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MIXED GENERALIZED FRACTIONAL BROWNIAN MOTION

SHAYKHAH ALAJMI AND EZZEDINE MLIKI*

ABSTRACT. To extend several known centered Gaussian processes, we introduce a new centered mixed self-similar Gaussian process called the mixed generalized fractional Brownian motion, which could serve as a good model for a larger class of natural phenomena. This process generalizes both the well-known mixed fractional Brownian motion introduced by Cheridito [7] and the generalized fractional Brownian motion introduced by Zili [29]. We study its main stochastic properties, its non-Markovian and non-stationarity characteristics and the conditions under which it is not a semimartingale. We prove the long-range dependence properties of this process.

1. Introduction

Fractional Brownian motion on the whole real line (fBm for short) $B^H = \{B_t^H, t \in \mathbb{R}\}$ of Hurst parameter H is the best known centered Gaussian process with long-range dependence. Its covariance function is

$$\text{Cov}(B_t^H, B_s^H) = \frac{1}{2}[|t|^{2H} + |s|^{2H} - |t - s|^{2H}], \quad (1.1)$$

where H is a real number in $(0, 1)$ and the case $H = \frac{1}{2}$ corresponds to the Brownian motion. It is the unique continuous Gaussian process starting from zero, the self-similarity and stationarity of the increments are two main properties for which fBm enjoyed successes as modeling tool in finance and telecommunications. Researchers have applied fractional Brownian motion to a wide range of problems, such as bacterial colonies, geophysical data, electrochemical deposition, particle diffusion, DNA sequences and stock market indicators [20, 22]. In particular, computer science applications of fBm include modeling network traffic and generating graphical landscapes [21]. The fBm was investigated in many papers (e.g. [2, 12, 16, 17, 18, 19]). The main difference between fBm and regular Brownian motion is that the increments in Brownian motion are independent, increments for fBm are not.

In [4], the authors suggested another kind of extension of the Brownian motion, called the sub-fractional Brownian motion (sfBm for short), which preserves most properties of the fBm, but not the stationarity of the increments. It is a centered

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Gaussian process $\xi^H = \{\xi_t^H, t \geq 0\}$, defined by:

$$\xi_t^H = \frac{B_t^H + B_{-t}^H}{\sqrt{2}}, \quad t \geq 0, \quad (1.2)$$

where $H \in (0, 1)$. The case $H = \frac{1}{2}$ corresponds to the Brownian motion.

The sfBm is intermediate between Brownian motion and fractional Brownian motion in the sense that it has properties analogous to those of fBm, self-similarity, not Markovian but the increments on nonoverlapping intervals are more weakly correlated, and their covariance decays polynomially at a higher rate in comparison with fBm (for this reason in [4] is called sfBm). So, the sfBm does not generalize the fBm. The sfBm was investigated in many papers (e.g. [3, 4, 24, 26]).

An extension of the sfBm was introduced by Zili in [28] as a linear combination of a finite number of independent sub-fractional Brownian motions. It was called the mixed sub-fractional Brownian motion (msfBm for short). The msfBm is a centered mixed self-similar Gaussian process and does not have stationary increments. The msfBm do not generalize the fBm.

In [29], Zili introduced new model called the generalized fractional Brownian motion (gfBm for short) which is an extension of both sub-fractional Brownian motion and fractional Brownian motion. A gfBm with parameters a, b , and H , is a process $Z^H = \{Z_t^H(a, b), t \geq 0\}$ defined by

$$Z_t^H(a, b) = aB_t^H + bB_{-t}^H, \quad t \geq 0. \quad (1.3)$$

The gfBm was investigated in [10, 30]. The gfBm generalize the sfBm but not the mixed fractional Brownian motion.

The mixed fractional Brownian motion (mfBm for short) is a linear combination between a Brownian motion and an independent fractional Brownian motion of Hurst parameter H . It was introduced by Cheridito [7] to present a stochastic model of the discounted stock price in some arbitrage-free and complete financial markets. The mfBm is a centered Gaussian process starting from zero with covariance function

$$\text{Cov}(N_t^H(a, b), N_s^H(a, b)) = a^2(t \wedge s) + \frac{b^2}{2} (t^{2H} + s^{2H} - |t - s|^{2H}), \quad (1.4)$$

with $H \in (0, 1)$. When $a = 1$ and $b = 0$, the mfBm is the Brownian motion and when $a = 0$ and $b = 1$, is the fBm. We refer also to [1, 7, 9, 25, 27] for further information on this process.

In this paper, we introduce a new stochastic model, which we call the mixed generalized fractional Brownian motion.

Definition 1.1. A *mixed generalized fractional Brownian motion* (mgfBm for short) of parameters a, b, c and $H \in (0, 1)$ is a centered Gaussian process

$$M^H(a, b, c) = \{M_t^H(a, b, c), t \geq 0\},$$

defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with the covariance function

$$C(t, s) = a^2(t \wedge s) + \frac{(b+c)^2}{2} (t^{2H} + s^{2H}) - bc(t+s)^{2H} - \frac{(b^2 + c^2)}{2} |t - s|^{2H}, \quad (1.5)$$

where $t \wedge s = \frac{1}{2} (t + s - |t - s|)$.

The mgfBm is completely different from all the extensions mentioned above. The process $M^H(a, b, c)$ is motivated by the fact that this process already introduced for specific values of a, b and c . Indeed $M^H(a, b, 0)$ is the mixed fractional Brownian motion and $M^H(0, b, c)$, is the generalized fractional Brownian motion. This why we will name $M^H(a, b, c)$ the mixed generalized fractional Brownian motion. It allows to deal with a larger class of modeled natural phenomena, including those with stationary or non-stationary increments.

Our goal is to study the main stochastic properties of this new model, paying attention to the long-range dependence, self-similarity, increment stationary, Markovity and semi-martingale properties.

2. The Main Properties

Existence of the mixed generalized fractional Brownian motion $M^H(a, b, c)$ for any $H \in (0, 1)$ can be shown in the following way: consider the process

$$M_t^H(a, b, c) = aB_t + bB_t^H + cB_{-t}^H, \quad t \geq 0, \quad (2.1)$$

where $B = \{B_t, t \in \mathbb{R}\}$ is a Brownian motion and $B^H = \{B_t^H, t \in \mathbb{R}\}$ is an independent fractional Brownian motion with Hurst parameter $H \in (0, 1)$.

Using (1.1) and since B and B^H are independent we obtain the following lemma.

Lemma 2.1. *For all $s, t \geq 0$, the process (2.1) is a centered Gaussian process with covariance function given by (1.5).*

Proof. Let $s, t \geq 0$ and $C(t, s) = \text{Cov}(M_t^H(a, b, c), M_s^H(a, b, c))$. Then

$$\begin{aligned} C(t, s) &= \text{Cov}(aB_t + bB_t^H + cB_{-t}^H, aB_s + bB_s^H + cB_{-s}^H) \\ &= a^2(t \wedge s) + b^2 \text{Cov}(B_t^H, B_s^H) + bc \text{Cov}(B_t^H, B_{-s}^H) + cb \text{Cov}(B_{-t}^H, B_s^H) \\ &\quad + c^2 \text{Cov}(B_{-t}^H, B_{-s}^H) \\ &= a^2(t \wedge s) + \frac{b^2}{2}(t^{2H} + s^{2H} - |t - s|^{2H}) + \frac{bc}{2}(t^{2H} + s^{2H} - |t + s|^{2H}) \\ &\quad + \frac{cb}{2}(t^{2H} + s^{2H} - |-(t + s)|^{2H}) + \frac{c^2}{2}(t^{2H} + s^{2H} - |-(t - s)|^{2H}) \\ &= a^2(t \wedge s) + \frac{b^2}{2}t^{2H} + \frac{b^2}{2}s^{2H} - \frac{b^2}{2}|t - s|^{2H} + \frac{bc}{2}t^{2H} + \frac{bc}{2}s^{2H} \\ &= a^2(t \wedge s) + \frac{(b + c)^2}{2}(t^{2H} + s^{2H}) - bc|t + s|^{2H} - \frac{(b^2 + c^2)}{2}|t - s|^{2H}. \end{aligned}$$

Hence the covariance function of the process (2.1) is precisely $C(t, s)$ given by (1.5). Therefore the $M^H(a, b, c)$ exists. $\square$

Remark 2.1. *Some special cases of the mixed generalized fractional Brownian motion:*

- (1) *If $a = 0, b = 1, c = 0$, then $M^H(0, 1, 0)$ is a fBm.*
- (2) *If $a = 0, b = c = \frac{1}{\sqrt{2}}$, then $M^H(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ is a sfBm.*
- (3) *If $a = 1, b = 0, c = 0$, then $M^H(1, 0, 0)$ is a Bm.*
- (4) *If $a = 0$, then $M^H(0, b, c)$, is a gfBm.*

- (5) If $c = 0$, then $M^H(a, b, 0)$, is a $mfBm$.
 (6) If $b = c$, then $M^H(a, \frac{b}{\sqrt{2}}, \frac{b}{\sqrt{2}})$, is a $smfBm$.

So the mixed generalized fractional Brownian motion is, at the same, a generalization of the fractional Brownian motion, sub-fractional Brownian motion, the sub-mixed fractional Brownian motion, generalized fractional Brownian motion, mixed fractional Brownian motion and of course of the standard Brownian motion.

Proposition 2.1. *The $mgfBm$ satisfies the following properties:*

- (1) For all $t \geq 0$,

$$E \left(M_t^H(a, b, c) \right)^2 = a^2 t + (b^2 + c^2 - (2^{2H} - 2)bc) t^{2H}.$$

- (2) Let $0 \leq s < t$ and $\alpha(t, s) = E \left(M_t^H(a, b, c) - M_s^H(a, b, c) \right)^2$. Then

$$\begin{aligned} E \left(M_t^H(a, b, c) - M_s^H(a, b, c) \right)^2 &= a^2 |t - s| - 2^{2H} bc (t^{2H} + s^{2H}) \\ &\quad + (b^2 + c^2) |t - s|^{2H} + 2bc |t + s|^{2H}. \end{aligned}$$

- (3) We have for all $0 \leq s < t$,

$$a^2(t - s) + \gamma_{(b, c, H)}(t - s)^{2H} \leq \alpha(t, s) \leq a^2(t - s) + \nu_{(b, c, H)}(t - s)^{2H},$$

where

$$\gamma_{(b, c, H)} = (b^2 + c^2 - 2bc(2^{2H-1} - 1)) \mathbf{1}_{\mathcal{C}}(b, c, H) + (b^2 + c^2) \mathbf{1}_{\mathcal{D}}(b, c, H),$$

$$\nu_{(b, c, H)} = (b^2 + c^2) \mathbf{1}_{\mathcal{C}}(a, b, H) + (b^2 + c^2 - 2bc(2^{2H-1} - 1)) \mathbf{1}_{\mathcal{D}}(b, c, H),$$

$$\mathcal{C} = \{(b, c, H) \in \mathbb{R}^2 \times]0, 1[; (H > \frac{1}{2}, bc \geq 0) \text{ or } (H < \frac{1}{2}, bc \leq 0)\},$$

and

$$\mathcal{D} = \{(b, c, H) \in \mathbb{R}^2 \times]0, 1[; (H > \frac{1}{2}, bc \leq 0) \text{ or } (H < \frac{1}{2}, bc \geq 0)\}.$$

Proof. (1) It is a direct consequence of (1.5).

- (2) Let $0 \leq s < t$ and $\alpha(t, s) = E \left(M_t^H(a, b, c) - M_s^H(a, b, c) \right)^2$. Then

$$\begin{aligned} \alpha(t, s) &= E \left(M_t^H(a, b, c) \right)^2 + E \left(M_s^H(a, b, c) \right)^2 - 2E \left(M_t^H(a, b, c) M_s^H(a, b, c) \right) \\ &= a^2 t + b^2 t^{2H} + 2bct^{2H} - 2^{2H} bct^{2H} + c^2 t^{2H} + a^2 s + b^2 s^{2H} + 2bcs^{2H} \\ &\quad - 2^{2H} bcs^{2H} + c^2 s^{2H} - 2a^2(t \wedge s) - b^2 t^{2H} - b^2 s^{2H} + b^2 |t - s|^{2H} \\ &\quad - bct^{2H} - bcs^{2H} + bc|t + s|^{2H} - cbt^{2H} - cbs^{2H} + cb|t + s|^{2H} - c^2 t^{2H} \\ &\quad - c^2 s^{2H} + c^2 |t - s|^{2H} \\ &= a^2(t + s) - 2^{2H} bc(t^{2H} + s^{2H}) - 2a^2(t \wedge s) + (b^2 + c^2) |t - s|^{2H} \\ &\quad + 2bc|t + s|^{2H} \\ &= a^2 |t - s| - 2^{2H} bc(t^{2H} + s^{2H}) + (b^2 + c^2) |t - s|^{2H} + 2bc|t + s|^{2H}. \end{aligned}$$

- (3) It is a direct consequence of the second item of Proposition 2.1 and Lemma 3 in [29].

□

Proposition 2.2. *For all $(a, b, c) \in \mathbb{R}^3 \setminus \{(0, 0, 0)\}$ and $H \in (0, 1) \setminus \{\frac{1}{2}\}$, the mgfBm is not a self-similar process.*

Proof. This follows from the fact that, for fixed $h > 0$, the processes $\{M_{ht}^H(a, b, c), t \geq 0\}$ and $\{h^H M_t^H(a, b, c), t \geq 0\}$ are Gaussian, centered, but don't have the same covariance function. Indeed

$$\begin{aligned}
C(ht, hs) &= a^2(ht \wedge hs) + \frac{b^2}{2}((ht)^{2H} + (hs)^{2H} - |ht - hs|^{2H}) \\
&\quad + \frac{bc}{2}((ht)^{2H} + (hs)^{2H} - |ht + hs|^{2H}) \\
&\quad + \frac{cb}{2}((ht)^{2H} + (hs)^{2H} - |-(ht + hs)|^{2H}) \\
&\quad + \frac{c^2}{2}((ht)^{2H} + (hs)^{2H} - |-(ht - hs)|^{2H}) \\
&= a^2(ht \wedge hs) + \frac{b^2}{2}(ht)^{2H} + \frac{b^2}{2}(hs)^{2H} - \frac{b^2}{2}|ht - hs|^{2H} \\
&\quad + \frac{bc}{2}(ht)^{2H} + \frac{bc}{2}(hs)^{2H} - \frac{bc}{2}|ht + hs|^{2H} + \frac{bc}{2}(ht)^{2H} + \frac{bc}{2}(hs)^{2H} \\
&\quad - \frac{bc}{2}|ht + hs|^{2H} + \frac{c^2}{2}(ht)^{2H} + \frac{c^2}{2}(hs)^{2H} - \frac{c^2}{2}|ht - hs|^{2H} \\
&= a^2h(t \wedge s) + h^{2H} \frac{(b+c)^2}{2}((t)^{2H} + (s)^{2H}) - bch^{2H}|t + s|^{2H} \\
&\quad - h^{2H} \frac{(b^2 + c^2)}{2}|t - s|^{2H}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
Cov(h^H M_t^H(a, b, c), h^H M_s^H(a, b, c)) &= h^{2H} Cov(M_t^H(a, b, c), M_s^H(a, b, c)) \\
&= a^2 h^{2H}(t \wedge s) \\
&\quad + h^{2H} \frac{(b+c)^2}{2}(t^{2H} + s^{2H}) \\
&\quad - bch^{2H}|t + s|^{2H} \\
&\quad - h^{2H} \frac{(b^2 + c^2)}{2}|t - s|^{2H}.
\end{aligned}$$

Then the mgfBm is not a self-similar process for all $(a, b, c) \in \mathbb{R}^3 \setminus \{(0, 0, 0)\}$. $\square$

Remark 2.2. *As a consequence of Proposition 2.2, we see that:*

- (1) $M^H(0, b, c)$ is a self-similar process for all $(b, c) \in \mathbb{R}^2$.
- (2) $M^{\frac{1}{2}}(a, b, c)$ is a self-similar process for all $(a, b, c) \in \mathbb{R}^3$.

Now, we will study the Markovian property.

Theorem 2.1. *Assume $H \in (0, 1) \setminus \{\frac{1}{2}\}$, $a \in \mathbb{R}$ and $(b, c) \in \mathbb{R}^2 \setminus \{(0, 0)\}$. Then $M^H(a, b, c)$ is not a Markovian process.*

Proof. The process $M^H(a, b, c)$ is a centered Gaussian. Then, if $M_t^H(a, b, c)$ is a Markovian process, according to Revuz and Yor [23], for all $s < t < u$, we would

have

$$C(s, u)C(t, t) = C(s, t)C(t, u).$$

We will only prove the theorem in the case where $a \neq 0$, the result with $a = 0$ is known in [29]. For the proof we follow the proof of Proposition 1 given in [29]. Using Proposition 2.1, we get

$$\begin{aligned} C(s, u) &= a^2s + \frac{(b+c)^2}{2}(u^{2H} + s^{2H}) - bc|u+s|^{2H} - \frac{(b^2+c^2)}{2}|u-s|^{2H}, \\ C(t, t) &= a^2t + (b^2+c^2 - (2^{2H}-2)bc)t^{2H}, \\ C(s, t) &= a^2s + \frac{(b+c)^2}{2}(t^{2H} + s^{2H}) - bc|t+s|^{2H} - \frac{(b^2+c^2)}{2}|t-s|^{2H}, \\ C(t, u) &= a^2t + \frac{(b+c)^2}{2}(u^{2H} + t^{2H}) - bc|u+t|^{2H} - \frac{(b^2+c^2)}{2}|u-t|^{2H}. \end{aligned}$$

In the particular case where $1 < s = \sqrt{t} < t < u = t^2$, we have

$$\begin{aligned} C(\sqrt{t}, t^2) &= a^2t^{\frac{1}{2}} + \frac{(b+c)^2}{2}(t^{4H} + t^H) - bc|t^2 + t^{\frac{1}{2}}|^{2H} - \frac{(b^2+c^2)}{2}|t^2 - t^{\frac{1}{2}}|^{2H}, \\ C(t, t) &= a^2t + (b^2+c^2 - (2^{2H}-2)bc)t^{2H}, \\ C(\sqrt{t}, t) &= a^2t^{\frac{1}{2}} + \frac{(b+c)^2}{2}(t^{2H} + t^H) - bc|t + t^{\frac{1}{2}}|^{2H} - \frac{(b^2+c^2)}{2}|t - t^{\frac{1}{2}}|^{2H}, \\ C(t, t^2) &= a^2t + \frac{(b+c)^2}{2}(t^{4H} + t^{2H}) - bc|t^2 + t|^{2H} - \frac{(b^2+c^2)}{2}|t^2 - t^{\frac{1}{2}}|^{2H}. \end{aligned}$$

Then by using that,

$$C(\sqrt{t}, t^2)C(t, t) = C(\sqrt{t}, t)C(t, t^2),$$

we have

$$\begin{aligned} &\left[a^2t^{\frac{1}{2}} + \frac{(b+c)^2}{2}(t^{4H} + t^H) - bc|t^2 + t^{\frac{1}{2}}|^{2H} - \frac{(b^2+c^2)}{2}|t^2 - t^{\frac{1}{2}}|^{2H} \right] \\ &\times [a^2t + (b^2+c^2 - (2^{2H}-2)bc)t^{2H}] \\ &= \left[a^2t^{\frac{1}{2}} + \frac{(b+c)^2}{2}(t^{2H} + t^H) - bc|t + t^{\frac{1}{2}}|^{2H} - \frac{(b^2+c^2)}{2}|t - t^{\frac{1}{2}}|^{2H} \right] \\ &\times \left[a^2t + \frac{(b+c)^2}{2}(t^{4H} + t^{2H}) - bc|t^2 + t|^{2H} - \frac{(b^2+c^2)}{2}|t^2 - t^{\frac{1}{2}}|^{2H} \right]. \end{aligned}$$

It follows that

$$\begin{aligned} &\left[a^2t^{\frac{1}{2}} + t^{4H} \left(\frac{(b+c)^2}{2}(1 + t^{-3H}) - bc|1 + t^{-\frac{3}{2}}|^{2H} - \frac{(b^2+c^2)}{2}|1 - t^{-\frac{3}{2}}|^{2H} \right) \right] \\ &\times [a^2t + (b^2+c^2 - (2^{2H}-2)bc)t^{2H}] \\ &= \left[a^2t^{\frac{1}{2}} + t^{2H} \left(\frac{(b+c)^2}{2}(1 + t^{-H}) - bc|1 + t^{-\frac{1}{2}}|^{2H} - \frac{(b^2+c^2)}{2}|1 - t^{-\frac{1}{2}}|^{2H} \right) \right] \\ &\times \left[a^2t + t^{4H} \left(\frac{(b+c)^2}{2}(1 + t^{-2H}) - bc|1 + t^{-1}|^{2H} - \frac{(b^2+c^2)}{2}|1 - t^{-1}|^{2H} \right) \right]. \end{aligned}$$

Hence

$$\begin{aligned}
& a^4 t^{\frac{3}{2}} + a^2 (b^2 + c^2 - (2^{2H} - 2)bc) t^{2H+\frac{1}{2}} \\
& + a^2 \left(\frac{(b+c)^2}{2} (1+t^{-3H}) - bc |1+t^{-\frac{3}{2}}|^{2H} - \frac{(b^2+c^2)}{2} |1-t^{-\frac{3}{2}}|^{2H} \right) \\
& \times t^{4H+1} \\
& + \left(\frac{(b+c)^2}{2} (1+t^{-3H}) - bc |1+t^{-\frac{3}{2}}|^{2H} - \frac{(b^2+c^2)}{2} |1-t^{-\frac{3}{2}}|^{2H} \right) \\
& \times (b^2 + c^2 - (2^{2H} - 2)bc) t^{6H} \\
= & a^4 t^{\frac{3}{2}} + a^2 \left(\frac{(b+c)^2}{2} (1+t^{-2H}) - bc |1+t^{-1}|^{2H} - \frac{(b^2+c^2)}{2} |1-t^{-1}|^{2H} \right) \\
& \times t^{4H+\frac{1}{2}} \\
& + a^2 \left(\frac{(b+c)^2}{2} (1+t^{-H}) - bc |1+t^{-\frac{3}{2}}|^{2H} - \frac{(b^2+c^2)}{2} |1-t^{-\frac{3}{2}}|^{2H} \right) \\
& \times t^{2H+1} \\
& + \left(\frac{(b+c)^2}{2} (1+t^{-H}) - bc |1+t^{-\frac{3}{2}}|^{2H} - \frac{(b^2+c^2)}{2} |1-t^{-\frac{3}{2}}|^{2H} \right) \\
& \times \left(\frac{(b+c)^2}{2} (1+t^{-2H}) - bc |1+t^{-1}|^{2H} - \frac{(b^2+c^2)}{2} |1-t^{-1}|^{2H} \right) t^{6H}.
\end{aligned}$$

Take t^{6H} as a common factor, we get

$$\begin{aligned}
& a^2 (b^2 + c^2 - (2^{2H} - 2)bc) t^{-4H+\frac{1}{2}} \\
& + a^2 \left(\frac{(b+c)^2}{2} (1+t^{-3H}) - bc |1+t^{-\frac{3}{2}}|^{2H} - \frac{(b^2+c^2)}{2} |1-t^{-\frac{3}{2}}|^{2H} \right) t^{-2H+1} \\
& + \left(\frac{(b+c)^2}{2} (1+t^{-3H}) - bc |1+t^{-\frac{3}{2}}|^{2H} - \frac{(b^2+c^2)}{2} |1-t^{-\frac{3}{2}}|^{2H} \right) \\
& \times (b^2 + c^2 - (2^{2H} - 2)bc) \\
= & a^2 \left(\frac{(b+c)^2}{2} (1+t^{-2H}) - bc |1+t^{-1}|^{2H} - \frac{(b^2+c^2)}{2} |1-t^{-1}|^{2H} \right) t^{-2H+\frac{1}{2}} \\
& + a^2 \left(\frac{(b+c)^2}{2} (1+t^{-H}) - bc |1+t^{-\frac{1}{2}}|^{2H} - \frac{(b^2+c^2)}{2} |1-t^{-\frac{1}{2}}|^{2H} \right) t^{-4H+1} \\
& + \left(\frac{(b+c)^2}{2} (1+t^{-H}) - bc |1+t^{-\frac{1}{2}}|^{2H} - \frac{(b^2+c^2)}{2} |1-t^{-\frac{1}{2}}|^{2H} \right) \\
& \times \left(\frac{(b+c)^2}{2} (1+t^{-2H}) - bc |1+t^{-1}|^{2H} - \frac{(b^2+c^2)}{2} |1-t^{-1}|^{2H} \right).
\end{aligned}$$

Therefore

$$\begin{aligned}
& a^2 (b^2 + c^2 - (2^{2H} - 2)bc) t^{-4H+\frac{1}{2}} \\
& + a^2 \left[\frac{1}{2} (b+c)^2 (1+t^{-3H}) - bc \left(1 + 2Ht^{-\frac{3}{2}} + H(2H-1)t^{-3} + o(t^{-3}) \right) \right. \\
& \quad \left. - \frac{1}{2} (b^2 + c^2) \left(1 - 2Ht^{-\frac{3}{2}} + H(2H-1)t^{-3} + o(t^{-3}) \right) \right] t^{-2H+1} \\
& + \left[\frac{1}{2} (b+c)^2 (1+t^{-3H}) - bc \left(1 + 2Ht^{-\frac{3}{2}} + H(2H-1)t^{-3} + o(t^{-3}) \right) \right. \\
& \quad \left. - \frac{1}{2} (b^2 + c^2) \left(1 - 2Ht^{-\frac{3}{2}} + H(2H-1)t^{-3} + o(t^{-3}) \right) \right] \\
& \quad \times [b^2 + c^2 - (2^{2H} - 2)bc] \\
= & a^2 \left[\frac{1}{2} (b+c)^2 (1+t^{-2H}) - bc (1 + 2Ht^{-1} + H(2H-1)t^{-2} + o(t^{-2})) \right. \\
& \quad \left. - \frac{1}{2} (b^2 + c^2) (1 - 2Ht^{-1} + H(2H-1)t^{-2} + o(t^{-2})) \right] t^{-2H+\frac{1}{2}} \\
& + a^2 \left[\frac{1}{2} (b+c)^2 (1+t^{-H}) - bc \left(1 + 2Ht^{-1/2} + H(2H-1)t^{-1} + o(t^{-1}) \right) \right. \\
& \quad \left. - \frac{1}{2} (b^2 + c^2) \left(1 - 2Ht^{-\frac{1}{2}} + H(2H-1)t^{-1} + o(t^{-1}) \right) \right] t^{-4H+1} \\
& + \left[\frac{1}{2} (b+c)^2 (1+t^{-H}) - bc \left(1 + 2Ht^{-\frac{1}{2}} + H(2H-1)t^{-1} + o(t^{-1}) \right) \right. \\
& \quad \left. - \frac{1}{2} (b^2 + c^2) \left(1 - 2Ht^{-\frac{1}{2}} + H(2H-1)t^{-1} + o(t^{-1}) \right) \right] \\
& \quad \times \left[\frac{1}{2} (b+c)^2 (1+t^{-2H}) - bc (1 + 2Ht^{-1} + H(2H-1)t^{-2} + o(t^{-2})) \right. \\
& \quad \left. - \frac{1}{2} (b^2 + c^2) (1 - 2Ht^{-1} + H(2H-1)t^{-2} + o(t^{-2})) \right].
\end{aligned}$$

First case: $0 < H < \frac{1}{2}$, $a \neq 0$ and $b+c \neq 0$. By Taylor's expansion we get, as $t \rightarrow \infty$,

$$\begin{aligned}
& a^2 (b^2 + c^2 - (2^{2H} - 2)bc) t^{-4H+\frac{1}{2}} + a^2 \frac{1}{2} (b+c)^2 t^{-5H+1} \\
& + \frac{1}{2} (b+c)^2 [b^2 + c^2 - (2^{2H} - 2)bc] t^{-3H} \\
\approx & a^2 \frac{1}{2} (b+c)^2 t^{-4H+\frac{1}{2}} + a^2 \frac{1}{2} (b+c)^2 t^{-5H+1} + \frac{1}{4} (b+c)^4 t^{-3H}.
\end{aligned}$$

Therefore

$$\begin{aligned}
& a^2 (b^2 + c^2 - (2^{2H} - 2)bc) t^{-4H+\frac{1}{2}} + \frac{1}{2} (b+c)^2 [b^2 + c^2 - (2^{2H} - 2)bc] t^{-3H} \\
\approx & a^2 \frac{1}{2} (b+c)^2 t^{-4H+\frac{1}{2}} + \frac{1}{4} (b+c)^4 t^{-3H},
\end{aligned}$$

which is true if and only if

$$\frac{(b-c)^2}{2} - (2^{2H} - 2)bc = 0 \quad \text{and} \quad a^2 \frac{(b-c)^2}{2} - a^2(2^{2H} - 2)bc = 0.$$

However, it is easy to check that

$$\frac{(b-c)^2}{2} - (2^{2H} - 2)bc > 0 \quad \text{and} \quad a^2 \frac{(b-c)^2}{2} - a^2(2^{2H} - 2)bc > 0,$$

for fixed c, b and every real a .

Second case: $0 < H < \frac{1}{2}$, $a \neq 0$ and $b + c = 0$. By Taylor's expansion we get, as $t \rightarrow \infty$,

$$\begin{aligned} & a^2[b^2 + c^2 - (2^{2H} - 2)bc]t^{-4H+\frac{1}{2}} + a^2[-2Hbc + (b^2 + c^2)H]t^{-2H-\frac{1}{2}} \\ & + [-2Hbc + (b^2 + c^2)H] \times [b^2 + c^2 - (2^{2H} - 2)bc]t^{-\frac{3}{2}} \\ \approx & a^2[-2Hbc + (b^2 + c^2)H]t^{-2H-\frac{1}{2}} + a^2[-2Hbc + (b^2 + c^2)H]^2t^{-4H+\frac{1}{2}} \\ & + [-2Hbc + (b^2 + c^2)H]^2t^{-\frac{3}{2}}. \end{aligned}$$

Hence

$$\begin{aligned} & a^2[b^2 + c^2 - (2^{2H} - 2)bc]t^{-4H+\frac{1}{2}} \\ & + [-2Hbc + (b^2 + c^2)H] \times [b^2 + c^2 - (2^{2H} - 2)bc]t^{-\frac{3}{2}} \\ \approx & a^2[-2Hbc + (b^2 + c^2)H]t^{-4H+\frac{1}{2}} + [-2Hbc + (b^2 + c^2)H]^2t^{-\frac{3}{2}}, \end{aligned}$$

which is true if and only if $b = c = 0$. This is a contradiction.

Third case: $\frac{1}{2} < H < 1$, $a \neq 0$ and $b - c \neq 0$. By Taylor's expansion we get, as $t \rightarrow \infty$,

$$\begin{aligned} & a^2[b^2 + c^2 - (2^{2H} - 2)bc]t^{-4H+\frac{1}{2}} + a^2H(b-c)^2t^{-2H-\frac{1}{2}} \\ & + H(b-c)^2[b^2 + c^2 - (2^{2H} - 2)bc]t^{-\frac{3}{2}} \\ \approx & a^2H(b-c)^2t^{-2H-\frac{1}{2}} + a^2H(b-c)^2t^{-4H+\frac{1}{2}} + H^2(b-c)^4t^{-\frac{3}{2}}. \end{aligned}$$

Then

$$\begin{aligned} & a^2[b^2 + c^2 - (2^{2H} - 2)bc]t^{-4H+\frac{1}{2}} + H(b-c)^2[b^2 + c^2 - (2^{2H} - 2)bc]t^{-\frac{3}{2}} \\ \approx & a^2H(b-c)^2t^{-4H+\frac{1}{2}} + H^2(b-c)^4t^{-\frac{3}{2}}, \end{aligned}$$

which is true if and only if

$$[b^2(1-H) + c^2(1-H) + (2-2^{2H}+2H)bc] = 0.$$

However, it is easy to check that $b^2(1-H) + c^2(1-H) + (2-2^{2H}+2H)bc > 0$ for fixed c, b and every real a .

Fourth case: $\frac{1}{2} < H < 1$, $a \neq 0$ and $b - c = 0$. By Taylor's expansion we get, as $t \rightarrow \infty$,

$$\begin{aligned} & a^2(b^2 + c^2 - (2^{2H} - 2)bc)t^{-4H+\frac{1}{2}} + \frac{1}{2}(b+c)^2[b^2 + c^2 - (2^{2H} - 2)bc]t^{-3H} \\ \approx & a^2\frac{1}{2}(b+c)^2t^{-4H+\frac{1}{2}} + \frac{1}{4}(b+c)^4t^{-3H}, \end{aligned}$$

which is true if and only if $2 - 2^{2H} = 0$. This contradicts the fact that $H \neq \frac{1}{2}$. $\square$

Let us check the mixed self-similarity property of the mgfBm. This property was introduced in [27] for the mfBm and investigated to show the Hölder continuity of the mfBm. See also [11] for the sfBm case.

Proposition 2.3. *For any $h > 0$, $\{M_{ht}^H(a, b, c)\} \triangleq \{M_t^H(ah^{\frac{1}{2}}, bh^H, ch^H)\}$, where $\triangleq$ "to have the same law".*

Proof. For fixed $h > 0$, the processes $\{M_{ht}^H(a, b, c)\}$ and $\{M_t^H(ah^{\frac{1}{2}}, bh^H, ch^H)\}$ are Gaussian and centered. Therefore, one only have to prove that they have the same covariance function. But, for any $s, t \geq 0$, since B and B^H are independent, then

$$\begin{aligned} C(ht, hs) &= a^2h(t \wedge s) + \frac{(b^2 + c^2)}{2} [h^{2H}(t^{2H} + s^{2H} - |t - s|^{2H})] \\ &\quad + bc[h^{2H}(t^{2H} + s^{2H} - |t + s|^{2H})] \\ &= \text{Cov}\left(M_t^H(ah^{\frac{1}{2}}, bh^H, ch^H), M_s^H(ah^{\frac{1}{2}}, bh^H, ch^H)\right). \end{aligned}$$

□

Proposition 2.4. *For all $a \in \mathbb{R}$ and $(b, c) \in \mathbb{R}^2 \setminus \{(0, 0)\}$, the increments of the mgfBm are not stationary.*

Proof. Let $a \in \mathbb{R}$ and $(b, c) \in \mathbb{R}^2 \setminus \{(0, 0)\}$. For a fixed $t \geq 0$ consider the processes $\{P_t, t \geq 0\}$ define by $P_t = M_{t+s}^H(a, b, c) - M_s^H(a, b, c)$. Using Proposition 2.1, we get

$$\begin{aligned} \text{Cov}(P_t, P_t) &= E\left(M_{t+s}^H(a, b, c) - M_s^H(a, b, c)\right)^2 \\ &= a^2(t + s + s) - 2^{2H}bc((t + s)^{2H} + s^{2H}) - 2a^2s \\ &\quad + (b^2 + c^2)|t + s - s|^{2H} + 2bc|t + s + s|^{2H} \\ &= a^2(t + 2s) - 2^{2H}bc((t + s)^{2H} + s^{2H}) - 2a^2s + (b^2 + c^2)t^{2H} \\ &\quad + 2bc|t + 2s|^{2H}. \end{aligned}$$

Using Proposition 2.1, we get

$$\text{Cov}(M_t^H(a, b, c), M_t^H(a, b, c)) = a^2t + (b^2 + c^2 - (2^{2H} - 2)bc)t^{2H}.$$

Since both processes are centered Gaussian, the inequality of covariance functions implies that P_t does not have the same distribution as $M_t^H(a, b, c)$. Thus, the incremental behavior of $M^H(a, b, c)$ at any point in the future is not the same. Hence the increments of $M^H(a, b, c)$ are not stationary. □

Remark 2.3. *As a consequence of Proposition 2.4, we see that:*

- (1) *the increments of $M^H(0, b, c)$ are not stationary for all $(b, c) \in \mathbb{R}^2 \setminus \{(0, 0)\}$.*
- (2) *the increments of $M^H(a, b, 0)$ are stationary for all $(a, b) \in \mathbb{R}^2$.*

Proposition 2.5. (1) *Let $H \in (0, 1)$. The mgfBm admits a version whose sample paths are almost Hölder continuous of order strictly less than $\frac{1}{2} \wedge H$.*
 (2) *When b or c not zero and $H \in (0, 1) \setminus \{\frac{1}{2}\}$ the mgfBm is not a semi-martingale.*

Proof. (1) Let $s, t \geq 0$ and $\alpha = 2$. The proof follows by Kolmogorov criterion from Lemma 3 in [29] and using Proposition 2.1 we get

$$\begin{aligned} E(|M_t^H(a, b, c) - M_s^H(a, b, c)|^\alpha) &= a^2|t - s| - 2^{2H}bc(t^{2H} + s^{2H}) \\ &\quad + (b^2 + c^2)|t - s|^{2H} + 2bc|t + s|^{2H} \\ &\leq C_\alpha|t - s|^{\alpha(\frac{1}{2} \wedge H)}, \end{aligned}$$

where $C_\alpha = (a^2 + \nu(b, c, H))$ and $\nu_{(b, c, H)}$ is given in Lemma 3 in [29].

(2) Suppose first that $H < \frac{1}{2}$. We get from Proposition 2.1

$$\alpha(t, s) \geq \gamma_{(b, c, H)}(t - s)^{2H}.$$

Since $2H < 1$ and $\gamma_{(b, c, H)} > 0$ then the assumption of Corollary 2.1 in [5] is satisfied, and consequently the mgfBm is not a semi-martingale. Suppose now that $H > \frac{1}{2}$. We get from Proposition 2.1

$$a^2(t - s) + \gamma_{(b, c, H)}(t - s)^{2H} \leq \alpha(t, s) \leq (a^2 + \nu_{(b, c, H)})(t - s)^{1 \wedge 2H},$$

then

$$\gamma_{(b, c, H)}(t - s)^{2H} \leq \alpha(t, s) \leq (a^2 + \nu_{(b, c, H)})(t - s)^{2H}.$$

Since $1 < 2H < 2$ and $\nu_{(b, c, H)} > 0$ then the assumption of Lemma 2.1 in [5] is satisfied, and consequently the mgfBm is not a semi-martingale. $\square$

3. Long-range Dependence of the mgfBm Increments

Definition 3.1. We say that the increments of a stochastic process X are *long-range dependent* if for every integer $p \geq 1$, we have

$$\sum_{n \geq 1} R_X(p, p + n) = \infty,$$

where

$$R_X(p, p + n) = E((X_{p+1} - X_p)(X_{p+n+1} - X_{p+n})).$$

This property was investigated in many papers (e.g. [3, 6, 8, 12, 20]).

Theorem 3.1. For every $a \in \mathbb{R}$ and $(b, c) \in \mathbb{R}^2 \setminus \{(0, 0)\}$, the increments of $M^H(a, b, c)$ are long-range dependent if and only if $H > \frac{1}{2}$ and $b \neq c$.

Proof. For all $n \geq 1$ and $p \geq 1$, we have

$$\begin{aligned}
& R_M(p, p+n) \\
&= E \left((M_{p+1}^H(a, b, c) - M_p^H(a, b, c)) (M_{p+n+1}^H(a, b, c) - M_{p+n}^H(a, b, c)) \right) \\
&= E \left(M_{p+1}^H(a, b, c) M_{p+n+1}^H(a, b, c) \right) - E \left(M_{p+1}^H(a, b, c) M_{p+n}^H(a, b, c) \right) \\
&\quad - E \left(M_p^H(a, b, c) M_{p+n+1}^H(a, b, c) \right) + E \left(M_p^H(a, b, c) M_{p+n}^H(a, b, c) \right) \\
&= C(p+1, p+n+1) - C(p+1, p+n) - C(p, p+n+1) + C(p, p+n) \\
&= a^2(p+1) + \frac{(b+c)^2}{2} ((p+1)^{2H} + (p+n+1)^{2H}) - bc(2p+n+2)^{2H} \\
&\quad - \frac{(b^2+c^2)}{2} n^{2H} - a^2(p+1) - \frac{(b+c)^2}{2} ((p+1)^{2H} + (p+n)^{2H}) \\
&\quad + bc(2p+n+1)^{2H} + \frac{(b^2+c^2)}{2} |n-1|^{2H} - a^2p \\
&\quad - \frac{(b+c)^2}{2} (p^{2H} + (p+n+1)^{2H}) + bc(2p+n+1)^{2H} \\
&\quad + \frac{(b^2+c^2)}{2} |n+1|^{2H}.
\end{aligned}$$

Hence

$$\begin{aligned}
R_M(p, p+n) &= \frac{(b^2+c^2)}{2} ((n+1)^{2H} - 2n^{2H} + (n-1)^{2H}) \\
&\quad - bc((2p+n+2)^{2H} - 2(2p+n+1)^{2H} + (2p+n)^{2H}).
\end{aligned}$$

Then for every integer $p \geq 1$, by Taylor's expansion, as $n \rightarrow \infty$, we have

$$\begin{aligned}
R_M(p, p+n) &= \frac{b^2+c^2}{2} n^{2H} \left[\left(1 + \frac{1}{n}\right)^{2H} - 2 + \left(1 - \frac{1}{n}\right)^{2H} \right] \\
&\quad - bc n^{2H} \left[\left(1 + \frac{2p+2}{n}\right)^{2H} - 2 \left(1 + \frac{2p+1}{n}\right) + \left(1 + \frac{2p}{n}\right)^{2H} \right] \\
&= H(2H-1) n^{2H-2} (b-c)^2 \\
&\quad - 4H(2H-1)(H-1) bc(2p+1) n^{2H-3} (1 + o(1)).
\end{aligned}$$

If $b \neq c$, we see that as $n \rightarrow \infty$,

$$R_M(p, p+n) \approx H(2H-1) n^{2H-2} (b-c)^2.$$

Then

$$\sum_{n \geq 1} R_M(p, p+n) = \infty \Leftrightarrow 2H-2 > -1 \Leftrightarrow H > \frac{1}{2}.$$

If $b = c$, then, as $n \rightarrow \infty$,

$$R_M(p, p+n) \approx 4H(2H-1)(H-1)a^2(2p+1)n^{2H-3}.$$

For every $H \in (0, 1)$, we have $2H-3 < -1$ and, consequently,

$$\sum_{n \geq 1} R_M(p, p+n) < \infty.$$

□

- Remark 3.1.** (1) For all $a \in \mathbb{R}$ and $b \in \mathbb{R} \setminus \{0\}$, the increments of $M^H(a, b, 0)$ are long-range dependent if and only if $H > \frac{1}{2}$.
- (2) If $b = c = \frac{1}{\sqrt{2}}$, the increments of $M^H(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ are short-range dependent if and only if $H \in (0, 1)$. But if $b \neq c$, the increments of $M^H(0, b, c)$ are long-range dependent if and only if $H > \frac{1}{2}$.
- (3) From [4], the increments of $M^H(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ on intervals $[u, u + r], [u + r, u + 2r]$ are more weakly correlated than those of $M^H(0, 1, 0)$.
- (4) From [30], If $H > \frac{1}{2}$, $b^2 + c^2 = 1$ and $bc \geq 0$, the increments of $M^H(0, b, c)$ are more weakly correlated than those of $M^H(0, 1, 0)$, but more strongly correlated than those of $M^H(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.
- (5) From [30], If $H \geq \frac{1}{2}$, $(bc \leq 0$ and $(b - c)^2 \leq 1)$ or $(bc \geq 0$ and $b^2 + c^2 \leq 1)$, the increments of $M^H(0, b, c)$ are more strongly correlated than those of both $M^H(0, 1, 0)$ and $M^H(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

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