that none of $x, y, z$ is a cut vertex of $G'_A$.

The only remaining possibility is that there is a cut vertex of $G'_A$ which is not any of $v, x, y, z$. There exists a proper separation $(A', B')$ of $G'_A$ such that $A' \cap B' = \{u\}$. However, this would imply that $x, y, z$ are all contained within $A' \backslash B'$, which would imply that $u$ is a cut vertex of $G$. This again derives a contradiction, and we know that $G$ does not have a cut vertex.

We suppose then, that $G'_A$ is only 2-connected. There exists a proper separation $(A'B')$ such that $A' \cap B' = \{u_1, u_2\}$. If $x, y, z$ are all contained within one of $A'$ or $B'$, then $\{u_1, u_2\}$ would be a 2-cut in $G$, which we know not to exist. We may assume then that $x \in A' \backslash B'$ and $y \in B' \backslash A'$. By necessity, $v$ is one of $u_1$ or $u_2$, say $v = u_1$.

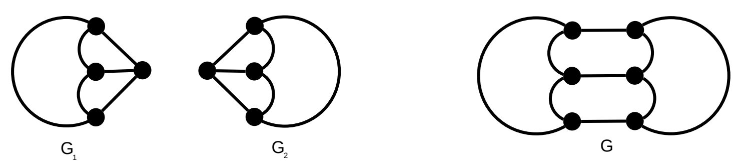
By symmetry, suppose that $z \in A'$. As $y$ has degree at least 3, there exists a vertex other than $y$ in $B' \backslash A'$. This would then imply that $\{y, u_2\}$ is a 2-cut of $G$, a contradiction. As $G'_A$ is known not to be disconnected or have a 1- or 2-cut, $G'_A$ is 3-connected.

□

We now have several results on the exclusion of certain graphs from T-sums.

**Lemma 3.6.** *Let $H$ be a q-4-c and $\Delta$-less graph. Let $G_1$ and $G_2$ be 3-connected $H$-free graphs. Then any T-sum of $G_1$ and $G_2$ is $H$-free.*

*Proof.* Let $G_1$ and $G_2$ be $H$-free 3-connected graphs. Suppose that $u_i$ is a cubic vertex of $G_i$ with neighbors $x_i, y_i, z_i$, $i = 1, 2$. Let $G$ be a T$(u_1, u_2)$-sum of $G_1$ and $G_2$ shown below. Note that we use the T-sum with all edges of the matching present because it contains each of the other cases as a minor. Hence, if it is $H$-free, all T-sums of $G_1$ and $G_2$ are $H$-free.




Define $G_i \backslash u_i = G'_i$, $i = 1, 2$. Now, suppose $G$ contains an $H$ minor, and let $\{V_1, V_2, ...V_n\}$ be the blocks of the $H$ minor in $G$. Let $n_i =$ the number of $V_j$ for which $V_j \cap V(G'_i) \neq \varnothing$. We may assume that $n_2 \geq n_1$, and that $n_1 + n_2 \geq 5$.

Suppose $n_1 \geq 5$, and then suppose that $V(G'_1)$ has a non-empty intersection with $V_1, V_2, V_3, V_4, V_5$. Then $x_1, y_1, z_1$ are contained in at most three different blocks, say $V_1, V_2, V_3$. Then $\{v_1, v_2, v_3\}$ contains a cut of $H$ which separates $v_4$ and $v_5$ from at least 2 other vertices. This is a contradiction, since $H$ is q-4-c.

We suppose then that $n_1 = 4$, and that $V(G_1')$ has a non-empty intersection with $V_1, V_2, V_3, V_4$. Suppose $\{x_1, y_1, z_1\} \subseteq V_j \cup V_k$ for some $j, k = 1, 2, 3, 4$. Then $\{v_j, v_k\}$ would contain a 1- or 2-cut of $H$. As this would imply that $H$ is not 3-connected, we may assume that $x_1, y_1, z_1$ are contained in separate blocks, say $x_1 \in V_1$, $y_1 \in V_2$, and $z_1 \in V_3$. If $x_2 \in V_j$, $j \neq 1, 2, 3, 4$, then $\{v_j, v_2, v_3\}$ is a nontrivial 3-cut of $H$. If $x_2$ is an element of $V_j$, $j = 2, 3$, then $\{v_2, v_3\}$ would be a 2-cut of $H$. As $x_2$ is clearly not an element of $V_4$, we may assume that $x_2 \in V_1$. By using a similar argument, we may assume that $y_2 \in V_2$ and $z_2 \in V_3$. Now, as $H$ is 3-connected, we know that $v_4$ is cubic in $H$. This implies, however, that $\{V_j'\}$ are the blocks of an $H$ minor in $G_2$, where $V_4' = \{u_2\}$ and $V_j' = V_j \cap V(G_2')$ for $j \neq 4$.

Suppose that $n_1 = 3$, and that $V(G_1')$ has a non-empty intersection with $V_1, V_2, V_3$. We may assume that $x_1 \in V_1$, $y_1 \in V_2$, and $z_1 \in V_3$ by the same argument as in the previous case. Suppose $\{x_2, y_2, z_2\} \subseteq V_k \cup V_m$. Then $\{v_k, v_m\}$ contains a 1- or 2-cut of $H$. We may assume then, that each of $x_2, y_2, z_2$ is in a different block. Therefore, we know that $x_2$ is not in either $V_2$ or $V_3$. We now suppose that $x_2 \in V_k$, $k \neq 1, 2, 3$, say $k = 4$. Then since $H$ is 3-connected, we know $v_1$ to be cubic in $H$ and that the edges $v_1v_2$ and $v_1v_3$ exist. Now, if $y_2 \in V_m$, $m \neq 1, 2, 3, 4$, then $v_2v_3$ or $v_2v_4$ must be an edge in $H$. However, this would imply $H$ contains a triangle, which is known to be false. We may assume then, that $y_2 \in V_2$ and $z_2 \in V_3$. This implies that $\{V_j'\}$ are the blocks of an $H$ minor in $G_2$, where $V_1' = \{u_2\}$ and $V_j' = V_j \cap V(G_2')$ for $j \neq 1$.

By symmetry, we may assume that $x_2 \in V_1$, $y_2 \in V_2$, and $z_2 \in V_3$. We know, since $H$ is triangle-free, that $v_1, v_2, v_3$ are not pairwise adjacent. So at most two of $v_1v_2$, $v_2v_3$, $v_1v_3$ exist. Suppose two of the edges exist, $v_1v_2$ and $v_1v_3$. Then $\{V_j'\}$ are the blocks of an $H$ minor in $G_2$, where $V_1' = V_1 \cap V(G_2') \cup \{u_2\}$ and $V_j' = V_j \cap V(G_2')$

for $j \neq 1$. Suppose only one of the edges exists, $v_1v_2$. Then $\{V'_j\}$ are the blocks of an $H$ minor in $G_2$, where $V'_1 = V_1 \cap V(G'_2) \cup \{u_2\}$ and $V'_j = V_j \cap V(G'_2)$ for $j \neq 1$.

Suppose $n_1 = 2$, and that $V(G'_1)$ has a non-empty intersection with $V_1, V_2$. The vertices $x_1, y_1, z_1$ do not all belong to the same block, else $H$ has a cut vertex. We may assume then, that $x_1 \in V_1$ and $y_1, z_1 \in V_2$. Suppose $x_2$ is not an element of $V_1$. Then $v_1$ has degree $\leq 2$ in $H$, and $H$ is not 3-connected. Hence $x_2 \in V_1$. If $\{y_2, x_2\} \subseteq V_j$, $j \neq 2$, then $v_2$ will have degree $\leq 2$ in $H$. We may assume that if $y_2$ and $z_2$ belong to the same block, that block is $V_2$.

In the case where $z_2 \in V_2$, then $\{V'_j\}$ are the blocks of an $H$ minor in $G_2$, where $V'_2 = V_2 \cap V(G'_2) \cup \{u_2\}$ and $V'_j = V_j \cap V(G'_2)$ for $j \neq 2$. We assume then, by symmetry, that neither $z_2$ nor $y_2$ is in $V_2$, and instead that $y_2 \in V_3$ and $z_2 \in V_4$. Then $\{V'_j\}$ are the blocks of an $H$ minor in $G_2$, where $V'_2 = \{u_2\}$ and $V'_j = V_j \cap V(G'_2)$ for $j \neq 2$.

Finally, suppose $n_1 = 1$, $V(G'_1) \subseteq V_1$. Suppose $x_2 \in V_1$. Then $\{V'_j\}$ are the blocks of an $H$ minor in $G_2$, where $V'_1 = V_1 \cap V(G'_2) \cup \{u_2\}$ and $V'_j = V_j \cap V(G'_2)$ for $j \neq 1$. We may assume that none of $x_2, y_2, z_2$ are in $V_1$. Then $\{V'_j\}$ are the blocks of an $H$ minor in $G_2$, where $V'_1 = \{u_2\}$ and $V'_j = V_j \cap V(G'_2)$ for $j \neq 1$.

Clearly $n_1 = 0$ also implies that $G_2$ has an $H$ minor. Hence $G$ must be $H$-free.

□

These results lead us to some very nice Corollaries about families of $H$-free graphs.

**Corollary 3.7.** *Let $H$ be q-4-c and $\Delta$-less. Then $\{$ H-free 3-connected graphs $\} = \{$ graphs constructed by repeated T-sums of $K_4$ and H-free q-4-c graphs $\}$.*

*Proof.* Let $H$ be q-4-c and triangle free, and let $G$ be a T-sum of $H$-free q-4-c graphs. Suppose $G$ is constructed by a single T-sum of 2 such graphs $G_1$ and $G_2$.

Then by Lemma 3.5 $G$ is 3-connected, and by Lemma 3.8 $G$ is $H$-free. Suppose now that $G$ is constructed by $n$ such T-sums, and that after $n-1$ sums, the resulting graphs $G_1$ and $G_2$ are 3-connected and $H$-free. Then $G$ is also 3-connected and $H$-free. By induction, then $G$ is always 3-connected and $H$-free.

We consider the other direction then. Let $H$ be q-4-c and triangle free, and suppose further that $G$ is a counterexample of minimum order. That is, let $G$ be $H$-free, 3-connected but not q-4-c graph that can not be written as the T-sums of $H$-free q-4-c graphs and $K_4$'s. Then $G$ has a proper separation $(A, B)$ such that $A \cap B = \{x, y, z\}$, $|A \backslash B| \geq 2$, and $|B \backslash A| \geq 2$.

It is possible that some of $\{x, y, z\}$ could have only a single neighbor in exactly one of $A$ or $B$. For ease of notation then, we make the following adjustments to our sets $A$ and $B$. For each $u \in \{x, y, z\}$, we do the following: If $u$ has a single neighbor in $A$, we label that neighbor $u_A$, remove $u$ from $A$, and relabel vertex $u$ to $u_B$ in $B$. If $u$ has a single neighbor in $B$, we label that neighbor $u_B$, remove $u$ from $B$, and relabel $u$ to $u_A$ in $A$. If $u$ has 2 or more neighbors in both $A$ and $B$, then we relabel $u$ to $u_A$ in $A$ and to $u_B$ in $B$. We note that exactly one of these three scenarios must be true since $G$ is known to be 3-connected. These new sets we name $A'$ and $B'$, respectively.

Now, we define $G_{A'}$ to be the graph on vertex set $V(G_{A'}) = A' \cup \{v_A\}$ where two vertices in $A'$ are adjacent if and only if they are adjacent in $G$ and vertex $v_A$ is adjacent to $\{x_A, y_A, z_A\}$. Define $G_{B'}$ similarly. We note that through this process, it is possible that $G_{A'}$ or $G_{B'}$ has only 4 vertices. As they are known to be 3-connected, they would be isomorphic to $K_4$.

Now clearly $G$ is a $\mathrm{T}(v_A, v_B)$-sum of 3-connected graphs $G_{A'}$ and $G_{B'}$, and by our assumption then, at least one of $\{G_{A'}, G_{B'}\}$ is not $K_4$ and is either not $H$-free or not q-4-c. However, since each of $\{G_{A'}, G_{B'}\}$ is a minor of $G$, and $G$ is $H$-free, each of them must also be $H$-free. This leaves us with the conclusion that at least one of $\{G_{A'}, G_{B'}\}$ is not $K_4$ and is not q-4-c. We assume $G_{A'}$ is not q-4-c.

If $G_{A'}$ had a 0-, 1-, or 2-cut, then $G$ would also have a 0-, 1-, or 2-cut. We may assume then that $G_{A'}$ is at least 3-connected. But, we know since $|B \backslash A| \geq 2$ that $|G_{A'}| < |G|$. As $G$ was a counterexample of minimum order, we must be able to write $G_{A'}$ as a T-sum of $H$-free q-4-c graphs $H_1, H_2, ...H_n$. If $G_{B'}$ is q-4-c or $K_4$, this would derive a contradiction since it implies $G$ can be written as a T-sum of $H_1, H_2, ...H_n$, and $G_{B'}$. We may assume then that $G_{B'}$ is not q-4-c and not $K_4$.

As this same argument can be used for $G_{B'}$ as $G_{A'}$, counterexample $G$ must not exist, and hence the equality holds.

□

**Corollary 3.8.** *Let $H$ be i-4-c and $\Delta$-less. Then $\{$ $H$-free 3-connected graphs $\}$ = $\{$ graphs constructed by repeated T-sums of $K_4$ and $H$-free i-4-c graphs $\}$.*

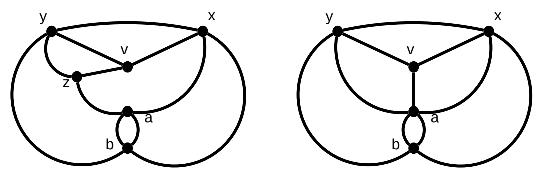
*Proof.* Let $H$ be q-4-c and triangle free, and let $G$ be a T-sum of $H$-free i-4-c graphs and $K_4$'s. Suppose $G$ is constructed by a single T-sum of 2 such graphs $G_1$ and $G_2$. Then by Lemma 3.5 $G$ is 3-connected, and by Lemma 3.8 $G$ is $H$-free. Suppose now that $G$ is constructed by $n$ such T-sums, and that after $n-1$ sums, the resulting graphs $G_1$ and $G_2$ are 3-connected and $H$-free. Then $G$ is also 3-connected and $H$-free. By induction, then $G$ is always 3-connected and $H$-free.

Now, suppose that $G$ is $H$-free and 3-connected. By Corollary 3.9, $G$ is the repeated T-sum of q-4-c $H$-free graphs and $K_4$'s. The corollary then holds if we can write any q-4-c $H$-free graph as T-sums of i-4-c $H$-free graphs and $K_4$'s.

Suppose $G$ is a minimal counterexample, i.e. $G$ is q-4-c, but not i-4-c, and cannot be written as a T-sum of i-4-c graphs and $K_4$'s, but each graph $G'$ which is a minor of $G$ is not a counterexample. Then there exists in $G$ a cubic vertex $v$ with neighbors

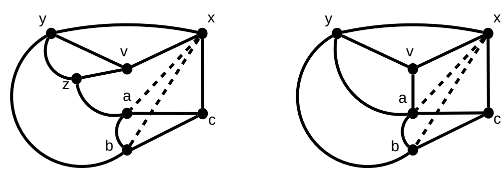
$\{x, y, z\}$ such that $xy$ is an edge of $G$. If $G\backslash xy$ is still q-4-c, then $G$ is the T-sum of $K_4$ and $G\backslash xy$.

Suppose then, $G\backslash xy$ is not q-4-c. Then $G\backslash xy$ has a proper separation $(A, B)$ such that $x \in A\backslash B$ and $y \in B\backslash A$. As $(A, B)$ is a proper separation, and $v$ is adjacent to both $x$ and $y$; $v \in A \cap B$. Let $A \cap B = \{v, a, b\}$. Then, by symmetry, either (1) $z \in B\backslash A$ or (2) $z \in \{a, b\}$. The diagrams below illustrate cases (1) and (2).



(1) (2)

In either case $|A\backslash B| \geq 2$ and $\{x, a, b\}$ is a 3-cut of $G$. Hence $|A\backslash B| = 2$, and there exists $c \in A\backslash B$ such that $c$ is cubic and adjacent to $x, a, b$. In further diagrams dashed edges may or may not exist.

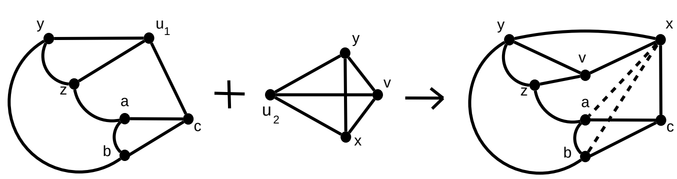


(1) (2)

From here the cases diverge.

Case (1)

In (1), the following diagram demonstrates the construction of $G$ by the T-sum of $G'$ and $K_4$. Note that if $xa$ or $xb$ exists, they may be added using subsequent T-sums of $K_4$'s.

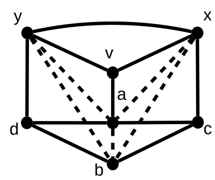


(1)

As $G$ was assumed to be minimal, and $G'$ is a minor of $G$, $G'$ may be written as T-sums of i-4-c $H$-free graphs and $K_4$'s.

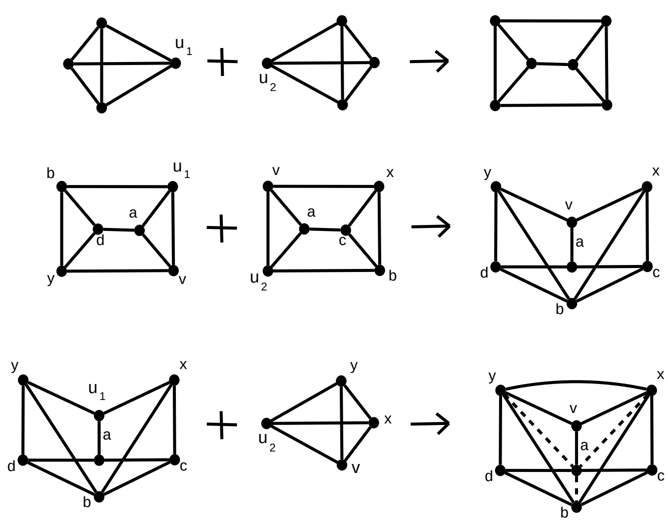
Case (2)

In (2), $\{y, a, b\}$ is also a 3-cut of $G$. Hence $|B\backslash A| = 2$, and there exists $d \in B\backslash A$, a cubic vertex adjacent to $y, a, b$.




(2)

Vertex $b$ must be adjacent to at least one other vertex. It must be adjacent to $y$, otherwise $\{x, a, d\}$ is a nontrivial 3-cut of $G$. Similarly, $bx$ must exist, otherwise $\{y, a, c\}$ is a nontrivial 3-cut of $G$. We demonstrate below, how this graph is the T-sum of five $K_4$'s.




(2)

Note that edges $xa$, $ya$, and $ab$ may be added using more T-sums with $K_4$.

□

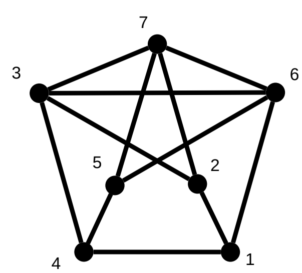
This T-sum structure allows for an alternative to clique sums for the decomposition of 3-connected graphs. We will use this decomposition to classify 3-connected graphs without $P_3$ minors.

# Chapter 4
# Exclusion of $P_3$

Our goal in this chapter is to prove the following result. A graph $G - nK_2$ is obtained from $G$ by deleting $n$ non-adjacent edges.

**Theorem 4.1.** *A 3-connected graph $G$ has no $P_3$ minor if and only if $G$ is one of $\{V_8, K_{3,3}^{2,2}, K_{3,3}^{3,2}, K_6, K_5\}$ or $G$ is constructible in the following manner: Let $H$ be constructed from 3-connected planar graphs by repeated 3-sums, then 3-sum copies of $K_5$ to $H$.*

To do this we consider that to characterize those graphs which are $P_3$-free, we only need to consider those graphs which are 3-connected and $P_3$-free.




$P_3$

From Theorem 3.1 we consider that to characterize those graphs which are $P_3$-free, we only need to consider those graphs which are 3-connected and $P_3$-free, since $P_3$ itself is 3-connected.

**Lemma 4.2.** *{ $P_3$-free graphs } = 0-,1-, and 2-sums of $\{K_1, K_2, K_3\}$ and { $P_3$-free 3-connected graphs }.*

As mentioned earlier, our primary motivation for focusing on 3-connected graphs is that more tools are available to use. To characterize $P_3$-free graphs we will need one of these: the Splitter Theorem [18]. For this theorem we need define a couple of graph operations. Graph $G'$ is created by adding edge $e$ to $G$, if $V(G') = V(G)$ and $E(G') = E(G) \cup \{e\}$. In the cases where we do not care in what position the edge is added, we will denote the graph with the added edge $G + e$. A graph $G'$ is obtained by *splitting a vertex* of $G$, if $G$ can be obtained from $G'$ by contracting an edge $uv$ where both $u$ and $v$ have degree at least 3 in $G$. As the definition implies that the graph created by the vertex split is 3-connected, we only split vertices of degree at least 4.

**Theorem 4.3** (Splitter Theorem). *Suppose a 3-connected graph $H$ (not $K_4$) is a proper minor of a 3-connected graph $G$ (not a wheel), then $G$ has a 3-connected minor $F$ which is obtained from $H$ by either adding an edge or splitting a vertex.*

We note here that since $H$ is 3-connected, any graph $F$ created by adding an edge or splitting a vertex must also must be 3-connected. We also note that this process is iterative. In this way, if $H$ and $G$ are graphs as described above, then there is an entire sequence of 3-connected graphs that can be 'grown' from $H$ to $G$ using the processes of adding edges and splitting vertices. It is with this in mind, that we consider the graph $V_8$.

### 4.1 $V_8$-free Graphs

Let $V_8$ be the graph consisting of an eight vertex cycle, where each vertex in the cycle is adjacent to the diametrically opposed vertex. A picture of $V_8$ can be found on page 4 of this dissertation.

**Lemma 4.4.** *The following are true:*

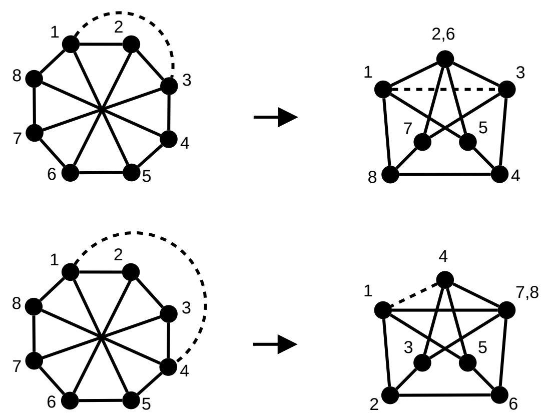
1. *$V_8$ does not contain $P_3$ as a minor.*
2. *$V_8 + e$ does contain $P_3$ as a minor.*

*Proof.* Proof of (1)

$V_8$ has 8 vertices and 12 edges. $P_3$ has 7 vertices and 12 edges. Any deletion of a vertex or contraction of an edge in $V_8$ will result in a graph without the requisite number of edges to contain $P_3$ as a minor.

Proof of (2)

By symmetry, there are exactly 2 non-isomorphic ways to add an edge to $V_8$ and each contain $P_3$ as a minor. These are shown below.



□

These observations give a very nice place to start when considering $P_3$-free 3-connected graphs.

The Splitter Theorem gives much information on the relationship between $V_8$-free and $P_3$-free graphs. Since $V_8$ is 3-connected, 3-connected graphs that contain $V_8$ as a proper minor must have as minors a sequence of 3-connected graphs that can be grown from $V_8$ by edge additions and vertex splits. However, $V_8$ is cubic. Meaning no vertex can be split, and any addition of an edge to $V_8$ contains $P_3$ as a minor. This leads to the following conclusion.

**Corollary 4.5.** { *3-connected $P_3$-free graphs* } $=$ $\{V_8\} \cup$ { *3-connected graphs which contain neither $P_3$ nor $V_8$*}.

From this corollary, the collection of $P_3$-free graphs can be found by first observing which $V_8$-free graphs contain $P_3$ minors. For this we need to define a few graphs.

Let $n \geq 3$. Let $A_n$, the *alternating double wheel*, have vertices $u_1, u_2$, called hub vertices, and vertices $r_1, r_2, ..., r_{2n}$, called rim vertices, where $u_1$ is adjacent to all rim vertices of odd index, $u_2$ is adjacent to all rim vertices of even index, and $r_i$ is adjacent to $r_{i+1}$ mod $2n$ for all $i = 1, 2, ...2n$. The graph $B_n$ is graph $A_n$ with an additional edge between the two hub vertices. The graph $DW_n$ is the graph with hub vertices $u_1, u_2$ and rim vertices $r_1, r_2, ...r_n$ where each hub vertex is adjacent to each rim vertex, $u_1$ is adjacent to $u_2$, and $r_i$ is adjacent to $r_{i+1}$ mod $n$ for all $i = 1, 2, ...n$. The *line graph* of $G$, denoted $L(G)$ is the graph with vertex set $E(G)$ where $e_1$ and $e_2$ are adjacent in $L(G)$ if and only if they are incident with a shared vertex in $G$.

With these definitions we consider an unpublished result of Robertson, which has been used by others, in [3] for example.

**Lemma 4.6.** *If $G$ is a $V_8$-free, i-4-c graph, then $G$ is one of the following:*

- *a planar graph*
- *a graph with fewer than 8 vertices*
- *$DW_n$ or $B_n$, $n \geq 3$*
- *the line graph of $K_{3,3}$*
- *all edges of $G$ are covered by exactly 4 vertices (these graphs are all minors of $K_{4,n}$ for some $n$)*

Since $V_8$ is i-4-c and $\Delta$-less, applying Corollary 3.10 to $V_8$ gives the following Corollary.

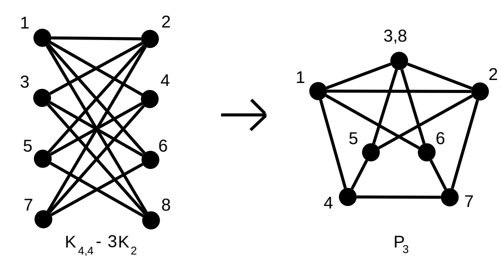
**Corollary 4.7.** { *$V_8$-free 3-connected graphs* } = { *graphs constructed by repeated T-sums of $K_4$ and $V_8$-free i-4-c graphs* }

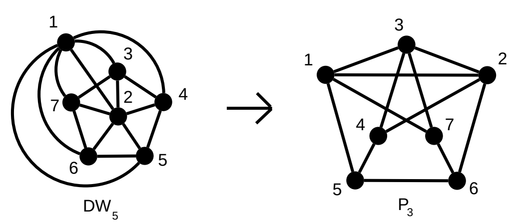
Given this equivalence, we examine the possible structure of a $V_8$-free, i-4-c graph that is also $P_3$-free.

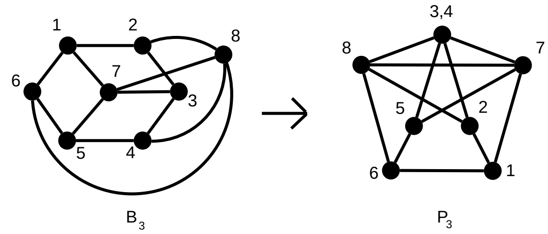
So, considering a $V_8$-free graph $G$ defined as in Lemma 4.6, we can ask: Does $G$ contain $P_3$ as a minor? In the case where $G$ is planar, the answer is clearly no, but the other possibilities for $G$ require a little more analysis. To further explore this, here are some more observations regarding the $V_8$-free graphs with 8 or more vertices.

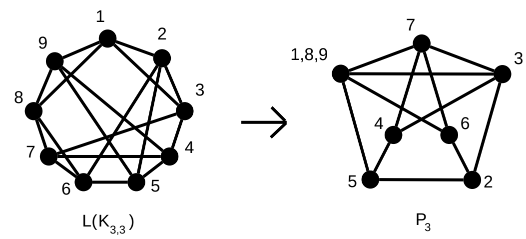
**Lemma 4.8.** *$P_3$ is a minor of $K_{4,4} - 3K_2$, $DW_5$, $B_3$, and the line graph of $K_{3,3}$.*

*Proof.* The following diagrams show $P_3$ is a minor of $K_{4,4} - 3K_2$, $DW_5$, $B_3$ and the line graph of $K_{3,3}$.






□

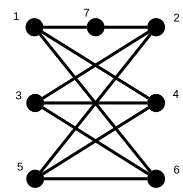
We note as well that further reduction of $K_{4,4} - 3K_2$ results in a graph either planar, in possession of fewer than 8 vertices, or not 3-connected.

In general, if a $V_8$-free graph would contain $P_3$ as a minor, it would need at least 7 vertices and be non-planar. Among the non-planar $V_8$-free graphs with at least 8 vertices, $P_3$ is a minor of $K_{4,4} - 3K_2$, $DW_6$, $B_3$, and $L(K_{3,3})$. As these are all of the minimal graphs of their respective categories with at least 8 vertices, we know that $P_3$ is a minor of every non-planar graph in each of those categories with at least 8 vertices. The only possibilities left to consider are those graphs which are $V_8$-free and have fewer than 8 vertices. In order to contain a $P_3$ minor, such a graph would by necessity have at least 7, and hence exactly 7, vertices and be non-planar. We consider such graphs.

**Lemma 4.9.** *Let $G$ be i-4-c, non-planar, and have 7 vertices. Then $G$ contains a $P_3$ minor.*

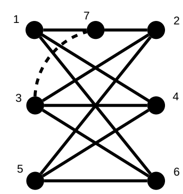
*Proof.* We assume that a graph $G$ has these qualities. Then, we know $G$ must contain either $K_5$ or $K_{3,3}$ as a minor by virtue of its non-planarity.

As $G$ is 3-connected, non-planar, and has 7 vertices, $G$ is known to contain a subgraph isomorphic to a subdivision of $K_{3,3}$ [5]. By symmetry then, $G$ contains the subgraph $D$ shown below.



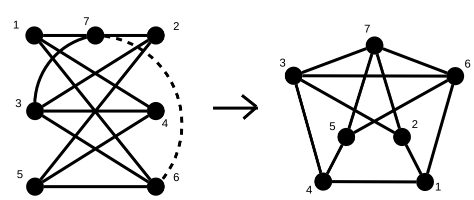

D

We can say more than this when it is considered that $G$ is i-4-c. By this connectivity it is known that vertex 7 must be incident with at least 1 more edge. By symmetry, there is only one possibility for the placement of this edge.




D

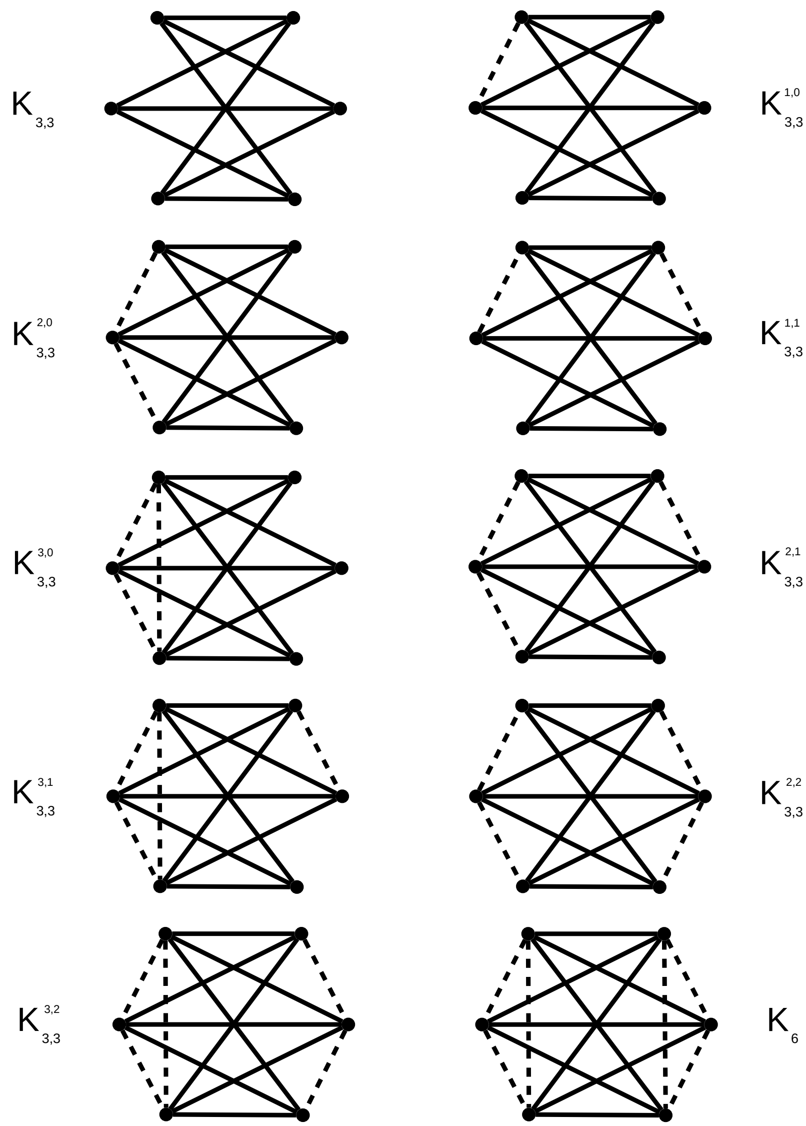
However, this graph is still not i-4-c, as the 3-cut $\{1, 3, 5\}$ separates too much of $G$. Hence, there must exist another edge incident with 7 or 2 and either 4 or 6. Any edge of this type is isomorhpic to any other. The diagram below shows that adding one such of these edges results in a $P_3$ minor.




D $\qquad$ $P_3$

$\square$

We know then, that if $G$ is i-4-c, non-planar, and contains 7 vertices, then $G$ contains $P_3$. It stands to reason, that if a graph $G$ is i-4-c, non-planar, and $P_3$-free, then $G$ has 6 vertices or $G$ is $K_5$. We can construct all of the non-planar, 3-connected graphs with 6 vertices by adding edges to $K_{3,3}$. Let $(X_1, X_2)$ be a partition of the vertex set of $K_{3,3}$ such that each vertex of $X_1$ is adjacent to each vertex of $X_2$. Define $K_{3,3}^{i,j}$, $i, j = 0, 1, 2, 3$, to be the graph obtained by adding $i$ edges between vertices of $X_1$ and $j$ edges between vertices of $X_2$. As every vertex of $K_{3,3}$ is symmetric to each other vertex, this process is well-defined. We note a few things about this notation. First, $K_{3,3}^{i,j}$ is isomorphic to $K_{3,3}^{j,i}$, and second, $K_{3,3}^{3,3}$ is $K_6$. Together, we will call these 10 graphs $\mathbf{K}_{3,3}$. For clarity, diagrams of each of the graphs in $\mathbf{K}_{3,3}$ is shown below.




Not all of the graphs in $\mathbf{K}_{3,3}$ are i-4-c. Inspecting them gives the following lemma.

**Lemma 4.10.** *The set of i-4-c $\{P_3, V_8\}$-free non-planar graphs is $\{K_5,\ K_6,\ K_{3,3}^{3,2},\ K_{3,3}^{2,2},\ K_{3,3}\}$.*

Additionally, every graph in $\mathbf{K}_{3,3}$ other than $K_6$, $K_{3,3}^{3,2}$, and $K_{3,3}^{2,2}$ can be obtained by T-summing copies of $K_4$ to $K_{3,3}$. We also notice that $K_5$, $K_6$, $K_{3,3}^{3,2}$, and $K_{3,3}^{2,2}$ do not participate in any T-sums as these graphs do not contain a cubic vertex. With this we can make the following claim about 3-connected $P_3$-free graphs.

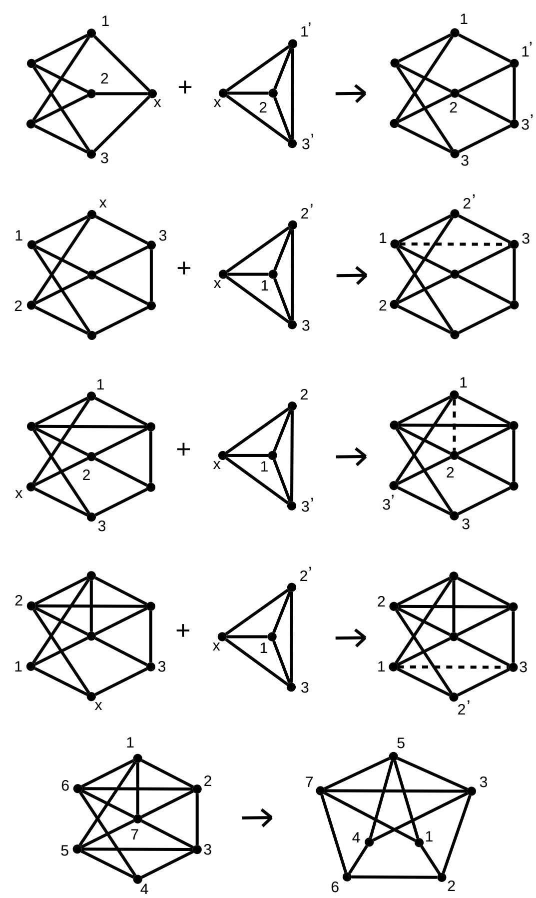
**Lemma 4.11.** *$\{$ $P_3$-free 3-connected graphs $\}$ $\subseteq$ $\{$ $V_8$, $K_5$, $K_6$, $K_{3,3}^{3,2}$, $K_{3,3}^{2,2}$ $\}$ $\cup$ $\{$ graphs constructible from $K_4, K_{3,3}$ and i-4-c planar graphs by repeated T-sums $\}$.*

*Proof.* Suppose $G$ is $P_3$-free and 3-connected. Furthermore, suppose $G$ contains $V_8$. If $G$ is not $V_8$, then the Splitter Theorem tells us that $G$ contains $V_8 + e$ as a minor, and hence $P_3$. Therefore $G$ is $V_8$.

We may assume then, that $G$ is also $V_8$-free. Then by Corollary 4.7 and Lemma 4.9, the lemma holds.

□

It is interesting to note that the above lemma is not, in fact, an equality. All of our T-sum technology works for those graphs that are i-4-c and $\Delta$-less. Though $P_3$ is q-4-c, it is not i-4-c, and it possesses several triangles. That we can say anything at all about those graphs which are $P_3$-free using T-sums is owed entirely to the fact that $P_3$ is very close in structure to a graph that is i-4-c and $\Delta$-less, $V_8$. Below is an example of the construction of a graph that contains a $P_3$ minor from T-sums of $K_{3,3}$ and several $K_4$'s.

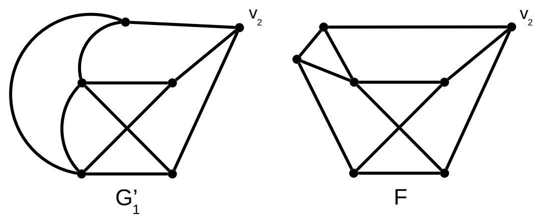


In this vein, we can examine what condition with our T-sums results in the creation of a $P_3$ minor.

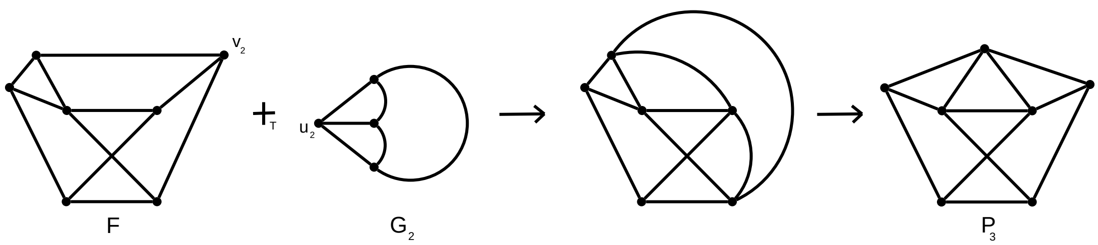
**Lemma 4.12.** *Let $K \in \boldsymbol{K}_{3,3}$ such that $v_1, v_2$ are two adjacent cubic vertices of $K$, and $G_i$ be a 3-connected graph with cubic vertex $u_i$, for $i = 1, 2$. If $G_1'$ is a T$(u_1, v_1)$-sum of $G_1$ and $K$ such that $|G_1'| > |K|$, and $G_2'$ is a T$(u_2, v_2)$-sum of $G_2$ and $G_1'$ such that $|G_2'| > |G_1'|$, then $G_2'$ contains a $P_3$ minor.*

*Proof.* First, if the lemma holds for $K_{3,3}$, then clearly the lemma holds for all members of $\mathbf{K}_{3,3}$. We assume $K = K_{3,3}$.

Let $G_1$ and $G_2$ be 3-connected graphs with cubic vertices $u_1$ and $u_2$, respectively. Then $G_1'$ is the T$(u_1, v_1)$-sum of $G_1$ and $K$. In order to maintain the degree of the vertex $v_2$, in the T-sum of $G_1$ and $K$ we know that the matching edge incident with $v_2$ must not have been contracted. Then $G_1'$ has the structure shown below, and contains the minor $F$ also shown below.




$G_2'$, the T$(u_2, v_2)$-sum of $G_2$ and $G_1'$, would then contain as a minor the T$(u_2, v_2)$-sum of $G_1'$ and $F$, which contains $P_3$.




$\square$

As an immediate consequence of this lemma, we know that if two cubic vertices of a graph in $\mathbf{K}_{3,3}$ are used in the construction of a $P_3$-free graph $G$, then those two cubic vertices are non-adjacent. Due to the construction of $K_{3,3}$, any two non-adjacent cubic vertices used in T-sums to create a $P_3$-free graph would have the exact same set of neighbors. As such, T-sums involving graphs in $\mathbf{K}_{3,3}$ can be simulated with 3-sums of either planar graphs, in the case of $K_{3,3}$ or $K_{3,3}^{i,0}$, or with a 3-sum to a single triangle of $K_5$, in the case of $K_{3,3}^{i,1}$, where $i \geq 1$. Additionally, since only a single triangle of any of these replacement graphs is to be used for simulated 3-sums, we may 3-sum any required replacement graph last. That is to say, perform all other required 3-sums in the construction, then 3-sum any graphs that have replaced members of $\mathbf{K}_{3,3}$.

It is possible however, that a 3-sum of $K_4$ and one of our graphs in $\mathbf{K}_{3,3}$ will contain a $P_3$ minor. In the proof of Theorem 4.1, we will need to consider these when identifying a construction for $P_3$-free graphs. The following Lemma will be useful in those arguments.

**Lemma 4.13.** *Let $G$ be a 3-connected graph with a $P_3$ minor and a triangle. Then any 3-sum $G'$ of $G$ and $K_4$ contains a $P_3$ minor, or is isomorphic to $V_8$.*

*Proof.* Let $G$ be 3-connected and contain a $P_3$ minor. Let $G'$ be the graph obtained from a 3-sum of $G$ and $K_4$. It is clear that $G$ is a minor of $G'$ unless all of the triangle edges over which the 3-sum was done are deleted during the sum. Hence, if two or fewer of the summing edges are deleted after the sum, then $G'$ contains a $P_3$ minor.

Suppose that all three of the of the summing edges are deleted, and that $G'$ does not contain a $P_3$ minor. Since any two of the summing edges may be recovered by contracting the newly added vertex to one of its neighbors, it must be true that all three of the summing edges participated in the $P_3$ minor of $G$. Then $G'$ contains a $V_8$ minor.

Since $G$ contains at least 12 edges by virtue of the fact that it has a $P_3$ minor, $G'$ must have at least 12 edges. If $G'$ has exactly 12 edges, $G' = V_8$. If $G'$ has more than 12 edges, then it contains a $P_3$ minor by Lemma 4.4.

□

## 4.2 Proof of Theorem 4.1

The "*graphs constructible from $K_4$, $K_{3,3}$, and i-4-c planar graphs by repeated T-sums*" part of Lemma 4.10 can be simplified with a few observations. This part of the Lemma can be restated as "*constructible from $K_{3,3}$ and 3-connected planar graphs by repeated T-sums*" since $K_4$ and i-4-c planar graphs are all 3-connected.

Each graph in $\mathbf{K}_{3,3}$ with a cubic vertex is of this type, as a T-sum of $K_{3,3}$ and $K_4$'s can serve to add up to four edges to $K_{3,3}$.

We can see that 3-sums of planar graphs are of this type. Let $G'$ be a graph obtained from a 3-connected planar graph with triangle $\{x, y, z\}$ by adding a cubic vertex $v$ adjacent to $\{x, y, z\}$. Then $G'$ is planar unless $G$ has a proper separation $(X_1, X_2)$ such that $X_1 \cap X_2 = \{x, y, z\}$. However, $G'$ is then the T-sum of $K_{3,3}$, $G[X_1]'$, and $G[X_2]'$. Where $G[X_i]'$ is $G[X_i]$ with an extra vertex $v_i$ adjacent to $\{x, y, z\}$ for $i = 1, 2$.

T-sums with members of $\mathbf{K}_{3,3}$ that have a cubic vertex correspond directly to 3-sums involving $K_5$, where only a single triangle of $K_5$ may ever be used in any 3-sums. These finally give rise to the classification of $P_3$-free 3-connected graphs mentioned in Theorem 1.6.

**Theorem 4.14.** *$\{$ 3-connected $P_3$-free graphs $\} = \{ V_8, K_5, K_6, K_{3,3}^{3,2}, K_{3,3}^{2,2} \} \cup \{$ graphs $G$ such that $G$ is constructed by first taking repeated 3-sums of planar graphs to form $H$, and then 3-summing copies of $K_5$ to $H$ $\}$.*

To prove this theorem, we will prove both directions of the inclusion. These are given below in Lemmas 4.13 and 4.14. For the forward inclusion, Lemma 4.13 is used in conjunction with Lemma 4.10.

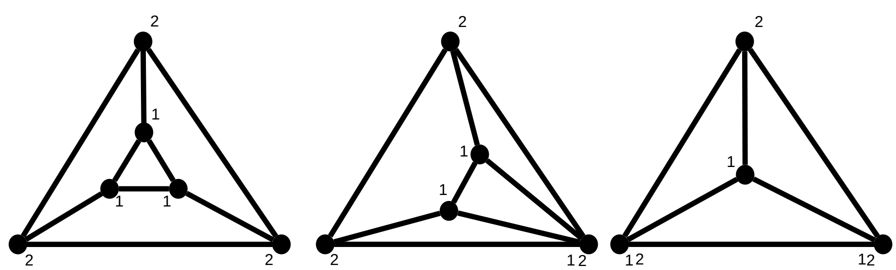
**Lemma 4.15.** *Let $G$ be constructible from T-sums of $K_{3,3}$ and 3-connected planar graphs. If $G$ is $P_3$-free, then $G$ may be constructed by first taking repeated 3-sums of planar graphs to form $H$, and then 3-summing copies of $K_5$ to $H$.*

*Proof.* Suppose $G$ is constructible from T-sums of $K_{3,3}$ and 3-connected planar graph. Then clearly $G$ is constructible from T-sums of members of $\mathbf{K}_{3,3}$ and 3-connected planar graphs. This assumption will help to simplify some of the following arguments.

Suppose $G$ is $P_3$-free, 3-connected, and not one of $\{V_8, K_6, K_{3,3}^{3,2}, K_{3,3}^{-2,2}, K_5\}$. We know by Corollary 4.10 and our prior assumption that $G$ is constructible from T-sums of 3-connected planar graphs and members of $\mathbf{K}_{3,3}$. If we can show that this implies $G$ can be obtained by first taking repeated 3-sums of planar graphs to form $H$, and then 3-summing copies of $K_5$ to $H$, then we have shown the forward inclusion.

Let $G$ be the T-sum of $G_1$ and $G_2$ where $G_1$ and $G_2$ are both either 3-connected and planar or a member of $\mathbf{K}_{3,3}$. Let $v_i$ in $G_i$ be the two cubic vertices to be summed over. Then let $G_i'$ be the graph obtained from $G_i$ by deleting $v_i$ and then adding edges so that each neighbor of $v_i$ in $G_i$ is now adjacent in $G_i'$. We see that if $G_i$ is planar, then $G_i'$ is planar. If $G_i \in \mathbf{K}_{3,3}$, then $G_i'$ is either planar or $K_5$. $G$ is then the 3-sum of $G_1'$ and $G_2'$, each either planar or $K_5$.

Then the $T(v_1, v_2)$-sum of $G_1$ and $G_2$ can be simulated by a 3-sum of planar graphs. Suppose in the T-sum, all matching edges were contracted. Then the T-sum is merely a 3-sum of $G_1'$ and $G_2'$. However, if some of the matching edges were not contracted, then the T-sum is the 3-sum of $G_1'$, $G_2'$ and a bridging graph $H$, where $H$ is one of the following graphs, depending on the number of matching edges contracted. $G_1'$ is summed to the triangle labeled with 1's, and $G_2'$ is summed to the triangle labeled with 2's.




No matching edges contracted

One matching edge contracted

Two matching edges contracted

We suppose then, that $G$ is constructed by the T-sum of several graphs $\{G_i\}$, $i \geq 3$, all of which are either planar or belong to $\mathbf{K}_{3,3}$.

Suppose $G$ is a counterexample of minimum order. That is, suppose $G$ is 3-connected, $P_3$-free, non-planar, and constructible from T-sums of 3-connected planar graphs and members of $\mathbf{K}_{3,3}$, but $G$ cannot be constructed from 3-sums of planar graphs and then copies of $K_5$.

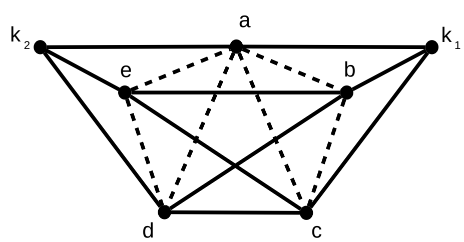
Suppose $G$ is q-4-c. Then $G$ is the 3-sum of an i-4-c graph $G'$ and $n$ $K_4$'s, $n \geq 0$. To see this, suppose $G$ is not i-4-c. Then since $G$ is q-4-c, $G$ has a cubic vertex $v$ such that two of the neighbors of $v$ are adjacent. $G$ is then the 3-sum of $G_1$ and $K_4$ where $G_1$ is the graph $G \backslash v$ with additional edges such that $\{a, b, c\}$, all former neighbors of $v$, are now pairwise adjacent. Clearly $G_1$ is 3-connected, but suppose $G_1$ is not q-4-c. Then $G_1$ has a 3-separation $(X_1, X_2)$ such that $|X_i \backslash X_j| \geq 2$, $i \neq j$. However, all edges of triangle $\{a, b, c\}$ are in either $G_1[X_1]$ or $G_1[X_2]$. This would, in turn, imply that $G$ was not q-4-c. Inductively then, $G$ is the 3-sum of an i-4-c graph $G'$ and $n$ $K_4$'s.

Suppose then, that $G = G' + nK_4$. We will use this short-hand to represent that $G$ is the 3-sum of an i-4-c graph $G'$ and $n$ copies of $K_4$. Additionally, we may assume that $G'$ is non-planar. If $G'$ is planar then we have a required construction for $G$. Since $G$ is q-4-c, it must also be true that all of the $K_4$'s are summed to some triangle of $G'$.

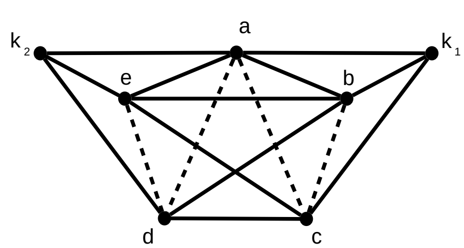
If $G'$ is i-4-c, non-planar, and a minor of $G$, and hence $P_3$-free, then by Lemma 4.10, $G' \in \{V_8,\ K_5,\ K_6,\ K_{3,3}^{3,2},\ K_{3,3}^{2,2}\}$. If $G' = V_8$, then $n = 0$, since $V_8$ has no triangles. We may assume that $G \in \{K_5,\ K_6,\ K_{3,3}^{3,2},\ K_{3,3}^{2,2}\}$. However, any 3-sum of one of $\{K_6,\ K_{3,3}^{3,2},\ K_{3,3}^{2,2}\}$ and $K_4$ contains $P_3$, and by Lemma 4.13, $G$ would then contain $P_3$ or be isomorphic to $V_8$. It must be true that $G' = K_5$.

If $n = 0, 1$, then $G$ is a 3-sum of planar graphs and then $K_5$. So we assume $n \geq 2$. If all of the $K_4$'s are summed to the same triangle of $G'$, then again, we have the required construction $G$. We assume then, that there are at least two different triangles of $G'$ that are used to 3-sum copies of $K_4$.

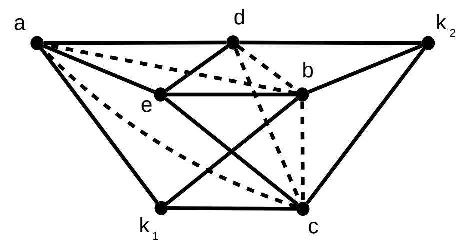
There are two possibilities for the two triangles to be summed over. Either they share a single vertex, or they share two. Suppose they share exactly one vertex. Let $V(G') = \{a, b, c, d, e\}$. Suppose that $K1$ is a copy of $K_4$ with vertex set $\{a, b, c, k_1\}$, and that $K1$ is 3-summed to $G'$ over triangle $\{a, b, c\}$. Call this graph $G_1$. Suppose further that $K2$ is summed to $G_1$ over a triangle that shares only a single vertex with $\{a, b, c\}$, say $\{a, d, e\}$. Then $G_2$ is the graph shown below, where the dashed edges may or may not be present.



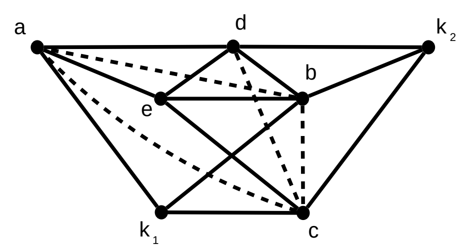

Suppose that in the process of creating $G_2$, edges $ae$ and $ad$ were both deleted. Then $\{k_2, b, c\}$ would be a non-trivial 3-cut of $G$, since all future $K_4$'s would by necessity be summed to one side of the separation or the other. Since the pair $ae$ and $ad$ and the pair $ab$ and $ac$ are symmetric, we may assume that edges $ab$ and $ae$ survived the 3-sums with $K1$ and $K2$. Then $G_2$ is the graph shown below, where the dashed edges may or may not be present.



We see then, that $G_2$ contains a $P_3$ minor. Inductively, using Lemma 4.13, $G$ is either $V_8$ or contains a $P_3$ minor. We may assume then that $K2$ was summed to $G_1$ over a triangle that shared two vertices with $\{a, b, c\}$, say $\{b, c, d\}$. Then $G_2$ is the graph shown below where the dashed edges may or may not be present.




Suppose that in the process of constructing $G_2$ all of $ab, ac, bd, cd$ were deleted. Then $\{k_1, k_2, e\}$ would be a non-trivial 3-cut of $G$, since all future $K_4$'s would by necessity be summed to one side of the separation or the other. Since all four of these edges are symmetric to one another, we may assume that $bd$ exists.




Again, we can see that $G_2$ contains a $P_3$ minor. Inductively, with Lemma 4.13, this would imply that $G$ is either $V_8$ or contains a $P_3$ minor. We may assume then, that if $G$ is a q-4-c, $P_3$-free graph, then $G$ can be written as the 3-sum of planar graphs and then $K_5$.

We assume then, that $G$ is not q-4-c. Then $G$ has a separation $(X_1, X_2)$ such that $|X_1 \backslash X_2|$ and $|X_2 \backslash X_1|$ are both at least 2.

Let $X_1 \cap X_2 = \{v_1, v_2, v_3\}$. For each $v_j$, $j = 1, 2, 3$, consider if $v_j$ has exactly 1 neighbor in either $X_1$ or $X_2$. If $v_j$ has exactly one neighbor in $X_i$, then delete $v_j$ from that $X_i$. Since each vertex has degree at least 3, we will potentially remove each $v_j$ from at most one $X_i$. After doing this, label the new sets $X'_1$ and $X'_2$. Clearly then, $G$ is the T-sum of $G[X'_1]^+$ and $G[X'_2]^+$, where $G[X'_i]^+$ is the graph of $G[X'_i]$ with an additional vertex $x_i$ added such that $x_i$ is adjacent to each of $v_j$ if they are still in $X'_i$ and adjacent to the lone neighbor of $v_j$, if $v_j$ was removed from $X_i$.

Let $G'_i$ be the graph of $G[X'_i]^+ \backslash x_i$ with edges added so that all neighbors of $x_i$ are pairwise adjacent. Each $G'_i$ is 3-connected unless $|G[X'_i]^+| = 4$, which can only happen if at least two $v_j$ were removed from $X_i$. Suppose $|G[X'_1]^+| = 4$. Then $G$ is the 3-sum of $G'_2$ and a bridging graph. The bridging graph required depends on whether 2 or 3 vertices were removed from $X_1$ to create $X'_1$.

So, $G$ is the 3-sum of $G'_1$, $G'_2$ , and potentially a bridging graph. As each of the bridging graphs are 3-connected and planar, if we can show that each $G'_i$ can be constructed by 3-sums of planar graphs and $K_5$'s, then we are done.

Consider $G'_1$. We know that $G'_1$ is 3-connected and $P_3$-free. Then $G'_1$ is either one of $\{K_6, K^{3,2}_{3,3}, K^{2,2}_{3,3}\}$ or is constructible from 3-sums of planar graphs and $K_5$'s by the minimality of $G$. However, if $G'_1 \in \{K_6, K^{3,2}_{3,3}, K^{2,2}_{3,3}\}$, then $G[X'_1]^+$ contains a $P_3$ minor. This would imply that $G$ contains a $P_3$ minor, which is a contradiction since $G$ was assumed to be $P_3$-free.

We may assume then, that each $G'_i$ is constructible from 3-sums of planar graphs and then $K_5$'s. We consider then, which triangle of $G'_1$ and $G'_2$ are being summed over to create $G$. If the triangles to be used in the sum came from planar components of the decompositions of each $G'_i$, then we have a construction for $G$ by 3-sums of planar graphs and then $K_5$'s since we could always 3-sum any $K_5$'s last, as any given $K_5$ has a single triangle over which any sums would take place.

Suppose then that $G'_1$ is the 3-sum of planar graphs and at least one $K_5$. Let the vertices of this $K_5$ be $\{a, b, c, d, e\}$, and let the triangle used in the construction of

$G_1'$ be $\{a, b, c\}$. Suppose further that the triangle over which $G_1'$ and $G_2'$ are to be summed is $\{a, d, e\}$.

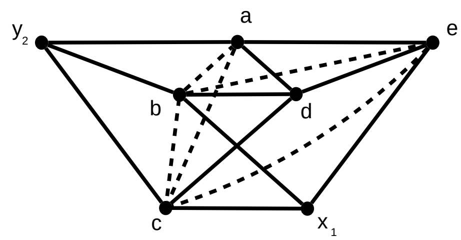
The graph $H = G[X_1']^+$ can be thought of as the graph of $G_1'$ with an additional vertex $x_1$ adjacent to $\{a, d, e\}$ with the possible deletion of some of the edges among $\{a, d, e\}$. $H$ has a separation $(Y_1, Y_2)$ such that $Y_1 \cap Y_2 = \{a, b, c\}$. By symmetry, suppose $d \in Y_2$. Define vertex sets $Y_i'$ from $Y_i$ according to the same methods as used for $X_i'$. Then define $H[Y_i']^+$ in the same manner as used for $G[X_i']+$ and label the added vertices $y_i$. $G$ is then a $\mathrm{T}(y_1, y_2)$-sum of $H[Y_1']^+$ and $H[Y_2']^+$, and then a $\mathrm{T}(x_1, x_2)$-sum with $G[X_2']^+$.

By this construction, $H[Y_2']^+$ has either 6 or 7 vertices depending on whether vertex $a$ was removed from $Y_2$ during the creation of $Y_2'$. If it has order 7, then $H[Y_i']^+$ is the T-sum of a member of $\mathbf{K}_{3,3}$ and another graph. In this case, however, the T-sum of $H[Y_2']^+$ and $G[X_2']^+$ would contain at least 8 vertices, and imply by Lemma 4.12 that $G$ contains a $P_3$ minor.

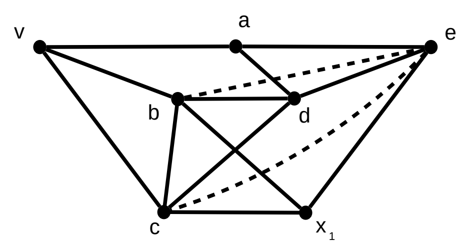
If $H[Y_2']^+$ has only 6 vertices, then it is a member of $\mathbf{K}_{3,3}$, and the T-sums constructing $G$ would imply by Lemma 4.12 that $G$ has a $P_3$ minor unless $H[Y_1']^+ = K_4$ and the two matching edges not incident with $x_1$ in the $\mathrm{T}(y_1, y_2)$-sum are contracted. This would imply that $G_1'$ has order 5, and hence is either planar or $K_5$. In either case, we have an acceptable construction for $G$.

We may assume then that instead $G_2'$ is to be summed to $G_1'$ over a triangle that shares 2 vertices with $\{a, b, c\}$, say $\{b, c, d\}$.

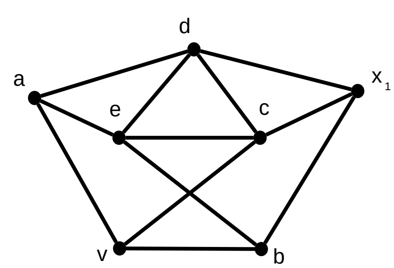
The graph $H = G[X_1']^+$ can be thought of as the graph of $G_1'$ with an additional vertex $x_1$ adjacent to $\{b, c, d\}$ with the possible deletion of some of the edges among $\{b, c, d\}$. $H$ has a separation $(Y_1, Y_2)$ such that $Y_1 \cap Y_2 = \{a, b, c\}$. By symmetry, suppose $e \in Y_2$. Define vertex sets $Y_i'$ from $Y_i$ according to the same methods as used for $X_i'$. Then define $H[Y_i']^+$ in the same manner as used for $G[X_i']+$ and label the added vertices $y_i$. $G$ is then a $\mathrm{T}(y_1, y_2)$-sum of $H[Y_1']^+$ and $H[Y_2']^+$, and then a $\mathrm{T}(x_1, x_2)$-sum with $G[X_2']^+$. $H[Y_2']^+$ is shown below where the dashed edges may or may not be present.




We see from the diagram that if the $\mathrm{T}(y_1, y_2)$-sum of $H[Y_1']^+$ and $H[Y_2']^+$ allows for the creation of edges $ab$ or $ac$, then $G[X_1']^+$, and hence $G$, would contain a $P_3$ minor. It must be true then that $H[Y_1']^+ = K_4$ and all matching edges not incident with vertex $a$ in the $\mathrm{T}(y_1, y_2)$-sum are contracted. $G[X_1']^+$ must then be the following graph where the dashed edges may or may not be present.




Now since $G$ has at least one more vertex than $G[X_1']^+$, at least one of the edges added with the $\mathrm{T}(x_1, x_2)$-sum may be contracted to create edge $bd$ or $cd$. As these are symmetric, we will assume $cd$. This, however, derives another contradiction since it demonstrates a $P_3$ minor, and $G$ was assumed to be $P_3$-free.



It must be true then, that all 3-sums involving $K_5$ components are done over a single triangle, and hence $G$ is constructible from 3-sums of planar graphs and then $K_5$'s.

$\square$

**Lemma 4.16.** *Let $G$ be a graph obtained by first taking repeated 3-sums of planar graphs to form $H$, and then 3-summing copies of $K_5$ to $H$. Then $G$ is $P_3$-free.*

*Proof.* Suppose a counterexample $G$ exists. That is, there exists a graph $G$ such that $G$ is obtained by first taking repeated 3-sums of planar graphs to form $H$, and then 3-summing copies of $K_5$ to $H$. Suppose such a $G$ contains a $P_3$ minor. If such a counterexample exists, then there is a counterexample $H$ such that $H$ is the 3-sum of $H_1, H_2, ..., H_n$, where each $H_i$ is either $K_5$ or 3-connected and planar, and $n$ is a minimum. Among all graphs of this type we may also assume that $H$ is the graph with minimum order. We note that $n \geq 2$ as clearly $P_3$ is not a minor of any planar graph nor of $K_5$.

As $H$ contains $P_3$, $H$ contains $K_5$ as a minor. Since 3-sums of planar graphs do not contain $K_5$, $H$ is then the 3-sum of $H'$ and $K_5$. Let $V_1, V_2, ..., V_7$ be the blocks of the $P_3$ minor in $H$ corresponding to vertices $v_1, v_2, ..., v_7$ of $P_3$.

Let $\{x, y, z\}$ be the triangle vertices of $H'$ and $K_5$ over which the 3-sum occurs. We claim that each of $x$, $y$, and $z$ belong to distinct $V_j$, $j = 1, 2, ...7$. If that were not the case, then either $P_3$ would have a vertex cut smaller than three vertices, which it does not, or $H'$ would contain a $P_3$ minor, which it cannot by our minimum assumption of $n$.

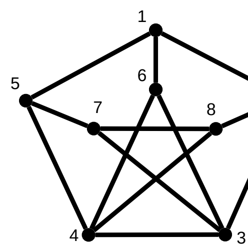
We can say then, that $K_5$ meets at most four of $V_j$. Should it meet five $V_j$, this would imply that $P_3$ was not q-4-c. Suppose $K_5$ meets only three $V_j$. Then, since $x$, $y$, and $z$ belong to separate $V_j$, and the corresponding $v_j$ can be at most pairwise adjacent, $H'$ would contain a $P_3$ minor. This contradicts our assumption of minimality for $n$. Hence $K_5$ meets exactly four $V_j$, say $V_1, V_2, V_3, V_4$.

Again, the vertices $v_1, v_2, v_3, v_4$ can be at most pairwise adjacent. We may assume that $x \in V_1$, $y \in V_2$, and $z \in V_3$. Then let $H''$ be the three sum of $H'$ and $K_4$ over $x, y, z$, where $x, y, z, a$ are the vertices of $K_4$. Then $\{V'_j\}$, where $V'_4 = \{a\}$ and $V'_j = V_j \cap V(H')$ for $j \neq 4$, are the blocks of a $P_3$ minor in $H''$. However, as $|H''| < |H|$, $H$ was not of minimum order. This contradicts our assumption of minimality, and hence $H$ does not contain $P_3$, and no such counterexample exists.

$\square$

This gives a complete characterization of those graphs which are $P_3$-free. Hopefully knowledge of this structure can some day be used to characterize those graphs which are $P_2$-free. As of yet, we have not been completely successful. Though in Chapter 5 of this paper, we were able to construct a characterization of those sufficiently large graphs which are $P_2$-free.

# Chapter 5
# Exclusion of $P_2$




$P_2$

While the author was unable to produce a complete characterization of graphs without a $P_2$ (shown above) minor, we were able to characterize those graphs without $P_2$ minors that are sufficiently large. For the result we need to define some classes of graphs.

There are a few theorems dealing with unavoidable minors of large graphs which we will be using. We will be using similar terminology as used in the papers referenced, but for convenience we will give the definitions of the commonly referenced graphs here as well. Let $A_n$, $B_n$, and $DW_n$ be defined as in Chapter 4. Let $n \geq 3$. The *n-rung ladder*, $L_n$, has vertices $v_1, v_2, ..., v_n$ and $u_1, u_2, ..., u_n$, where $v_1, v_2, ..., v_n$ and $u_1, u_2, ..., u_n$ form paths and each $u_i$ is adjacent to $v_i$ for $i = 1, 2, ..., n$. The *n-rung circular ladder*, $O_n$, is obtained from $L_n$ by adding edges joining $v_1$ to $v_n$ and $u_1$ to $u_n$. The *n-rung Mobius ladder*, $M_n$, is obtained from $L_n$ by adding edges joining $v_1$ to $u_n$ and $v_n$ to $u_1$. $K_{4,n}$ has vertices $u_1, u_2, u_3, u_4$ and vertices $a_1, a_2, ...a_n$ where each $u_i$, $i = 1, 2, 3, 4$ is adjacent to each $a_j$, $j = 1, 2, ...n$. $K'_{4,n}$ is the graph with vertices $x, x', y, y'$ and vertices $a_1, a'_1, a_2, a'_2, ...a_n, a'_n$ where each $a_j$ is adjacent to $x, y$, and $a'_j$, and each $a'_j$ is adjacent to $x', y'$, and $a_j$ for $j = 1, 2, ...n$. The graph $SM_n$ is obtained from $DW_n$ by adding some number of triads to triangles of $DW_n$

such that every triad is adjacent to at least two $r_i$ and no two triads are adjacent to the same pair of $r_i$. We note that $M_4 = V_8$, and $P_2$ is, in fact, $V_8$ plus an edge.

Let $\mathbf{K}_{4,n}$ be the collection of all q-4-c minors of any $K_{4,n}$. Let $\mathbf{M}_n$ be the collection of all q-4-c minors of any $M_n$, and let $\mathbf{SM}_n$ be the collection of all q-4-c minors of some $SM_n$. The following is the main result of this chapter.

**Theorem 5.1.** *For every integer $n \geq 6$, there exists a number $N$ such that every non-planar q-4-c graph $G$ of order at least $N$ contains a $P_2$ minor, unless $G$ is a member of $\boldsymbol{K}_{4,n}$, $\boldsymbol{M}_n$, or $\boldsymbol{SM}_n$.*

## 5.1 Bridges

We will be examining unavoidable subgraphs in large q-4-c graphs to determine which of those graphs contain $P_2$ as a minor. We need to define a few terms first.

If we consider a subgraph $H$ of a graph $G$, it is natural to ask, "Where were all of the deleted portions of $G$?" We define a *bridge* of $H$ in $G$ to be either an edge $xy$ such that $x, y \in V(H)$ and $xy \in E(G) - E(H)$, or a component $B$ of $G - H$ along with all of the edges that are incident with exactly one vertex of $V(B)$ and one vertex of $V(H)$. Bridges of the first type are called *trivial* bridges. We will call the edges from a bridge $B$ to $H$ the *legs* of $B$ and the vertices of $H$ to which they are incident the *feet* of $B$. The following is a result by Tutte [19] regarding subgraphs of 3-connected graphs. It has been restated to suit our needs.

**Theorem 5.2.** *Suppose a 3-connected graph $G$ contains a subgraph $H$, which is isomorphic to a subdivision of a 3-connected graph $M$. Then it is possible to find a subgraph $H'$ of $G$ such that $H'$ is isomorphic to a subdivision of $M$ and the bridges of $H'$ in $G$ have feet that are not all incident with a single branch of $H'$.*

Subgraphs chosen in accordance to Tutte's theorem will be called *rigid subgraphs.*

## 5.2 Unavoidable Minors in Large q-4-c Graphs

To gain a better understanding of what large q-4-c graphs look like there have been several results on unavoidable minors in q-4-c graphs. The following theorem [12] by Oporowski, Oxley, and Thomas gives a list of unavoidable minors for q-4-c graphs.

**Theorem 5.3.** *For every integer $n \geq 4$, there is an integer $N$ such that every quasi 4-connected graph with at least $N$ vertices contains a subgraph isomorphic to a subdivision of one of $A_n$, $O_n$, $M_n$, $K_{4,n}$, and $K'_{4,n}$.*

This theorem includes all q-4-c graphs. However, as we are only concerned about graphs which could potentially contain $P_2$ as a minor, and $P_2$ is non-planar, we would like to consider only those graphs which are non-planar, q-4-c, and large. Fortunately, the above result was improved upon in [2] for non-planar graphs.

**Theorem 5.4.** *For every integer $n \geq 4$, there is an integer $N$ such that every quasi 4-connected non-planar graph with at least $N$ vertices has a subgraph isomorphic to a subdivision of one of $B_n$, $M_n$, $K_{4,n}$, and $K'_{4,n}$.*

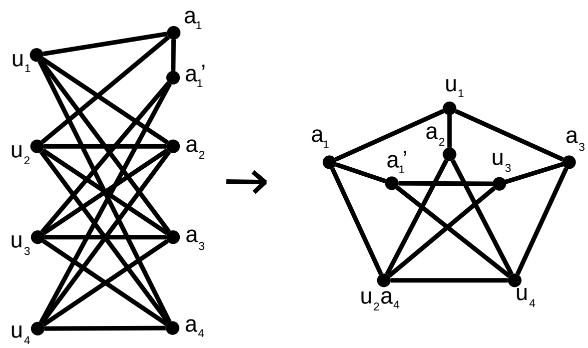
We will assume each of these unavoidable minors in turn, in an attempt to ultimately determine the structure of large $P_2$-free graphs.

## 5.3 Extensions of $K'_{4,n}$, $K_{4,n}$

The $K'_{4,n}$ case will be handled first as it is both trivial and extremely useful when handling the $K_{4,n}$ case.

**Lemma 5.5.** *Suppose that $G$ is q-4-c and contains a subgraph $H$ which is isomorphic to a subdivision of $K'_{4,n}$, $n \geq 3$. Then $G$ contains $P_2$ as a minor.*

*Proof.* We notice the following.




Since the above graph is clearly contained within $K'_{4,n}$ as a minor for every $n \geq 4$, we can conclude that $G$ is not $P_2$-free.

$\square$

This is not incredibly surprising as $K'_{4,3}$ contains $P_0$. With this in mind, we turn our attentions to those graphs with $K_{4,n}$ minors.

We first must consider whether $K_{4,n}$ contains $P_2$ as a minor. This is relatively clear. $K_{4,n}$ has the property that all its edges are covered by at most 4 vertices. We can see that if a graph $G$ has this property, and $\{a, b, c, d\}$ is an edge cover, clearly $\{a, b, c, d\}$ is still an edge cover of $G \backslash e$. If we consider the deletion of a vertex $v$, then $\{a, b, c, d\} - \{v\}$ is a vertex cover of $G \backslash v$. And, since $\{a, b, c, d\}$ is an edge cover of $G$, the edges of $G/e$ are covered by either $\{a, b, c, d\}$, or three of these vertices together with the new vertex created by the contraction. Hence, this property is preserved under the minor operation. By inspection, we can see $P_2$ does not have this property, and hence $K_{4,n}$ is $P_2$-free for all $n$.

Suppose then that $G$ is large, q-4-c, and contains a subgraph $H$ which is isomorphic to a subdivision of $K_{4,n}$, $n \geq 5$. Then we can define $H$ as $\bigcup_{i,j} P(u_i, a_j)$, $i = 1, 2, 3, 4$, $j = 1, 2, ..., n$ where $P(u_i, a_j)$ is a path from $u_i$ to $a_j$ and all $P(u_i, a_j)$ are interior disjoint. Furthermore, suppose that we choose $H$ in such a way that

$H$ is a rigid subgraph of $G$ and $H$ is maximal with regard to $n$. We then claim the following.

**Lemma 5.6.** *All feet of any bridge $B$ of $H$ must be incident with vertices $u_i$ for some $i$.*

*Proof.* Suppose false, then consider the possibility that $B$ has a foot $a_j$ for some $j$, without loss of generality say $j = 1$. As $H$ is a rigid subgraph of $G$, we know that either $B$ also must have a foot incident with $P(u_i, a_j) - \{u_i\}$ for $j \neq 1$ at a vertex we will call $v$, or $B$ must have two additional feet, one incident with $P(u_i, a_1) - \{a_1\}$ and one incident with $P(u_k, a_1) - \{a_1\}$, $i \neq k$.

In either case, a contradiction is derived. From Lemma 5.5 we can see that both cases imply that $G$ would contain $P_2$ as a minor. We are left to assume then that $B$ does not have a leg incident with $a_j$ for any $j$, but instead has a in $P(u_i, a_j) - \{u_i, a_j\}$ for some $i, j$. By symmetry, say $B$ has a foot incident with $P(u_1, a_1) - \{u_1, a_1\}$ at a vertex $v$. Then either $B$ has a foot incident with $P(u_i, a_1) - \{a_1\}$, $i \neq 1$, or $B$ has a foot incident with $P(u_i, a_j) - \{u_1, a_j\}$, $j \neq 1$. Again, however, both of these cases imply that $G$ would contain a $P_2$ minor. We are left to assume then, that all feet of any bridge $B$ are incident with $u_i$ for some $i$.

□

**Theorem 5.7.** *Let $G$ be a q-4-c, $P_2$-free graph such that $G$ contains a subgraph that is isomorphic to a subdivision of $K_{4,n}$, $n \geq 5$. Then $G \in \boldsymbol{K}_{4,n}$.*

*Proof.* From Lemma 5.6, we know that there are no bridges with feet in any $P(u_i, a_j) - \{u_i\}$. Hence, all $P(u_i, a_j)$ of $H$ are actually of length 1. We also may conclude that every trivial bridge of $H$ is an edge incident with two $u_i$, and therefore there are at most 6 such bridges since we assume $G$ to be simple.

With regard to the non-trivial bridges of $H$, we can say that each non-trivial bridge has feet incident with exactly 3 of $\{u_1, u_2, u_3, u_4\}$. Fewer $u_i$ feet would contradict the 3-connected property of $G$, more would contradict the maximality of $H$. From this, we can assume that every non-trivial bridge of $H$ is, in fact, a triad since it could be separated from $G$ by its 3 $u_i$ feet and $G$ is q-4-c. Lastly, we can say that there are at most 4 such bridges, one each with feet incident with each 3 element subset of $\{u_1, u_2, u_3, u_4\}$, also due to the q-4-c property of $G$.

Let $K$ be the graph $H$ with all 10 bridges added. Then $K$ is a minor of $K_{4,n+7}$, since the contraction of 3 edges of $K_{4,n+7}$ can make each $u_i$ adjacent, and with the deletion of 4 edges our graph can have 4 triads. Since $G$ is a minor of $K$ and $K$ is a minor of $K_{4,n+7}$, $G \in \mathbf{K}_{4,n}$.

□

### 5.4 Extensions of $M_n$

Suppose that $G$ contains an $M_n$ minor. Again, we begin by checking if the case is trivial, i.e. if $M_n$ contains $P_2$ as a minor for some $n$. We notice that every vertex of $M_n$ is symmetric and by that symmetry there are only two types of edges, $v_i u_i$ and $v_i v_{i+1}$, $i = 1, 2, ...n$, reducing the indices modulo $n$. $M_n$ has the interesting property that for every edge $e$ of $M_n$, either $M_n \backslash e$ or $M_n / e$ is planar. Specifically, contraction of any $v_i u_i$ edge results in a planar graph, and likewise for the deletion of any $v_i v_{i+1}$ edge. We call this property the Contraction/Deletion property. We observe that this property is also preserved under taking minors of $M_n$. From diagram of $P_2$ at the beginning of this chapter, we can see that contraction of edge 16 from $P_2$ contains $K_5$, whereas deletion of 16 from $P_2$ contains $K_{3,3}$. We can conclude then that $M_n$ does not contain $P_2$ for any $n$.

We suppose now that $G$ is q-4-c, and contains a subgraph $H$ that is isomorphic to a subdivision of $M_n$, $n \geq 5$. Then $H$ can be defined as $\bigcup_i P(v_i, v_{i+1}) \cup \bigcup_i P(u_i, u_{i+1}) \cup \bigcup_i P(v_i, u_i) \cup P(v_n, u_1) \cup P(v_1, u_n)$, where $i = 1, 2, ...n$ and each path is interior disjoint with all other paths. Additionally, we choose $H$ in such a way that $H$ is a rigid subgraph of $G$ and such that $n$ is maximal. For ease of notation, we also define a *quadrangle* of $H$, $Q(v_i, u_{i+1}) = P(v_i, v_{i+1}) \cup P(u_i, u_{i+1}) \cup P(v_i, u_i) \cup P(v_{i+1}, u_{i+1})$, reducing all indices modulo $n$ if necessary. The following figures are some observations about potential additions to $M_4$ and $M_5$ and will be helpful in the arguments to follow.

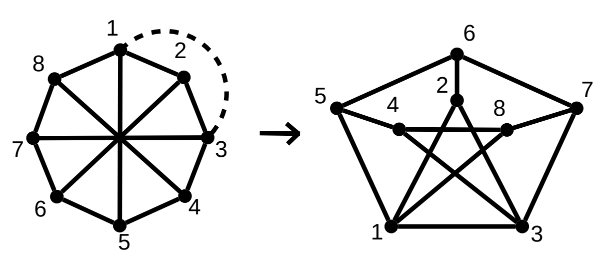
We may equivalently define $M_n$ as graph consisting of a cycle of vertices $v_1, v_2, ...v_{2n}$ where each $v_i$ is also adjacent to $v_{i+n}$, indices mod $2n$. This is equivalent to our previous definition by defining $u_i = v_{n+i}$, $i = 1, 2, ...n$.

**Lemma 5.8.** *The following additions to $M_n$ contain $P_2$:*

*1. If $n \geq 4$, the addition of any edge between non-adjacent $v_i, v_j$ where $j \neq i + (n-1), i + n, i + (n+1)$*

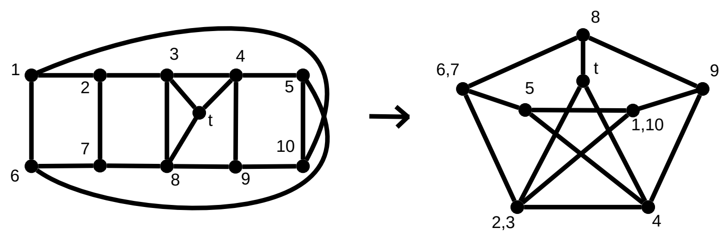
*2. If $n \geq 5$, the addition of a triad $t$ adjacent to vertices $v_i, v_{i+1}$ and one of $v_{i+n}$ or $v_{i+(n+1)}$*

*Proof.* For 1, we begin by noticing the following about $M_4$.




As any two vertices of $M_n$ are symmetric, the addition of any $v_iv_j$ edge, $j \neq i + (n-1), i + n, i + (n+1)$, would contain the above graph as a minor, and hence $P_2$.

To show 2, we have the following figure.




For any $n \geq 5$, the addition of such a triad to $M_n$ would contain the above graph as a minor, and hence $P_2$.

□

With this information, we can narrow down the location of any feet of a bridge incident with $H$.

**Lemma 5.9.** *Any bridge $B$ of $H$ must have all of its feet incident with a single quadrangle of $H$.*

*Proof.* Suppose $B$ is a bridge of $H$ and $B$ has two feet which are not incident with the same quadrangle of $H$, say with $Q(v_i, u_{i+1})$ and $Q(v_j, u_{j+1})$, $i \neq j$. By potentially relabeling the vertices, we may assume that $i < j$ and $(j - i) \leq (i - j)$ mod $n$. We can then classify the possibilities for $i$ and $j$ to the three cases $i+2 < j$, $i + 2 = j$, and $i + 1 = j$.

Suppose first that $i + 2 < j$. Let us assume that $B$ has a foot incident with $P(v_i, v_{i+1})$. Then $B$ must also have a foot incident with one of the paths in $Q(v_j, u_{j+1})$. However, incidence with any of the four paths of $Q(v_j, u_{j+1})$ would result in $G$ containing a graph isomorphic to the first graph in figure 5.8 as a minor, and hence $P_2$. We may assume then, by symmetry, that $B$ does not have a foot incident with any of $P(v_i, v_{i+1})$, $P(v_j, v_{j+1})$, $P(u_i, u_{i+1})$, or $P(u_j, u_{j+1})$. In other words, $B$ has a foot incident with either $P(v_i, u_i) - \{v_i, u_i\}$ or $P(v_{i+1}, u_{i+1}) - \{v_{i+1}, u_{i+1}\}$ and a second incident with either $P(v_j, u_j) - \{v_j, u_j\}$ or $P(v_{j+1}, u_{j+1}) - \{v_{j+1}, u_{j+1}\}$.

Again though, all these possibilities imply that $G$ has a minor isomorphic to the first graph in Lemma 5.8.

Let $i+2=j$. Suppose $B$ has a foot incident with $Q(v_i, u_{i+1}) - P(v_{i+1}, u_{i+1})$. In this case all of the arguments from the $i+2<j$ case hold and hence $G$ would contain $P_2$ as a minor. We assume then, that each foot of $B$ incident with $Q(v_i, u_{i+1})$ is incident with $P(v_{i+1}, u_{i+1})$. By our assumption, $B$ would necessarily also have a foot incident with $Q(v_j, u_{j+1}) - P(v_j, u_j)$. This is similar to the previous case and for the same reasons can not exist.

Lastly, we suppose that $i+1=j$. By our assumption, $B$ must have at least one foot incident with $Q(v_i, u_{i+1}) - P(v_{i+1}, u_{i+1})$ and one foot incident with $Q(v_j, u_{j+1}) - P(v_j, u_j)$. Yet again, this case reduces to $G$ containing a $P_2$ minor.

□

This lemma severely restricts the possible locations for any bridges of $H$ in $G$. As we will now show, it also completely restricts the structure of any bridges.

**Lemma 5.10.** *All bridges of $H$ in $G$ are trivial.*

*Proof.* Suppose $B$ is a bridge of $H$ and $B$ is non-trivial. By connectivity of $G$, $B$ must have at least 3 feet, and by our previous lemma, those feet must be contained in a single quadrangle of $H$. Since $H$ is a rigid subgraph, these feet may not be contained within a single branch of the quadrangle. This leaves as possibilities that $B$ has at least one foot incident with each of $P(v_i, v_{i+1})$ and $P(v_i, u_i)$, $P(v_i, v_{i+1})$ and $P(u_i, u_{i+1})$, or $P(v_i, u_i)$ and $P(v_{i+1}, u_{i+1})$. In any of these cases, $B$ must have a third foot incident with the quadrangle. Regardless where, each case implies that $G$ contains a minor isomorphic to the second graph in Lemma 5.8, and hence $P_2$.

□

**Theorem 5.11.** *Let $G$ be q-4-c, $P_2$-free, and contain a subgraph that is isomorphic to a subdivision of $M_n$, $n \geq 5$. Then $G \in \boldsymbol{M}_n$.*

*Proof.* We know then that all bridges of $H$ are single edges between vertices on two paths of a quadrangle of $H$. We observe then that no bridge is incident with $P(v_i, u_i) - \{v_i, u_i\}$ for any $i$. Otherwise, $G$ would again contain the second graph from Lemma 5.8, and hence $P_2$ as a minor.

By connectivity, we can say then that every path $P(v_i, u_i)$ is of length 1 for every $i$, or equivalently, that every bridge of $H$ is an edge from $P(v_i, v_{i+1})$ to $P(u_i, u_{i+1})$ for some $i$. Suppose then, that an edge $e$ exists between $P(v_i, v_{i+1}) - \{v_i, v_{i+1}\}$ and $P(u_i, u_{i+1}) - \{u_i, u_{i+1}\}$. Our choice for $G$ would then contain a subgraph isomorphic to a subdivision of $M_{n+1}$ and our choice for $H$ would not have been maximal. So we can say that every bridge of $H$ is an edge incident with a vertex $v_i$ or $u_i$ for some $i$.

Suppose $e$ is such an edge, and by symmetry suppose $e$ is incident with $v_i$ and incident with $P(u_i, u_{i+1}) - \{u_i, u_{i+1}\}$ at vertex $w$. We see that this graph is not q-4-c since the vertex set $\{u_{i-1}, u_{i+1}, v_i\}$ separates both $u_i$ and $w$ from the rest of $H$. So, either $w = u_{i+1}$ or there is another edge $e'$ incident with $P(u_i, w)$.

Suppose $e'$ is incident with $P(u_i, w) - \{u_i, w\}$. Then $e'$ is also incident with $P(v_i, v_{i+1}) - \{v_i\}$, and then $G$ would contain a minor isomorphic to the first graph in Lemma 5.8, and hence would not be $P_2$ free. We can conclude then that $e'$ would need be incident with either $u_i$ or $w$. If $e'$ is incident with $w$, then by the maximality of $H$, $e'$ is an edge between $w$ and $v_{i+1}$.

If $e'$ is incident with $u_i$, then $e'$ must also be incident with $P(v_{i-1}, v_i) - \{v_i\}$. If $e'$ is not incident with $v_{i-1}$, then this would also the maximal choice of $H$. Hence, if $e'$ is incident with $u_i$, then it is a $u_i v_{i-1}$ edge. These observations give us guidelines for the structure of $G$ if it is q-4-c, $P_3$-free and contains a subgraph isomorphic to a subdivision of $M_n$, $n \geq 5$, as a minor.

From the above observations, we may assume that $G$ is $H$ with trivial bridges added inside quadrangles according to the following guidelines:

1. The minimum degree of any vertex in $G$ is 3.

2. The maximum degree of any vertex in $G$ is 5.

3. If two bridges $a_1b_1$ and $a_2b_2$ have distinct feet, then the feet do not alternate along the quadrangle.

4. Any vertex with a degree 4 neighbor $w$ on the opposite path whose two same path neighbors are not also adjacent to $w$ shall have minimum degree 4.

5. Vertex $v_1$ may be adjacent with $v_n$. Vertex $u_1$ may be adjacent with $u_m$.

According to these conditions $G$ may also be constructed from some $M_k$, by contracting some of the $v_iv_{i+1}$ and $u_ju_{j+1}$ edges so that no two cubic vertices belong to a triangle. Hence $G \in \mathbf{M}_n$.

□

## 5.5 Extensions of $B_n$

We define the class of graphs $\mathbf{SM}_n = \{$ all graphs which are q-4-c minors of $SM_n$ for some $n\}$. We note in particular, $B_n \in \mathbf{SM}_n$. We claim that all graphs in this class are $P_2$-free. Let $DW_n + kt$ be the graph obtained from $DW_n$ by adding $k$ triads to triangles of $DW_n$ such that each triad has at least two $r_i$ neighbors, and no two triads have the same two $r_i$ neighbors.

**Lemma 5.12.** *$DW_n + kt$, $n \geq 4$, $k = 0, 1, 2, ...n$, is $P_2$-free for all $n, k$.*

*Proof.* First, we claim if $DW_n$ contains $P_2$ as a minor for any $n$, then $DW_6$ contains $P_2$. This is fairly easy to see. There are only three types of edges in $DW_n$: $u_1u_2$,

$u_j, r_i$, and $r_i, r_{i+1}$. Contraction of any edge of the first two types results in a planar graph, while contraction of any edge of the third type will yield $DW_{n-1}$. As $P_2$ has 8 vertices, the claim holds.

First, we note that $DW_n$ is the graph with no triads added, and hence is labeled $DW_n + 0t$, and is known to be $P_2$-free as all $DW_n$ have the same Deletion/Contraction property as $M_n$.

Let $n + k = 6$, $k \geq 1$. Then we suppose $DW_n + kt$ has a $P_2$ minor. Let $t_1$ be one of the triads, and assume $t_1$ is adjacent to $r_1, r_2, u_1$. Since $DW_n + kt$ has 8 vertices, $t_1$ is a vertex of the $P_2$ minor, we know each edge incident with $t_1$ is an edge of the minor. Additionally, if $r_1 r_2$ is not an edge of the minor then this would imply that $P_2$ is a minor of $DW_{n+1} + (k-1)t$. Consider then, if $DW_5 + t$ contains $P_2$, then $t$ must be vertex 6 in the diagram of $P_2$ at the beginning of the chapter, since $P_2$ contains only a single vertex that is both cubic and in a triangle. It would also be true that $r_1$ and $r_2$ are vertices 3 and 4, and vertex $u_1$ is vertex 1. However, this would imply that $r_1$ and $r_2$ are collectively adjacent to all of $\{u_2, r_3, r_4, r_5\}$. Since neither is adjacent to $r_4$, we have a contradiction, and $DW_5 + t$ must then be $P_2$-free.

Since $P_2$ contains only a single cubic vertex in a triangle, and by our reasoning, all edges of all triangles containing triads must participate in a $P_2$ minor, we know that $DW_6$, $DW_5 + t$, $DW_4 + 2t$, and $DW_3 + 3t$ are all $P_2$-free.

Now let $DW_n + kt$ contain a $P_2$ minor such that $n + k$ is a minimum and then $k$ is a minimum. We know then that $DW_n + (k-1)t$ does not contain $P_2$ as a minor. Hence the $P_2$ minor in question must have been created with the addition of the $k$th triad $t_k$. By symmetry, let the neighbors of $t_k$ be $u_1, r_1, r_2$. Since the contraction of any edge incident with $t_k$ creates only parallel edges, and $t_k$ is cubic, we know that each of the edges incident with $t_k$ participate in the $P_2$ minor. Additionally, edge $r_1r_2$ also participates in the $P_2$ minor, as $(DW_n + kt)\backslash r_1r_2$ is a minor of $DW_{n+1} + (k-1)t$ which we know to be $P_2$-free by our induction hypothesis.

To create the $P_2$ minor, we may first contract some of the remaining edges of $DW_n + kt$ and then delete some of the edges that remain. However, contraction of $u_1u_2$ or any $u_jr_i$ edge results in a planar graph. We know then that if any edges are contracted, they must be either rim edges or triad edges. Contraction of any triad edge would imply that $P_2$ is a minor of $DW_n + (k-1)t$, which we know to be false, and contraction of any rim edge would imply that $P_2$ is a minor of $DW_{n-1} + kt$ or $DW_{n-1} + (k-1)t$, which we also know to be false. From this, we can say that every vertex of the $P_2$ minor is, in fact, a vertex of $DW_n + kt$. This would imply that $P_2$ is a minor of $DW_3 + 3t$, $DW_4 + 2t$, $DW_5 + t$, or $DW_6$, a contradiction.

□

As $DW_n + nt = SM_n$, Lemma 5.12 implies that all graphs in $\mathbf{SM}_n$ are $P_2$-free.

Suppose then, $G$ is q-4-c, $P_2$-free, and contains a subgraph isomorphic to a subdivision $B_n$. Let $H$ be such a rigid subgraph of $G$, with $n$ maximum. For the purposes of this section, we require that $n \geq 6$. We may then define $H$ similarly to the previous cases as a union of internal vertex disjoint paths $P(r_i, r_{i+1})$ for $i = 1, 2, ...2n$ where each index of $r$ is reduced mod $2n$, $P(r_i, u_k)$ for $k = 1, 2$ where $i = k$ mod 2, and $P(u_1, u_2)$. We note that as $H$ is a subdivision of $B_n$ and $B_n \backslash u_1u_2$ is planar, that $H\backslash(P(u_1, u_2) - \{u_1, u_2\})$ is also planar. We can define the faces of $H$ to be the quadrangles $P(r_i, r_{i+1}) \cup P(r_{i+1}, r_{i+2}) \cup P(r_i, u_k) \cup P(r_{i+2}, u_k)$.

Before analyzing the bridges of $H$, we observe the following variations of $DW_6$ which contain $P_2$ as a minor. As $B_n$ contains $DW_n$ as a minor for all $n$, these will be useful in eliminating some possible bridge positions.

**Lemma 5.13.** *The following contain $P_2$.*

Figure 1

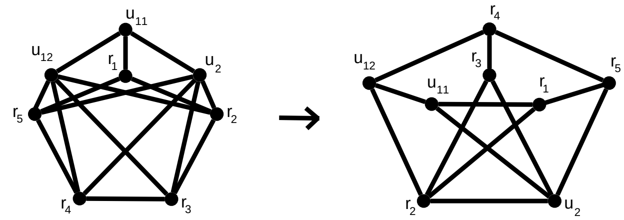



Figure 2

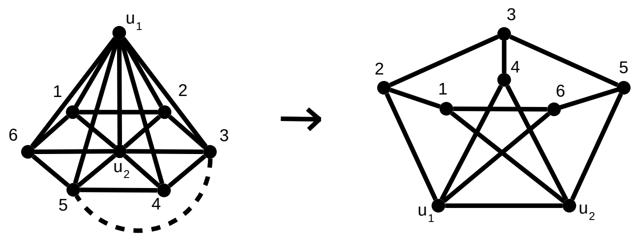



Figure 3

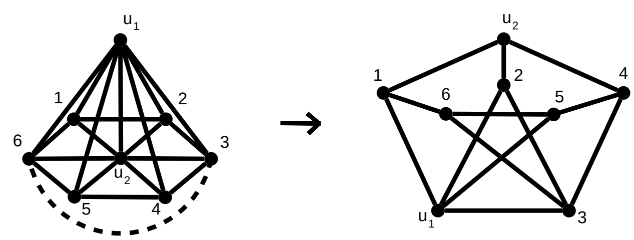
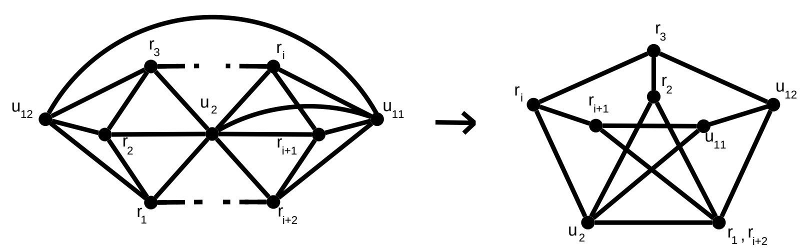



Figure 4




*Proof.* We note here that the implication of figure 1 above is that if any bridge added to $H$ allows the contraction of edges to where there is a vertex adjacent to $u_1$, $u_2$, and a rim vertex, then $H$ contains $P_2$. Figures 2 and 3 above imply that no bridge may have feet on rim paths such that there are at least two $r_i$ between them, else $H$ has a $P_2$ minor. While figure 4 implies that if an added bridge allows contraction to where a vertex $v$ is adjacent to rim vertices $r_i, r_{i+1}, r_{i+2}$ and some $u_k$ while $r_{i+1}$ is adjacent to $u_{k+1}$, then $H$ would contain $P_2$. With these observations, we analyze the possible locations for bridges of $H$ in $G$.

□

**Lemma 5.14.** *No bridge $B$ of $H$ in $G$ is incident with $P(u_1, u_2) - \{u_1, u_2\}$.*

*Proof.* Suppose there exists bridge $B$ such that $B$ has a foot in $P(u_1, u_2) - \{u_1, u_2\}$. Then $B$ must have another foot $x$ in $P(r_i, r_{i+1})$ or $P(r_i, u_k) - \{u_k\}$ for some $i, k$. In either of these cases, $x$ is contractible to some $r_i$. $G$ would then contain figure 1 in Lemma 5.13, and hence $P_2$. Since $G$ is assumed to be $P_2$-free, we may assume that $P(u_1, u_2)$ is, in fact, just the edge $u_1u_2$.

□

To facilitate the bridge analysis of a $B_n$ subgraph of $G$, we use the following result of Norin and Thomas [11].

**Theorem 5.15.** *Let $G$ be q-4-c and non-planar and $H'$ be a q-4-c planar graph. If $G$ contains a subgraph $H$ isomorphic to a subdivision of $H'$, then one of the following is true:*

*i. $H$ has a bridge $B$ such that no face of $H$ contains all feet of $B$.*

*ii. There exist two bridges $B_1, B_2$ with feet $s_1, t_1$ and $s_2, t_2$ respectively, such that the vertices $s_1, s_2, t_1, t_2$ are distinct and belong to some face boundary of $H$ in the order listed. Moreover, for $i = 1, 2$, the vertices $s_i$ and $t_i$ do not belong to the same branch of $H$ and if two branches of $H$ contain all of $s_1, s_2, t_1, t_2$, then those two branches are vertex disjoint.*

We aim to use this theorem to demonstrate that our graph $G\backslash u_1u_2$ is planar. This would imply that all bridges of $H$ in $G$ would have feet contained within a single face of $H\backslash u_1u_2$, and that when added do not create crossings within that face.

**Theorem 5.16.** *$G\backslash u_1u_2$ is planar.*

*Proof.* Suppose $G \backslash u_1 u_2$ is non-planar. We know that $H \backslash u_1 u_2$ is planar, and since $n \geq 4$ we know that $G \backslash u_1 u_2$ and $H \backslash u_1 u_2$ are q-4-c. It follows that (i) or (ii) from theorem 5.15 must hold.

Suppose (i) holds. Then there exists a bridge $B$ such that the feet of $B$ are not all contained within a single face of $H \backslash u_1 u_2$. Then $H$ has a bridge $B$ in $G$ with feet $x$ and $y$ such that one of the following is true:

1. $x = u_k$ and $y \in P(r_j, u_{k+1}) - \{r_j, u_{k+1}\}$ with indices of $u$ reduced mod 2.
2. $x \in P(r_i, u_k) - \{r_i, u_k\}$ and $y \in P(r_j, u_{k+1}) - \{r_j, u_{k+1}\}$
3. $x \in P(r_i, u_k) - \{r_i, u_k\}$ and $y \in P(r_j, u_k) - \{u_k\}$ where $j \neq i+2, i-2$
4. $x \in P(r_i, u_k) - \{r_i, u_k\}$ and $y \in P(r_j, r_{j+1})$ where $j \neq i, i+1, i-1, i-2$ and $y \neq r_{i+2}, r_{i-2}$
5. $x \in P(r_i, r_{i+1})$ and $y \in P(r_j, r_{j+1})$ where $j \neq i, i-1, i+1$ and $y \neq r_{i-1}, r_{i+2}$

However, cases 1 and 2 imply that $G$ contractible to figure 1 in Lemma 5.13, while cases 3, 4, and 5 imply that $G$ is contractible to figure 2 or 3 in Lemma 5.13. Hence, if (i) holds, then $G$ is not $P_2$-free. By our assumptions then, (ii) must hold.

Suppose (ii) holds, then there exists two bridges $B_1, B_2$ with distinct feet $s_1, t_1$ and $s_2, t_2$ respectively, such that the bridges and their feet satisfy the conditions for the theorem. By symmetry, we assume that the feet $s_1, t_1, s_2, t_2$ are in the quadrangle $P(r_1, r_2) \cup P(r_2, r_3) \cup P(r_1, u_1) \cup P(r_3, u_1)$.

It is possible that $s_1, t_1, s_2, t_2$ are contained in exactly two branches. By symmetry, then we may assume the branches to be $P(r_1, r_2)$ and $P(r_3, u_1)$, and $s_1$ is between $r_1$ and $s_2$ on $P(r_1, r_2)$. However, as each of the feet is distinct, this would imply that the face of $H$ could be contracted to where $s_1, t_1, s_2, t_2$ are identified with the vertices $r_1, r_2, r_3, u_1$ respectively. This, however would imply that $G$ is contractible to figure 1 in Lemma 5.13, a contradiction.

We may assume then, that $s_1, t_1, s_2, t_2$ are not contained within only two branches of $H$. Suppose then, that they are not contained within three branches of $H$. By symmetry, we may assume $s_1 \in P(r_1, r_2) - \{r_2\}$, $s_2 \in P(r_2, r_3) - \{r_3\}$, $t_1 \in P(r_3, u_1) - \{u_1\}$, and $t_2 \in P(r_1, u_1) - \{r_1\}$. Similarly to the previous case, this would also imply that the feet are contractible to $r_1, r_2, r_3$, and $u_1$, respectively.

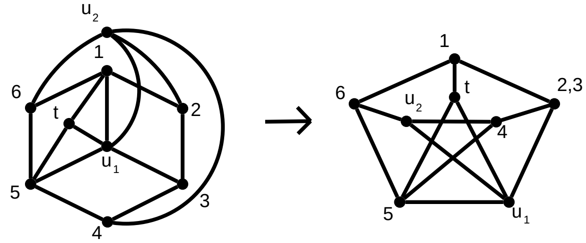
It must be true then, that $s_1, t_1, s_2, t_2$ are contained in exactly three branches of $H$. As these branches are by necessity contiguous, then $s_1, t_1, s_2, t_2$ are contained in paths of the form $P(w, x) \cup P(x, y) \cup P(y, z)$. We may assume by symmetry that $s_1 \in P(w, x) - \{x\}$ and $t_2 \in P(y, z) - \{y\}$. We see then that for three branches to be required to contain $s_1, t_1, s_2, t_2$, that the vertices may be contracted to $w, x, y, z$ respectively. Again, this implies a $P_2$ minor in $G$, which we have assumed to be $P_2$-free. Since neither (i) nor (ii) holds, we may assume that if $G$ is $P_2$-free, $G \backslash u_1 u_2$ is planar.

□

We can now restrict our observations to those bridges added within a single face of $H \backslash u_1 u_2$ without crossings. We begin with an analysis of those bridges larger than a triad. The following arguments involve checking many cases. We use the following lemma to simplify some of them.

**Lemma 5.17.** *$B_n$ with a triad $t$ added to some $r_i$, $r_{i+2}$, and $u_k$, contains $P_2$.*

*Proof.* We observe the following.




□

From this observation, we can see that for any non-trivial bridge, the feet of that bridge may not lie on the paths of a quadrangle of $H$ such that 3 of those feet may be contracted to $r_i$, $r_{i+2}$, and $u_k$. With this in mind, we narrow down the possible locations for feet of some non-trivial bridge $B$. Additionally, we have the following Corollary.

**Corollary 5.18.** *Let $B$ be a bridge of $H$ in $G$ such that $B$ has a foot in $P(r_i, u_k) - \{r_i, u_k\}$ for some $i, k$. Then $B$ does not have a foot which is contractible to either $r_{i+2}$ or $r_{i-2}$.*

This is easily implied by Lemma 5.17, as such a bridge would then imply $G$ contained $P_2$.

**Lemma 5.19.** *Let $C$ be a cycle consisting of four paths $P(x_1, x_2)$, $P(x_2, x_3)$, $P(x_3, y)$, and $P(y, x_1)$, each of length at least two, in that clockwise order. Let $B$ be a non-trivial bridge with all of its feet on $C$. Then $C$ can be contracted to a 4-cycle with vertices $x_1, x_2, x_3, y$ such that $x_1, x_3, y$ are feet of $B$ unless*

*1. All feet of $B$ are contained within two adjacent paths such that $y$ is not a vertex of both paths.*

*2. $B$ has a single foot in the interior of $P(x_i, y)$, and all other feet are contained in $P(x_j, x_k)$, $j, k \neq i$.*

*3. $B$ has exactly 3 feet. One each in the interior of $P(x_1, y)$ and $P(x_3, y)$, and vertex $x_2$.*

*Proof.* If $B$ has no feet in $P(x_1, y) - \{x_1\}$ or $P(x_3, y) - \{x_3\}$, then 1 is true.

Since $P(y, x_1)$ is symmetric to $P(x_3, y)$, we will argue from the $x_1$ side.

Suppose $B$ has two feet in $P(y, x_1)$, as these feet are contractible to $x_1$ and $y$, all other feet of $B$ must not be contractible to $x_3$. Hence all other feet of $B$ must be contained in $P(y, x_1) \cup P(x_1, x_2)$, and 1 is true.

We assume then, that $B$ has exactly one foot in $P(y, x_1) - \{x_1\}$. Suppose $y$ is a foot of $B$. Then $B$ must have another foot $x$, and by symmetry, we may assume $x \in P(x_1, x_2) - \{x_2\}$. If $B$ then has a foot in $P(x_2, x_3) - \{x_2\}$, then the feet of $B$ may be contracted to $x_1$ and $x_3$. Hence all other feet of $B$ are within either $P(x_1, x_2)$ or $P(x_2, x_3)$, and 1 is true.

Suppose that $y$ is not a foot of $B$, and suppose that $B$ has exactly one foot $x \in P(y, x_1) - \{y, x_1\}$. Since $x$ is contractible to either $x_1$ or $y$, $B$ must not have two more feet that are contractible to either $x_1$ and $x_3$ or $x_3$ and $y$. Hence it must be the case that all other feet of $B$ are contained in either $P(x_1, x_2)$ or $P(x_2, x_3)$, in which case 1 or 2 is true respectively, or $B$ has exactly two more feet, one in $P(x_3, y) - \{x_3, y\}$ and $x_2$, and 3 is true.

□

**Corollary 5.20.** *All non-trivial bridges of $H$ in $G$ have all feet contained in paths of the following forms:*

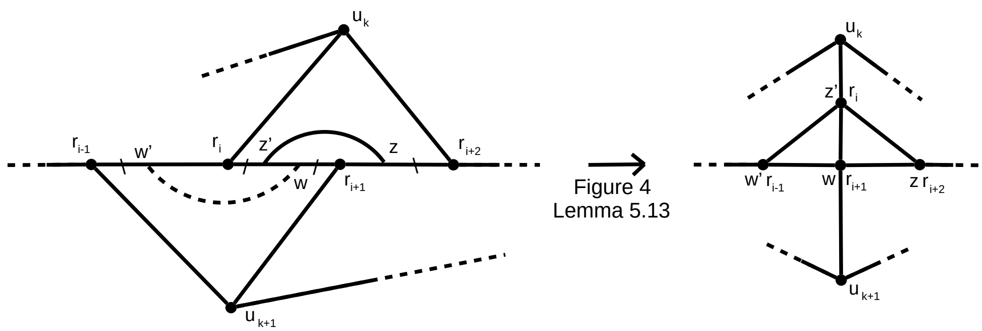
1. $P(r_i, u_k) \cup P(r_i, r_{i+1})$ *for some* $i$
2. $P(r_i, u_k) \cup P(r_i, r_{i-1})$ *for some* $i$
3. $P(r_i, r_{i+1}) \cup P(r_{i+1}, r_{i-1})$ *for some* $i$.

*Proof.* This is fairly clear. By Lemma 5.17, we know that no bridge may have feet contractible to $u_k$, $r_i$, and $r_{i+2}$ for any $k, i$. Since each bridge must attach to a quadrangle face of $H$, they must then attach in the positions listed in Lemma 5.19. However, bridges of type 3 in Lemma 5.19, would imply that $G$ has a $P_2$ minor by Corollary 5.18.

□

**Lemma 5.21.** *Let $B$ and $B'$ be bridges with feet in $P(r_i, r_{i+1}) \cup P(r_{i+1}, r_{i+2})$ and $P(r_{i+1}, r_{i+2}) \cup P(r_{i+2}, r_{i+3})$ respectively. Then there exists $a \in P(r_{i+1}, r_{i+2})$ such* *that all feet of $B$ are contained within $P(r_i, r_{i+1}) \cup P(r_{i+1}, a)$ and all the feet of $B'$ are contained within $P(a, r_{i+2}) \cup P(r_{i+2}, r_{i+3})$.*

*Proof.* Suppose false. Then there exists $B$ and $B'$ with feet contained in paths as described above. There is a foot $b$ of $B$ and $b'$ of $B'$ such that $b \in P(b', r_{i+2}) - \{b'\}$. This, however, would imply that $G$ contains a $P_2$ minor.




□

We introduce a few definitions here to ease the notation of the following proofs. Let $B_i$ be the collection of all bridges with feet contained in $P(r_{i-1}, r_i) \cup P(r_i, r_{i+1}) \cup P(r_i, u_k)$ that do not have two of $\{r_{i-1}, r_{i+1}, u_k\}$ as feet. Let $\bar{B}_i$ be the collection of all bridges with feet contained in $P(r_{i-1}, r_i) \cup P(r_i, r_{i+1}) \cup P(r_i, u_k)$ that are not trivial bridges of the form $r_{i-1}u_k$, $r_{i+1}u_k$, or $r_{i-1}r_{i+1}$. We note here that $|\bar{B}_i \backslash B_i| \leq 3$ by Lemma 5.16.

Let $\{a, b, c\}$ be a nontrivial 3-cut of $G$, and let the separation defined by this cut be $(A, B)$. Then a *rescuing bridge* for $\{a, b, c\}$ is a bridge with a foot in $A \backslash B$ and a foot in $B \backslash A$.

**Lemma 5.22.** *For every $r_i$, there is a vertex $a \in P(r_{i-1}, r_i)$ and a vertex $b \in P(r_i, r_{i+1})$ such that every bridge in $B_i$ has all of its feet contained within $P(a, r_i) \cup P(r_i, b) \cup P(r_i, u_k)$, and any bridge with a foot in $P(a, r_i) \cup P(r_i, b) - \{a, b\}$ and a* *foot not in* $P(r_{i-1}, r_i) \cup P(r_i, r_{i+1}) \cup P(r_i, u_k)$ *has only the single foot* $r_i$ *in* $P(a, r_i) \cup P(r_i, b) - \{a, b\}$.

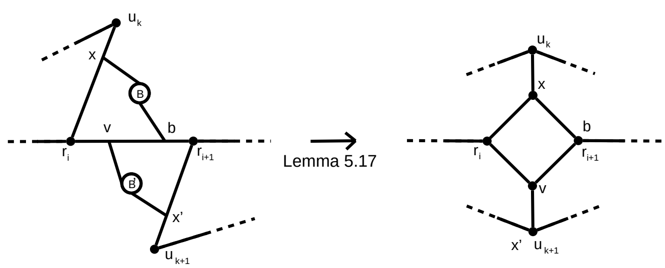
*Proof.* Suppose $B_i$ is empty. Then let $a = b = r_i$, and the Lemma holds trivially.

We assume then, that $B_i$ is not empty. Among all feet of the bridges in $B_i$ that lie in $P(r_{i-1}, r_i)$, let $a$ be the foot nearest $r_{i-1}$, if one exists, and $a = r_i$ otherwise. Among all feet of the bridges in $B_i$ that lie in $P(r_i, r_{i+1})$, let $b$ be the foot nearest $r_{i+1}$, if one exists, and $b = r_i$ otherwise. Clearly by this definition all feet of bridges in $B_i$ are contained within $P(a, r_i) \cup P(r_i, b) \cup P(r_i, u_k)$.

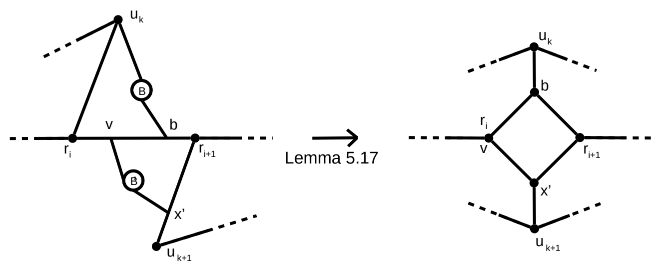
Suppose that there exists a bridge $B'$ with a foot $b' \in P(a, r_i) \cup P(r_i, b) - \{a, b\}$, and a foot $x'$ not in $P(r_{i-1}, r_i) \cup P(r_i, r_{i+1}) \cup P(r_i, u_k)$. We note that $B'$ does not have all of its feet contained within $P(r_i, r_{i+1}) \cup P(r_{i+1}, r_{i+2})$, or by symmetry $P(r_{i-2}, r_{i-1}) \cup P(r_{i-1}, r_i)$. Such a position for $B'$ would imply that $b$ is the foot of some bridge $B \in \bar{B}_i$ with all feet in $P(r_{i-1}, r_i) \cup P(r_i, r_{i+1})$, and hence by Lemma 5.21 would imply $G$ has a $P_2$ minor.

We know then that all feet of $B'$ must either be contained in $P(r_{i-1}, r_i) \cup P(r_{i-1}, u_{k+1})$ or $P(r_i, r_{i+1}) \cup P(r_{i+1}, u_{k+1})$. As these are symmetric, we will assume $P(r_i, r_{i+1}) \cup P(r_{i+1}, u_{k+1})$ and $b \neq r_i$. We know that $b$ is the foot of some bridge $B \in B_i$, and that the feet of $B$ are contained in $P(r_i, b) \cup P(r_i, u_k)$. Let $x$ be the foot of $B$ nearest $u_k$.

Now, suppose $x \neq u_k$. If $B'$ has a foot $v \in P(r_i, b) - \{r_i, b\}$, then $G$ would contain $P_2$.



Now suppose $x = u_k$, and hence $b \neq r_{i+1}$. Let $x'$ be the foot of $B'$ nearest $u_{k+1}$. If $x' = u_{k+1}$, then our choice for $n$ was not a maximum. So we may assume that $x' \neq u_{k+1}$. However, this also derives a contradiction since it would imply that $G$ contains a $P_2$ minor.




$\square$

**Lemma 5.23.** *Let $B \in \bar{B}_i \backslash B_i$ such that $u_k$ is a foot of $B$, and all feet of $B$ are in $P(r_i, r_{i+1}) \cup P(r_i, u_k)$. Then the following are true:*

*1. If $B$ has a foot $v \in P(r_i, u_k) - \{u_k\}$ and $B'$ is a bridge with feet in $P(r_i, r_{i+1}) \cup P(r_{i+1}, u_{k+1})$, then $B'$ has exactly one foot in $P(r_i, r_{i+1}) - \{r_{i+1}\}$, $r_i$.*

*2. If $B$ has no feet in $P(r_i, u_k) - \{r_i, u_k\}$ and exactly one foot $v \in P(r_i, r_{i+1}) - \{r_{i+1}\}$, then all feet in $P(r_i, r_{i+1}) - \{r_{i+1}\}$ of any bridge $B'$ with feet contained in $P(r_i, r_{i+1}) \cup P(r_{i+1}, u_{k+1})$ are in $\{r_i, v\}$.*

*3. If $B$ has no feet in $P(r_i, u_k) - \{r_i, u_k\}$ and more than one foot in $P(r_i, r_{i+1}) - \{r_{i+1}\}$, then any bridge $B'$ with feet contained in $P(r_i, r_{i+1}) \cup P(r_{i+1}, u_{k+1})$, has only the single foot in $P(r_i, r_{i+1}) - \{r_{i+1}\}$, $r_i$.*

*Proof.* Proof of (1). Suppose $B$ is as described in (1). Let $B'$ be a bridge with foot $b' \in P(r_i, r_{i+1}) - \{r_i, r_{i+1}\}$. Then $G$ contains a $P_2$ minor.

Proof of (2) and (3).

Let $v'$ be a foot of $B'$ in $P(r_i, r_{i+1}) - \{r_i, r_{i+1}\}$, and let $x'$ be the foot of $B'$ nearest $u_{k+1}$. Since $B$ is non-trivial, we know that $B$ has a foot $v \in P(r_i, r_{i+1}) - \{r_i, r_{i+1}\}$.

First, we suppose $v \in P(v', r_{i+1}) - \{v'\}$. Then either $x' = u_{k+1}$ or $x' \neq u_{k+1}$. If $x' = u_{k+1}$, our choice for $n$ would not have been a maximum.

However, if $x' \neq u_{k+1}$, then $G$ would contain a $P_2$ minor.

We assume then that $v \in P(r_i, v') - \{v'\}$. This also would imply that $G$ has a $P_2$ minor.

Hence, either $v' = r_i$, or $v' = v$ and $B$ has only the three feet $u_k, r_{i+1}$, and $v$.

□

**Lemma 5.24.** *No bridge in $\bar{B}_i \backslash B_i$ has a foot $v \in P(r_i, u_k) - \{r_i, u_k\}$ for any $i, k$.*

*Proof.* Let $B$ be a bridge of $H$ in $G$ such that $B$ has a foot $v \in P(r_i, u_k) - \{r_i, u_k\}$.

Suppose $B \in \bar{B}_i \backslash B_i$. By symmetry, $B$ at least has feet $r_{i+1}, u_k$, and $v$. Since $\{r_i, r_{i+1}, u_k\}$ would be a non-trivial 3-cut of $G$, there is a rescuing bridge for this cut, $B'$. We know that $B' \notin \bar{B}_{i+1}$ by Lemma 5.23, and hence $B' \in \bar{B}_i$.

Suppose first that the feet of $B'$ are contained in $P(r_{i-1}, r_i) \cup P(r_i, r_{i+1})$. If $B' \in \bar{B}_i \backslash B_i$, then $\{r_{i-1}, r_{i+1}, u_k\}$ would be a non-trivial 3-cut of $G$, and by our previous lemmas, there can exist no rescuing bridge. We can assume then, that $B' \in B_i$, and there is no bridge that as both $r_{i+1}$ and $r_{i-1}$ as feet.

Let $a$ and $b$ be vertices chosen according to Lemma 5.22. Then $\{a, r_{i+1}, u_k\}$ would be a non-trivial 3-cut of $G$, and therefore there must exist a rescuing bridge $B''$. We can say that $B'' \in\notin \bar{B}_{i+1}$ by Lemma 5.23, and hence $B'' \in \bar{B}_i \backslash B_i$. More specifically, $B''$ is a bridge with all feet contained within $P(r_{i-1}, r_i) \cup P(r_i, u_k)$ and $B''$ has a foot $v'' \in P(a, r_i) \cup P(r_i, u_k) - \{a, u_k\}$.

Now $\{r_{i-1}, r_{i+1}, u_k\}$ would be a non-trivial 3-cut of $G$, and hence there is another rescuing bridge $B'''$. However, by our construction, $B''' \in \bar{B}_i$. It must be true then that $B''' \in B_{i-1}$ or $B''' \in B_{i+1}$. In either case, the existence of $B'''$ implies a contradiction (of either the maximality of $n$ or the fact that $G$ is $P_2$-free) by

Lemma 5.23. By this contradiction, we may say that there are no bridges with all feet contained within $P(r_{i-1}, r_i) \cup P(r_i, r_{i+1})$.

Suppose instead that all feet of $B'$ are contained within $P(r_{i-1}, r_i) \cup P(r_i, u_k)$ and there are no bridges for which all feet are contained within $P(r_{i-1}, r_i) \cup P(r_i, r_{i+1})$. Let $v'$ be the foot of $B'$ that lies in $P(r_i, u_k) - \{r_i, u_k\}$

If $B' \in \bar{B}_i \backslash B_i$, then $\{r_{i-1}, r_{i+1}, u_k\}$ would be a non-trivial 3-cut of $G$, and hence there is a rescuing bridge for this cut, $B''$. By Lemma 5.23, $B''$ must be a member of $\bar{B}_{i-1}$ or $\bar{B}_{i+1}$ and must have $r_i$ as a foot. This derives a contradiction, since this structure would imply a $P_2$ minor by Lemma 5.13 (figure 4).

Finally, we may assume then, that $B' \in B_i$, and that there is no bridge with feet $r_{i-1}, u_k$, and $v$, where $v \in P(r_i, u_k) - \{r_i, u_k\}$. Let vertices $a$ and $b$ be defined as in Lemma 5.22. Then $\{a, r_{i+1}, u_k\}$ would be a non-trivial 3-cut of $G$, and any rescuing bridge $B''$ for this cut would be a member of $\bar{B}_{i-1}$, a member of $\bar{B}_{i+1}$, or the trivial bridge $r_i u_k$. With Lemma 5.22 then, we know that any rescuing bridge would have $r_i$ as a foot and would have a second foot contractible to $u_{k+1}$. This arrives at another contradiction, since it implies a $P_2$ minor by Lemma 5.13 (figure 4).

□

**Lemma 5.25.** *No non-trivial bridge has a foot $v \in P(r_i, u_k) - \{r_i, u_k\}$ for any $i, k$.*

*Proof.* Let $B \in B_i$ be a non-trivial bridge, and by symmetry let $x \neq u_k$ be the foot of $B$ nearest $u_k$ and $v$ be the foot of $B$ nearest $r_{i+1}$.

Suppose there exists a bridge $B' \in \bar{B}_i$ such that all feet of $B'$ are contained in $P(r_{i-1}, r_i) \cup P(r_i, r_{i+1})$. Then by Lemmas 5.21, 5.22, and 5.23, the existence of a rescuing bridge for the cut $\{r_{i-1}, r_{i+1}, u_k\}$ would imply that either $n$ was not a maximum or $G$ contains a $P_2$ minor, both contradictions. We may assume then, that there are no bridges for which all feet are contained within $P(r_{i-1}, r_i) \cup P(r_i, r_{i+1})$.

Suppose there exists no bridge $B' \in \bar{B}_i$ such that all feet of $B'$ are contained in $P(r_{i-1}, r_i) \cup P(r_i, u_k)$ and $B'$ has at least one foot in $P(r_i, u_k) - \{r_i, u_k\}$. Let $a$ and $b$ be vertices defined as in Lemma 5.22. Then $\{r_i, b, u_k\}$ is a non-trivial 3-cut of $G$ and by Lemma 5.22, there has no rescuing bridge. This is impossible since $G$ is q-4-c.

We may assume then, that there exists $B' \in B_i$ such that all feet of $B'$ are contained in $P(r_{i-1}, r_i) \cup P(r_i, u_k)$ and $B'$ has at least one foot in $P(r_i, u_k) - \{r_i, u_k\}$. Let vertices $a$ and $b$ be chosen as in Lemma 5.22. The 3-cut $\{a, u_k, b\}$ is non-trivial, and must have a rescuing bridge. By 5.24, the rescuing bridge $B''$, must be a member of $\bar{B}_{i-1}$, a member of $\bar{B}_{i+1}$, or the trivial bridge $r_i u_k$. In any of these cases $B''$ has $r_i$ as a foot by Lemma 5.22 and a foot contractible to $u_{k+1}$. This derives a contradiction however, as it implies that $G$ has a $P_2$ minor.

□

**Lemma 5.26.** *Let $H'$ be the graph of $H$ with each bridge with feet contained in $P(r_{i-1}, r_i) \cup P(r_i, r_{i+1})$ added, if such bridges exist. Of all feet of these bridges, let $x$ be the foot nearest $r_{i-1}$ and let $y$ be the foot nearest $r_{i+1}$. Then $\{x, y, u_k\}$ separates at most one vertex from $H'$.*

*Proof.* Suppose there exists at least one such bridge, $\{x, y, u_k\}$ separates at least the vertex $r_i$. Suppose the lemma is false. Let $v \neq r_i$ be a vertex in $P(x, r_i) \cup P(r_i, y) \cup P(r_i, u_k) - \{x, y, u_k\}$.

Due to the planarity of the bridges, $x$ and $y$ are two feet of some bridge $B$. If $B \in \bar{B}_i \backslash B_i$, then by our previous lemmas no rescuing bridge can exist for $\{x, y, u_k\}$. We can assume then, that $B \in B_i$. Choose vertices $a$ and $b$ as in Lemma 5.22. The vertices $\{a, b, u_k\}$ separate more than one vertex, and hence this cut has a rescuing bridge $B'$. By Lemma 5.22, $B' \in \bar{B}_i \backslash B_i$.

There are at most two such bridges, one each with feet contained in $P(r_{i-1}, r_i) \cup P(r_i, u_k)$ and $P(r_i, r_{i+1}) \cup P(r_i, u_k)$. The existence of any such bridges would then necessitate a rescuing bridge $B''$ for $\{r_{i-1}, r_{i+1}, u_k\}$. The bridge $B''$ would be a member of $B_{i-1}$ or $B_{i+1}$. It's existence however, would derive a contradiction of either the maximality of $n$ or the fact that $G$ is $P_2$-free by Lemma 5.23. Since no such rescuing bridge can exist, $\{x, y, u_k\}$ must not be a non-trivial 3-cut of $G$ and the lemma holds.

□

There are some direct corollaries that we can make from this lemma regarding some of the bridges of $G$.

**Corollary 5.27.** *No non-trivial bridge has all of its feet contained within $P(r_{i-1}, r_i) \cup P(r_i, r_{i+1})$ for any $i$.*

**Corollary 5.28.** *For each $i$, there is at most one bridge with all of its feet contained within $P(r_{i-1}, r_i) \cup P(r_i, r_{i+1})$.*

**Lemma 5.29.** *Let $B$ be a trivial bridge with feet in $P(r_{i-1}, r_i) \cup P(r_i, r_{i+1})$. Then there exists no bridge with $r_i$ as a foot.*

*Proof.* Suppose $B$ exists, and let $B'$ be a bridge with foot $r_i$. By Lemma 5.21 and the fact that bridges are added without crossings, we know that $B' \in \bar{B}_i$ and $B'$ is non-trivial.

If $B \in \bar{B}_i \backslash B_i$, then $\{r_{i-1}, r_{i+1}, u_k\}$ is a non-trivial 3-cut of $G$, and there can exist no rescuing bridge by our previous lemmas.

We can assume that $B \in B_i$. Choose vertices $a$ and $b$ according to Lemma 5.22. If $B' \in B_i$, then there exists a rescuing bridge $B''$ for $\{a, b, u_k\}$. Also according to Lemma 5.22, $B'' \in \bar{B}_i \backslash B_i$.

There are at most two such bridges, one each with feet contained in $P(r_{i-1}, r_i) \cup P(r_i, u_k)$ and $P(r_i, r_{i+1}) \cup P(r_i, u_k)$. The existence of any such bridges would then necessitate a rescuing bridge $B'''$ for $\{r_{i-1}, r_{i+1}, u_k\}$. The bridge $B'''$ would be a member of $B_{i-1}$ or $B_{i+1}$. It's existence however, would derive a contradiction of either the maximality of $n$ or the fact that $G$ is $P_2$-free by Lemma 5.23.

We suppose then, that $B' \in \bar{B}_i \backslash B_i$. By symmetry, we assume that the feet of $B'$ are contained within $P(r_i, r_{i+1}) \cup P(r_i, u_k)$. There is then a rescuing bridge $B''$ for $\{a, r_{i+1}, u_k\}$. By our previous lemmas $B'' \in \bar{B}_i \backslash B_i$ and has all feet contained within $P(r_{i-1}, r_i) \cup P(r_i, u_k)$. Now however, there must be a rescuing bridge for $\{r_{i-1}, r_{i+1}, u_k\}$, and that bridge must be a member of either $B_{i-1}$ or $B_{i+1}$. Either case would derive a contradiction of either the maximality of $n$ or the fact that $G$ is $P_2$-free by our previous lemmas.

□

**Lemma 5.30.** *For each $r_i$, there is at most one bridge with a foot in $P(r_i, u_k) - \{r_i, u_k\}$.*

*Proof.* Let $B$ and $B'$ be bridges that each have at least one foot in $P(r_i, u_k) - \{r_i, u_k\}$. By Lemma 5.25, we know they are both trivial, and hence $B, B' \in B_i$. By Lemma 5.26, we know that there are no bridges with all feet contained within $P(r_{i-1}, r_i) \cup P(r_i, r_{i+1})$.

Choose vertices $a$ and $b$ according to Lemma 5.22. Suppose that one of $a$ or $b$ is $r_i$, by symmetry say $a$. The cut $\{a, b, u_k\}$ has a rescuing bridge $B''$, but by Lemma 5.22 $B' \in \bar{B}_i$. Since there are no bridges attached to $P(r_{i-1}, r_i) \cup P(r_i, r_{i+1})$, we may assume that $B''$ has a foot in $P(r_i, u_k) - \{r_i, u_k\}$ and all feet contained within $P(r_{i-1}, r_i) \cup P(r_i, u_k)$. By Lemma 5.25, $B''$ would be trivial. This contradicts our assumption that $a = r_i$, since $B'' \in B_i$.

We may assume then, that neither $a$ nor $b$ is $r_i$. Suppose that the feet of $B$ and $B'$ are contained within the same two paths, say $P(r_i, r_{i+1})$ and $P(r_i, u_k)$. Then $\{r_i, b, u_k\}$ has a rescuing bridge $B''$. By our previous arguments $B'' \in B_i$, and the feet of $B''$ are contained within $P(r_{i-1}, r_i) \cup P(r_i, u_k)$. There is also a rescuing bridge $B'''$ for $\{a, b, u_k\}$, and $B'''$ is a member of either $\bar{B}_{i-1}$ or $\bar{B}_{i+1}$ and has $r_i$ as a foot. This derives a contradiction, as it would imply that $G$ has a $P_2$ minor.

Finally, suppose the feet of $B$ and $B'$ are not contained within the same two paths. Again there is a rescuing bridge $B''$ for $\{a, b, u_k\}$. By our previous arguments $B''$ is a member of either $\bar{B}_{i-1}$ or $\bar{B}_{i+1}$ and has $r_i$ as a foot. Again we derive a contradiction to the assumption that $G$ is $P_2$-free.

□

**Lemma 5.31.** *Every non-trivial bridge is a triad.*

*Proof.* Let $B$ be a non-trivial bridge of $H$ in $G$. By our previous lemmas and symmetry, we know that $B$ has foot $u_k$, and all other feet of $B$ are contained in $P(r_i, r_{i+1})$. Suppose $B$ has at least four feet. Let $x$ be the foot of $B$ nearest $r_i$, $z$ be the foot nearest $r_{i+1}$, and let $y$ be a foot of $B$ in $P(r_i, r_{i+1})$.

Then there is a rescuing bridge for $\{u_k, x, z\}$, $B'$. By Lemmas 5.22 and 5.23, $B' \notin \bar{B}_i$. The only remaining possibility for $B'$ is that all feet of $B'$ are contained within $P(r_{i-1}, r_i) \cup P(r_i, r_{i+1})$. By Corollary 5.27, $B'$ is trivial, and hence $B' \in B_i$.

Choose vertices $a$ and $b$ according to Lemma 5.22. There is a rescuing bridge for $\{a, z, u_k\}$, $B''$. The bridge $B'' \in \bar{B}_i \backslash B_i$, since any other location would derive contradictions to either the maximality of $n$ or the $P_2$-free property of $G$ by previous lemmas. $B''$ is then non-trivial and has a foot $v \in P(a, r_i) - \{a\}$. However, now there must be a rescuing bridge for $\{r_{i-1}, z, u_k\}$, $B'''$, and $B'''$ must be a member of either $B_{i-1}$ or $B_{i+1}$. Either case would derive a contradiction by our previous lemmas.

□

**Lemma 5.32.** *Let $t$ be a triad bridge of $H$ in $G$ with feet $u_k, a,$ and $b$. Then there exists no bridge with a foot $v \in P(a, b) - \{a, b\}$.*

*Proof.* Let $t$ be a triad bridge of $H$ with feet $u_k$ and $a, b \in P(r_i, r_{i+1})$. Let $B$ be a bridge with foot $v \in P(a, b) - \{a, b\}$. We know that $B \in B_{i+1}$ by 5.16, 5.26, and 5.29. Let $x$ be the foot of $B$ nearest $u_{k+1}$. Then $G$ would contain a $P_2$ minor.

□

With these observations, we prove our claim.

**Theorem 5.33.** *Let $G$ be a q-4-c, $P_2$-free graph such that $G$ contains a subgraph that is isomorphic to a subdivision of $B_n$, $n \geq 6$. Then $G \in \boldsymbol{SM}_n$.*

*Proof.* Let $G$ be a q-4-c, $P_2$-free graph such that $G$ contains $H$ as a minor, where $H$ is isomorphic to a subdivision of $B_n$, $n \geq 6$. There are only two types of bridges which may exist: trivial bridges and triads.

There are three types of trivial bridges $v_1v_2$:

1. $v_1 = u_k$ for some $k$;
2. $v_1 \in P(r_i, u_k) - \{r_i, u_k\}$ for some $i$ and $v_2 \in P(r_j, r_{j+1})$ for some $j = i, i-1$;
3. $v_1 \in P(r_{i-1}, r_i)$ and $v_2 \in P(r_i, r_{i+1})$ for some $i$.

Additions of bridges of the first type are additions of more 'spoke' edges to our $B_n$ structure, and as such are always present in some member of $\mathbf{SM}_n$. Suppose there is a bridge of the second type. By Lemmas 5.22, 5.26, and 5.29, the vertex $v_1$

functions as a triad with feet $u_k, r_i$, and $v_2$. Suppose there is a bridge of the third type. Then by Lemmas 5.22, 5.26, and 5.29, and Corollary 5.28, the addition of this bridge again simulates a triad. The vertex $r_i$ is a triad with feet $u_k, v_1$, and $v_2$, and we refine the rim cycle of our graph as $..., r_{i-2}, r_{i-1}, v_1, v_2, r_{i+1}, ....$

Any triad $t$ must be added such that $t$ is adjacent to some $u_k$ and two rim vertices by Lemma 5.25. Any triads with feet contained within the same two branches of $H$, may not cross by Thoerem 5.16, and must be added to where no two triads share the same two rim feet per Lemma 5.17. Additionally, no bridge may have a foot between the two rim feet of any triad, by Lemma 5.32. In this manner, all bridges added serve to create triads within a $DW_m$ for some $m$. We can then think of this equivalently as some graph $SM_m$ from which some number of triads and $u_k r_i$ edges have been deleted so as to remain q-4-c.

□

We have then a characterization for the structure of those sufficiently large graphs which do not contain a $P_2$ minor. The author hopes to one day give a complete characterization of those graphs that are $P_2$-free. And perhaps $P$-free.

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# Vita

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