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Robert J. Elliott

University of South Australia, Campus Central - City West, GPO Box 2471, Robert.Elliott@unisa.edu.au

W. P. Malcolm

Defence Science and Technology Group Australia, Russell Offices Canberra, Constitution Ave, Russell ACT 2600, Australia, Paul.Malcolm2@defence.gov.au

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NEW REPRESENTATIONS FOR A SEMI-MARKOV CHAIN AND RELATED FILTERS

ROBERT ELLIOTT* AND W. P. MALCOLM

ABSTRACT. In this article we investigate estimation for a partially observed semi-Markov chain, or a Hidden semi-Markov Model (HsMM). We derive semimartingale dynamics for a semi-Markov chain and give them in a new vector form which explicitly exhibits the times at which jump-events occur and the probabilities of state transitions. However, the most important result is the new vector lattice state-space representation for a general finite-state, discrete-time semi-Markov chain. On this space the semi-Markov chain and its occupation times are a Markov process with dynamics described by finite matrices. These representations are new. Finite dimensional recursive filters are derived for a HsMM.

1. Introduction

Semi-Markov chains are related to renewal processes and have been used in applications since their introduction over 60 years ago. Their general occupation-time distributions offer a far richer class of models than standard Markov chains. The two main contributions of this paper are;

- (1) a new vector state-space representation for a general finite-state semi-Markov chain which exhibits it as a Markov chain,
- (2) the consequent extension to semi-Markov chains of the filtering, smoothing and estimation results,
- (3) the matrix and vector semimartingale dynamics for the semi-Markov chain.

Earlier references on semi-Markov processes include the books by Koski [8], Barbu and Limnios [2], and van der Hoek and Elliott [10]. References on filtering include, Krishnamurthy, Moore and Chung [9] and Elliott, Limnios and Swishchuk [5]. Filters for Markov modulated time series were obtained in the PhD Thesis [1]. The matrix representation in this paper of the dynamics is new.

2. Stochastic Dynamics

All processes are defined on a probability space (Ω, \mathcal{F}, P) . Our process of interest is a semi-Markov chain $X = \{X_k, k = 0, 1, 2, \dots\}$ with arbitrary state sojourn distributions. As is now standard the finite state space for the process

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* Corresponding author.

X is identified with the set of unit vectors $S = \{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_N\}$, where $\mathbf{e}_i := (0, \dots, 0, 1, 0, \dots, 0)' \in \mathbb{R}^N$. We also write $m \in \{1, 2, 3, \dots\}$ exclusively as a time index for state sojourns.

Notation 2.1. The initial state $X_0 \in S$, is taken as given, or its probability distribution $p_0 = (p_0^1, p_0^2, \dots, p_0^N)' \in [0, 1]^N$ is known. The chain will change state at random discrete times τ_n . State transitions at these times are of the type $\mathbf{e}_i \rightarrow \mathbf{e}_j$, with $i \neq j$. We set $\tau_0 := 0$. Successive jump event times form a strictly increasing sequence $\tau_0 < \tau_1 < \tau_2 < \tau_3 \dots$. Write $\mathcal{F}_k := \sigma\{X_u, u \leq k\}$ and $\mathbb{F} = \{\mathcal{F}_u\}_{u \geq 0}$ for the filtration generated by X .

We now define a time-homogeneous semi-Markov chain.

Definition 2.2. The stochastic process X is a time-homogeneous semi-Markov process if

$$P(X_{\tau_{n+1}} = \mathbf{e}_j, \tau_{n+1} - \tau_n = m \mid \mathcal{F}_{\tau_n}) = P(X_{\tau_{n+1}} = \mathbf{e}_j, \tau_{n+1} - \tau_n = m \mid X_{\tau_n} = \mathbf{e}_i).$$

If $X_{\tau_n} = \mathbf{e}_i$ we write this as $q(\mathbf{e}_j, \mathbf{e}_i, m)$.

This can be factorized as

$$P(\tau_{n+1} - \tau_n = m \mid X_{\tau_{n+1}} = \mathbf{e}_j, X_{\tau_n} = \mathbf{e}_i) P(X_{\tau_{n+1}} = \mathbf{e}_j \mid X_{\tau_n} = \mathbf{e}_i) = f_{j,i}(m) p_{j,i},$$

say. Here

$$f_{j,i}(m) := P(\tau_{n+1} - \tau_n = m \mid X_{\tau_{n+1}} = \mathbf{e}_j, X_{\tau_n} = \mathbf{e}_i) \quad \text{and} \\ p_{j,i} := P(X_{\tau_{n+1}} = \mathbf{e}_j \mid X_{\tau_n} = \mathbf{e}_i).$$

Consequently

$$q(\mathbf{e}_i, \mathbf{e}_j, m) = f_{j,i}(m) p_{j,i}. \quad (2.1)$$

We can also consider the factorization

$$P(X_{\tau_{n+1}} = \mathbf{e}_j, \tau_{n+1} - \tau_n = m \mid X_{\tau_n} = \mathbf{e}_i) = P(\tau_{n+1} - \tau_n = m \mid X_{\tau_n} = \mathbf{e}_i) \times \\ P(X_{\tau_{n+1}} = \mathbf{e}_j \mid \tau_{n+1} - \tau_n = m, X_{\tau_n} = \mathbf{e}_i) \quad (2.2) \\ = \pi_i(m) p_{j,i}(m), \text{ say.}$$

Here

$$\pi_i(m) := P(\tau_{n+1} - \tau_n = m \mid X_{\tau_n} = \mathbf{e}_i) \quad \text{and} \\ p_{j,i}(m) := P(X_{\tau_{n+1}} = \mathbf{e}_j \mid \tau_{n+1} - \tau_n = m, X_{\tau_n} = \mathbf{e}_i).$$

Approximations 2.3. If $f_{j,i}(m)$ does not depend upon \mathbf{e}_j we can write

$$P(\tau_{n+1} - \tau_n = m \mid X_{\tau_{n+1}} = \mathbf{e}_j, X_{\tau_n} = \mathbf{e}_i) = P(\tau_{n+1} - \tau_n = m \mid X_{\tau_n} = \mathbf{e}_i) \quad (2.3) \\ = \pi_i(m).$$

That is, for each i , $1 \leq i \leq N$, $\{\pi_i(m), m = 1, 2, 3, \dots\}$ is a probability distribution on the positive integers. Then under this simplification

$$q(\mathbf{e}_j, \mathbf{e}_i, m) = \pi_i(m) p_{j,i}. \quad (2.4)$$

Note that, as we assumed X is homogeneous in time, all these probabilities are independent of n . If $p_{j,i}(m)$ does not depend upon m then from (2.2) we again have:

$$q(\mathbf{e}_j, \mathbf{e}_i, m) = \pi_i(m)p_{j,i}. \quad (2.5)$$

The approximation given by equation (2.4) or equation (2.5) is that used by Ferguson [7] However, in this paper we shall not discuss any approximations but use the general decomposition

$$q(\mathbf{e}_j, \mathbf{e}_i, m) = \pi_i(m)p_{j,i}(m). \quad (2.6)$$

Notation 2.4. Write

$$G_i(m) := P(\tau_{n+1} - \tau_n \leq m \mid X_{\tau_n} = \mathbf{e}_i) = \sum_{\ell=1}^m \pi_i(\ell),$$

$$F_i(m) := P(\tau_{n+1} - \tau_n > m \mid X_{\tau_n} = \mathbf{e}_i) = 1 - G_i(m).$$

We now provide the conditional probability for a state-transition to occur at the next discrete time. This probability plays an important role in subsequent calculations and is denoted by $\Delta^i(m)$. Given some discrete-time k , write τ_n for the most recent transition-event time prior to k , (or at k), that is, $\tau_n := \max_{\ell} \{\tau_\ell \leq k\}$. Further, suppose that for some m , $X_{\tau_n+m-1} = \mathbf{e}_i$. The probability of a transition-event occurring at the next time $\tau_n + m$ is

$$P(\tau_{n+1} = \tau_n + m \mid X_{\tau_{n+k-1}} = X_{\tau_n} = \mathbf{e}_i) =$$

$$P(\tau_{n+1} = \tau_n + m \mid \tau_{n+1} > \tau_n + m - 1, X_{\tau_n} = \mathbf{e}_i) = \frac{\pi_i(m)}{F_i(m-1)}.$$

This result is from the definition of conditional probability.

Write $A := \{\tau_{n+1} = \tau_n + m\}$, $B := \{\tau_{n+1} > \tau_n + m - 1\}$ and $C := \{\tau_n = \mathbf{e}_i\}$. Then

$$P(\tau_{n+1} = \tau_n + m \mid X_{\tau_n+m-1} = X_{\tau_n} = \mathbf{e}_i)$$

$$= P(A \mid B \cap C)$$

$$= \frac{P(A \cap B \mid C)}{P(B \mid C)},$$

(but $A \cap B = A$ as $A \subset B$, so it equals)

$$= \frac{P(\tau_{n+1} = \tau_n + m \mid X_{\tau_n} = \mathbf{e}_i)}{P(\tau_{n+1} > \tau_n + m - 1 \mid X_{\tau_n+m-1} = \mathbf{e}_i)},$$

$$= \frac{\pi_i(m)}{F_i(m-1)}.$$

Write $\Delta^i(m) := \frac{\pi_i(m)}{F_i(m-1)}$.

Definition 2.5. For each index i , $1 \leq i \leq N$, we define the recursive process $h_k^i := \langle X_k, \mathbf{e}_i \rangle + \langle X_k, \mathbf{e}_i \rangle \langle X_k, X_{k-1} \rangle h_{k-1}^i$, with $h_0^i := \langle X_0, \mathbf{e}_i \rangle \in \{0, 1\}$. The h^i processes are non-zero only at times when $X = \mathbf{e}_i$. The process h^i returns the

cumulative time spent in state e_i .

If $h_k = \sum_{i=1}^N h_k^i$ then $h_0 = 1$ and $h_k = 1 + \langle X_k, X_{k-1} \rangle h_{k-1}$. The process h_k measures the amount of time since the last transition event. This process is never zero.

2.1. Transition-Event Probabilities.

Lemma 2.6. *Suppose $i \neq j$, $1 \leq i, j \leq N$. Then $P(X_{k+1} = e_j | X_k = e_i, h_k^i) = p_{j,i}(h_k^i) \Delta^i(h_k^i)$.*

Proof. Write $A := \{X_{k+1} = e_j\}$, $B' := \{\tau_{n+1} - \tau_n = h_k^i\}$, $B'' := \{\tau_{n+1} > \tau_n + h_k^i - 1\}$ and $C := \{X_{\tau_n} = e_i = X_k\}$. Then

$$\begin{aligned} P(X_{k+1} = e_j | X_k = e_i, h_k^i) &= P(A \cap B' | B'' \cap C) \\ &= \frac{P(A \cap B' \cap B'' \cap C)}{P(B'' \cap C)} = \frac{P(A \cap B' \cap C)}{P(B'' \cap C)} \\ &= P(A | B' \cap C) \frac{P(B' \cap C)}{P(B'' \cap C)} \end{aligned}$$

(as $B' \cap B'' = B'$)

$$\begin{aligned} &= p_{j,i}(h_k^i) \frac{\pi^i(h_k^i)}{F_i(h_k^i - 1)} \\ &= p_{j,i}(h_k^i) \Delta^i(h_k^i). \end{aligned}$$

□

Remark 2.7. We are assuming there is a jump from e_i to a different e_j , $i \neq j$, at

time $k + 1$. So, $\sum_{\substack{j=1 \\ j \neq i}}^N p_{j,i}(k + 1) = 1$.

Corollary 2.8. *Under the same hypotheses,*

$$\begin{aligned} P(X_{k+1} = e_i | X_k = e_i, h_k^i) &= 1 - \Delta^i(h_k^i) \\ &= 1 - \left(\Delta^i(h_k^i) \sum_{\substack{j=1 \\ j \neq i}}^N p_{j,i}(h_k^i) \right) \\ &= 1 - \sum_{\substack{j=1 \\ j \neq i}}^N (p_{j,i}(h_k^i) \Delta^i(h_k^i)). \end{aligned}$$

Notation 2.9. For $m = 1, 2, \dots$, write $A(m)$ for the $N \times N$ matrix with entries $a_{i,i}(m) = 1 - \Delta^i(m)$ and $a_{j,i}(m) = p_{j,i}(m) \Delta^i(m)$.

Example 2.10. Then for $N = 3$ and some $h_k = m$,

$$A(m) := \begin{bmatrix} 1 - \Delta^1(m) & p_{1,2}(m) \Delta^2(m) & p_{1,3}(m) \Delta^3(m) \\ p_{2,1}(m) \Delta^1(m) & 1 - \Delta^2(m) & p_{2,3}(m) \Delta^3(m) \\ p_{3,1}(m) \Delta^1(m) & p_{3,2}(m) \Delta^2(m) & 1 - \Delta^3(m) \end{bmatrix}.$$

Notation 2.11. Define the matrices: $\Pi(m) := (p_{i,j}(m), 1 \leq i, j \leq N)$ where $p_{i,i}(m) = -1$ and $p_{j,i}(m) = P(X_{\tau_{n+1}} = \mathbf{e}_j \mid \tau_{n+1} - \tau_n = m, X_{\tau_n} = \mathbf{e}_i)$, for $i \neq j$. Write $D(m) := \text{diag}(\Delta^1(m), \Delta^2(m), \dots, \Delta^N(m))$. Then $A(m) = I + \Pi(m)D(m)$, where I is the $N \times N$ identity matrix.

For the case when $N = 3$,

$$\Pi(m) = \begin{bmatrix} -1 & p_{1,2}(m) & p_{1,3}(m) \\ p_{2,1}(m) & -1 & p_{2,3}(m) \\ p_{3,1}(m) & p_{3,2}(m) & -1 \end{bmatrix}, \quad D(m) = \begin{bmatrix} \Delta^1(m) & 0 & 0 \\ 0 & \Delta^2(m) & 0 \\ 0 & 0 & \Delta^3(m) \end{bmatrix}$$

and so $A(m) = I + \Pi(m)D(m)$. This decomposition nicely separates the probabilities of when the jump occurs and where it goes. A key result is the following representation of the semi-Markov chain X .

Theorem 2.12. *The semi-Markov chain X has the following semi-martingale dynamics:*

$$X_{k+1} = A(h_k)X_k + M_{k+1} \in \mathbb{R}^N.$$

Here M_{k+1} is a martingale increment: $E[M_{k+1} \mid X_k, h_k] = \mathbf{0} \in \mathbb{R}^N$.

Proof. For $i \neq j$ $E[\langle X_{k+1}, \mathbf{e}_j \rangle \mid X_k = \mathbf{e}_i, h_k^i] = P(X_{k+1} = \mathbf{e}_j \mid X_k = \mathbf{e}_i, h_k^i) = a_{j,i}(h_k^i)$ from Lemma 2.6 and the definition of $a_{j,i}(h_k^i)$. For the transition $X_k = \mathbf{e}_i \rightarrow X_{k+1} = \mathbf{e}_i$,

$E[\langle X_{k+1}, \mathbf{e}_i \rangle \mid X_k = \mathbf{e}_i, h_k^i] = P(X_{k+1} = \mathbf{e}_i \mid X_k = \mathbf{e}_i, h_k^i) = a_{i,i}(h_k^i)$ from Corollary 2.8 and the definition of $a_{i,i}(h_k^i)$. So $E[X_{k+1} \mid X_k, h_k] = A(h_k)X_k \in \mathbb{R}^N$ and

$$\begin{aligned} E[M_{k+1} \mid X_k, h_k] &= E[X_{k+1} - A(h_k)X_k \mid X_k, h_k] \\ &= \mathbf{0} \in \mathbb{R}^N. \end{aligned}$$

That is M_{k+1} is a (vector) martingale increment. \square

3. Lattice-based State-Space Dynamics

In this section we describe a countably infinite state space for a general semi-Markov chain. In this state space the process (X, h) is in fact a Markov chain. This property is known but the matrix representations are new.

3.1. Lattice-based State-Space. The complete description of the state of our semi-Markov chain X at time k is given by the state of the chain $X_k \in \{\mathbf{e}_1, \dots, \mathbf{e}_N\}$ and the number of time steps h_k the chain has been in that state since the last jump. To simplify the discussion here we suppose $N = 3$ so $S = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

A state space \bar{S} for the chain $\bar{X}_k := (X_k, h_k)$ can be identified with countably many copies of S as follows: Elements of \bar{S} can be thought of as infinite column vectors so, for example,

$$\begin{aligned} (\mathbf{e}_1, 1) &\text{ corresponds to } \underbrace{(1, 0, 0 \mid 0, 0, 0 \mid 0, \dots)'}_{h=1} \text{ and} \\ (\mathbf{e}_2, \ell) &\text{ corresponds to } (0, 0, 0 \mid \dots \mid \underbrace{0, 1, 0 \mid 0, \dots})'_{h_k=\ell} \end{aligned}$$

with $\mathbf{e}_2 = (0, 1, 0)'$ in the ℓ^{th} block. As a basis of unit vectors for this $\bar{X} = (X, h)$ process we take unit vectors $\mathbf{e}_{i,n}$, $1 \leq i \leq 3$, $n = 1, 2, \dots$. Here the i denotes the state in $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and the n corresponds to the sojourn time in state \mathbf{e}_i since the last jump at τ_n . Recall $S = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, so $\mathbf{e}_{i,n}$ is in the n^{th} block of $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$. Write $\bar{S} = \{\mathbf{e}_{i,n}, 1 \leq i \leq 3, n = 1, 2, \dots\}$. There is a map from \mathbb{R}^3 to $\mathbb{R}^{3 \times \mathbb{N}}$ given by

$T : (\alpha_1, \alpha_2, \alpha_3)' \rightarrow ((\alpha_1, \alpha_2, \alpha_3), (\alpha_1, \alpha_2, \alpha_3), \dots)'$. With I_N the $N \times N$ unit matrix this is given by the $\mathbb{N} \times \mathbb{N}$ matrix: $T = (I_N, I_N, I_N, \dots)'$. The adjoint of this is a map from $\mathbb{R}^{\mathbb{N} \times \mathbb{N}}$ to $\mathbb{R}^{\mathbb{N}}$ given by $T^* = (I_N, I_N, I_N, \dots)$.

3.2. State Transition Events. Note the counter $h_k = h_k(X_k)$ starts at 1, the first time X jumps to a new state. With the above notation $(\mathbf{e}_i, r) = \mathbf{e}_{i,r} \rightarrow (\mathbf{e}_i, r+1) = \mathbf{e}_{i,(r+1)}$ with probability $(1 - \Delta^i(r))$, or $(\mathbf{e}_i, r) = \mathbf{e}_{i,r} \rightarrow (\mathbf{e}_j, 1) = \mathbf{e}_{j,i}$, $j \neq i$, with probability $p_{j,i}(r)\Delta^i(r)$. For example, suppose at time 0 the chain is in state $(\mathbf{e}_1, 1) = \mathbf{e}_{1,1} = (1, 0, 0 \mid 0, 0, 0 \mid \dots)'$. This can become either $\mathbf{e}_{1,2} = (\mathbf{e}_1, 2) = (0, 0, 0 \mid 1, 0, 0 \mid 0, \dots)'$ with probability $(1 - \Delta^1(1))$, or $\mathbf{e}_{2,1} = (\mathbf{e}_2, 1) = (0, 1, 0 \mid 0, 0, 0 \mid 0, \dots)'$ with probability $p_{2,1}(r)\Delta^1(1)$, or $\mathbf{e}_{3,1} = (\mathbf{e}_3, 1) = (0, 0, 1 \mid 0, 0, 0 \mid 0, \dots)'$ with probability $p_{3,1}(r)\Delta^1(1)$. There is then an infinite matrix C which describes these transitions.

3.3. Dynamics for $\bar{X}_k := (X_k, h_k)$. In the $N = 3$ state case and for some value of $m \in \{1, 2, \dots\}$, write

$$\Pi(m) = \begin{bmatrix} 0 & p_{1,2}(m)\Delta^2(m) & p_{1,3}(m)\Delta^3(m) \\ p_{2,1}(m)\Delta^1(m) & 0 & p_{2,3}(m)\Delta^3(m) \\ p_{3,1}(m)\Delta^1(m) & p_{3,2}(m)\Delta^2(m) & 0 \end{bmatrix}$$

and $D(m) = \text{diag}\{1 - \Delta^1(m), 1 - \Delta^2(m), 1 - \Delta^3(m)\}$. With $\mathbf{0}$ representing the 3×3 zero matrix

$$C = \begin{bmatrix} \Pi(1) & \Pi(2) & \Pi(3) & \dots \\ D(1) & \mathbf{0} & \mathbf{0} & \dots \\ \mathbf{0} & D(2) & \mathbf{0} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \quad (3.1)$$

If we write the enlarged vectors as \bar{X}_k then the semi-martingale dynamics of the Markov chain can be written as $\bar{X}_{k+1} = C\bar{X}_k + \bar{M}_{k+1} \in \bar{S}$. This gives $E[\bar{X}_{k+1} \mid \bar{X}_k] = C\bar{X}_k$ and $E[\bar{X}_{k+1} \mid \bar{X}_0] = C^{k+1}\bar{X}_0$. At time $k \in \{0, 1, 2, \dots\}$ the sojourn time h_k^i cannot be more than $k+1$ and the next possible value of h_k^i is $k+2$. Consequently the size of C at time k is at most $(k+2)N \times (k+1)N$.

For example, at time 0 the C matrix has the form $C = \begin{bmatrix} \Pi(1) \\ D(1) \end{bmatrix}$. At time 1 the C matrix has the form

$$C = \begin{bmatrix} \Pi(1) & \Pi(2) \\ D(1) & \mathbf{0} \\ \mathbf{0} & D(2) \end{bmatrix} \quad \text{and so on.} \quad (3.2)$$

Consequently, at any finite time the C matrix is finite. Also the state space of \bar{X} at time k only has $(k+1)N$ elements. The size of the state space and the corresponding matrices will also remain finite if the sojourn distributions all have finite support.

4. Observation Dynamics

The filtering results of [6] are now adapted to this situation. Note that if $\bar{X}_k \in \bar{S}$ then $T^* \bar{X}_k = X_k \in S$. We suppose the Markov chain \bar{X} is not observed directly. Instead there is an observation sequence $y = \{y_0, y_1, \dots, y_k, \dots\}$ where

$$y_k = c(X_k) + d(X_k)w_k. \quad (4.1)$$

The observations are of $X_k = T^* \bar{X}_k$ rather than \bar{X}_k . $\{w_k, k = 0, 1, 2, \dots\}$ is a sequence of i.i.d. $N(0, 1)$ random variables. $c(\cdot)$ and $d(\cdot)$ are known real valued functions. Note that any real function $g(X_k)$ takes only the finite number of values $g(\mathbf{e}_1), g(\mathbf{e}_2), \dots, g(\mathbf{e}_N)$. Write $g_k = g(\mathbf{e}_k)$ and $\mathbf{g} = (g_1, g_2, \dots, g_N)' \in \mathbb{R}^N$. Then $g(X_k) = \langle \mathbf{g}, X_k \rangle$. Consequently there are vectors $\mathbf{c} = (c_1, c_2, \dots, c_N)$, $\mathbf{d} = (d_1, d_2, \dots, d_N)$ such that $c(X_k) = \langle \mathbf{c}, X_k \rangle$ and $d(X_k) = \langle \mathbf{d}, X_k \rangle$. We suppose $d_k > 0$ for $k = 1, \dots, N$.

Remark 4.1. We suppose the observation process y is scalar-valued. The extension to a vector-valued y is straight forward.

5. Finite-Dimensional Recursive Filters

5.1. Change of Probability Measure Formulation. We suppose there is a second ‘reference’ probability measure, \bar{P} , under which 1.) the process \bar{X} is still a Markov chain with dynamics $\bar{X}_{k+1} = C\bar{X}_k + \bar{M}_k$ and 2.) the process $y = \{y_0, y_1, \dots\}$ is a sequence of i.i.d. $N(0, 1)$ random variables. From \bar{P} we now construct the original probability P under which; 1.) the process $X = T^* \bar{X}$ is a semi-Markov chain with dynamics as above so $X_{k+1} = A(h(k))X_k + M_{k+1}$ and 2.) The process $w = (w_0, w_1, \dots)$ is a sequence of i.i.d. $N(0, 1)$ random variables where $w_k = \frac{y_k - \langle \mathbf{c}, X_k \rangle}{\langle \mathbf{d}, X_k \rangle}$.

Definition 5.1. For $k = 0, 1, 2, \dots$ write $\lambda_k := \frac{\phi\left(\frac{y_k - \langle \mathbf{c}, X_k \rangle}{\langle \mathbf{d}, X_k \rangle}\right)}{\langle \mathbf{d}, X_k \rangle \phi(y_k)}$,

where $\phi(x)$ is the $N(0, 1)$ density $\frac{1}{\sqrt{2\pi}} \exp -\frac{1}{2}x^2$, and

$$\Lambda_{0,k} := \prod_{\ell=0}^k \lambda_\ell. \quad (5.1)$$

Recall $\mathcal{F}_k = \sigma\{X_0, X_1, \dots, X_k\}$ and write $\mathcal{Y}_k = \sigma\{y_0, y_1, \dots, y_k\}$ and $\mathcal{G}_k = \sigma\{X_0, \dots, X_k, y_0, \dots, y_k\}$. We consider the related filtrations $\{\mathcal{F}_k\}$, $\{\mathcal{Y}_k\}$ and $\{\mathcal{G}_k\}$.

Definition 5.2. The original ‘real world’ probability P is defined in terms of \bar{P} by setting

$$\frac{dP}{d\bar{P}} \Big|_{\mathcal{G}_k} = \Lambda_{0,k}.$$

We can then prove

Lemma 5.3. *Under P X is a semi-Markov chain with dynamics $X_{k+1} = A(h(k))X_k + M_{k+1}$ and $\{w_k, k = 0, 1, \dots\}$ is a sequence of i.i.d. $N(0, 1)$ random variables where $w_k = (y_k - \langle \mathbf{c}, X_k \rangle) / \langle \mathbf{d}, X_k \rangle$.*

That is, under P $y_k = \langle \mathbf{c}, X_k \rangle + \langle \mathbf{d}, X_k \rangle w_k$.

Proof. For a proof see [4]. □

Recall from §3.1, that the chain \bar{X} has dynamics $\bar{X}_{k+1} = C\bar{X}_k + \bar{M}_{k+1} \in \bar{S}$. We suppose, as in §4, that the observation process is $y_k = c(X_k) + d(X_k)w_k$, where $X_k = T^* \bar{X}_k$. As above

$$\lambda_k = \frac{\phi\left(\frac{y_k - \langle \mathbf{c}, X_k \rangle}{\langle \mathbf{d}, X_k \rangle}\right)}{\langle \mathbf{d}, X_k \rangle \phi(y_k)}.$$

Write $\lambda_k^i = \frac{\phi\left(\frac{y_k - c_i}{d_i}\right)}{d_i \phi(y_k)}$ and $\Lambda_{0,k} = \prod_{\ell=0}^k \lambda_\ell$. However, for any $n = 1, 2, \dots$, $T^* \mathbf{e}_{i,n} = \mathbf{e}_i$ so, for example, $\langle \mathbf{c}, \mathbf{e}_i \rangle = c_i = \langle \mathbf{c}, T^* \mathbf{e}_{i,n} \rangle = \langle T\mathbf{c}, \mathbf{e}_{i,n} \rangle$ and λ_k can be written in terms of the full state \bar{X}_k :

$$\lambda_k = \frac{\phi\left(\frac{y_k - \langle T\mathbf{c}, \bar{X}_k \rangle}{\langle T\mathbf{d}, \bar{X}_k \rangle}\right)}{\langle T\mathbf{d}, \bar{X}_k \rangle \phi(y_k)}.$$

5.2. A Finite-dimensional recursive filter for X . Write $\gamma_k = \bar{E}[\Lambda_k \bar{X}_k | \mathcal{Y}_k]$ for the unnormalized conditional expected value of \bar{X}_k given the observations \mathcal{Y}_k to time k . Again suppose $N = 3$, write $\Gamma_3(y_{k+1}) := \text{diag}\{\lambda_{k+1}^1(y_{k+1}), \lambda_{k+1}^2(y_{k+1}), \lambda_{k+1}^3(y_{k+1})\}$ and $\Gamma(y_{k+1}) := \text{diag}\{\Gamma_3(y_{k+1}), \Gamma_3(y_{k+1}), \Gamma_3(y_{k+1}), \dots\}$. We then have the recursion.

Theorem 5.4. $\gamma_{k+1} = \Gamma(y_{k+1})C\gamma_k$ with γ_0 given by X_0 , or its probability distribution.

Proof.

$$\begin{aligned} \gamma_{k+1} &= \bar{E}[\Lambda_{k+1} \bar{X}_{k+1} | \mathcal{Y}_{k+1}] \\ &= \sum_{n=1}^{\infty} \sum_{i=1}^N \bar{E}[\Lambda_k \lambda_{k+1}^i \langle \bar{X}_{k+1}, \mathbf{e}_{i,n} \rangle | \mathcal{Y}_{k+1}] \mathbf{e}_{i,n} \\ &= (y_{k+1}) \sum_{n=1}^{\infty} \sum_{i=1}^N \lambda_{k+1}^i(y_{k+1}) \bar{E}[\Lambda_k \langle C \bar{X}_k, \mathbf{e}_{i,n} \rangle | \mathcal{Y}_k] \mathbf{e}_{i,n} \\ &= \sum_{n=1}^{\infty} \sum_{i=1}^N \lambda_{k+1}^i(y_{k+1}) \langle C \gamma_k, \mathbf{e}_{i,n} \rangle \mathbf{e}_{i,n} = \Gamma(y_{k+1})C\gamma_k. \end{aligned}$$

□

Remark 5.5. As noted above, at any finite time, or if the sojourn distributions have finite support, the matrices C are of finite dimension.

6. Parameter Estimation

Recall that with $a_{i,i}(m) = 1 - \Delta^i(m)$, $a_{j,i}(m) = p_{j,i}(m)\Delta^i(m)$ and $N = 3$, then $D(m) = \text{diag}\{a_{1,1}(m), a_{2,2}(m), a_{3,3}(d)\}$, $\Pi(m) = \mathbf{0} + \sum_{\substack{i,j \in \mathcal{M} \\ i \neq j}} a_{i,j}(m)$ and

$$C = \begin{bmatrix} \Pi(1) & \Pi(2) & \Pi(3) & \dots \\ D(1) & 0 & 0 & \dots \\ 0 & D(2) & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The dynamics of the chain $\bar{X}_k \in \bar{S}$ are given by $\bar{X}_{k+1} = C\bar{X}_k + \bar{M}_{k+1} \in \bar{S}$. With $X_k := T^* \bar{X}_k$ the observation process is given by

$$y_k = \langle \mathbf{c}, X_k \rangle + \langle \mathbf{d}, X_k \rangle w_k$$

and for some $\mathbf{e}_{i,n}$, this is

$$\begin{aligned} y_k &= \langle \mathbf{c}, T^* \mathbf{e}_{i,n} \rangle + \langle \mathbf{d}, T^* \mathbf{e}_{i,n} \rangle w_k \\ &= \langle T\mathbf{c}, \mathbf{e}_{i,n} \rangle + \langle T\mathbf{d}, \mathbf{e}_{i,n} \rangle w_k, \quad \text{for } i \in \{1, 2, \dots, N\} \text{ and } n \in \{1, 2, \dots\}. \end{aligned}$$

We wish to estimate the parameters of the model, that is the $\mathbf{c}, \mathbf{d} \in \mathbb{R}^N$ and the $a_{j,i}(k)$, $1 \leq i, j \leq N$, $i \neq j$. Note we need only estimate the off-diagonal elements $a_{j,i}(k)$ of the matrices $\Pi(k)$.

Now $N_k^{j,i}(m) = \sum_{\ell=1}^k \langle \bar{X}_{\ell-1}, \mathbf{e}_{i,m} \rangle \langle \bar{X}_{\ell}, \mathbf{e}_{j,i} \rangle$ gives the number of jumps from state $\mathbf{e}_{i,m}$ to state $\mathbf{e}_{j,i}$ up to time k . $J_k^i(m) = \sum_{\ell=1}^k \langle \bar{X}_{\ell-1}, \mathbf{e}_{i,m} \rangle$ gives the amount of time spent in state $\mathbf{e}_{i,m}$ up to time k . We also need estimates for sums of the form

$$G_k^i = \sum_{\ell=1}^k f(y_\ell) \langle X_{\ell-1}, \mathbf{e}_i \rangle = \sum_{\ell=1}^k \sum_{m=1}^l f(y_\ell) \langle X_{\ell-1}, T^* \mathbf{e}_{i,m} \rangle.$$

Here the function $f(\cdot)$ is any bounded mapping. As in [6] we first consider the un-normalized vector estimate $\sigma(N_k^{j,i}(m)\bar{X}_k) := \bar{E} [\Lambda_k N_k^{j,i}(m) \bar{X}_k \mid \mathcal{Y}_k]$, A recursion for this quantity is given by:

Lemma 6.1.

$$\sigma(N_k^{j,i}(m)\bar{X}_k) = \Gamma(y_{k+1})C\sigma(N_k^{j,i}(m)X_k) + a_{j,i}(m) \langle \gamma_k, \mathbf{e}_{i,m} \rangle \mathbf{e}_{j,i}$$

where γ_k is determined by Theorem 5.4.

Proof. Suppose γ_k and $\sigma(N_k^{j,i}(m)X_k)$ have been determined. Then

$$\begin{aligned} & \overline{E} \left[\Lambda_{k+1} N_{k+1}^{j,i}(m) \overline{X}_{k+1} \mid \mathcal{Y}_{k+1} \right] \\ &= \overline{E} \left[\Lambda_k \lambda_{k+1} (N_k^{j,i}(n) + \langle \overline{X}_{k+1}, \mathbf{e}_{j,i} \rangle \langle \overline{X}_k, \mathbf{e}_{i,m} \rangle) \overline{X}_{k+1} \mid \mathcal{Y}_{k+1} \right] \\ & \text{(similarly to Theorem 5.4 this is)} \\ &= \Gamma(y_{k+1}) \sum_{p,q=1}^m \overline{E} \left[\Lambda_k N_k^{j,i}(m) \langle C \overline{X}_k, \mathbf{e}_{p,q} \rangle \mid \mathcal{Y}_{k+1} \right] \mathbf{e}_{p,q} + \\ & \overline{E} \left[\Lambda_k \lambda_{k+1} \langle \overline{X}_{k+1}, \mathbf{e}_{j,i} \rangle \langle \overline{X}_k, \mathbf{e}_{i,m} \rangle \mid \mathcal{Y}_{k+1} \right] \mathbf{e}_j. \end{aligned}$$

The result follows. \square

Similarly we can establish:

Lemma 6.2. *With $\sigma(J_k^i(m) \overline{X}_k) = \overline{E} [\Lambda_k J_k^i(m) \overline{X}_k \mid \mathcal{Y}_k]$*

$$\sigma(J_{k+1}^i(m) \overline{X}_{k+1}) = \Gamma(y_{k+1}) C \sigma(J_k^i(m) \overline{X}_k) + \langle \gamma_k, \mathbf{e}_{i,m} \rangle \Gamma(y_{k+1}) C \mathbf{e}_{i,m}.$$

Proof.

$$\begin{aligned} & \overline{E} [\Lambda_{k+1} J_{k+1}^i(m) \overline{X}_{k+1} \mid \mathcal{Y}_{k+1}] \\ &= \overline{E} [\Lambda_k \lambda_{k+1} (J_k^i(m) + \langle \overline{X}_k, \mathbf{e}_{i,m} \rangle) \overline{X}_{k+1} \mid \mathcal{Y}_{k+1}] \\ &= \Gamma(y_{k+1}) C \sigma(J_k^i(m) \overline{X}_k) + \langle \gamma_k, \mathbf{e}_{i,m} \rangle \Gamma(y_{k+1}) C \mathbf{e}_{i,m}. \end{aligned}$$

\square

In general, with $\sigma(G_k^i \overline{X}_k) = \overline{E} [\Lambda_k G_k^i \overline{X}_k \mid \mathcal{Y}_k]$ we have

Lemma 6.3.

$$\sigma(G_{k+1}^i \overline{X}_{k+1}) = \Gamma(y_{k+1}) C \sigma(G_k^i \overline{X}_k) + f(y_{k+1}) \sum_{m=1}^k \langle \gamma_k, \mathbf{e}_{i,m} \rangle C \mathbf{e}_{i,m}.$$

Remark 6.4. Now $\langle \overline{X}_k, \mathbf{1} \rangle = 1$ for all k , where $\mathbf{1}$ is an infinite column vector of 1s. Therefore, for example, $\langle \sigma(N_k^{j,i}(m) \overline{X}_k), \mathbf{1} \rangle$ gives an unnormalized estimate for $\sigma(N_k^{j,i}(m))$.

In turn these provide estimates such as $\hat{a}_{j,i}(m) = \frac{\sigma(N_k^{j,i}(m))}{\sigma(J_k^i(m))}$ for $i \neq j$, and for the other parameters of the model as in [4].

7. Smoothers

Suppose $0 \leq k \leq T$ and we have observed $\{y_0, y_1, \dots, y_T\}$. We wish to find $E[X_k \mid \mathcal{Y}_T]$. Write $\Lambda_{k+1,T} = \prod_{\ell=k+1}^T \lambda_\ell$. Using Bayes' theorem again, (see [4]) and the reference measure of §5.1.

$$E[\overline{X}_k \mid \mathcal{Y}_T] = \frac{\overline{E}[\Lambda_{0,T} \overline{X}_k \mid \mathcal{Y}_T]}{\overline{E}[\Lambda_{0,T} \mid \mathcal{Y}_T]}.$$

Now $\Lambda_{0,t} = \Lambda_{0,k} \Lambda_{k+1,T}$ and $\bar{E} [\Lambda_{0,T} \bar{X}_k | \mathcal{Y}_T] = \bar{E} [\Lambda_{0,k} X_k \bar{E} [\Lambda_{k+1,T} | \mathcal{Y}_T, \mathcal{F}_k] \mathcal{Y}_T]$. However, \bar{X} is Markov so $\bar{E} [\Lambda_{k+1,T} | \mathcal{Y}_T, \mathcal{F}_k] = \bar{E} [\Lambda_{k+1,T} | \mathcal{Y}_T, \bar{X}_k]$.

Definition 7.1. For $1 \leq i \leq N$ and $n \in \{1, 2, \dots, k+1\}$ write $v_{k,T}^i(n) = 1$ and $v_{k,T}^i(n) = \bar{E} [\Lambda_{k+1,T} | \mathcal{Y}_T, \bar{X}_k = \mathbf{e}_{i,n}]$. Set

$$v_{k,T} := \left(v_{k,T}^1(1) \dots v_{k,T}^N(1) \mid v_{k,T}^1(2) \dots v_{k,T}^N(2) \mid \times \dots \mid v^1(k+1) \dots v_{k,T}^N(k+1) \right).$$

Theorem 7.2. *The process v satisfies the backward dynamics*

$$v_{k,T} = C^* \Gamma(y_{k+1}) v_{k,T}, \text{ with } v_{T,T} = (1, 1, \dots, 1)' \in \mathbb{R}^{(k+1)N}. \quad (7.1)$$

Proof. For $\mathbf{e}_{i,n} \in \bar{S}$ consider

$$\begin{aligned} \langle v_{k,T}, \mathbf{e}_{i,n} \rangle &= v_{k,T}^i(n) \\ &= \bar{E} \left[\Lambda_{k+2,T}, \lambda_{k+1} \mid \mathcal{Y}_T, \bar{X}_k = \mathbf{e}_{i,n} \right] \\ &= \sum_{m=1}^{k+1} \sum_{i=1}^N \bar{E} \left[\langle \bar{X}_{k+1}, \mathbf{e}_{j,m} \rangle \Lambda_{k+2,T} \mid \mathcal{Y}_T, \bar{X}_k = \mathbf{e}_{i,n} \right] \lambda_{k+1}^j(y_{k+1}) \\ &= \sum_{m=1}^{k+1} \sum_{i=1}^N \bar{E} \left[\langle \bar{X}_{k+1}, \mathbf{e}_{j,m} \rangle \bar{E} [\Lambda_{k+2,T} \mid \mathcal{Y}_T, \bar{X}_{k+1} = \mathbf{e}_{j,m}, \bar{X}_k = \mathbf{e}_{i,n}] \mid \right. \\ &\quad \left. \mathcal{Y}_T, \bar{X}_k = \mathbf{e}_{i,n} \right] \lambda_{k+1}^j(y_{k+1}) \\ &= \sum_{m=1}^{k+1} \sum_{i=1}^N \bar{E} \left[\langle \bar{X}_{k+1}, \mathbf{e}_{j,m} \rangle \langle v_{k+1,T}, \mathbf{e}_{j,m} \rangle \mid \mathcal{Y}_T, \bar{X}_k = \mathbf{e}_{i,n} \right] \lambda_{k+1}^j(y_{k+1}) \\ &= \sum_{m=1}^{k+1} \sum_{i=1}^N C_{jm,in}(k) \langle v_{k+1,T}, \mathbf{e}_{j,m} \rangle \lambda_{k+1}^j(y_{k+1}). \end{aligned}$$

and the result follows. \square

Theorem 7.3. *An unnormalized smoothed estimate for \bar{X}_k given observations $\{y_0, y_1, \dots, y_T\}$ is*

$$q_{k,T} = \bar{E} [\Lambda_{0,T} \bar{X}_k | \mathcal{Y}_T] = \text{diag } \lambda_k \cdot v_{k,T}.$$

Proof.

$$\begin{aligned} \bar{E} [\Lambda_{0,T} \langle \bar{X}_k, \mathbf{e}_{i,n} \rangle | \mathcal{Y}_T] &= \bar{E} \left[\Lambda_{0,k} \langle \bar{X}_k, \mathbf{e}_{i,n} \rangle \bar{E} [\Lambda_{k+1,T} | \mathcal{Y}_T, \bar{X}_k = \mathbf{e}_{i,n}] \mathcal{Y}_T \right] \\ &= \langle \gamma_k, \mathbf{e}_{i,n} \rangle \langle v_{k,T}, \mathbf{e}_{i,n} \rangle. \end{aligned}$$

Therefore

$$\bar{E} [\Lambda_{0,T} \bar{X}_k | \mathcal{Y}_T] = \sum_{m=1}^{k+1} \sum_{i=1}^N \langle \gamma_k, \mathbf{e}_{i,n} \rangle \langle v_{k,T}, \mathbf{e}_{i,n} \rangle \mathbf{e}_{i,n} = \text{diag } \gamma_k \cdot v_{k,T}.$$

\square

8. Simulation Study

8.1. Example Stochastic Dynamics. The indirectly observed X -process we consider is a three-state semi Markov chain with distinct classes for its sojourn distributions. The distributions for the states e_1 , e_2 and e_3 are, respectively, a Dirac distribution with full mass on 5, a finite distribution on the natural numbers $\{2, 3, 5\}$ with corresponding probabilities $\{0.3, 0.5, 0.2\}$ and a geometric distribution with parameters 0.35. The (column stochastic) transition matrix used to simulate an embedded Markov chain, (from which we construct a semi Markov

chain realisation) has the form $\begin{bmatrix} 0 & 3/10 & 1/4 \\ 1/3 & 0 & 3/4 \\ 2/3 & 7/10 & 0 \end{bmatrix}$. The initial distribution used

for X_0 was uniform across the state space. The $\Delta^i(h)$ probabilities for the model we describe here are listed in Table 1. Given that the sojourns distributions for

TABLE 1. End-of-Sojourn Probabilities

Sojourn :	h=1	h=2	h=3	h=4	h=5	h=6	...
$\Delta^1(h)$	0	0	0	0	1	0	...
$\Delta^2(h)$	0	0.3	0.71	0	1	0	...
$\Delta^3(h)$	0.35	0.35	0.35	0.35	0.35	0.35	...

the states e_1 and e_2 are both finite, the corresponding support for $\Delta^1(h)$ and $\Delta^2(h)$ is also finite. However, the state e_3 has a geometric sojourn, this means $\Delta^3(h)$ is constant on \mathbb{N} . The observation dynamics used were given above by equation (4.1), with parameter values $c(e_1) = -1$, $c(e_2) = 1$ and $c(e_3) = 2$, each determined by the state of X at time k , and $d(e_1) = 1.2$, $d(e_2) = 0.4$ and $d(e_3) = 0.2$, also determined by the state of X at time k . The inclusion of one or more geometric sojourns in a semi Markov model (or indeed any other candidate sojourn distribution defined on \mathbb{N}) means that the matrix C defined at (3.1) will be an infinite matrix. Consequently a suitable truncation of the matrix C must be used. For the simulation study we assumed that the maximum realized state sojourn (for the geometric distribution) was no more $h_{\text{Max}} = 50$, (for a geometric distribution parameter of 0.35, the event that $h > 50$ has measure approximately equal to $4.42250e-10$). Consequently our C matrix had dimensions 150×150 .

8.2. Results. The recursive filter given in Theorem 5.4 generates a sequence of unnormalized probabilities distributions $\{\gamma_\ell\}_{\ell \geq 0}$. These unnormalized probabilities are joint distributions for the random variables $X \in \{e_1, e_2, e_3\}$ and $h \in \{1, 2, \dots, h_{\text{Max}}\}$. The corresponding normalized estimated densities for X and for h are easily recovered from the normalized version of γ_k by marginalisation. In our example we compute Maximum a Posteriori (MAP) estimates from these densities. In Figure 1 we show a realization of the observed process $\{y_\ell\}_{\ell \geq 0}$, the partially observed semi Markov process X and the filtered estimate of X . For clarity, the independent variable on these plots is marked only at the embedded

chain transition times, τ_0, τ_1, \dots . Similarly, in Figure 2, we plot the exact h -process and the MAP estimates of h marginalized from each γ_k . Comparing the estimates of X in Figure 1 with the estimates of h in Figure 2, we can see that the expected dependence between X and h is clear, as errors in these estimators appear in the same time regions, for example $k \in \{7, 8, 9, 10\}$ and in $k \in \{28, 29, 30\}$. It is encouraging that at the times following these regions the filter has recovered.

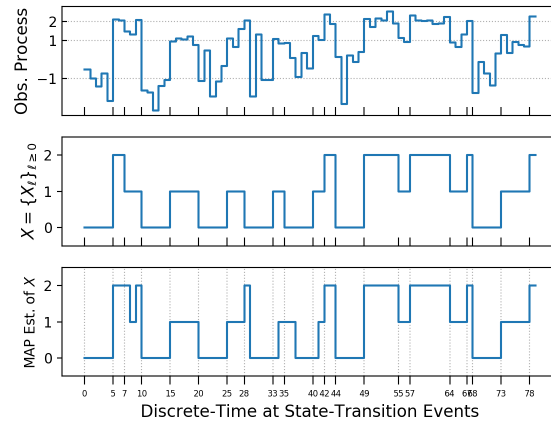


FIGURE 1. The uppermost plot shows the observation process. The middle plot shows the exact X process. The bottom plot is the filtered estimate of X .

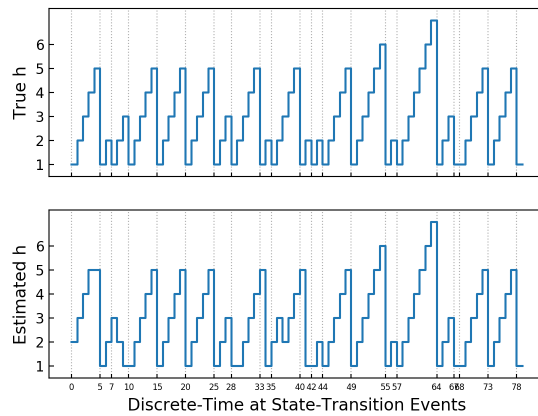


FIGURE 2. The uppermost plot is the exact h -process. The bottom plot in this figure is the filtered estimate of h .

9. An Exact Filter for the State and State-Sojourn Time

In this section we provide a second, direct, derivation for the recursive filter of Theorem 5.4. Given the semi-Markov chain X and the observation process y of Section 4 we wish to obtain joint conditional estimates of X_k and h_k given $\mathcal{Y}_k = \sigma\{y_0, y_1, \dots, y_k\}$. Suppose $F : \{1, 2, \dots\} \rightarrow \mathbb{R}$ is an arbitrary function. We consider $E[\langle X_k, \mathbf{e}_i \rangle F(h_k^i) \mid \mathcal{Y}_k]$ for any $i \in \{1, 2, \dots, N\}$. We wish to find $E[\langle X_k, \mathbf{e}_i \rangle F(h_k^i) \mid \mathcal{Y}_k]$. Using the Bayes' rule of [4], this equals $\frac{\overline{E}[\Lambda_k \langle X_k, \mathbf{e}_i \rangle F(h_k^i) \mid \mathcal{Y}_k]}{\overline{E}[\Lambda_k \mid \mathcal{Y}_k]}$. The denominator here is derived from the numerator by taking $F = 1$ and summing over i .

Notation 9.1. Suppose there are unnormalized probabilities $\gamma_k^i(n)$ such that

$$\overline{E}[\Lambda_k \langle X_k, \mathbf{e}_i \rangle F(h_k^i) \mid \mathcal{Y}_k] = \sum_{n=1}^{\infty} F(n) \gamma_k^i(n).$$

However, as noted in Section 3.3, $h_k^i \leq k+1$ so $\gamma_k^i(n) = 0$ for $n > k+1$ and the sum here is only up to $n = k+1$. As in section 5 write $\lambda_k^i(y_k) = \frac{\phi((y_k - c_i)/d_i)}{d_i \phi(y_k)}$.

We shall obtain the following recursions for the γ

Theorem 9.2. For $n = 1$

$$\gamma_k^i(1) = \lambda_k^i(y_k) \sum_{\substack{j=1 \\ j \neq i}}^N \sum_{n=1}^{k+1} a_{i,j}(n) \gamma_{k-1}^j(n). \quad (9.1)$$

For $1 < n \leq k+1$

$$\gamma_k^i(n) = \lambda_k^i(y_k) a_{i,i}(n-1) \gamma_{k-1}^i(n-1). \quad (9.2)$$

Proof. Suppose $i \in \{1, 2, \dots, N\}$ and $F : \{1, 2, \dots\} \rightarrow \mathbb{R}$ is an arbitrary function so, as above,

$$\begin{aligned} \overline{E}[\Lambda_k \langle X_k, \mathbf{e}_i \rangle F(h_k^i) \mid \mathcal{Y}_k] &= \sum_{n=1}^{k+1} F(n) \gamma_k^i(n) \\ &= \overline{E}[\Lambda_{k-1} \lambda_k \langle X_k, \mathbf{e}_i \rangle F(\langle X_k, \mathbf{e}_i \rangle \langle X_{k-1}, \mathbf{e}_i \rangle h_{k-1}^i) \mid \mathcal{Y}_k] \\ &= \lambda_k^i(y_k) \overline{E}[\Lambda_{k-1} \langle X_k, \mathbf{e}_i \rangle F(1 + \langle X_{k-1}, \mathbf{e}_i \rangle h_{k-1}^i) \mid \mathcal{Y}_{k-1}] \\ &= \lambda_k^i(y_k) \sum_{j=1}^N \overline{E}[\Lambda_{k-1} \langle X_{k-1}, \mathbf{e}_j \rangle \langle X_k, \mathbf{e}_i \rangle F(1 + \langle X_{k-1}, \mathbf{e}_i \rangle h_{k-1}^i) \mid \mathcal{Y}_{k-1}] \\ &= \lambda_k^i(y_k) \sum_{\substack{j=1 \\ j \neq i}}^N \overline{E}[\Lambda_{k-1} \langle X_{k-1}, \mathbf{e}_j \rangle \langle X_k, \mathbf{e}_i \rangle F(1) \mid \mathcal{Y}_{k-1}] \\ &\quad + \lambda_k^i(y_k) \overline{E}[\Lambda_{k-1} \langle X_{k-1}, \mathbf{e}_i \rangle \langle X_k, \mathbf{e}_i \rangle F(1 + h_{k-1}^i) \mid \mathcal{Y}_{k-1}] \end{aligned}$$

$$\begin{aligned}
&= \lambda_k^i(y_k) \sum_{\substack{j=1 \\ j \neq i}}^N \overline{E}[\Lambda_{k-1} \langle X_{k-1}, \mathbf{e}_j \rangle \langle A(h_{k-1}^j) X_{k-1}, \mathbf{e}_i \rangle \times \\
&\quad F(1) \mid \mathcal{Y}_{k-1}] + \lambda_k^i(y_k) \overline{E}[\Lambda_{k-1} \langle X_{k-1}, \mathbf{e}_i \rangle \times \\
&\quad \langle A(h_{k-1}^i) X_{k-1}, \mathbf{e}_i \rangle F(1 + h_{k-1}^i) \mid \mathcal{Y}_k] \\
&= \lambda_k^i(y_k) \sum_{\substack{j=1 \\ j \neq i}}^N F(1) \overline{E}[\Lambda_{k-1} \langle X_{k-1}, \mathbf{e}_j \rangle a_{ij}(h_{k-1}^j) \mid \mathcal{Y}_{k-1}] \\
&\quad + \lambda_k^i(y_k) \overline{E}[\Lambda_{k-1} \langle X_{k-1}, \mathbf{e}_i \rangle a_{i,i}(h_{k-1}^i) F(1 + h_{k-1}^i) \mid \mathcal{Y}_{k-1}] \\
&= \lambda_k^i(y_k) F(1) \sum_{\substack{j=1 \\ j \neq i}}^n \overline{E}[\Lambda_{k-1} \langle X_{k-1}, \mathbf{e}_j \rangle a_{i,j}(h_{k-1}^j) \mid \mathcal{Y}_{k-1}] \\
&\quad + \lambda_k^i(y_k) \overline{E}[\Lambda_{k-1} \langle X_{k-1}, \mathbf{e}_i \rangle a_{i,i}(h_{k-1}^i) F(1 + h_{k-1}^i) \mid \mathcal{Y}_{k-1}] \\
&= \lambda_k^i(y_k) F(1) \sum_{\substack{j=1 \\ j \neq i}}^N \sum_{n=1}^{k+1} a_{i,j}(n) \gamma_{k-1}^j(n) + \lambda_k^i(y_k) \sum_{n=1}^{k+1} a_{ii}(n) F(1+n) \gamma_{k-1}^i(n) \\
&= \lambda_k^i(y_k) F(1) \sum_{\substack{j=1 \\ j \neq i}}^N \sum_{n=1}^{k+1} a_{i,j}(n) \gamma_{k-1}^j(n) + \lambda_k^i(y_k) \sum_{m=2}^{k+1} a_{i,i}(m-1) F(m) \gamma_{k-1}^i(m-1) \\
&= \lambda_k^i(y_k) F(1) \sum_{\substack{j=1 \\ j \neq i}}^N a_{i,j}(n) \gamma_{k-1}^j(n) + \lambda_k^i(y_k) \sum_{n=2}^{k+1} a_{i,i}(n-1) F(n) \gamma_{k-1}^i(n-1).
\end{aligned} \tag{9.3}$$

Now F is an arbitrary function $F : \{1, 2, \dots\} \rightarrow \mathbb{R}$. Consider an F such that $F(1) = 1$ and $F(n) = 0$ if $n \neq 1$. Then from (9.3)

$$\gamma_k^i(1) = \lambda_k^i(y_k) \sum_{\substack{j=1 \\ j \neq i}}^N \sum_{n=1}^{k+1} a_{i,j}(n) \gamma_{k-1}^j(n).$$

This is the recursion for $\gamma_k^i(1)$, the unnormalized conditional probability given \mathcal{Y}_k that at time k $h_k^i(X_k) = 1$ and $x_k = \mathbf{e}$. Now consider another F which is such that $F(m) = 1$ for some $m > 1$ and $F(m) = 0$ otherwise. Then from (A.3) and (5.4):

$$\gamma_k^i(m) = \lambda_k^i(y_k) a_{i,i}(m-1) \gamma_{k-1}^i(m-1).$$

This is the recursion for $\gamma_k^i(k-1)$, the unnormalized conditional probability given \mathcal{Y}_k that, at time k , $h_k^i = m$ and $X_k = \mathbf{e}_i$. This provides a coordinate-wise version of Theorem 5.4. \square

Remark 9.3. Note that, as in the earlier results, the recursions only involve finite sums.

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ROBERT J. ELLIOTT: UNISA BUSINESS, UNIVERSITY OF SOUTH AUSTRALIA, CAMPUS CENTRAL - CITY WEST, GPO BOX 2471 AND UNIVERSITY OF CALGARY, CALGARY T2N 1 N4, CANADA
E-mail address: Robert.Elliott@unisa.edu.au and relliott@ucalgary.ca
URL: <https://people.unisa.edu.au/Robert.Elliott> and <http://www.ucalgary.ca/~relliott>

W. P. MALCOLM: DEFENCE SCIENCE AND TECHNOLOGY GROUP AUSTRALIA, RUSSELL OFFICES CANBERRA, CONSTITUTION AVE, RUSSELL ACT 2600, AUSTRALIA
E-mail address: Paul.Malcolm2@defence.gov.au