Journal of Stochastic Analysis

Volume 2 | Number 2 Article 2

May 2021

Mixed Generalized Fractional Brownian Motion

Shaykhah Alajmi
Imam Abdulrahman Bin Faisal University, P. O. Box 1982, Dammam, Saudi Arabia, sho192010@hotmail.com

Ezzedine Mliki
Imam Abdulrahman Bin Faisal University, P. O. Box 1982, Dammam, Saudi Arabia, ermliki@iau.edu.sa

Recommended Citation
Alajmi, Shaykhah and Mliki, Ezzedine (2021) "Mixed Generalized Fractional Brownian Motion," *Journal of Stochastic Analysis*: Vol. 2 : No. 2 , Article 2.
DOI: 10.31390/josa.2.2.02
Available at: https://digitalcommons.lsu.edu/josa/vol2/iss2/2

Journal of Stochastic Analysis
Vol. 2, No. 2 (2020) Article 2 (14 pages)
DOI: 10.31390/josa.2.2.02

MIXED GENERALIZED FRACTIONAL BROWNIAN MOTION

SHAYKHAH ALAJMI AND EZZEDINE MLIKI*

Abstract. To extend several known centered Gaussian processes, we introduce a new centered mixed self-similar Gaussian process called the mixed generalized fractional Brownian motion, which could serve as a good model for a larger class of natural phenomena. This process generalizes both the well-known mixed fractional Brownian motion introduced by Cheridito [7] and the generalized fractional Brownian motion introduced by Zili [29]. We study its main stochastic properties, its non-Markovian and non-stationarity characteristics and the conditions under which it is not a semimartingale. We prove the long-range dependence properties of this process.

1. Introduction

Fractional Brownian motion on the whole real line (fBm for short) $B^H = \{B_t^H,\, t \in \mathbb{R}\}$ of Hurst parameter H is the best known centered Gaussian process with long-range dependence. Its covariance function is

$$\mathrm{Cov}(B_t^H,\, B_s^H) = \frac{1}{2}[|t|^{2H} + |s|^{2H} - |t-s|^{2H}], \tag{1.1}$$

where H is a real number in $(0,1)$ and the case $H = \frac{1}{2}$ corresponds to the Brownian motion. It is the unique continuous Gaussian process starting from zero, the self-similarity and stationarity of the increments are two main properties for which fBm enjoyed successes as modeling tool in finance and telecommunications. Researchers have applied fractional Brownian motion to a wide range of problems, such as bacterial colonies, geophysical data, electrochemical deposition, particle diffusion, DNA sequences and stock market indicators [20, 22]. In particular, computer science applications of fBm include modeling network traffic and generating graphical landscapes [21]. The fBm was investigated in many papers (e.g. [2, 12, 16, 17, 18, 19]). The main difference between fBm and regular Brownian motion is that the increments in Brownian motion are independent, increments for fBm are not.

In [4], the authors suggested another kind of extension of the Brownian motion, called the sub-fractional Brownian motion (sfBm for short), which preserves most properties of the fBm, but not the stationarity of the increments. It is a centered

Received 2020-7-16; Accepted 2021-3-10; Communicated by the editors.
2010 *Mathematics Subject Classification.* 60G15, 60G17, 60G18, 60G20.
Key words and phrases. Mixed fractional Brownian motion, generalized fractional Brownian motion, long-range dependence, stationarity, Markovity, semimartingale.
* Corresponding author.

Gaussian process $\xi^H = \left\{\xi_t^H,\ t \geq 0\right\}$, defined by:

$$\xi_t^H = \frac{B_t^H + B_{-t}^H}{\sqrt{2}}, \quad t \geq 0, \tag{1.2}$$

where $H \in (0,1)$. The case $H = \frac{1}{2}$ corresponds to the Brownian motion.

The sfBm is intermediate between Brownian motion and fractional Brownian motion in the sense that it has properties analogous to those of fBm, self-similarity, not Markovian but the increments on nonoverlapping intervals are more weakly correlated, and their covariance decays polynomially at a higher rate in comparison with fBm (for this reason in [4] is called sfBm). So, the sfBm does not generalize the fBm. The sfBm was investigated in many papers (e.g. [3, 4, 24, 26]).

An extension of the sfBm was introduced by Zili in [28] as a linear combination of a finite number of independent sub-fractional Brownian motions. It was called the mixed sub-fractional Brownian motion (msfBm for short). The msfBm is a centered mixed self-similar Gaussian process and does not have stationary increments. The msfBm do not generalize the fBm.

In [29], Zili introduced new model called the generalized fractional Brownian motion (gfBm for short) which is an extension of both sub-fractional Brownian motion and fractional Brownian motion. A gfBm with parameters $a, b,$ *and* H, is a process $Z^H = \left\{Z_t^H(a,b), t \geq 0\right\}$ defined by

$$Z_t^H(a,b) = aB_t^H + bB_{-t}^H, \quad t \geq 0. \tag{1.3}$$

The gfBm was investigated in [10, 30]. The gfBm generalize the sfBm but not the mixed fractional Brownian motion.

The mixed fractional Brownian motion (mfBm for short) is a linear combination between a Brownian motion and an independent fractional Brownian motion of Hurst parameter H. It was introduced by Cheridito [7] to present a stochastic model of the discounted stock price in some arbitrage-free and complete financial markets. The mfBm is a centered Gaussian process starting from zero with covariance function

$$\mathrm{Cov}(N_t^H(a,b), N_s^H(a,b)) = a^2(t \wedge s) + \frac{b^2}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \tag{1.4}$$

with $H \in (0,1)$. When $a = 1$ and $b = 0$, the mfBm is the Brownian motion and when $a = 0$ and $b = 1$, is the fBm. We refer also to [1, 7, 9, 25, 27] for further information on this process.

In this paper, we introduce a new stochastic model, which we call the mixed generalized fractional Brownian motion.

Definition 1.1. A *mixed generalized fractional Brownian motion* (mgfBm for short) of parameters a, b, c and $H \in (0,1)$ is a centered Gaussian process

$$M^H(a,b,c) = \{M_t^H(a,b,c),\ t \geq 0\},$$

defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with the covariance function

$$C(t,s) = a^2(t \wedge s) + \frac{(b+c)^2}{2}(t^{2H} + s^{2H}) - bc(t+s)^{2H} - \frac{(b^2+c^2)}{2}|t-s|^{2H}, \tag{1.5}$$

where $t \wedge s = \frac{1}{2}\left(t + s - |t-s|\right)$.

The mgfBm is completely different from all the extensions mentioned above. The process $M^H(a,b,c)$ is motivated by the fact that this process already introduced for specific values of a, b and c. Indeed $M^H(a,b,0)$ is the mixed fractional Brownian motion and $M^H(0,b,c)$, is the generalized fractional Brownian motion. This why we will name $M^H(a,b,c)$ the mixed generalized fractional Brownian motion. It allows to deal with a larger class of modeled natural phenomena, including those with stationary or non-stationary increments.

Our goal is to study the main stochastic properties of this new model, paying attention to the long-range dependence, self-similarity, increment stationary, Markovity and semi-martingale properties.

2. The Main Properties

Existence of the mixed generalized fractional Brownian motion $M^H(a,b,c)$ for any $H \in (0,1)$ can be shown in the following way: consider the process

$$M_t^H(a,b,c) = aB_t + bB_t^H + cB_{-t}^H, \quad t \geq 0, \tag{2.1}$$

where $B = \{B_t,\, t \in \mathbb{R}\}$ is a Brownian motion and $B^H = \{B_t^H,\, t \in \mathbb{R}\}$ is an independent fractional Brownian motion with Hurst parameter $H \in (0,1)$.

Using (1.1) and since B and B^H are independent we obtain the following lemma.

Lemma 2.1. *For all $s,\ t \geq 0$, the process (2.1) is a centered Gaussian process with covariance function given by (1.5).*

Proof. Let $s,\ t \geq 0$ and $C(t,s) = Cov\left(M_t^H(a,b,c), M_s^H(a,b,c)\right)$. Then

$$\begin{aligned}
C(t,s) &= Cov\Big(\left(aB_t + bB_t^H + cB_{-t}^H\right), \left(aB_s + bB_s^H + cB_{-s}^H\right) \Big) \\
&= a^2(t \wedge s) + b^2 Cov(B_t^H, B_s^H) + bc Cov(B_t^H, B_{-s}^H) + cb Cov(B_{-t}^H, B_s^H) \\
&\quad + c^2 Cov(B_{-t}^H, B_{-s}^H) \\
&= a^2(t \wedge s) + \frac{b^2}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right) + \frac{bc}{2}\left(t^{2H} + s^{2H} - |t+s|^{2H}\right) \\
&\quad + \frac{cb}{2}\left(t^{2H} + s^{2H} - |-(t+s)|^{2H}\right) + \frac{c^2}{2}\left(t^{2H} + s^{2H} - |-(t-s)|^{2H}\right) \\
&= a^2(t \wedge s) + \frac{b^2}{2}t^{2H} + \frac{b^2}{2}s^{2H} - \frac{b^2}{2}|t-s|^{2H} + \frac{bc}{2}t^{2H} + \frac{bc}{2}s^{2H} \\
&= a^2(t \wedge s) + \frac{(b+c)^2}{2}(t^{2H} + s^{2H}) - bc|t+s|^{2H} - \frac{(b^2+c^2)}{2}|t-s|^{2H}.
\end{aligned}$$

Hence the covariance function of the process (2.1) is precisely $C(t,s)$ given by (1.5). Therefore the $M^H(a,b,c)$ exists. □

Remark 2.1. *Some special cases of the mixed generalized fractional Brownian motion:*

1. *If $a = 0, b = 1, c = 0$, then $M^H(0,1,0)$ is a fBm.*
2. *If $a = 0, b = c = \frac{1}{\sqrt{2}}$, then $M^H(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ is a sfBm.*
3. *If $a = 1, b = 0, c = 0$, then $M^H(1,0,0)$ is a Bm.*
4. *If $a = 0$, then $M^H(0,b,c)$, is a gfBm.*

(5) *If* $c=0$, *then* $M^H(a,b,0)$, *is a mfBm.*
(6) *If* $b=c$, *then* $M^H(a,\frac{b}{\sqrt{2}},\frac{b}{\sqrt{2}})$, *is a smfBm.*

So the mixed generalized fractional Brownian motion is, at the same, a generalization of the fractional Brownian motion, sub-fractional Brownian motion, the sub-mixed fractional Brownian motion, generalized fractional Brownian motion, mixed fractional Brownian motion and of course of the standard Brownian motion.

Proposition 2.1. *The mgfBm satisfies the following properties:*

(1) *For all* $t\geq 0$,
$$E\left(M_t^H(a,b,c)\right)^2=a^2t+\left(b^2+c^2-(2^{2H}-2)bc\right)t^{2H}.$$
(2) *Let* $0\leq s<t$ *and* $\alpha(t,s)=E\left(M_t^H(a,b,c)-M_s^H(a,b,c))^2\right)$. *Then*
$$\begin{array}{rcl}E\left(M_t^H(a,b,c)-M_s^H(a,b,c)\right)^2 &=& a^2|t-s|-2^{2H}bc(t^{2H}+s^{2H})\\ && +(b^2+c^2)|t-s|^{2H}+2bc|t+s|^{2H}.\end{array}$$
(3) *We have for all* $0\leq s<t$,
$$a^2(t-s)+\gamma_{(b,c,H)}(t-s)^{2H}\leq\alpha(t,s)\leq a^2(t-s)+\nu_{(b,c,H)}(t-s)^{2H},$$
where
$$\gamma_{(b,c,H)}=(b^2+c^2-2bc(2^{2H-1}-1))\,\mathbf{1}_{\mathcal{C}}(b,c,H)+(b^2+c^2)\,\mathbf{1}_{\mathcal{D}}(b,c,H),$$
$$\nu_{(b,c,H)}=(b^2+c^2)\,\mathbf{1}_{\mathcal{C}}(a,b,H)+(b^2+c^2-2bc(2^{2H-1}-1))\,\mathbf{1}_{\mathcal{D}}(b,c,H),$$
$$\mathcal{C}=\{(b,c,H)\in\mathbb{R}^2\times]0,1[;\ (H>\frac{1}{2},\ bc\geq 0)\ or\ (H<\frac{1}{2},\ bc\leq 0)\},$$
and
$$\mathcal{D}=\{(b,c,H)\in\mathbb{R}^2\times]0,1[;\ (H>\frac{1}{2},\ bc\leq 0)\ or\ (H<\frac{1}{2},\ bc\geq 0)\}.$$

Proof. (1) It is a direct consequence of (1.5).

(2) Let $0\leq s<t$ and $\alpha(t,s)=E\left(M_t^H(a,b,c)-M_s^H(a,b,c)\right)^2$. Then
$$\begin{array}{rcl}\alpha(t,s) &=& E\left(M_t^H(a,b,c)\right)^2+E\left(M_s^H(a,b,c)\right)^2-2E\left(M_t^H(a,b,c)M_s^H(a,b,c)\right)\\ &=& a^2t+b^2t^{2H}+2bct^{2H}-2^{2H}bct^{2H}+c^2t^{2H}+a^2s+b^2s^{2H}+2bcs^{2H}\\ && -2^{2H}bcs^{2H}+c^2s^{2H}-2a^2(t\wedge s)-b^2t^{2H}-b^2s^{2H}+b^2|t-s|^{2H}\\ && -bct^{2H}-bcs^{2H}+bc|t+s|^{2H}-cbt^{2H}-cbs^{2H}+cb|t+s|^{2H}-c^2t^{2H}\\ && -c^2s^{2H}+c^2|t-s|^{2H}\\ &=& a^2(t+s)-2^{2H}bc(t^{2H}+s^{2H})-2a^2(t\wedge s)+(b^2+c^2)|t-s|^{2H}\\ && +2bc|t+s|^{2H}\\ &=& a^2|t-s|-2^{2H}bc(t^{2H}+s^{2H})+(b^2+c^2)|t-s|^{2H}+2bc|t+s|^{2H}.\end{array}$$
(3) It is a direct consequence of the second item of Proposition 2.1 and Lemma 3 in [29].

□

Proposition 2.2. *For all* $(a,b,c) \in \mathbb{R}^3 \setminus \{(0,0,0)\}$ *and* $H \in (0,1) \setminus \{\frac{1}{2}\}$, *the mgfBm is not a self-similar process.*

Proof. This follows from the fact that, for fixed $h > 0$, the processes $\left\{M^H_{ht}(a,b,c),\ t \geq 0\right\}$ and $\left\{h^H M^H_t(a,b,c),\ t \geq 0\right\}$ are Gaussian, centered, but don't have the same covariance function. Indeed

$$
\begin{aligned}
C\left(ht,hs\right) &= a^2(ht \wedge hs) + \frac{b^2}{2}\left((ht)^{2H} + (hs)^{2H} - |ht-hs|^{2H}\right)\\
&\quad + \frac{bc}{2}\left((ht)^{2H} + (hs)^{2H} - |ht+hs|^{2H}\right)\\
&\quad + \frac{cb}{2}\left((ht)^{2H} + (hs)^{2H} - |-(ht+hs)|^{2H}\right)\\
&\quad + \frac{c^2}{2}\left((ht)^{2H} + (hs)^{2H} - |-(ht-hs)|^{2H}\right)\\
&= a^2(ht \wedge hs) + \frac{b^2}{2}(ht)^{2H} + \frac{b^2}{2}(hs)^{2H} - \frac{b^2}{2}|ht-hs|^{2H}\\
&\quad + \frac{bc}{2}(ht)^{2H} + \frac{bc}{2}(hs)^{2H} - \frac{bc}{2}|ht+hs|^{2H} + \frac{bc}{2}(ht)^{2H} + \frac{bc}{2}(hs)^{2H}\\
&\quad - \frac{bc}{2}|ht+hs|^{2H} + \frac{c^2}{2}(ht)^{2H} + \frac{c^2}{2}(hs)^{2H} - \frac{c^2}{2}|ht-hs|^{2H}\\
&= a^2h(t \wedge s) + h^{2H}\frac{(b+c)^2}{2}\left((t)^{2H} + (s)^{2H}\right) - bch^{2H}|t+s|^{2H}\\
&\quad - h^{2H}\frac{(b^2+c^2)}{2}|t-s|^{2H}.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
Cov\left(h^H M^H_t(a,b,c), h^H M^H_s(a,b,c)\right) &= h^{2H} Cov\left(M^H_t(a,b,c), M^H_s(a,b,c)\right)\\
&= a^2h^{2H}(t \wedge s)\\
&\quad + h^{2H}\frac{(b+c)^2}{2}\left(t^{2H} + s^{2H}\right)\\
&\quad - bch^{2H}|t+s|^{2H}\\
&\quad - h^{2H}\frac{(b^2+c^2)}{2}|t-s|^{2H}.
\end{aligned}
$$

Then the mgfBm is not a self-similar process for all $(a,b,c) \in \mathbb{R}^3 \setminus \{(0,0,0)\}$. □

Remark 2.2. *As a consequence of Proposition 2.2, we see that:*

1. $M^H(0,b,c)$ *is a self-similar process for all* $(b,c) \in \mathbb{R}^2$.
2. $M^{\frac{1}{2}}(a,b,c)$ *is a self-similar process for all* $(a,b,c) \in \mathbb{R}^3$.

Now, we will study the Markovian property.

Theorem 2.1. *Assume* $H \in (0,1) \setminus \left\{\frac{1}{2}\right\}, a \in \mathbb{R}$ *and* $(b,c) \in \mathbb{R}^2 \setminus \{(0,0)\}$. *Then* $M^H(a,b,c)$ *is not a Markovian process.*

Proof. The process $M^H(a,b,c)$ is a centered Gaussian. Then, if $M^H_t(a,b,c)$ is a Markovian process, according to Revuz and Yor [23], for all $s < t < u$, we would have

$$C(s,u)C(t,t) = C(s,t)C(t,u).$$

We will only prove the theorem in the case where $a \neq 0$, the result with $a = 0$ is known in [29]. For the proof we follow the proof of Proposition 1 given in [29]. Using Proposition 2.1, we get

$$\begin{aligned}
C(s,u) &= a^2 s + \frac{(b+c)^2}{2}(u^{2H} + s^{2H}) - bc|u+s|^{2H} - \frac{(b^2+c^2)}{2}|u-s|^{2H},\\
C(t,t) &= a^2 t + \big(b^2 + c^2 - (2^{2H}-2)bc\big)\, t^{2H},\\
C(s,t) &= a^2 s + \frac{(b+c)^2}{2}(t^{2H} + s^{2H}) - bc|t+s|^{2H} - \frac{(b^2+c^2)}{2}|t-s|^{2H},\\
C(t,u) &= a^2 t + \frac{(b+c)^2}{2}(u^{2H} + t^{2H}) - bc|u+t|^{2H} - \frac{(b^2+c^2)}{2}|u-t|^{2H}.
\end{aligned}$$

In the particular case where $1 < s = \sqrt{t} < t < u = t^2$, we have

$$\begin{aligned}
C(\sqrt{t},t^2) &= a^2 t^{\frac{1}{2}} + \frac{(b+c)^2}{2}(t^{4H} + t^{H}) - bc|t^2+t^{\frac{1}{2}}|^{2H} - \frac{(b^2+c^2)}{2}|t^2-t^{\frac{1}{2}}|^{2H},\\
C(t,t) &= a^2 t + \big(b^2 + c^2 - (2^{2H}-2)bc\big)\, t^{2H},\\
C(\sqrt{t},t) &= a^2 t^{\frac{1}{2}} + \frac{(b+c)^2}{2}(t^{2H} + t^{H}) - bc|t+t^{\frac{1}{2}}|^{2H} - \frac{(b^2+c^2)}{2}|t-t^{\frac{1}{2}}|^{2H},\\
C(t,t^2) &= a^2 t + \frac{(b+c)^2}{2}(t^{4H} + t^{2H}) - bc|t^2+t|^{2H} - \frac{(b^2+c^2)}{2}|t^2-t^{\frac{1}{2}}|^{2H}.
\end{aligned}$$

Then by using that,

$$C(\sqrt{t},t^2)C(t,t) = C(\sqrt{t},t)C(t,t^2),$$

we have

$$\begin{aligned}
&\left[a^2 t^{\frac{1}{2}} + \frac{(b+c)^2}{2}(t^{4H} + t^{H}) - bc|t^2+t^{\frac{1}{2}}|^{2H} - \frac{(b^2+c^2)}{2}|t^2-t^{\frac{1}{2}}|^{2H}\right]\\
&\times \left[a^2 t + \big(b^2 + c^2 - (2^{2H}-2)bc\big)\, t^{2H}\right]\\
= &\left[a^2 t^{\frac{1}{2}} + \frac{(b+c)^2}{2}(t^{2H} + t^{H}) - bc|t+t^{\frac{1}{2}}|^{2H} - \frac{(b^2+c^2)}{2}|t-t^{\frac{1}{2}}|^{2H}\right]\\
&\times \left[a^2 t + \frac{(b+c)^2}{2}(t^{4H} + t^{2H}) - bc|t^2+t|^{2H} - \frac{(b^2+c^2)}{2}|t^2-t^{\frac{1}{2}}|^{2H}\right].
\end{aligned}$$

It follows that

$$\begin{aligned}
&\left[a^2 t^{\frac{1}{2}} + t^{4H}\left(\frac{(b+c)^2}{2}(1 + t^{-3H}) - bc|1+t^{-\frac{3}{2}}|^{2H} - \frac{(b^2+c^2)}{2}|1-t^{-\frac{3}{2}}|^{2H}\right)\right]\\
&\times \left[a^2 t + \big(b^2 + c^2 - (2^{2H}-2)bc\big)\, t^{2H}\right]\\
= &\left[a^2 t^{\frac{1}{2}} + t^{2H}\left(\frac{(b+c)^2}{2}(1 + t^{-H}) - bc|1+t^{-\frac{1}{2}}|^{2H} - \frac{(b^2+c^2)}{2}|1-t^{-\frac{1}{2}}|^{2H}\right)\right]\\
&\times \left[a^2 t + t^{4H}\left(\frac{(b+c)^2}{2}(1 + t^{-2H}) - bc|1+t^{-1}|^{2H} - \frac{(b^2+c^2)}{2}|1-t^{-1}|^{2H}\right)\right].
\end{aligned}$$

Hence

$$\begin{aligned}
a^4t^{\frac{3}{2}} &+ a^2\left(b^2+c^2-(2^{2H}-2)bc\right)t^{2H+\frac{1}{2}}\\
&+ a^2\left(\frac{(b+c)^2}{2}(1+t^{-3H})-bc|1+t^{-\frac{3}{2}}|^{2H}-\frac{(b^2+c^2)}{2}|1-t^{-\frac{3}{2}}|^{2H}\right)\\
&\times t^{4H+1}\\
&+ \left(\frac{(b+c)^2}{2}(1+t^{-3H})-bc|1+t^{-\frac{3}{2}}|^{2H}-\frac{(b^2+c^2)}{2}|1-t^{-\frac{3}{2}}|^{2H}\right)\\
&\times \left(b^2+c^2-(2^{2H}-2)bc\right)t^{6H}\\
=a^4t^{\frac{3}{2}} &+ a^2\left(\frac{(b+c)^2}{2}(1+t^{-2H})-bc|1+t^{-1}|^{2H}-\frac{(b^2+c^2)}{2}|1-t^{-1}|^{2H}\right)\\
&\times t^{4H+\frac{1}{2}}\\
&+ a^2\left(\frac{(b+c)^2}{2}(1+t^{-H})-bc|1+t^{-\frac{3}{2}}|^{2H}-\frac{(b^2+c^2)}{2}|1-t^{-\frac{3}{2}}|^{2H}\right)\\
&\times t^{2H+1}\\
&+ \left(\frac{(b+c)^2}{2}(1+t^{-H})-bc|1+t^{-\frac{3}{2}}|^{2H}-\frac{(b^2+c^2)}{2}|1-t^{-\frac{3}{2}}|^{2H}\right)\\
&\times \left(\frac{(b+c)^2}{2}(1+t^{-2H})-bc|1+t^{-1}|^{2H}-\frac{(b^2+c^2)}{2}|1-t^{-1}|^{2H}\right)t^{6H}.
\end{aligned}$$

Take t^{6H} as a common factor, we get

$$\begin{aligned}
&a^2\left(b^2+c^2-(2^{2H}-2)bc\right)t^{-4H+\frac{1}{2}}\\
&+a^2\left(\frac{(b+c)^2}{2}(1+t^{-3H})-bc|1+t^{-\frac{3}{2}}|^{2H}-\frac{(b^2+c^2)}{2}|1-t^{-\frac{3}{2}}|^{2H}\right)t^{-2H+1}\\
&+\left(\frac{(b+c)^2}{2}(1+t^{-3H})-bc|1+t^{-\frac{3}{2}}|^{2H}-\frac{(b^2+c^2)}{2}|1-t^{-\frac{3}{2}}|^{2H}\right)\\
&\times\left(b^2+c^2-(2^{2H}-2)bc\right)\\
=\;&a^2\left(\frac{(b+c)^2}{2}(1+t^{-2H})-bc|1+t^{-1}|^{2H}-\frac{(b^2+c^2)}{2}|1-t^{-1}|^{2H}\right)t^{-2H+\frac{1}{2}}\\
&+a^2\left(\frac{(b+c)^2}{2}(1+t^{-H})-bc|1+t^{-\frac{1}{2}}|^{2H}-\frac{(b^2+c^2)}{2}|1-t^{-\frac{1}{2}}|^{2H}\right)t^{-4H+1}\\
&+\left(\frac{(b+c)^2}{2}(1+t^{-H})-bc|1+t^{-\frac{1}{2}}|^{2H}-\frac{(b^2+c^2)}{2}|1-t^{-\frac{1}{2}}|^{2H}\right)\\
&\times\left(\frac{(b+c)^2}{2}(1+t^{-2H})-bc|1+t^{-1}|^{2H}-\frac{(b^2+c^2)}{2}|1-t^{-1}|^{2H}\right).
\end{aligned}$$

Therefore

$$\begin{aligned}
& a^2\left(b^2+c^2-(2^{2H}-2)bc\right)t^{-4H+\frac{1}{2}} \\
& +a^2\Big[\frac{1}{2}(b+c)^2\left(1+t^{-3H}\right)-bc\Big(1+2Ht^{-\frac{3}{2}}+H(2H-1)t^{-3}+\circ(t^{-3})\Big) \\
& -\frac{1}{2}(b^2+c^2)\Big(1-2Ht^{-\frac{3}{2}}+H(2H-1)t^{-3}+\circ(t^{-3})\Big)\Big]t^{-2H+1} \\
& +\Big[\frac{1}{2}(b+c)^2\left(1+t^{-3H}\right)-bc\Big(1+2Ht^{-\frac{3}{2}}+H(2H-1)t^{-3}+\circ(t^{-3})\Big) \\
& -\frac{1}{2}(b^2+c^2)\Big(1-2Ht^{-\frac{3}{2}}+H(2H-1)t^{-3}+\circ(t^{-3})\Big)\Big] \\
& \times[b^2+c^2-(2^{2H}-2)bc] \\
= \; & a^2\Big[\frac{1}{2}(b+c)^2\left(1+t^{-2H}\right)-bc\left(1+2Ht^{-1}+H(2H-1)t^{-2}+\circ(t^{-2})\right) \\
& -\frac{1}{2}(b^2+c^2)\left(1-2Ht^{-1}+H(2H-1)t^{-2}+\circ(t^{-2})\right)\Big]t^{-2H+\frac{1}{2}} \\
& +a^2\Big[\frac{1}{2}(b+c)^2\left(1+t^{-H}\right)-bc\Big(1+2Ht^{-1/2}+H(2H-1)t^{-1}+\circ(t^{-1})\Big) \\
& -\frac{1}{2}(b^2+c^2)\Big(1-2Ht^{-\frac{1}{2}}+H(2H-1)t^{-1}+\circ(t^{-1})\Big)\Big]t^{-4H+1} \\
& +\Big[\frac{1}{2}(b+c)^2\left(1+t^{-H}\right)-bc\Big(1+2Ht^{-\frac{1}{2}}+H(2H-1)t^{-1}+\circ(t^{-1})\Big) \\
& -\frac{1}{2}(b^2+c^2)\Big(1-2Ht^{-\frac{1}{2}}+H(2H-1)t^{-1}+\circ(t^{-1})\Big)\Big] \\
& \times\Big[\frac{1}{2}(b+c)^2\left(1+t^{-2H}\right)-bc\left(1+2Ht^{-1}+H(2H-1)t^{-2}+\circ(t^{-2})\right) \\
& -\frac{1}{2}(b^2+c^2)\left(1-2Ht^{-1}+H(2H-1)t^{-2}+\circ(t^{-2})\right)\Big].
\end{aligned}$$

First case: $0 < H < \frac{1}{2}$, $a \neq 0$ and $b+c \neq 0$. By Taylor's expansion we get, as $t \to \infty$,

$$\begin{aligned}
& a^2\left(b^2+c^2-(2^{2H}-2)bc\right)t^{-4H+\frac{1}{2}}+a^2\frac{1}{2}(b+c)^2t^{-5H+1} \\
& +\frac{1}{2}(b+c)^2[b^2+c^2-(2^{2H}-2)bc]t^{-3H} \\
\approx \; & a^2\frac{1}{2}(b+c)^2t^{-4H+\frac{1}{2}}+a^2\frac{1}{2}(b+c)^2t^{-5H+1}+\frac{1}{4}(b+c)^4t^{-3H}.
\end{aligned}$$

Therefore

$$\begin{aligned}
& a^2\left(b^2+c^2-(2^{2H}-2)bc\right)t^{-4H+\frac{1}{2}}+\frac{1}{2}(b+c)^2[b^2+c^2-(2^{2H}-2)bc]t^{-3H} \\
\approx \; & a^2\frac{1}{2}(b+c)^2t^{-4H+\frac{1}{2}}+\frac{1}{4}(b+c)^4t^{-3H},
\end{aligned}$$

which is true if and only if

$$\frac{(b-c)^2}{2} - (2^{2H}-2)bc = 0 \quad \text{and} \quad a^2\frac{(b-c)^2}{2} - a^2(2^{2H}-2)bc = 0.$$

However, it is easy to check that

$$\frac{(b-c)^2}{2} - (2^{2H}-2)bc > 0 \quad \text{and} \quad a^2\frac{(b-c)^2}{2} - a^2(2^{2H}-2)bc > 0,$$

for fixed c, b and every real a.
Second case: $0 < H < \frac{1}{2}$, $a \neq 0$ and $b + c = 0$. By Taylor's expansion we get, as $t \to \infty$,

$$\begin{aligned} & a^2[b^2+c^2-(2^{2H}-2)bc]t^{-4H+\frac{1}{2}} + a^2\left[-2Hbc+(b^2+c^2)H\right]t^{-2H-\frac{1}{2}} \\ & + \left[-2Hbc+(b^2+c^2)H\right] \times \left[b^2+c^2-(2^{2H}-2)bc\right]t^{-\frac{3}{2}} \\ \approx\ & a^2\left[-2Hbc+(b^2+c^2)H\right]t^{-2H-\frac{1}{2}} + a^2\left[-2Hbc+(b^2+c^2)H\right]t^{-4H+\frac{1}{2}} \\ & + \left[-2Hbc+(b^2+c^2)H\right]^2 t^{-\frac{3}{2}}. \end{aligned}$$

Hence

$$\begin{aligned} & a^2[b^2+c^2-(2^{2H}-2)bc]t^{-4H+\frac{1}{2}} \\ & + \left[-2Hbc+(b^2+c^2)H\right] \times \left[b^2+c^2-(2^{2H}-2)bc\right]t^{-\frac{3}{2}} \\ \approx\ & a^2\left[-2Hbc+(b^2+c^2)H\right]t^{-4H+\frac{1}{2}} + \left[-2Hbc+(b^2+c^2)H\right]^2 t^{-\frac{3}{2}}, \end{aligned}$$

which is true if and only if $b = c = 0$. This is a contradiction.
Third case: $\frac{1}{2} < H < 1$, $a \neq 0$ and $b - c \neq 0$. By Taylor's expansion we get, as $t \to \infty$,

$$\begin{aligned} & a^2[b^2+c^2-(2^{2H}-2)bc]t^{-4H+\frac{1}{2}} + a^2H(b-c)^2t^{-2H-\frac{1}{2}} \\ & + H(b-c)^2[b^2+c^2-(2^{2H}-2)bc]t^{-\frac{3}{2}} \\ \approx\ & a^2H(b-c)^2t^{-2H-\frac{1}{2}} + a^2H(b-c)^2t^{-4H+\frac{1}{2}} + H^2(b-c)^4t^{-\frac{3}{2}}. \end{aligned}$$

Then

$$\begin{aligned} & a^2[b^2+c^2-(2^{2H}-2)bc]t^{-4H+\frac{1}{2}} + H(b-c)^2[b^2+c^2-(2^{2H}-2)bc]t^{-\frac{3}{2}} \\ \approx\ & a^2H(b-c)^2t^{-4H+\frac{1}{2}} + H^2(b-c)^4t^{-\frac{3}{2}}, \end{aligned}$$

which is true if and only if

$$\left[b^2(1-H) + c^2(1-H) + (2-2^{2H}+2H)bc\right] = 0.$$

However, it is easy to check that $b^2(1-H) + c^2(1-H) + (2-2^{2H}+2H)bc > 0$ for fixed c, b and every real a.
Fourth case: $\frac{1}{2} < H < 1$, $a \neq 0$ and $b - c = 0$. By Taylor's expansion we get, as $t \to \infty$,

$$\begin{aligned} & a^2\left(b^2+c^2-(2^{2H}-2)bc\right)t^{-4H+\frac{1}{2}} + \frac{1}{2}(b+c)^2[b^2+c^2-(2^{2H}-2)bc]t^{-3H} \\ \approx\ & a^2\frac{1}{2}(b+c)^2t^{-4H+\frac{1}{2}} + \frac{1}{4}(b+c)^4t^{-3H}, \end{aligned}$$

which is true if and only if $2 - 2^{2H} = 0$. This contradicts the fact that $H \neq \frac{1}{2}$. □

Let us check the mixed self-similarity property of the mgfBm. This property was introduced in [27] for the mfBm and investigated to show the Hölder continuity of the mfBm. See also [11] for the sfBm case.

Proposition 2.3. *For any $h>0$, $\left\{M^H_{ht}(a,b,c)\right\} \triangleq \left\{M^H_t(ah^{\frac{1}{2}}, bh^H, ch^H)\right\}$, where $\triangleq$ "to have the same law".*

Proof. For fixed $h>0$, the processes $\{M^H_{ht}(a,b,c)\}$ and $\{M^H_t(ah^{\frac{1}{2}}, bh^H, ch^H)\}$ are Gaussian and centered. Therefore, one only have to prove that they have the same covariance function. But, for any $s,\ t\geq 0$, since B and B^H are independent, then

$$\begin{aligned} C\left(ht,hs\right) &= a^2h(t\wedge s)+\frac{(b^2+c^2)}{2}\left[h^{2H}\left(t^{2H}+s^{2H}-|t-s|^{2H}\right)\right] \\ &\quad +bc\left[h^{2H}\left(t^{2H}+s^{2H}-|t+s|^{2H}\right)\right] \\ &= Cov\left(M^H_t(ah^{\frac{1}{2}}, bh^H, ch^H), M^H_s(ah^{\frac{1}{2}}, bh^H, ch^H)\right). \end{aligned}$$

□

Proposition 2.4. *For all $a\in\mathbb{R}$ and $(b,c)\in\mathbb{R}^2\setminus\{(0,0)\}$, the increments of the mgfBm are not stationary.*

Proof. Let $a\in\mathbb{R}$ and $(b,c)\in\mathbb{R}^2\setminus\{(0,0)\}$. For a fixed $t\geq 0$ consider the processes $\{P_t,\ t\geq 0\}$ define by $P_t = M^H_{t+s}(a,b,c)-M^H_s(a,b,c)$. Using Proposition 2.1, we get

$$\begin{aligned} Cov(P_t,P_t) &= E\left(M^H_{t+s}(a,b,c)-M^H_s(a,b,c)\right)^2 \\ &= a^2(t+s+s)-2^{2H}bc((t+s)^{2H}+s^{2H})-2a^2s \\ &\quad +(b^2+c^2)|t+s-s|^{2H}+2bc|t+s+s|^{2H} \\ &= a^2(t+2s)-2^{2H}bc((t+s)^{2H}+s^{2H})-2a^2s+(b^2+c^2)t^{2H} \\ &\quad +2bc|t+2s|^{2H}. \end{aligned}$$

Using Proposition 2.1, we get

$$Cov(M^H_t(a,b,c), M^H_t(a,b,c)) = a^2t+\left(b^2+c^2-(2^{2H}-2)bc\right)t^{2H}.$$

Since both processes are centered Gaussian, the inequality of covariance functions implies that P_t does not have the same distribution as $M^H_t(a,b,c)$. Thus, the incremental behavior of $M^H(a,b,c)$ at any point in the future is not the same. Hence the increments of $M^H(a,b,c)$ are not stationary. □

Remark 2.3. *As a consequence of Proposition 2.4, we see that:*

1. *the increments of $M^H(0,b,c)$ are not stationary for all $(b,c)\in\mathbb{R}^2\setminus\{(0,0)\}$.*
2. *the increments of $M^H(a,b,0)$ are stationary for all $(a,b)\in\mathbb{R}^2$.*

Proposition 2.5.
1. *Let $H\in(0,1)$. The mgfBm admits a version whose sample paths are almost Hölder continuous of order strictly less than $\frac{1}{2}\wedge H$.*
2. *When b or c not zero and $H\in(0,1)\setminus\{\frac{1}{2}\}$ the mgfBm is not a semi-martingale.*

Proof. (1) Let $s,\ t \geq 0$ and $\alpha = 2$. The proof follows by Kolmogorov criterion from Lemma 3 in [29] and using Proposition 2.1 we get

$$\begin{aligned} E\left(|M_t^H(a,b,c) - M_s^H(a,b,c)|^{\alpha}\right) &= a^2|t-s| - 2^{2H}bc(t^{2H}+s^{2H}) \\ &\quad +(b^2+c^2)|t-s|^{2H} + 2bc|t+s|^{2H} \\ &\leq C_{\alpha}|t-s|^{\alpha(\frac{1}{2}\wedge H)}, \end{aligned}$$

where $C_\alpha = \big(a^2 + \nu(b,c,H)\big)$ and $\nu_{(b,c,H)}$ is given in Lemma 3 in [29].

(2) Suppose first that $H < \frac{1}{2}$. We get from Proposition 2.1

$$\alpha(t,s) \geq \gamma_{(b,c,H)}(t-s)^{2H}.$$

Since $2H < 1$ and $\gamma_{(b,c,H)} > 0$ then the assumption of Corollary 2.1 in [5] is satisfied, and consequently the mgfBm is not a semi-martingale. Suppose now that $H > \frac{1}{2}$. We get from Proposition 2.1

$$a^2(t-s) + \gamma_{(b,c,H)}(t-s)^{2H} \leq \alpha(t,s) \leq (a^2 + \nu_{(b,c,H)})(t-s)^{1\wedge 2H},$$

then

$$\gamma_{(b,c,H)}(t-s)^{2H} \leq \alpha(t,s) \leq (a^2 + \nu_{(b,c,H)})(t-s)^{2H}.$$

Since $1 < 2H < 2$ and $\nu_{(b,c,H)} > 0$ then the assumption of Lemma 2.1 in [5] is satisfied, and consequently the mgfBm is not a semi-martingale.

□

3. Long-range Dependence of the mgfBm Increments

Definition 3.1. We say that the increments of a stochastic process X are *long-range dependent* if for every integer $p \geq 1$, we have

$$\sum_{n\geq 1} R_X(p, p+n) = \infty,$$

where

$$R_X(p,p+n) = E\left((X_{p+1} - X_p)(X_{p+n+1} - X_{p+n})\right).$$

This property was investigated in many papers (e.g. [3, 6, 8, 12, 20]).

Theorem 3.1. *For every $a \in \mathbb{R}$ and $(b,c) \in \mathbb{R}^2 \setminus \{(0,0)\}$, the increments of $M^H(a,b,c)$ are long-range dependent if and only if $H > \frac{1}{2}$ and $b \neq c$.*

Proof. For all $n \geq 1$ and $p \geq 1$, we have

$$\begin{aligned}
& R_M(p,p+n) \\
= \; & E\Big(\left(M^H_{p+1}(a,b,c) - M^H_p(a,b,c)\right) \left(M^H_{p+n+1}(a,b,c) - M^H_{p+n}(a,b,c)\right) \Big) \\
= \; & E\left(M^H_{p+1}(a,b,c)M^H_{p+n+1}(a,b,c)\right) - E\left(M^H_{p+1}(a,b,c)M^H_{p+n}(a,b,c)\right) \\
& -E\left(M^H_{p}(a,b,c)M^H_{p+n+1}(a,b,c)\right) + E\left(M^H_{p}(a,b,c)M^H_{p+n}(a,b,c)\right) \\
= \; & C(p+1,p+n+1) - C(p+1,p+n) - C(p,p+n+1) + C(p,p+n) \\
= \; & a^2(p+1) + \frac{(b+c)^2}{2}\left((p+1)^{2H} + (p+n+1)^{2H}\right) - bc(2p+n+2)^{2H} \\
& -\frac{(b^2+c^2)}{2}n^{2H} - a^2(p+1) - \frac{(b+c)^2}{2}\left((p+1)^{2H} + (p+n)^{2H}\right) \\
& +bc(2p+n+1)^{2H} + \frac{(b^2+c^2)}{2}|n-1|^{2H} - a^2 p \\
& -\frac{(b+c)^2}{2}\left(p^{2H} + (p+n+1)^{2H}\right) + bc(2p+n+1)^{2H} \\
& +\frac{(b^2+c^2)}{2}|n+1|^{2H}.
\end{aligned}$$

Hence

$$\begin{aligned}
R_M(p,p+n) \; = \; & \frac{(b^2+c^2)}{2}\left((n+1)^{2H} - 2n^H + (n-1)^{2H}\right) \\
& -bc\left((2p+n+2)^{2H} - 2(2p+n+1)^{2H} + (2p+n)^{2H}\right).
\end{aligned}$$

Then for every integer $p \geq 1$, by Taylor's expansion, as $n \to \infty$, we have

$$\begin{aligned}
R_M(p,p+n) \; = \; & \frac{b^2+c^2}{2}n^{2H}\left[\left(1+\frac{1}{n}\right)^{2H} - 2 + \left(1-\frac{1}{n}\right)^{2H}\right] \\
& -bcn^{2H}\left[\left(1+\frac{2p+2}{n}\right)^{2H} - 2\left(1+\frac{2p+1}{n}\right) + \left(1+\frac{2p}{n}\right)^{2H}\right] \\
= \; & H(2H-1)n^{2H-2}(b-c)^2 \\
& -4H(2H-1)(H-1)bc(2p+1)n^{2H-3}(1+\circ(1)).
\end{aligned}$$

If $b \neq c$, we see that as $n \to \infty$,

$$R_M(p,p+n) \approx H(2H-1)n^{2H-2}(b-c)^2.$$

Then

$$\sum_{n\geq 1} R_M(p,p+n) = \infty \quad \Leftrightarrow \; 2H-2 > -1 \quad \Leftrightarrow H > \frac{1}{2}.$$

If $b = c$, then, as $n \to \infty$,

$$R_M(p,p+n) \approx 4H(2H-1)(H-1)a^2(2p+1)n^{2H-3}.$$

For every $H \in (0,1)$, we have $2H-3 < -1$ and, consequently,

$$\sum_{n\geq 1} R_M(p,p+n) < \infty.$$

□

Remark 3.1. (1) *For all* $a \in \mathbb{R}$ *and* $b \in \mathbb{R} \setminus \{0\}$*, the increments of* $M^H(a, b, 0)$ *are long-range dependent if and only if* $H > \frac{1}{2}$.

(2) *If* $b = c = \frac{1}{\sqrt{2}}$*, the increments of* $M^H(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ *are short-range dependent if and only if* $H \in (0, 1)$*. But if* $b \neq c$*, the increments of* $M^H(0, b, c)$ *are long-range dependent if and only if* $H > \frac{1}{2}$.

(3) *From* [4]*, the increments of* $M^H(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ *on intervals* $[u, u + r], [u + r, u + 2r]$ *are more weakly correlated than those of* $M^H(0, 1, 0)$.

(4) *From* [30]*, If* $H > \frac{1}{2}$, $b^2 + c^2 = 1$ *and* $bc \geq 0$*, the increments of* $M^H(0, b, c)$ *are more weakly correlated than those of* $M^H(0, 1, 0)$*, but more strongly correlated than those of* $M^H(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

(5) *From* [30]*, If* $H \geq \frac{1}{2}$, $(bc \leq 0$ *and* $(b-c)^2 \leq 1)$ *or* $(bc \geq 0$ *and* $b^2 + c^2 \leq 1)$*, the increments of* $M^H(0, b, c)$ *are more strongly correlated than those of both* $M^H(0, 1, 0)$ *and* $M^H(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

References

1. Alajmi, S. and Mliki, E.: On the mixed fractional Brownian motion time changed by inverse α-stable subordinator, *Applied Mathematical Sciences* **14** (2020), no. 16, 755–763.
2. Alòs, E., Mazet, O. and Nualart, D.: Stochastic calculus with respect to fractional Brownian motion with Hurst parameter lesser than $\frac{1}{2}$, *Stoch. Proc. Appl.* **86** (1999), no. 1, 121–139.
3. Bardina, X. and Es-Sebaiy, K.: An extension of bifractional Brownian motion, *Comm. Stoch. Analy.* **5** (2011), 333–340.
4. Bojdecki, T., Gorostiza, L. G. and Talarczyk, A.: Sub-fractional Brownian motion and its relation to occupation times, *Stat. Prob. Lett.* **69** (2004), no. 4, 405–419.
5. Bojdecki, T., Gorostiza, L. G. and Talarczyk, A.: Fractional Brownian density process and its self-intersection local time of order k, *Journal of Theoretical Probability* **17** (2004), no. 3, 717–739.
6. Casas, I. and Gao, J.: Econometric estimation in long-range dependent volatility models: theory and practice, *J. Econom.* **147** (2008), no. 1, 72–83.
7. Cheridito, P.: Mixed fractional Brownian motion, *Bernoulli* **7** (2001), no. 6, 913–934.
8. Dongwei, C., Fei, H., Jingjing, X. and Lei, L.: Long-range correlation analysis among non-stationary passive scalar series in the turbulent boundary layer, *Physica A: Statistical Mechanics and its Applications* **517** (2019), 290–296.
9. El-Nouty, C.: The fractional mixed fractional Brownian motion, *Stat. Prob. Lett.* **65** (2003), no. 2, 111–120.
10. El-Nouty, C.: The Generalized Bifractional Brownian motion, *Int. J. for Computational Civil and Structural Engineering* **14** (2018), no. 4, 81–89.
11. El-Nouty, C. and Zili, M.: On the sub-mixed fractional Brownian motion, *Applied Mathematics-A Journal of Chinese Universities* **30** (2015), no. 1, 27–43.
12. Gao, J. B., Hu, J., tung, W. W., Cao, Y. H., Sarshar, N. and Roychowdhury V. P.: Assessement of long-range correlation in time series: How to avoid pitfalls, *Physical Review E* **73** (2006), 016117.
13. Hmissi, M., Mejri, H. and Mliki, E.: On the fractional powers of semidynamical systems, *Grazer Mathematische Berichte* **351** (2007), 66–78.
14. Hmissi, M., Mejri, H. and Mliki, E.: On the abstract exit equation, *Grazer Mathematische Berichte* **354** (2009), 84–98.
15. Hmissi, M. and Mliki, E.: On exit laws for subordinated semigroups by means of C^1-subordinators, *Comment. Math. Univ. Crolin.* **51** (2010), no. 4, 605–617.
16. Huy, D. P.: A remark on non-Markov property of a fractional Brownian motion, *Vietnam Journal of Mathematics* **31** (2003), no. 3, 237–240.

17. Kukush, A., Mishura, Y. and Valkeila, E.: Statistical inference with fractional Brownian motion, *Statistical Inference for Stochastic Processes* **8** (2005), no. 1, 71–93.
18. Majdoub, M. and Mliki, E.: Well-posedness for Hardy-Hénon parabolic equations with fractional Brownian noise, *Analysis and Mathematical Physics* **11** (2021), no. 1, 1–12.
19. Mandelbrot, B. B. and Van Ness, J. W.: Fractional Brownian motions, fractional noises and applications, *SIAM Review* **10** (1968), no. 4, 422–437.
20. Melnikov, A. and Mishura, Y.: On pricing and hedging in financial markets with long-range dependence, *Math. Finan. Econ.* **5** (2011), no. 1, 29–46.
21. Nane, E.: Laws of the iterated logarithm for a class of iterated processes, *Stat. Prob. Lett.* **79** (2009), no. 16, 1744–1751.
22. Önalan, Ö.: Time-changed generalized mixed fractional Brownian motion and application to arithmetic average Asian option pricing, *International journal of Applied Mathematical Research* **6** (2017), no. 3, 85–92.
23. Revuz, D. and Yor, M.: *Continuous martingales and Brownian motion*, Springer, 1999.
24. Sghir, A.: The generalized sub-fractional Brownian motion, *Commun. Stoch. Anal.* **7** (2013), no. 3, 373–382.
25. Thäle, C.: Further remarks on mixed fractional Brownian motion, *Applied Mathematical Sciences* **3** (2009), no. 38, 1885–1901.
26. Tudor, C.: Some properties of the sub-fractional Brownian motion, *Stochastics* **79** (2007), no. 5, 431–448.
27. Zili, M.: On the mixed-fractional Brownian motion, *J. Appl. Math. Stoch. Anal.* **2006** (2006).
28. Zili, M.: Mixed sub-fractional Brownian motion, *Random Oper. Stoch. Equ.* **22** (2014), no. 3, 163–178.
29. Zili, M.: Generalized fractional Brownian motion, *Mod. Stochast.: Theory Appl.* **4** (2017), no. 1, 15–24.
30. Zili, M.: On the Generalized fractional Brownian motion, *Mathematical Models and Computer Simulation* **10** (2018), no. 6, 759–769.

Shaykhah Alajmi: Department of Mathematics, College of Science, Imam Abdulrahman Bin Faisal University, P. O. Box 1982, Dammam, Saudi Arabia.
Department of Mathematics, University of Hafr Al-Batin, P. O. Box 1803, Hafr Al-Batin, 31991, Saudi Arabia
E-mail address: sho192010@hotmail.com

Ezzedine Mliki: Department of Mathematics, College of Science, Imam Abdulrahman Bin Faisal University, P. O. Box 1982, Dammam, Saudi Arabia.
Basic and Applied Scientific Research Center, Imam Abdulrahman Bin Faisal University, P. O. Box 1982, Dammam, 31441, Saudi Arabia.
E-mail address: ermliki@iau.edu.sa