Journal of Stochastic Analysis

Volume 2 | Number 1 Article 6

February 2021

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Journal of Stochastic Analysis
Vol. 2, No. 1 (2021) Article 6 (9 pages)
DOI: 10.31390/josa.2.1.06

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LINEAR DECOMPOSITION AND ANTICIPATING INTEGRAL FOR CERTAIN RANDOM VARIABLES

CHING-TANG WU AND JU-YI YEN*

ABSTRACT. In this paper, we construct an anticipating stochastic integral by linearly decompose a class of non $\mathcal{F}_t$-measurable random variables. The result is applied to the derivation of the Itô formula.

1. Introduction

Throughout the paper, B_t denotes the one-dimensional standard Brownian motion and $\mathcal{F}_t$ is the σ-field generated by $\{B_s; 0 \le s \le t\}$.

The Itô stochastic integral:

$$\int_0^\infty f(t)dB_t \tag{1.1}$$

is established for any $\mathcal{F}_t$-adapted function $f(t)$ such that $\int_0^\infty |f(t)|^2 dt < \infty$. A well-known example to extend the scope of the Itô integral was given in 1976 by Itô as follows:

$$\int_0^t B_1 dB_s, \quad 0 \le t \le 1. \tag{1.2}$$

Here, for $0 \le t < 1$, B_1 is not adapted to $\mathcal{F}_t$. The concept of the *enlargement of filtration* was then introduced to solve this problem; the basic idea is to build a σ-field $\mathcal{G}_t$ generated by $\mathcal{F}_t$ and B_1, in a way that the Brownian motion B_t is a semimartingale with respect to $\mathcal{G}_t$. Thus, (1.2) can be defined in the sense of the original Itô stochastic integral. However, the enlargement of filtration cannot be easily generalized, since for different enlarged filtrations, B_t may no longer be a semimartingale. See e.g. [4, 5, 6] for detailed studies in this direction.

About a decade ago, an alternate technique was introduced by Ayed and Kuo [1, 2] in order to solve stochastic integrals with non-adapted integrands such as (1.2). Specifically, Ayed and Kuo decompose the integrand into two components, i.e., the adapted parts and the anticipating part. In this work, we follow the method developed in [1, 2] to define the anticipating stochastic integral

$$(G)\int_0^t f(L)dB_s, \tag{1.3}$$

Received 2020-7-2; Accepted 2021-2-12; Communicated by the editors.
2010 *Mathematics Subject Classification.* Primary 60G20, 60H05; Secondary 60G48.
Key words and phrases. Brownian motion, adapted stochastic process, Itô Integral, instantly independent stochastic process, anticipating integral.

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where L is not necessarily $\mathcal{F}_t$-measurable, and f is a C^2-function with bounded second derivative.

This paper is organized as follows. In Section 2, we first introduce some necessary notions for our analysis. We then show that if the integrand in (1.3) can be decomposed into two parts as in [1, 2] (namely, the adapted parts and the anticipating part), then such decomposition is unique with certain properties (to be specified in Section 2). The stochastic integral (1.3) is properly defined in Section 3. In particular, we use the left-end point to compute the adapted part, and the right-end point for the anticipating part. In Section 4, we give the Itô formula for (1.3) to complete our study.

2. Decomposition of L

Before we define the stochastic integral (1.3), we shall first introduce two terminologies: *instant independence* and *near-martingale*. Ayed and Kuo [1] introduced the concept of instantly independent for certain anticipating stochastic processes. We recall the definition of *instant independence*.

Definition 2.1. A stochastic process $(Y_t)_{a\le t\le b}$, is said to be *instantly independent* with respect to the filtration $(\mathcal{F}_t)$, if $\sigma(Y_t)$ and $\mathcal{F}_t$ are independent for each t.

Moreover, in [8], the notion of *near-martingale* is defined:

Definition 2.2. Let $\mathbb{E}|X_t| < \infty$ for all t. We say that a stochastic process (X_t) is a *near-martingale* with respect to the filtration $(\mathcal{F}_t)$ if

$$\mathbb{E}[X_t - X_s|\mathcal{F}_s] = 0, \quad \forall s \le t.$$

Now, we decompose a random variable $\bar{L}$ as

$$\bar{L} = C_t + I_t,$$

where C_t takes the class of instantly independent stochastic process (called the counterpart), and I_t is an $(\mathcal{F}_t)$-adapted process (called the Itô part). Lemma 2.3 below shows the uniqueness of such decomposition if such decomposition exists.

Lemma 2.3. *Given a filtration $(\mathcal{F}_t)$, if a random variable $\bar{L}$ can be decomposed into*

$$\bar{L} = C_t + I_t \tag{2.1}$$

for all t, where (C_t) takes the class of instantly independent stochastic process with $\mathbb{E}[C_t] = 0$ for all t; and (I_t) is an $(\mathcal{F}_t)$-adapted process, then the linear decomposition in (2.1) is unique. Moreover, (I_t) is a martingale, and (C_t) is a near-martingale with respect to $(\mathcal{F}_t)$.

Proof. We first show the uniqueness of the decomposition. Suppose $\bar{L}$ has two decompositions:

$$\bar{L} = C_t + I_t = D_t + J_t,$$

where C_t and D_t denote the counterpart, and I_t and J_t denote the Itô part of the random variable $\bar{L}$, respectively. Then

$$(C_t - D_t) + (I_t - J_t) = 0.$$

Taking the conditional expectation with respect to $\mathcal{F}_t$ on both sides, we get

$$\mathbb{E}[(C_t - D_t)|\mathcal{F}_t] + \mathbb{E}[(I_t - J_t)|\mathcal{F}_t] = 0.$$

Since $I_t - J_t$ are $\mathcal{F}_t$-measurable, we have

$$\mathbb{E}[(C_t - D_t)|\mathcal{F}_t] + (I_t - J_t) = 0.$$

Moreover, since C_t and D_t are instantly independent with respect to $\mathcal{F}_t$ and $\mathbb{E}[C_t] = \mathbb{E}[D_t] = 0$, we get

$$\mathbb{E}[(C_t - D_t)|\mathcal{F}_t] = \mathbb{E}[C_t - D_t] = \mathbb{E}[C_t] - \mathbb{E}[D_t] = 0$$

Thus, $(I_t - J_t) = 0$. This implies that $I_t = J_t$ and $C_t = D_t$ for all t.

Furthermore, since C_t is instantly independent with respect to $(\mathcal{F}_t)$, C_t is independent of $\mathcal{F}_s$ for all $s \leq t$. Thus,

$$\mathbb{E}[C_t - C_s|\mathcal{F}_s] = \mathbb{E}[C_t - C_s] = 0.$$

We have (C_t) is a near-martingale. Moreover, due to $I_t - I_s = C_s - C_t$ for $s < t$,

$$\mathbb{E}[I_t - I_s|\mathcal{F}_s] = \mathbb{E}[C_s - C_t|\mathcal{F}_s] = \mathbb{E}[C_s - C_t] = 0, \qquad s < t.$$

Since (I_t) is adapted, we get clearly that $\mathbb{E}[I_t|\mathcal{F}_s] = I_s$, i.e., (I_t) is a martingale. □

Example 2.4. Clearly, the decomposition in (2.1) exists if $(\bar{L}, B_s)_{0\leq s\leq t}$ is Gaussian. In particular, we can write:

$$\begin{aligned} C_t &= \bar{L} - \mathbb{E}[\bar{L}|\mathcal{F}_t] \\ I_t &= \mathbb{E}[\bar{L}|\mathcal{F}_t]. \end{aligned}$$

Under this assumption, C_t and $\mathcal{F}_t$ are independent since $\mathbb{E}[C_t B_s] = 0$ for all $s \leq t$.

(a) Let $\bar{L} = B_1$, then $C_t = B_1 - B_t$, and $I_t = B_t$, for $0 \leq t \leq 1$.
(b) Suppose $\bar{L} = \int_0^1 \zeta(u)dB_u + \int_0^1 \eta(u)du$ for some deterministic functions ζ and η, then

$$C_t = \int_t^1 \zeta(u)dB_u, \qquad \text{and} \qquad I_t = \int_0^t \zeta(u)dB_u + \int_0^1 \eta(u)du,$$

for $0 \leq t \leq 1$.

However, if $(\bar{L}, B_s)_{0\leq s\leq t}$ is not Gaussian, then the existence of (C_t) and (I_t) is not guaranteed. For example, when $\bar{L} = B_1^2$, $C_t = \bar{L} - \mathbb{E}[\bar{L}|\mathcal{F}_t]$ and B_s are not independent for $s \leq t$, this implies that C_t is not instantly independent with respect to $\mathcal{F}_t$. Although, in this case, $\bar{L}$ cannot be decomposed into $C_t + I_t$, $\bar{L}$ can be written as

$$\bar{L} = B_1^2 = ((B_1 - B_t) + B_t)^2 = (C_t + I_t)^2.$$

Throughout this paper, we investigate the case where $\bar{L}$ can be written as $f(L)$, where $(L, B_s)_{0\leq s\leq t}$ is Gaussian.

3. The Extension of Itô Integral

We would like to study the properties of the stochastic integral defined in (1.3) under the construction of (2.1). Different from the definition of the Itô integral, we define the anticipating integral $(G)\int_0^t f(L)dB_s$ as follows:

$$\begin{aligned}(G)\int_0^t f(L)dB_s &= (G)\int_0^t f(C_s+I_s)dB_s\\ &= \lim_{\|\Delta\|\to 0}\sum_{i=1}^n f(C_{t_i}+I_{t_{i-1}})(B_{t_i}-B_{t_{i-1}}),\end{aligned} \tag{3.1}$$

where $\Delta=\{t_0,t_1,...,t_n\}$ is a partition of $[0,t]$. Notice that for the counter part, we take the left-end point; whereas for the Itô part, we take the right-end point as the usual Itô integral. Our goal is to analyze if the above limit exists.

In the sequel, we assume that I_t is a continuous process.

Proposition 3.1. *Suppose f is a C^2-function with bounded second derivative and the cross-variation of I and B, $[I,B]$, exists, and is defined as*

$$[I,B]_t=\lim_{\|\Delta\|\to 0}\sum_{i=1}^n (B_{t_i}-B_{t_{i-1}})(I_{t_i}-I_{t_{i-1}}).$$

Then the limit

$$\lim_{\|\Delta\|\to 0}\sum_{i=1}^n f(C_{t_i}+I_{t_{i-1}})(B_{t_i}-B_{t_{i-1}})$$

converges in probability. Moreover, this new integral can be written as

$$(G)\int_0^t f(L)dB_s=f(L)B_t-f'(L)[I,B]_t. \tag{3.2}$$

Proof. Due to (2.1), we have $C_{t_{i-1}}+I_{t_{i-1}}=L$. Consider

$$\begin{aligned}&\sum_{i=1}^n f(C_{t_i}+I_{t_{i-1}})(B_{t_i}-B_{t_{i-1}})\\ &=\sum_{i=1}^n \big(f(C_{t_i}+I_{t_{i-1}})-f(C_{t_{i-1}}+I_{t_{i-1}})\big)(B_{t_i}-B_{t_{i-1}})\\ &\quad+\sum_{i=1}^n f(L)(B_{t_i}-B_{t_{i-1}})\end{aligned}$$

Clearly,

$$\sum_{i=1}^n f(L)(B_{t_i}-B_{t_{i-1}})=f(L)B_t.$$

By Taylor expansion, we get

$$\begin{aligned}&\sum_{i=1}^{n}(f(C_{t_i}+I_{t_{i-1}})-f(C_{t_{i-1}}+I_{t_{i-1}}))(B_{t_i}-B_{t_{i-1}})\\&=\sum_{i=1}^{n}\Big(f'(C_{t_{i-1}}+I_{t_{i-1}})(C_{t_i}-C_{t_{i-1}})\\&\qquad+\frac{1}{2}f''(\xi_i+I_{t_{i-1}})(C_{t_i}-C_{t_{i-1}})^2\Big)(B_{t_i}-B_{t_{i-1}})\end{aligned}$$

for some ξ_i between $C_{t_{i-1}}$ and C_{t_i}. The first order term of the above equation is equal to

$$\sum_{i=1}^{n}f'(L)(I_{t_{i-1}}-I_{t_i})(B_{t_i}-B_{t_{i-1}}),$$

which tends to $-f'(L)[I,B]_t$ in probability as $\|\Delta\|\to 0$. Since f'' is bounded by a constant, say α, we have

$$\begin{aligned}&\left|\sum_{i=1}^{n}\frac{1}{2}f''(\xi_i+I_{t_{i-1}})(C_{t_i}-C_{t_{i-1}})^2(B_{t_i}-B_{t_{i-1}})\right|^2\\ \le\;&\frac{1}{2}\alpha^2\sum_{i=1}^{n}|C_{t_i}-C_{t_{i-1}}|^4|B_{t_i}-B_{t_{i-1}}|^2\\ =\;&\frac{1}{2}\alpha^2\sum_{i=1}^{n}|I_{t_i}-I_{t_{i-1}}|^4|B_{t_i}-B_{t_{i-1}}|^2\\ \le\;&\frac{1}{2}\alpha^2\max_j|I_{t_j}-I_{t_{j-1}}|^4\sum_{i=1}^{n}|B_{t_i}-B_{t_{i-1}}|^2\end{aligned}$$

which tends to 0 as $\|\Delta\|\to 0$. □

Remark 3.2. The definition of (3.1) is well-defined. Suppose that $f(L)$ can be also written as $g(\tilde{L})$, where both $(L,B_s)_{0\le s\le t}$ and $(\tilde{L},B_s)_{0\le s\le t}$ are Gaussian. Without loss of generality, we may assume that $L,\tilde{L}\sim\mathcal{N}(0,1)$. Since $f(L)=g(\tilde{L})$, the only possibilities are that $\tilde{L}=\pm L$.

Assume that $\tilde{L}=-L$, then $f(x)=g(-x)$. Suppose that $\tilde{L}$ can be decomposed to $\tilde{L}=-C_t-\tilde{I}_t$ for all t, then

$$\begin{aligned}&\sum f(C_{t_i}+I_{t_{i-1}})(B_{t_i}-B_{t_{i-1}})-\sum g(-C_{t_i}-I_{t_{i-1}})(B_{t_i}-B_{t_{i-1}})\\ =\;&\sum\big(f(C_{t_i}+I_{i_{i-1}})-f(C_{t_{i-1}}+I_{i_{i-1}})\big)(B_{t_i}-B_{t_{i-1}})\\&-\sum\big(g(-C_{t_i}-I_{t_{i-1}})-g(-C_{t_{i-1}}-I_{t_{i-1}})\big)(B_{t_i}-B_{t_{i-1}}),\end{aligned}$$

which approaches to $f'(L)[I,B]_t+g'(-L)[I,B]_t$ as $\|\Delta\|$ goes to 0. Since $f'(x)=-g'(-x)$, which results $(G)\int_0^t f(L)\,dB_u=(G)\int_0^t g(\tilde{L})\,dB_u$.

Remark 3.3. In [1], the authors consider the stochastic integral of the form

$$\int h(t)\varphi(t)dB_t, \tag{3.3}$$

where $h(t)$ is an adapted process, and $\varphi(t)$ is an instantly independent process. In this paper, we consider the stochastic integral of the form

$$\int f(C_t + I_t) dB_t. \tag{3.4}$$

Notice that if $f(x+y) = \sum_n h_n(x)\varphi_n(x)$, for example, when f is a polynomial, sine function, cosine function, or exponential function, then the two integrals (3.3) and (3.4) obtain the same results.

Remark 3.4. Using a similar argument as in the proof of Proposition 3.1, we get the following generalization.

(a) Suppose that $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is twice differentiable with bounded second partial derivatives and $(L^1, L^2, B_s)_{s \le t}$ is Gaussian. Assume that I^1 and I^2 are the Itô part of L^1 and L^2, respectively, and the cross-variation $[I^1, B]$ and $[I^2, B]$ exist. Then

$$(G)\int_0^t f(L^1, L^2) dB_s = f(L^1, L^2)B_t - \frac{\partial}{\partial x} f(L^1, L^2)[I^1, B]_t - \frac{\partial}{\partial y} f(L^1, L^2)[I^2, B]_t.$$

(b) Suppose $f : [0, T] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a $C^{1,2,2}$*function* and the Itô integral $\int_0^t f(s, L, B_s) dB_s$ exists in the sense of enlargement of filtration[1], then

$$(G)\int_0^t f(s, L, B_s) dB_s = \int_0^t f(s, L, B_s) dB_s - \int_0^t \frac{\partial}{\partial x} f(s, L, B_s) d[I, B]_s.$$

In particular, if $f(t, x, y) = f_1(x) f_2(y)$, then

$$(G)\int_0^t f_1(L) f_2(B_s) dB_s = f_1(L)\int_0^t f_2(B_s) dB_s - f_1'(L)\int_0^t f_2(B_s) d[I, B]_s.$$

(c) The above discussion holds also for general semimartingale X.

Example 3.5. (a) Let $L = B_1$, $f(x) = x$, then

$$C_t = \begin{cases} B_1 - B_t, & 0 \le t \le 1, \\ 0, & t > 1, \end{cases}$$

and

$$I_t = \begin{cases} B_t, & 0 \le t \le 1, \\ B_1, & t > 1, \end{cases} .$$

Thus,

$$(G)\int_0^t B_1 dB_s = \begin{cases} B_1 B_t - 1 \cdot [B, B]_t = B_1 B_t - t, & 0 \le t < 1, \\ B_1 B_t - 1 \cdot [B, B]_1 = B_1 B_t - 1, & t \ge 1. \end{cases}$$

(b) Let $L = B_1$, $f(x) = x^2$, then

$$(G)\int_0^t B_1^2 dB_s = \begin{cases} B_1^2 B_t - 2tB_1, & 0 \le t < 1, \\ B_1^2 B_t - 2B_1, & t \ge 1. \end{cases}$$

[1]The condition on L such that the integral exists is shown in [6].

(c) Let $L = B_1,\ f(x) = e^x$, then for $0 \le t \le 1$,

$$(G)\int_0^t e^{B_1} dB_s = e^{B_1}(B_t - t).$$

(d) Let $L = B_1,\ f(x) = \tan^{-1}(x)$, then for $0 \le t \le 1$,

$$(G)\int_0^t \tan^{-1}(B_1) dB_u = B_t \tan^{-1}(B_1) - \frac{t}{1+B_1^2}.$$

(e) Let $L = B_1,\ f(t,x,y) = \frac{y}{1+x^2}$, then for $0 \le t \le 1$,

$$\begin{aligned} G)\int_0^t \frac{B_u}{1+B_1^2} dB_u &= \frac{1}{1+B_1^2}\int_0^t B_u dB_u + \frac{2B_1}{(1+B_1^2)^2}\int_0^t B_u du \\ &= \frac{B_t^2 - t}{2(1+B_1^2)} + \frac{2B_1}{(1+B_1^2)^2}\int_0^t B_u du \end{aligned}$$

(f) Let $L = \int_0^1 B_u du = B_1 - \int_0^1 u dB_u,\ f(x) = x$, then

$$I_s = \mathbb{E}[L|\mathcal{F}_s] = \mathbb{E}[B_1 - \int_0^1 u dB_u|\mathcal{F}_s] = B_s - \int_0^s u dB_u,$$

Thus

$$\begin{aligned} \int_0^1 \left(\int_0^1 B_u du\right) dB_s &= \int_0^1 L dB_u \\ &= LB_1 - [I,B]_1 \\ &= B_1 \int_0^1 B_u du - \frac{1}{2}, \end{aligned}$$

since for $t < 1$, $dI_t = (1-t)dB_t$, $[I,B]_t = \int_0^t (1-u)du = t - \frac{1}{2}t^2$.

Proposition 3.6. *Let $f : [0,T] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a $C^{1,2,2}$-function with bounded second partial derivatives, then for $0 \le t \le T$,*

$$M_t = (G)\int_0^t f(u, L, B_u) dB_u$$

is a near-martingale.

Proof. Let $\Delta = \{s = t_0 < t_1 < \cdots < t_n = t\}$ be a partition of $[s,t]$. Then

$$\begin{aligned} &\mathbb{E}[M_t - M_s|\mathcal{F}_s] \\ &= \mathbb{E}\left[(G)\int_s^t f(u,L,B_u)dB_u \middle| \mathcal{F}_s\right] \\ &= \lim_{\|\Delta\|\to 0} \mathbb{E}\left[\sum_{i=1}^n f(t_{i-1}, C_{t_i} + L_{t_{i-1}}, B_{t_{i-1}})(B_{t_i} - B_{t_{i-1}}) \middle| \mathcal{F}_s\right] \\ &= \lim_{\|\Delta\|\to 0} \sum_{i=1}^n \mathbb{E}\left[\mathbb{E}\left[f(t_{i-1}, C_{t_i} + L_{t_{i-1}}, B_{t_{i-1}})(B_{t_i} - B_{t_{i-1}})\middle| \mathcal{F}_{t_{i-1}}\right]\middle| \mathcal{F}_s\right] \\ &= \lim_{\|\Delta\|\to 0} \sum_{i=1}^n \mathbb{E}\left[\mathbb{E}\left[f(t_{i-1}, C_{t_i} + L_{t_{i-1}}, B_{t_{i-1}})\middle| \mathcal{F}_{t_{i-1}}\right]\mathbb{E}[B_{t_i} - B_{t_{i-1}}]\middle| \mathcal{F}_s\right], \end{aligned}$$

since $B_{t_i} - B_{t_{i-1}}$ is independent of $\mathcal{F}_{t_{i-1}}$ and C_{t_i}, $L_{t_{i-1}}$ and $B_{t_{i-1}}$ are $\mathcal{F}_{t_{i-1}}$-measurable. Due to $\mathbb{E}[B_{t_i} - B_{t_{i-1}}] = 0$, we get (M_t) is a near-martingale. □

4. Itô Formula

In [3, 7], the authors discuss the Itô formula of the form (3.3); here, we derive the Itô formula in the sense of (3.1).

Theorem 4.1. *Suppose $f \in C^{1,2,2}$. Then, the Itô formula for the anticipating integral defined in* (3.1) *is:*

$$\begin{aligned} f(T,L,B_T) &= f(0,L,0) + \int_0^T \frac{\partial}{\partial t} f(u,L,B_u)du + (G)\int_0^T \frac{\partial}{\partial y} f(u,L,B_u)dB_u \\ & \frac{1}{2}\int_0^T \frac{\partial^2}{\partial y^2} f(u,L,B_u)du + \int_0^T \frac{\partial^2}{\partial x \partial y} f(u,L,B_u)d[I,B]_u; \end{aligned}$$

and the Itô formula for the anticipating integral defined in the sense of the enlargement filtration is:

$$\begin{aligned} & f(T,L,B_T) - f(0,L,0) \\ &= \int_0^T \frac{\partial}{\partial t} f(u,L,B_u)du + \int_0^T \frac{\partial}{\partial y} f(u,L,B_u)dB_u + \frac{1}{2}\int_0^T \frac{\partial^2}{\partial y^2} f(u,L,B_u)du. \end{aligned}$$

Proof. Suppose that $0 = t_0 < t_1 < t_2 < \cdots < t_n = T$. Then

$$f(T,L,B_T) - f(0,L,0) = \sum_{i=1}^n (f(t_i,L,B_{t_i}) - f(t_{i-1},L,B_{t_{i-1}})).$$

We have

$$\begin{aligned} & f(t_i,L,B_{t_i}) - f(t_{i-1},L,B_{t_{i-1}}) \\ = \quad & (f(t_i,L,B_{t_i}) - f(t_{i-1},L,B_{t_i})) + (f(t_{i-1},L,B_{t_i}) - f(t_{i-1},L,B_{t_{i-1}})) \end{aligned}$$

(i) By Taylor formula, there exists t_i^* between t_{i-1} and t_i, such that

$$f(t_i,L,B_{t_i}) - f(t_{i-1},L,B_{t_i}) = \frac{\partial}{\partial t} f(t_i^*,L,B_{t_i})(t_i - t_{i-1}).$$

Since f is continuously differentiable in t,

$$\begin{aligned} & \lim_{\|\Delta\| \to 0} \sum_{i=1}^n (f(t_i,L,B_{t_i}) - f(t_{i-1},L,B_{t_i})) \\ &= \lim_{\|\Delta\| \to 0} \sum_{i=1}^n \frac{\partial}{\partial t} f(t_i,L,B_{t_i})(t_i - t_{i-1}) \\ &= \int_0^T \frac{\partial}{\partial t} f(u,L,B_u)du. \end{aligned}$$

(ii) There exists $B^*_{t_i}$ between $B_{t_{i-1}}$ and B_{t_i} such that

$$\begin{aligned}
&f(t_{i-1}, L, B_{t_i}) - f(t_{i-1}, L, B_{t_{i-1}}) \\
&= \frac{\partial}{\partial y} f(t_{i-1}, L, B_{t_{i-1}})(B_{t_i} - B_{t_{i-1}}) \\
&\quad + \frac{1}{2}\ \frac{\partial^2}{\partial y^2} f(t_{i-1}, L, B_{t^*_i})(B_{t_i} - B_{t_{i-1}})^2.
\end{aligned}$$

Summing across i from $i = 1$ to n, the last equality tends to

$$\int_0^t \frac{\partial}{\partial y} f(u, L, B_u) dB_u + \frac{1}{2} \int_0^t \frac{\partial^2}{\partial y^2} f(u, L, B_u) du.$$

By Remark 3.4,

$$\int_0^t \frac{\partial}{\partial y} f(u, L, B_u) dB_u = (G) \int_0^t \frac{\partial}{\partial y} f(u, L, B_u) dB_u + \int_0^t \frac{\partial^2}{\partial x \partial y} f(u, L, B_u) d[I, B]_u.$$

Hence, from (i) and (ii), the result is obtained. □

Acknowledgments. C.-T. Wu is greatly indebted to the Ministry of Science and Technology, Taiwan for the research grant (MOST 108-2115-M-143-001-). J.-Y. Yen is grateful to the Academia Sinica, Institute of Mathematics (Taipei, Taiwan) for their hospitality and support during some extended visits.

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