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EXACT SOLUTIONS TO OPTIMAL CONTROL PROBLEMS FOR WIENER PROCESSES WITH EXPONENTIAL JUMPS

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Abstract. The LQG homing problem, in which a diffusion process is controlled until a certain event takes place, is considered for Wiener processes with jumps that are exponentially distributed. The objective is either to minimize (or maximize) the expected time spent by the controlled process in an interval $[a, b]$, or to try to make the process leave this interval through a given endpoint. The integro-differential equation satisfied by the value function is transformed into a non-linear ordinary differential equation and is solved exactly in particular cases.

1. Introduction

Let $\{B(t), t \geq 0\}$ be a one-dimensional standard Brownian motion and $\{N(t), t \geq 0\}$ be a Poisson process with rate $\lambda \geq 0$. The two stochastic processes are assumed to be independent. We consider the controlled jump-diffusion process $\{X_u(t), t \geq 0\}$ defined by

$$X_u(t) = X_u(0) + \mu t + \int_0^t b[X_u(s)]\, u[X_u(s)]\, \mathrm{d}s + \sigma B(t) + \sum_{i=1}^{N(t)} Y_i, \tag{1.1}$$

where $\mu \in \mathbb{R}$ and $\sigma > 0$ are constants, and $b(\cdot)$ is a non-zero function. The random variables $Y_1, Y_2, \ldots$ are independent and are all exponentially distributed with parameter α. If $\lambda = 0$ and the control variable u is chosen equal to zero, then the uncontrolled process $\{X_0(t), t \geq 0\}$ is a Wiener process with infinitesimal mean μ and variance σ^2.

We want to find the control that minimizes the expected value of the cost criterion

$$J(x) := \int_0^{T(x)} \left\{ \frac{1}{2}\, q[X_u(t)]\, u^2[X_u(t)] + \theta \right\} \mathrm{d}t + K[X_u(T(x))], \tag{1.2}$$

where θ is a real constant, $q(\cdot)$ is a positive function, $T(x)$ is the *first-passage time*

$$T(x) = \inf\{t \geq 0 : X_u(t) \notin (a, b) \mid X_u(0) = x \in [a, b]\}, \tag{1.3}$$

and K is a general terminal cost function.

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The above problem is a particular *LQG homing problem* (where LQG stands for Linear Quadratic Gaussian), in which one tries to optimally control an n-dimensional diffusion process until a certain event occurs; see Whittle [7] and/or Whittle [8]. When λ is equal to zero, so that there are no jumps, it is sometimes possible to transform our problem into a purely probabilistic problem for the uncontrolled process $\{X_0(t), t \geq 0\}$ obtained by setting $u(\cdot) \equiv 0$ in Eq. (1.1). More precisely, if certain conditions are fulfilled, we can show that the optimal control $u^*(x)$ can be expressed in terms of the moment-generating function of the random variable $T_0(x)$ that corresponds to $T(x)$.

In Lefebvre [3], the author extended LQG homing problems to the case when $\{X_u(t), t \geq 0\}$ is a jump-diffusion process with jumps of constant size $\epsilon > 0$; see also Theodorou and Todorov [6]. He proved that, if ϵ is small, the (approximate) optimal control can be deduced from a mathematical expectation for the uncontrolled process $\{X_0(t), t \geq 0\}$. A particular problem for a controlled standard Brownian motion with jumps was solved explicitly by making use of the results obtained by Abundo [1]. Moreover, Lefebvre and Moutassim [5] were able to obtain *exact* solutions to the problem considered in Lefebvre [3].

Next, Lefebvre [4] computed an approximation to the exact optimal control when the independent random variables $Y_1, Y_2, \ldots$ are uniformly distributed over the interval $[-c, c]$. As expected, he found that the approximate solution is more precise when c is small. In the present paper, we will see that it is actually possible to obtain exact solutions to the LQG homing problem defined above; that is, when Y_i has an exponential distribution with parameter α, for $i = 1, 2, \ldots$

Jump-diffusion processes are used extensively in financial mathematics, but there are other real-life applications of these processes. For example, they can be used to model the variations of river flows; see Konecny and Nachtnebel [2]. In this type of application, the jumps are generated by weather events such as thunderstorms, and are necessarily positive. An exponential distribution for the jump sizes is a very common assumption in statistical hydrology.

Let us define the *value function*

$$F(x) = \inf_{\substack{u[X_u(t)] \\ 0 \leq t \leq T(x)}} E[J(x)]. \tag{1.4}$$

That is, $F(x)$ is the expected cost obtained by choosing the optimal value of the control $u[X_u(t)]$ for $0 \leq t \leq T(x)$.

In the next section, the integro-differential equation satisfied by $F(x)$ will be derived by making use of dynamic programming. Then, the integro-differential equation will be transformed into a non-linear ordinary differential equation. Particular problems will be considered and solved explicitly in Section 3. Finally, we will end this paper with a few concluding remarks.

2. Integro-differential Equation

We first prove the following proposition that gives us the integro-differential equation satisfied by $F(x)$. In Lefebvre [3], the corresponding equation when the jump size is a constant ϵ was derived.

Proposition 2.1. *The value function $F(x)$ defined in Eq. (1.4) satisfies the second-order, non-linear integro-differential equation*

$$\begin{aligned} 0 &= \theta + \mu F'(x) - \frac{1}{2}\frac{b^2(x)}{q(x)}\,[F'(x)]^2 + \frac{1}{2}\sigma^2 F''(x) \\ &\quad + \lambda \int_0^\infty [F(x+y) - F(x)]\,\alpha e^{-\alpha y}\,\mathrm{d}y \end{aligned} \tag{2.1}$$

for $a < x < b$, subject to the boundary conditions

$$F(x) = K(x) \quad \text{if } x = a \text{ or } x \geq b. \tag{2.2}$$

Proof. Making use of Bellman's principle of optimality, we can write that

$$\begin{aligned} F(x) &= \inf_{\substack{u[X_u(t)]\\ 0 \leq t \leq \Delta t}} E\left[\int_0^{\Delta t} \left\{\frac{1}{2} q[X_u(t)]\, u^2[X_u(t)] + \theta\right\} \mathrm{d}t \right. \\ &\qquad \left. + F\left(x + [\mu + b(x)\,u(x)]\,\Delta t + \sigma B(\Delta t) + \sum_{i=1}^{N(\Delta t)} Y_i\right)\right] \\ &= \inf_{\substack{u[X_u(t)]\\ 0 \leq t \leq \Delta t}} E\left[\left\{\frac{1}{2} q(x)\, u^2(x) + \theta\right\} \Delta t \right. \\ &\qquad \left. + F\left(x + [\mu + b(x)\,u(x)]\,\Delta t + \sigma B(\Delta t) + \sum_{i=1}^{N(\Delta t)} Y_i\right) + o(\Delta)\right]. \end{aligned} \tag{2.3}$$

Next, we have

$$P[N(\Delta t) = 0] = e^{-\lambda \Delta t} = 1 - \lambda \Delta t + o(\Delta t) \tag{2.4}$$

and

$$P[N(\Delta t) = 1] = \lambda \Delta t\, e^{-\lambda \Delta t} = \lambda \Delta t + o(\Delta t). \tag{2.5}$$

It follows that

$$\begin{aligned} &E\left[F\left(x + [\mu + b(x)\,u(x)]\,\Delta t + \sigma B(\Delta t) + \sum_{i=1}^{N(\Delta t)} Y_i\right)\right] \\ &= E\left[F\left(x + [\mu + b(x)\,u(x)]\,\Delta t + \sigma B(\Delta t) + Y_1\right)\right] \lambda \Delta t \\ &\quad + E\left[F\left(x + [\mu + b(x)\,u(x)]\,\Delta t + \sigma B(\Delta t)\right)\right] (1 - \lambda \Delta t) + o(\Delta t). \end{aligned} \tag{2.6}$$

Moreover, $E[B(\Delta t)] = 0$ and $E[B^2(\Delta t)] = V[B(\Delta t)] = \Delta t$. Hence, assuming that $F(x)$ is twice differentiable with respect to x and making use of Taylor's formula, we can write that

$$\begin{aligned} &E\left[F\left(x + [\mu + b(x)\,u(x)]\,\Delta t + \sigma B(\Delta t)\right)\right] (1 - \lambda \Delta t) \\ &= \left\{F(x) + [\mu + b(x)\,u(x)]\,\Delta t\, F'(x) + \frac{1}{2}\sigma^2\,\Delta t\, F''(x)\right\} - \lambda \Delta t\, F(x) \\ &\quad + o(\Delta t) \end{aligned} \tag{2.7}$$

and

$$E\left[F\left(x+[\mu+b(x)\,u(x)]\,\Delta t+\sigma\,B(\Delta t)+Y_1\right)\right]\lambda\Delta t \tag{2.8}$$
$$= E[F(x+Y_1)]\,\lambda\Delta t+o(\Delta t)=\lambda\Delta t\int_0^{\infty}F(x+y)\,\alpha\,e^{-\alpha y}\mathrm{d}y+o(\Delta t).$$

Substituting (2.7) and (2.8) into Eq. (2.3), we find that

$$\begin{aligned} 0 \quad = \quad & \inf_{\substack{u[X_u(t)]\\ 0\le t\le \Delta t}}\left\{\left(\frac{1}{2}\,q(x)\,u^2(x)+\theta\right)\Delta t+[\mu+b(x)\,u(x)]\,\Delta t\,F'(x)\right. \\ & +\frac{1}{2}\,\sigma^2\,\Delta t\,F''(x)-\lambda\Delta t\,F(x) \\ & \left.+\,\lambda\,\Delta t\int_0^{\infty}F(x+y)\,\alpha\,e^{-\alpha y}\mathrm{d}y+o(\Delta t)\right\}. \end{aligned} \tag{2.9}$$

Now, dividing each side of the above equation by Δt and letting Δt decrease to zero, we obtain the following *dynamic programming equation*:

$$\begin{aligned} 0 \quad = \quad & \inf_{u(x)}\left\{\frac{1}{2}\,q(x)\,u^2(x)+\theta+[\mu+b(x)\,u(x)]\,F'(x)+\frac{1}{2}\,\sigma^2\,F''(x)\right. \\ & \left.-\,\lambda\,F(x)+\lambda\int_0^{\infty}F(x+y)\,\alpha\,e^{-\alpha y}\mathrm{d}y\right\}. \end{aligned} \tag{2.10}$$

We deduce from Eq. (2.10) that the optimal control $u^*(x)$ can be expressed as

$$u^*(x)=-\frac{b(x)}{q(x)}\,F'(x). \tag{2.11}$$

Substituting this expression into Eq. (2.10), and noticing that we can write

$$F(x)=\int_0^{\infty}F(x)\,\alpha\,e^{-\alpha y}\mathrm{d}y, \tag{2.12}$$

we obtain Eq. (2.1).

Finally, the boundary conditions in Eq. (2.2) follow from the fact that, the jumps being strictly positive, the controlled process $\{X_u(t), t\ge 0\}$ cannot take on a value smaller than the left-hand endpoint a of the interval $[a,b]$, but it can cross the boundary $x=b$. □

Next, we will transform the integro-differential equation (2.1) into an ordinary differential equation.

Proposition 2.2. *Assume that the ratio* $b^2(x)/q(x)$ *is a constant:*

$$\kappa := \frac{b^2(x)}{2\,q(x)} \quad (>0), \tag{2.13}$$

and that the value function $F(x)$ *is thrice differentiable. Then* $F(x)$ *satisfies the non-linear, third-order ordinary differential equation*

$$\begin{aligned} \alpha\,\theta \quad = \quad & -(\lambda+\alpha\,\mu)\,F'(x)+\left(\mu-\frac{1}{2}\,\alpha\,\sigma^2\right)F''(x)+\frac{1}{2}\,\sigma^2\,F'''(x) \\ & +\,\alpha\,\kappa\,[F'(x)]^2-2\,\kappa\,F'(x)\,F''(x). \end{aligned} \tag{2.14}$$

Proof. Let $z := x + y$. Then Eq. (2.1) can be rewritten as

$$0 = \theta + \mu F'(x) - \kappa \left[F'(x)\right]^2 + \frac{1}{2}\sigma^2 F''(x) - \lambda F(x) + \lambda \alpha e^{\alpha x} \int_x^\infty F(z) e^{-\alpha z} \mathrm{d}z. \quad (2.15)$$

Next, differentiating with respect to x, we obtain

$$\begin{aligned} 0 &= \mu F''(x) - 2\kappa F'(x) F''(x) + \frac{1}{2}\sigma^2 F'''(x) \\ &\quad - \lambda F'(x) - \lambda \alpha F(x) + \lambda \alpha^2 e^{\alpha x} \int_x^\infty F(z) e^{-\alpha z} \mathrm{d}z. \end{aligned} \quad (2.16)$$

Since we deduce from Eq. (2.1) that

$$\begin{aligned} \lambda \int_0^\infty F(x+y) \alpha e^{-\alpha y} \mathrm{d}y &= \lambda \alpha e^{\alpha x} \int_x^\infty F(z) e^{-\alpha z} \mathrm{d}z \\ &= -\left\{\theta + \mu F'(x) - \kappa \left[F'(x)\right]^2 + \frac{1}{2}\sigma^2 F''(x) - \lambda F(x)\right\}, \end{aligned} \quad (2.17)$$

we find that $F(x)$ indeed satisfies Eq. (2.14). □

Equation (2.14) is obviously not easy to solve exactly. The mathematical software *Maple* does not give any explicit solution to this equation. Nevertheless, it is generally easier to try to solve a (third-order) non-linear *ordinary* differential equation than a (second-order) non-linear integro-differential equation.

In the next section, we will try to determine whether a function of a given form could indeed be a solution to Eq. (2.14), subject to the boundary conditions in (2.2). Two particular cases will be considered.

3. Particular Cases

Case I. First, we will look for solutions of the form

$$F(x) = c x^m, \quad (3.1)$$

where $c \neq 0$ and $m \in \mathbb{R}$ are constants. Substituting into Eq. (2.14), we obtain the following equation:

$$\begin{aligned} \alpha\theta &= -(\lambda + \alpha\mu) c m x^{m-1} + \left(\mu - \frac{1}{2}\alpha\sigma^2\right) c m (m-1) x^{m-2} \\ &\quad + \frac{1}{2}\sigma^2 c m (m-1)(m-2) x^{m-3} + \alpha\kappa c^2 m^2 x^{2m-2} \\ &\quad - 2\kappa c^2 m^2 (m-1) x^{2m-3}. \end{aligned} \quad (3.2)$$

Because the above equation must be satisfied for any $x \in (a, b)$, we must conclude that the only admissible value of m is $m = 1$.

Remark 3.1. Actually, if $\theta = 0$, Eq. (3.2) is satisfied with $m = 0$. However, then $F(x) \equiv c$, so that $u^*(x) \equiv 0$. Moreover, we must have $K[X_u(T(x))] \equiv c$ as well. If θ is equal to zero and $K[X_u(T(x))]$ is a constant, it is obvious that the optimal control is also equal to zero.

When $m = 1$, Eq. (3.2) reduces to

$$\alpha\theta = -(\lambda + \alpha\mu)c + \alpha\kappa c^2. \tag{3.3}$$

It follows that there are two possible values of the constant c:

$$c = \frac{(\lambda + \alpha\mu) \pm \left[(\lambda + \alpha\mu)^2 + 4\alpha^2\kappa\theta\right]^{1/2}}{2\alpha\kappa}. \tag{3.4}$$

Proposition 3.2. *If*

$$(\lambda + \alpha\mu)^2 \geq -4\alpha^2\kappa\theta \tag{3.5}$$

and if

$$K[X_u(T(x))] = cX_u(T(x)), \tag{3.6}$$

where c is given in (3.4), then the value function is $F(x) = cx$ for $x \in [a, b]$. Furthermore, the optimal control is

$$u^*(x) = -\frac{b(x)}{q(x)}c. \tag{3.7}$$

Remark 3.3. (i) Notice that the value function and the optimal control do not depend on the parameter σ. In the case of a controlled standard Brownian motion with jumps, so that $\mu = 0$, Eq. (3.4) simplifies to

$$c = \frac{\lambda \pm \left(\lambda^2 + 4\alpha^2\kappa\theta\right)^{1/2}}{2\alpha\kappa}. \tag{3.8}$$

(ii) The sign that we must choose in Eq. (3.4) corresponds to the one in the function K defined in (3.6). There are therefore two particular LQG homing problems that we can solve explicitly and exactly under the above assumptions.

(iii) If $\theta > 0$, the optimizer wants the controlled process $\{X_u(t), t \geq 0\}$ to leave the interval (a, b) as soon as possible, whereas when $\theta < 0$ the optimizer receives a reward while the process is in (a, b), so that the objective should be to maximize the survival time in this interval. If the constant c is positive (respectively negative), then the optimizer should try to make the process leave the interval (a, b) through its left-hand endpoint a (respectively right-hand endpoint b) as well.

(iv) We see that Eq. (3.5) is satisfied for any $\theta > 0$. However, we must have

$$\theta \geq \theta_0 := -\frac{(\lambda + \alpha\mu)^2}{4\alpha^2\kappa}. \tag{3.9}$$

The value $\theta = 0$ is admissible; if $\mu = 0$ too, we then obtain that

$$c = \frac{\lambda}{\alpha\kappa}. \tag{3.10}$$

(v) We assumed in Proposition 2.2 that the ratio $b^2(x)/q(x)$ is a constant. Nevertheless, it is important to observe that the optimal control is not necessarily a constant.

(vi) In the case when there are no jumps, so that $\lambda = 0$, we find that the constant c is

$$c = \frac{\mu \pm \left(\mu^2 + 4\kappa\theta\right)^{1/2}}{2\kappa}, \tag{3.11}$$

in which we assume that $\mu^2 \geq -4\kappa\theta$. Hence, if $\mu = 0$, we have $c = \pm\sqrt{\theta/\kappa}$, so that θ must then be positive. Actually, when $\lambda = 0$, we can obtain the general solution of Eq. (2.1), under the assumption in (2.13). Let

$$\gamma := \sqrt{\mu^2 + 4\kappa\theta}. \tag{3.12}$$

We find that

$$F(x) = \frac{1}{4\kappa}\left\{2\sigma^2 \ln(\gamma/\kappa) - \sigma^2 \ln\left[\left(c_1 e^{2\gamma x/\sigma^2} + c_2\right)^2\right] + 2(\gamma + \mu)x\right\}, \tag{3.13}$$

where the constants c_1 and c_2 can be determined uniquely from the boundary conditions (since there are no jumps)

$$F(x) = K(x) \quad \text{if } x = a \text{ or } x = b. \tag{3.14}$$

We can check that the particular solution $F(x) = cx$ (for each value of c) is obtained by choosing the appropriate constants c_1 and c_2. More precisely, the solution with the "+" sign in Eq. (3.11) corresponds to the choice $c_1 = 0$ and $c_2 = \gamma/\kappa$, while the one with the "−" sign corresponds to $c_2 = 0$ and $c_1 = \gamma/\kappa$.

(vii) A simple case is the one for which $\mu = 0$ and $\sigma = \theta = \lambda = \alpha = \kappa = 1$. We must then solve the differential equation

$$1 = -F'(x) - \frac{1}{2}F''(x) + \frac{1}{2}F'''(x) + [F'(x)]^2 - 2F'(x)F''(x). \tag{3.15}$$

One can check that the function

$$F(x) = \frac{1 \pm \sqrt{5}}{2}x \tag{3.16}$$

does indeed satisfy the above equation, together with the boundary conditions

$$F(a) = \frac{1 \pm \sqrt{5}}{2}a \quad \text{and} \quad F(x) = \frac{1 \pm \sqrt{5}}{2}x \quad \text{for } x \geq b. \tag{3.17}$$

Case II. Next, we will try solutions of the form

$$F(x) = ce^{mx}, \tag{3.18}$$

where $c \neq 0$ and $m \in \mathbb{R}$ are constants. Equation (2.14) becomes

$$\begin{aligned} \alpha\theta &= cme^{mx}\left[-(\lambda + \alpha\mu) + m\left(\mu - \frac{1}{2}\alpha\sigma^2\right) + \frac{1}{2}m^2\sigma^2\right] \\ &\quad + c^2\kappa m^2 e^{2mx}(\alpha - 2m). \end{aligned} \tag{3.19}$$

We deduce from the above equation that θ must be equal to zero. Then, as in Case I, the value $m = 0$ (together with $K[X_u(T(x))] \equiv c$) leads to the obvious optimal control $u^*(x) \equiv 0$.

Proposition 3.4. *Let $m = \alpha/2$. If the relation*

$$\frac{\alpha}{2} = -\frac{\mu \pm \left(\mu^2 - 2\lambda\sigma^2\right)^{1/2}}{\sigma^2} \quad (> 0) \tag{3.20}$$

holds, and if

$$K[X_u(T(x))] = c\exp\{mX_u(T(x))\}, \tag{3.21}$$

then the value function is $F(x) = ce^{mx}$ *for* $x \in [a, b]$. *Moreover, the optimal control is given by*

$$u^*(x) = -\frac{b(x)}{q(x)} c m e^{mx}. \tag{3.22}$$

Proof. Because Eq. (3.19) must be satisfied for any $x \in (a, b)$, we must take $m = \alpha/2$. Then, the coefficient of e^{mx} will also vanish if

$$\begin{aligned} 0 &= -(\lambda + \alpha\mu) + \frac{\alpha}{2}\left(\mu - \frac{1}{2}\alpha\sigma^2\right) + \frac{1}{8}\alpha^2\sigma^2 \\ \Longleftrightarrow \quad 0 &= \lambda + \mu\frac{\alpha}{2} + \frac{1}{2}\sigma^2\left(\frac{\alpha}{2}\right)^2, \end{aligned} \tag{3.23}$$

from which Eq. (3.20) follows at once. □

Remark 3.5. (i) For the solution to be valid, we must have

$$\mu^2 \geq 2\lambda\sigma^2. \tag{3.24}$$

Therefore, we cannot take $\mu = 0$ in this case. Moreover, the value of α will be positive if and only if $\mu < 0$. Thus, there are again two possible solutions.

(ii) This time, the solution does not depend on the constant c, but it depends on σ^2.

(iii) When there are no jumps (i.e., $\lambda = 0$), the general solution of Eq. (2.1) (together with Eq. (2.13)) reduces to

$$F(x) = -\frac{\sigma^2}{2\kappa}\ln\left(c_1 e^{-2\mu x/\sigma^2} + c_2\right), \tag{3.25}$$

from which we deduce that there are no solutions of the form $F(x) = ce^{mx}$, except when $m = 0$. The solution $F(x) \equiv c$ is actually a special case of the one given in Proposition 3.4. Notice that, when $\lambda = 0$, we should set α equal to zero as well.

(iv) As a particular example, let us choose $\mu = -2$, $\sigma = 1$, $\lambda = 2$ and $\kappa = 1$. Then, Eq. (3.20) implies that α must be equal to 4. We can check that the function $F(x) = ce^{2x}$ does indeed satisfy Eq. (2.14) when $\theta = 0$.

4. Conclusion

In this paper, we were able to find explicit solutions to LQG homing problems for one-dimensional jump-diffusion processes. Contrary to previous papers on this type of problems, the solutions that were obtained are *exact* ones. The continuous part of the jump-diffusion processes was a controlled Wiener process, and the jumps occurred according to a Poisson process. In the previous work, the size of the jumps was either deterministic or uniformly distributed. Here, this size was exponentially distributed.

To obtain explicit solutions to our problems, we first transformed the integro-differential equation satisfied by the value function into a non-linear ordinary differential equation of order three. Then, we looked for solutions of a certain form, namely first a power function, and next an exponential function.

Depending on the sign of the parameter θ in the cost function defined in Eq. (1.2), the optimizer wants either the controlled process to leave the interval (a, b) as soon as possible, or to maximize its survival time in this interval. There is also a final cost function that must be of the same form as the value function (or the other way around) for the solutions to be valid. We saw that the optimal control $u^*(x)$ was not trivial, except when the value function is a constant, which leads to $u^*(x) \equiv 0$.

Finally, in some applications the jumps can be both positive or negative, but cannot be as large as we want. Moreover, in financial mathematics the value of a stock or of a market index like the NASDAQ can decrease a lot in a short amount of time, but it obviously cannot become negative. Therefore, there are other distributions for the jump sizes that it would be interesting to consider in LQG homing problems.

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