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BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS ASSOCIATED WITH LÉVY PROCESSES AND PARTIAL INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we deal with a class of backward stochastic differential equations driven by Teugels martingales associated with a Lévy process (BSDELs). The comparison theorem is obtained. It is also shown that the solution of BSDE provides a viscosity solution of the associated system with partial integro-differential equations.

1. Introduction

Backward stochastic differential equations (BSDEs), in the nonlinear case, have been introduced by Pardoux and Peng [17] in 1990. They gave the first existence and uniqueness result under suitable assumptions on the coefficient and the terminal value of the BSDE. Since then, BSDE have gradually become an important mathematical tool in mathematical finance [6, 7], optimal stochastic control and stochastic games [5, 8, 9]. BSDE also appear as a powerful tool to give probabilistic formulas for solution of partial differential equations [15, 16, 18, 19].

On another context, Situ [24] studied BSDEs driven by a Brownian motion and a Poisson point process, Ouknine [14] studied BSDEs driven by a Poisson random measure, Nualart and Schoutens established in [13] the existence and uniqueness of solutions for BSDEs driven by a Lévy process and later Bahlali *et al.* [1] treated the case where the BSDE is driven by a Brownian motion and the martingales of Teugels associated with an independent Lévy process.

The comparison theorem for BSDEs turns out to be one of the classic results of this theory. But is not true in general case (see Remark 2.7 in [2] for a counter-example). We refer the reader to [2, 12, 22, 25] for some particular cases of the comparison theorem for BSDEs with jumps.

The main results of this paper are the proof of a comparison theorem for a classes of BSDEs driven by a Lévy process and a representation that identifies any solution of these classes of BSDEs as a viscosity solution of the associated partial integro-differential equations (PIDEs).

The paper is organized as follows. In Section 2, we introduce some notations and we discuss a pricing problem by replication in a market controlled by a Lévy process: we verify that it can be written in terms of linear BSDELs. In Section 3,

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we prove the comparison theorem of BSDELs under some appropriate conditions on the generator f. By means of our comparison theorem, one can show in Section 4 that a certain function defined through the solution of Markovian BSDELs is a viscosity solution of the associated system of P.I-D.Es.

2. Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a completed probability space on which a real-valued Lévy process $(L_t)_{t\in[0,T]}$ with càdlàg paths is defined. Let $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ be the right-continuous filtration generated by L: $(\mathcal{F}_t = \sigma\{L_s; s \leq t\})$ and assume that $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets of $\mathcal{F}$. The process L is characterized by its so-called local characteristics in the Lévy-Khintchine formula. So that

$$\mathbb{E}e^{iuL_t} = e^{-t\Psi(u)} \text{ with } \Psi(u) = -ibu + \frac{\sigma^2}{2}u^2 - \int_{\mathbb{R}} \left(e^{iux} - 1 - iux\mathbb{I}_{(|x|\leq 1)}\right)\nu(dx).$$

Thus L is characterized by its Lévy triplet (b, σ, ν) where $b \in \mathbb{R},\ \sigma^2 \geq 0$ and ν is a measure defined in $\mathbb{R} \setminus \{0\}$ and satisfies

(i) $\int_{\mathbb{R}}(1 \wedge x^2)\nu(dx) < +\infty$,

(ii) $\exists\ \varepsilon > 0$ and $\lambda > 0$ as $\int_{(-\varepsilon,\varepsilon)^c} e^{\lambda|x|}\nu(dx) < +\infty$.

This implies that the random variables L_t have moments of all orders, i.e.

$$\int_{-\infty}^{+\infty} |x|^i \nu(dx) < \infty,\ \forall i \geq 2.$$

For background on Lévy processes, we refer the reader to [3, 23].

We denote by $L_{t-} = \lim_{s\nearrow t} L_s$ and $\Delta L_t = L_t - L_{t-}$. We define the power jumps of the Lévy process L by

$$L_t^{(1)} = L_t \quad \text{and} \quad L_t^{(i)} = \sum_{0<s\leq t} (\Delta L_s)^i,\ i \geq 2.$$

Let $m_1 = \mathbb{E}[L_1] = b + \int_{|x|\geq 1} x\nu(dx)$ and $m_i = \int_{-\infty}^{+\infty} x^i\nu(dx)$ for $i \geq 2$. Let us put $Y_t^{(i)} = L_t^{(i)} - m_i t,\ i \geq 1$, the so–called Teugels martingales. We associate with the Lévy process $(L_t)_{0\leq t\leq T}$ the family of processes $(H^{(i)})_{i\geq 1}$ defined by $H_t^{(i)} = \sum_{j=1}^{i} a_{ij} Y_t^{(j)}$. The coefficients a_{ij} correspond to the orthonormalization of the polynomials 1, x, x^2, ... with respect to the measure $\pi(dx) = x^2\nu(dx) + \sigma^2\delta_0(dx)$. Specifically, the polynomials q_n defined by $q_n(x) = \sum_{k=1}^{n} a_{nk}x^{k-1}$ are orthonormal with respect to the measure π:

$$\int_{\mathbb{R}} q_n(x)q_m(x)\pi(dx) = 0 \text{ if } n \neq m \quad \text{and} \quad \int_{\mathbb{R}} q_n(x)^2\pi(dx) = 1.$$

We set

$$p_n(x) = xq_{n-1}(x) = a_{n,n}x^n + a_{n,n-1}x^{n-1} + ... + a_{n,1}x.$$

The martingales $H^{(i)}$, called the orthonormalized ith-power-jump processes, are strongly orthogonal and its predictable quadratic variation process is

$$\langle H^{(i)}, H^{(j)}\rangle_t = \delta_{ij}t.$$

Let us introduce the following spaces:

- $\ell^2 = \Big\{x = (x_n)_{n\geq 1};\ \|x\|_{\ell^2} = \big(\sum_{k=1}^{\infty} |x_n|^2\big)^{\frac{1}{2}} < \infty.\Big\}$.
- $\mathcal{H}^2 = \{\varphi : \mathcal{F}_t$-progressively measurable process, real-valued process and square integrable s.t. $\mathbb{E}\int_0^T |\varphi_t|^2 dt < \infty\Big\}$.
- $\mathcal{P}^2$ will denote the subspace of $\mathcal{H}^2$ formed by predictable processes.
- $\mathcal{S}^2 = \{\varphi : \mathcal{F}_t$-progressively measurable process, real-valued process, s.t. $\mathbb{E}[\sup_{t\leq T} |\varphi_t|^2] < \infty\big\}$.
- We shall denote by $\mathcal{H}^2(\ell^2)$ and $\mathcal{P}^2(\ell^2)$ the corresponding spaces of ℓ^2-valued process equipped with the norm $\|\phi\|^2_{\mathcal{H}^2(\ell^2)} = \mathbb{E}\int_0^T \|\phi_t\|_{\ell^2} dt$.

We now give the "standard" data (ξ, f) defined as

($\mathcal{H}$.0) A terminal value $\xi \in \mathbb{L}^2(\Omega, \mathcal{F}_T, \mathbb{P})$.

($\mathcal{H}$.1) A map $f : [0,T] \times \Omega \times \mathbb{R} \times \ell^2 \to \mathbb{R}$ which is $\mathcal{F}_t$-progressively measurable. In addition we assume that:
(1) $\mathbb{E}\int_0^T |f(t,0,0)|^2 dt < \infty$.
(2) f is uniformly κ-Lipschitz with respect to (y,z): i.e. there exists a constant $\kappa > 0$ such that for any $y_1, y_2 \in \mathbb{R}, z_1, z_2 \in \ell^2$ and $t \in [0,T]$,

$$|f(t,y_1,z_1) - f(t,y_2,z_2)| \leq \kappa\,(|y_1 - y_2| + \|z_1 - z_2\|_{\ell^2}) \qquad \mathbb{P}\text{-a.s.}$$

A solution of the BSDEL is a pair of processes $(Y,Z) \in \mathcal{S}^2 \times \mathcal{P}^2(\ell^2)$ such that, for all $t \in [0,T]$,

$$Y_t = \xi + \int_t^T f(s, Y_{s-}, Z_s) ds - \sum_{k=1}^{\infty} \int_t^T Z_s^{(k)} dH_s^{(k)}. \tag{2.1}$$

Nualart and Schoutens proved the following theorem.

Theorem 2.1 ([13], Theorem 1)**.** *Given standard data (ξ, f), there exists a unique solution of the BSDEL (2.1).*

2.1. BSDEL in finance. Consider a market where the risk-neutral dynamics of the asset can be described by the Doléans-Dade exponential

$$dX_t = X_{t-} dL_t, \qquad X_0 = x.$$

We suppose additionally that $x > 0$ and the Lévy measure is supported on a subset of $[-1, +\infty)$. This ensures that $X_t > 0,\ \forall t \leq T$ a.s.

The value of the risk-free bond at time $t \leq T$ is given by $X_t^0 = e^{rt}$ where the constant r is the riskless rate of interest. We suppose that there exists a risk-neutral measure $\mathbb{Q}$ such that $\tilde{X}_t = e^{-rt} X_t$ is a martingale and $(\tilde{L}_t)_{t\leq T} = (L_t - rt)_{t\leq T}$ will be a Lévy process; moreover the process $\tilde{L}$ is a martingale. Let $(\tilde{H}^{(i)})_{i\geq 1}$ be the orthonormalized power jump processes for $\tilde{L}$ under the measure $\mathbb{Q}$. We Enlarge the market with price processes $\tilde{X}^{(i)}$ defined, for all $t \leq T$, by

$$\tilde{X}_t^{(i)} = e^{rt} \tilde{H}_t^{(i)}, \quad i \geq 2.$$

Note that

$$\mathbb{E}_{\mathbb{Q}}[e^{-rt} \tilde{X}_t^{(i)} / \mathcal{F}_s] = \mathbb{E}_{\mathbb{Q}}[\tilde{H}_t^{(i)} / \mathcal{F}_s] = \tilde{H}_s^{(i)} \quad \forall\ 0 \leq s \leq t \leq T.$$

It follows that the stock $\tilde{X}$ and the power jump assets $(\tilde{X}^{(i)})_{i\geq 2}$ remains arbitrage free.

The value of a European option with terminal payoff $\phi(\tilde{X}_T)$ is defined as a conditional expectation of its actualized terminal payoff under risk neutral probability $\mathbb{Q}$:

$$V_t = \mathbb{E}_{\mathbb{Q}}\left[e^{-r(T-t)}\phi(\tilde{X}_T)/\mathcal{F}_t\right].$$

Suppose that $\mathbb{E}_{\mathbb{Q}}[|\phi(\tilde{X}_T)|^2] < \infty$. For all $(t,x) \in [0,T] \times [0,+\infty)$, we define the function v as follows

$$v(t,x) = \mathbb{E}_{\mathbb{Q}}\left[e^{-r(T-t)}\phi(\tilde{X}_T)/\tilde{X}_t = x\right].$$

Clearly, $V_t = v(t,\tilde{X}_t)$. We suppose that

$$\sigma^2 > 0 \quad \text{or} \quad \exists \beta \in (0,2), \qquad \lim_{\varepsilon \downarrow 0} \varepsilon^{-\beta} \int_{-\varepsilon}^{\varepsilon} |x|^2 v(dx) > 0.$$

Under this regularity condition, the function v is continuous on $[0,T] \times [0,+\infty)$, $\mathcal{C}^{1,2}$ on $(0,T) \times [0,+\infty)$ and v solves the partial integro-differential equation (see e.g. [4], Proposition 2)

$$\begin{cases} \dfrac{\partial v}{\partial t}(t,x) + \mathcal{L}v(t,x) - rv(t,x) = 0, \quad \forall (t,x) \in [0,T) \times [0,+\infty), \\ v(T,x) = \phi(x), \quad \forall x \in [0,+\infty), \end{cases}$$

where

$$\begin{aligned} \mathcal{L}v(t,x) \quad &= \quad m_1 x \frac{\partial v}{\partial x}(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 v}{\partial x^2}(t,x) \\ &\quad + \int_{\mathbb{R}} \left(v(t,x(1+y)) - v(t,x) - \frac{\partial v}{\partial x}(t,x)xy \right) \nu(dy). \end{aligned}$$

Next, let us define the following precesses

$$\begin{cases} \tilde{Y}_t &= v(t,\tilde{X}_t), \\ \tilde{Z}_t^{(k)} &= \displaystyle\int_{\mathbb{R}} \left[v(t,\tilde{X}_t(1+y)) - v(t,\tilde{X}_t)\right] p_k(y)\tilde{\nu}(dy), \quad k \geq 1. \end{cases}$$

It is not hard to see that $\left(\tilde{Y}, (\tilde{Z}^{(k)})_{k\geq 1}\right) \in \mathcal{S}^2 \times \mathcal{P}^2(\ell^2)$. So, we have

Proposition 2.2. *The process* $\left(\tilde{Y}, (\tilde{Z}^{(k)})_{k\geq 1}\right)$ *is the unique solution of the following BSDEL*

$$\tilde{Y}_t = \phi(\tilde{X}_T) - \int_t^T r\tilde{Y}_s ds - \sum_{k=1}^{\infty} \int_t^T \tilde{Z}_s^{(k)} d\tilde{H}_s^{(k)}, \quad 0 \leq t \leq T.$$

Proof. This is a direct consequence Itô's formula and the uniqueness result of the solution for BSDEL (2.1). $\square$

3. The Comparison Theorem

The aim of this section is to establish the comparison theorem when the coefficient f made up of:

($\mathcal{H}.2$) (1) $f(t,\omega,y,z) = f^1(t,\omega,y) + \sum_{k=1}^{\infty} \gamma^{(k)}(s,\omega)z^{(k)}$.

(2) f^1 is progressively measurable and $\mathbb{E}\left[\int_0^T |f^1(t,0)|^2 dt\right] < \infty$.

(3) f^1 is uniformly κ-Lipschitz with respect to y.

(4) $\sum_{k=1}^{\infty}\gamma_t^{(k)}\Delta H_t^{(k)} > -1$ for all (t,ω).

Theorem 3.1 (Comparison Theorem)**.** *Let (Y,Z) and (Y',Z') be solution of the BSDEL (2.1) with data (ξ,f) and (ξ',f') respectively which satisfies $(\mathcal{H}.0)-(\mathcal{H}.2)$. We suppose that $\xi \le \xi'$ and $f^1(t,y) \le {f'}^1(t,y)$ for all (t,y) $d\mathbb{P}\times dt$–a.s. Then, $Y_t \le Y'_t$, $\forall t \le T$ $\mathbb{P}$–a.s.*

Proof. Using the Meyer-Itô formula with the convex function $x \mapsto (x^+)2$ with $Y - Y'$ implies that

$$\begin{aligned}(Y_t - Y'_t)^{+2} &= (Y_0 - Y'_0)^{+2} - 2\int_0^t (Y_r - Y'_r)^+ \left(f^1(r,Y_r) - {f'}^1(r,Y'_r)\right) dr \\ &\quad -2\sum_{k=1}^{\infty}\int_0^t (Y_r - Y'_r)^+ \gamma_r^{(k)}(Z_r^{(k)} - {Z'}_r^{(k)})dr \\ &\quad +2\sum_{k=1}^{\infty}\int_0^t (Y_r - Y'_r)^+ (Z_r^{(k)} - {Z'}_r^{(k)})dH_r^{(k)} + \mathcal{A}_t, \qquad (3.1)\end{aligned}$$

where $\mathcal{A}$ is a continuous non decreasing process (see e.g. [20], Theorem 66 P. 210). Let $(\Gamma_{t,s}, s\in[t,T])$ be solution of the linear stochastic differential equation

$$\Gamma_{t,s} = 1 + \sum_{k=1}^{\infty}\int_t^s \Gamma_{t,r-}\gamma_r^{(k)}dH_r^{(k)},$$

which we can write as (see e.g. [20], p. 84)

$$\begin{aligned}\Gamma_{t,s} &= \exp\left(\sum_{k=1}^{\infty}\int_t^s \gamma_r^{(k)}dH_r^{(k)}\right) \\ &\quad \prod_{t<r\le s}\left\{(1+\sum_{k=1}^{\infty}\gamma_r^{(k)}\Delta H_r^{(k)})\exp(-\sum_{k=1}^{\infty}\gamma_r^{(k)}\Delta H_r^{(k)})\right\} > 0.\end{aligned}$$

On the other hand,from the integration by part formula combining with (3.1), we have for all $t \le s \le T$

$$\begin{aligned}&(Y_t - Y'_t)^{+2} = \Gamma_{t,s}(Y_s - Y'_s)^{+2} \\ &+2\int_t^s \Gamma_{t,r}(Y_r - Y'_r)^+\left(f^1(r,Y_r) - {f'}^1(r,Y'_r)\right)dr + \int_t^s \Gamma_{t,r}d\mathcal{A}_r \\ &+\sum_{k=1}^{\infty}\int_t^s \Gamma_{t,r}(Y_r - Y'_r)^+\left\{(Y_r - Y'_r)^+\gamma_r^{(k)} + 2\left(Z_r^{(k)} - {Z'}_r^{(k)}\right)\right\}dH_r^{(k)} \\ &+2\sum_{i,j=1}^{\infty}\int_t^s \Gamma_{t,r}\gamma_r^{(i)}(Y_r - Y'_r)^+(Z_r^{(j)} - {Z'}_r^{(j)})\left(d[H^{(i)},H^{(j)}]_r - \langle H^{(i)},H^{(j)}\rangle_r\right).\end{aligned}$$

In particular for $s = T$. It follows that

$$\begin{aligned}
&\mathbb{E}\left[(Y_t - Y'_t)^{+2}\right] \\
&\leq \mathbb{E}[\Gamma_{t,T}(\xi - \xi')^{+2}] + 2\mathbb{E}\int_t^T \Gamma_{t,r}(Y_r - Y'_r)^+ \left(f^1(r, Y_r) - f'^1(r, Y_r)\right) dr \\
&+2\kappa\mathbb{E}\int_t^T \Gamma_{t,r}(Y_r - Y'_r)^+|Y_r - Y'_r|dr \\
&\leq 2\kappa\int_t^T \mathbb{E}\left[\Gamma_{t,r}(Y_r - Y'_r)^{+2}\right] dr.
\end{aligned}$$

Using Gronwall's lemma we conclude that $Y_t \leq Y'_t$ $\mathbb{P}$–a.s for all $0 \leq t \leq T$. This completes the proof of the theorem. □

4. Viscosity Solution of Partial Integro-differential Equation

In this section, we prove that, if the BSDEL (2.1) has an adapted solution, then it will provide a solution to a system of integro-partial differential equations in the sense of viscosity.

4.1. Forward-Backward SDE with Lévy process. Suppose that:

($\mathcal{H}$.3) The autonomous function $F : \mathbb{R} \mapsto \mathbb{R}$ is κ-Lipschitz; that is there exists a constant $\kappa > 0$ such that

$$|F(x) - F(x')| \leq \kappa|x - x'|, \quad x, x' \in \mathbb{R}.$$

Proposition 4.1. ([21], Lemma 5.1) *For every initial condition $(t, x) \in [0, T]\times\mathbb{R}$, the stochastic differential equation*

$$X^{t,x}_s = x + \int_t^{s\vee t} F(X^{t,x}_{r-})dL_r, \tag{4.1}$$

has a unique solution such that

$$\mathbb{E}\sup_{t\leq s\leq T}|X^{t,x}_s|^p \leq C(p, \kappa)(T - t)(1 + |x|^p), \quad \textit{for all} \;\; p \geq 1. \tag{4.2}$$

Proposition 4.2. *Assume that $X^{t,x}$ is the solution of the SDE (4.1). Then, there exists a constant $C > 0$ such that, for all $t, t' \in [0, T]$ and $x, x' \in \mathbb{R}$,*

- $\mathbb{E}|X^{t,x}_{s'} - X^{t,x}_s|^2 \leq C(1 + |x|^2)|s - s'|, \ \forall\ s, s' \in [t, T].$
- $\mathbb{E}\sup_{t\leq r\leq s}|X^{t,x}_r - x|^2 \leq C(1 + |x|^2)|t - s|.$
- $\mathbb{E}\sup_{t\vee t'\leq s\leq T}|X^{t,x}_s - X^{t',x'}_s|^2 \leq C\left\{|t - t'| + |x - x'|^2\right\}.$

Proof. Due to Lemma 4.1 in [21], we can write for $s < s'$

$$\begin{aligned}
\mathbb{E}|X^{t,x}_s - X^{t,x}_{s'}|^2 &\leq C(b, \sigma, m_1)\int_s^{s'} \mathbb{E}|F(X^{t,x}_{r-})|^2dr \\
&\leq C|s - s'|(1 + \mathbb{E}\sup_{t\leq r\leq T}|X^{t,x}_r|^2).
\end{aligned}$$

The first point follows by virtue (4.2). For the second, by using Burkholder-Davis-Gundy inequality, we get

$$\begin{aligned}\mathbb{E}\sup_{t\le r\le s}|X_r^{t,x}-x|^2 &\le 4\mathbb{E}\int_t^s |F(X_{r-}^{t,x})|^2 d[L,L]_r\\ &\le C(1+|x|^2)|t-s|.\end{aligned}$$

On the same way, For $t<t'$ we have

$$\begin{aligned}&\mathbb{E}\sup_{t'\le s\le T}|X_s^{t,x}-X_s^{t',x'}|^2 \le C\Bigg\{|x-x'|^2+\mathbb{E}\int_t^{t'}|F(X_{r-}^{t,x})|^2 d[L,L]_r\\ &+\mathbb{E}\int_t^T |F(X_{r-}^{t,x})-F(X_{r-}^{t',x'})|^2 d[L,L]_r\Bigg\}\\ \le\ & C\left\{|x-x'|^2+|t-t'|+\mathbb{E}\int_t^T |X_r^{t,x}-X_r^{t',x'}|^2 dr\right\}.\end{aligned}$$

The result then follows from Gronwall's lemma. □

Now, Let $f:[0,T]\times\mathbb{R}^2\times\ell^2\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ two real functions such that

($\mathcal{H}$.4) (1) The coefficient f is continuous and κ-Lipschitz respect to x.
(2) The map $f(.,X^{t,x},.,.)$ verifies the hypothesis ($\mathcal{H}$.2) with $\gamma(s,\omega)=\gamma,\ \forall(s,\omega)$ such that $0<\varepsilon<\|\gamma\|_{\ell^2}<\infty$.
(3) The function g is κ-Lipschitz.

For each (t,x), let $\{(Y_s^{t,x},Z_s^{t,x});\ t\le s\le T\}$ be the unique solution of the BSDEL described by

$$Y_s^{t,x}=g(X_T^{t,x})+\int_{s\vee t}^T f(r,X_{r-}^{t,x},Y_{r-}^{t,x},Z_r^{t,x})dr-\sum_{k=1}^\infty\int_{s\vee t}^T Z_r^{(k)t,x}dH_r^{(k)}. \tag{4.3}$$

Existence and uniqueness follow from Theorem 2.1. Moreover, we have

$$\mathbb{E}\sup_{0\le s\le T}|Y_s^{t,x}|^2+\mathbb{E}\int_0^T\|Z_s^{t,x}\|_{\ell^2}ds\le C\left(1+|x|^2\right). \tag{4.4}$$

Let us define the deterministic function u by

$$u(t,x)=Y_t^{t,x},\quad (t,x)\in[0,T]\times\mathbb{R}. \tag{4.5}$$

So, we have

Proposition 4.3. *The following inequality holds,*

$$|u(t,x)-u(t',x')|\le C\left(|x-x'|^2+|t-t'|\right).$$

Proof. First, we remark that

$$\begin{aligned}&|u(t,x)-u(t',x')|^2=|Y_t^{t,x}-Y_t^{t,x'}|^2\\ &\le C\left\{\mathbb{E}\sup_{t\vee t'\le s\le T}|Y_s^{t,x}-Y_s^{t',x'}|^2+\mathbb{E}|Y_t^{t,x}-Y_t^{t',x'}|^2+|Y_{t'}^{t',x'}-Y_{t'}^{t,x}|^2\right\}.\end{aligned}$$

If we apply Itô's formula, we compute that

$$
\begin{aligned}
&\mathbb{E}|Y_t^{t,x} - Y_{t'}^{t,x}|^2 + \mathbb{E}\int_t^{t'} \|Z_s^{t,x}\|_{\ell^2} ds \\
&= 2\mathbb{E}\int_t^{t'} \left(Y_s^{t,x} - Y_{t'}^{t,x}\right) f(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x}) ds \\
&\leq C(\kappa,\varepsilon)\mathbb{E}\int_t^{t'} \left|Y_s^{t,x} - Y_{t'}^{t,x}\right|^2 ds \\
&+C|t-t'|\left(1 + \mathbb{E}\sup_{0\leq s\leq T}[|X_s^{t,x}|^2 + |Y_s^{t,x}|^2]\right) + \frac{\|\gamma\|_{\ell^2}}{\varepsilon}\mathbb{E}\int_t^{t'} \|Z_s^{t,x}\|_{\ell^2} ds.
\end{aligned}
$$

Gronwall's lemma yields

$$\mathbb{E}|Y_t^{t,x} - Y_{t'}^{t,x}|^2 \leq C|t-t'|(1+|x|^2).$$

On the other hand, using $(\mathcal{H}.4)$, we can write for $t \vee t' \leq s \leq T$

$$
\begin{aligned}
&\mathbb{E}|Y_s^{t,x} - Y_s^{t',x'}|^2 + \mathbb{E}\int_s^T \|Z_r^{t,x} - Z_r^{t',x'}\|_{\ell^2}^2 dr \\
&\leq \mathbb{E}|g(X_T^{t,x}) - g(X_T^{t',x'})|^2 \\
&+2\mathbb{E}\int_s^T |Y_r^{t,x} - Y_r^{t',x'}| \left| f^1(r, X_r^{t,x}, Y_r^{t,x}) - f^1(r, X_r^{t',x'}, Y_r^{t',x'})\right| dr \\
&+2\mathbb{E}\int_s^T |Y_r^{t,x} - Y_r^{t',x'}| \left|\langle \gamma, Z_r^{t,x} - Z_r^{t',x'}\rangle_{\ell^2}\right| dr \\
&\leq 2\kappa^2\mathbb{E}\sup_{s\leq r\leq T}[|X_r^{t,x} - X_r^{t',x'}|^2] + C(\kappa,\varepsilon)\mathbb{E}\int_s^T \left|Y_r^{t,x} - Y_r^{t',x'}\right|^2 ds \\
&+\frac{\|\gamma\|_{\ell^2}}{\varepsilon}\mathbb{E}\int_s^T \|Z_r^{t,x} - Z_r^{t',x'}\|_{\ell^2}^2 dr.
\end{aligned}
$$

The result follows by Burkholder-Davis-Gundy inequality. □

4.2. Viscosity solution. To begin, let us consider the following system for parabolic integral-partial differential equation

$$
\begin{cases}
\partial_t u(t,x) + \mathcal{L}^\nu u(t,x) + f(t,x,u(t,x),(u_k^1(t,x))_{k=1}^\infty) = 0, \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall (t,x) \in [0,T)\times\mathbb{R}, \\
u(T,x) = g(x), \quad \forall x \in \mathbb{R},
\end{cases} \tag{4.6}
$$

where

$$
\begin{aligned}
\mathcal{L}^\nu \phi(t,x) &= m_1 F(x)\partial_x\phi(t,x) + \frac{1}{2}\sigma^2 F(x)^2\partial_{xx}\phi(t,x) \\
&\quad + \int_{\mathbb{R}} \left(\phi(t,x+F(x)y) - \phi(t,x) - \frac{\partial\phi}{\partial x}(t,x)F(x)y\right)\nu(dy)
\end{aligned}
$$

and

$$\phi_k^1(t,x) = \int_{\mathbb{R}} \left(\phi(t,x+F(x)y) - \phi(t,x)\right) p_k(y)\nu(dy) \quad \text{for} \quad k \geq 1.$$

Definition 4.4. *(i)* $u \in \mathcal{C}([0,T] \times \mathbb{R}, \mathbb{R})$ is a *viscosity subsolution* of (4.6), if $u(T,x) \leq g(x)$; $\forall x \in \mathbb{R}$, and for $\varphi \in \mathcal{C}^{1,2}([0,T] \times \mathbb{R})$ if $u - \varphi$ has a local maximum at (t,x) then

$$-\partial_t \varphi(t,x) - \mathcal{L}^\nu \varphi(t,x) - f(t,x,u(t,x),(\varphi_k^1(t,x))_{k=1}^\infty) \leq 0.$$

(ii) $u \in \mathcal{C}([0,T] \times \mathbb{R}, \mathbb{R})$ is a *viscosity supersolution* of (4.6), if $u(T,x) \geq g(x)$; $\forall x \in \mathbb{R}$, and for $\varphi \in \mathcal{C}^{1,2}([0,T] \times \mathbb{R})$ if $u - \varphi$ has a local minimum at (t,x) then

$$-\partial_t \varphi(t,x) - \mathcal{L}^\nu \varphi(t,x) - f(t,x,u(t,x),(\varphi_k^1(t,x))_{k=1}^\infty)) \geq 0.$$

(iii) $u \in \mathcal{C}([0,T] \times \mathbb{R}, \mathbb{R})$ is a *viscosity solution* of (4.6) if it is both a viscosity subsolution and supersolution.

We give now the main result of this part:

Theorem 4.5. *The function u given by the formula (4.5) is a viscosity solution of the system (4.6).*

Proof. First, by uniqueness of the solution of BSDEL (4.3), we can write for any $s \in [t,T]$ that $Y_s^{t,x} = Y_s^{s,X_s^{t,x}} = u(s, X_s^{t,x})$. Now, we suppose that (t,x) is a local maximum of $u - \varphi$ where $\varphi \in \mathcal{C}^{1,2}([0,T] \times \mathbb{R}, \mathbb{R})$ and $u(t,x) = \varphi(t,x)$. We can assume additionally that φ and its derivatives have at most polynomial growth in x uniformly on t. Let $h > 0$ and consider $(\bar{Y}^h, \bar{Z}^h)$ the unique solution of the BSDEL given on $[t, t+h]$ by,

$$\bar{Y}_s^h = \varphi(t+h, X_{t+h}^{t,x}) + \int_s^{t+h} f(r, X_{r-}^{t,x}, \bar{Y}_{r-}^h, \bar{Z}_r^h) dr - \sum_{k=1}^\infty \int_s^{t+h} \bar{Z}_r^{(k)h} dH_r^{(k)}.$$

However, $u(t+h, X_{t+h}^{t,x}) \leq \varphi(t+h, X_{t+h}^{t,x})$. Then Theorem 3.1 implies that

$$\bar{Y}_s^h \geq Y_s^{t,x}, \quad \forall\ s \in [t, t+h]. \tag{4.7}$$

On the other hand, from Itô's formula (see [20], p. 78), we have

$$\begin{aligned}
&\varphi(t+h, X_{t+h}^{t,x}) \\
&= \varphi(s, X_s^{t,x}) + \int_s^{t+h} \partial_t \varphi(r, X_r^{t,x}) dr + \int_s^{t+h} F(X_{r-}^{t,x}) \partial_x \varphi(r, X_{r-}^{t,x}) dL_r \\
&+ \int_s^{t+h} \frac{1}{2} \sigma^2 F(X_r^{t,x})^2 \partial_{xx} \varphi(r, X_r^{t,x}) dr \\
&+ \sum_{s<r\leq t+h} \left\{ \varphi(r, X_r^{t,x}) - \varphi(r, X_{r^-}^{t,x}) - \partial_x \varphi(r, X_{r^-}^{t,x}) F(X_{r-}^{t,x}) \Delta L_r \right\}.
\end{aligned} \tag{4.8}$$

So, using Lemma 5 in [13] and the fact that

$$\int_{\mathbb{R}} y p_k(y) \nu(dy) = \int_{\mathbb{R}} y^2 q_{k-1}(y) \nu(dy) = 0, \quad \text{for} \quad k \geq 2,$$

we can write

$$\sum_{t<r\leq T}\left[\varphi(s,X_{r-}+F(X_{r-}^{t,x})\Delta L_r)-\varphi(s,X_{r-})-\frac{\partial\varphi}{\partial x}(r,X_{r-})F(X_{r-})\Delta L_r\right]$$
$$=-\int_t^T\int_{\mathbb{R}}\frac{\partial\varphi}{\partial x}(r,X_{r-})F(X_{r-})yp_1(y)\nu(dy)dH_r^{(1)}$$
$$+\sum_{k=1}^{\infty}\int_t^T\int_{\mathbb{R}}\left[\varphi(r,X_{r-}+F(X_{r-})y)-\varphi(r,X_{r-})\right]p_k(y)\nu(dy)dH_r^{(k)}$$
$$+\int_t^T\int_{\mathbb{R}}\Big[\varphi(r,X_{r-}+F(X_{r-})y)-\varphi(r,X_{r-})$$
$$-\frac{\partial\varphi}{\partial x}(r,X_{r-})F(X_{r-})y\Big]\nu(dy)dr.$$

But $L_t=L_t^{(1)}=a_{11}^{-1}H_t^{(1)}+m_1t$ and $p_1(y)=a_{11}y$, if we plug the last equality in (4.8) we have

$$\begin{aligned}\varphi(s,X_s^{t,x}) &= \varphi(t+h,X_{t+h}^{t,x})-\int_s^{t+h}\left(\partial_t\varphi+\mathcal{L}^{\nu}\varphi\right)(r,X_r^{t,x})dr\\ &\quad-\sum_{k=1}^{\infty}\int_s^{t+h}\varphi_k^1(r,X_{r^-}^{t,x})dH_r^{(k)}.\end{aligned}$$

We denote $\hat{Y}_s^h=\bar{Y}_s^h-\varphi(s,X_s^{t,x})$ and $\hat{Z}_s^{(k)h}=\bar{Z}_s^{(k)h}-\varphi_k^1(s,X_s^{t,x})$. Notice that $(\hat{Y}^h,\hat{Z}^h)$ verifies the following BSDEL

$$\hat{Y}_s^h=\int_s^{t+h}f(r,\hat{\Theta}_{r-}^{t,x})dr+\int_s^{t+h}\left(\partial_t\varphi+\mathcal{L}^{\nu}\varphi\right)(r,X_r^{t,x})dr-\sum_{k=1}^{\infty}\int_s^{t+h}\hat{Z}_r^{(k)h}dH_r^{(k)}.$$

where

$$\hat{\Theta}_r^{t,x}=\left(X_r^{t,x},\hat{Y}_r^h+\varphi(r,X_r^{t,x}),\left(\hat{Z}_r^{(k)h}+\varphi_k^1(r,X_r^{t,x})\right)_{k=1}^{\infty}\right).$$

Applying Itô's formula, we can write

$$\mathbb{E}|\hat{Y}_s^h|^2+\mathbb{E}\int_s^{t+h}\|\hat{Z}_r^h\|^2dr$$
$$\leq Ch(1+\mathbb{E}\sup_{s\leq r\leq t+h}|X_r^{t,x}|^2)+\mathbb{E}\int_s^{t+h}|\hat{Y}_r^h|^2dr+\frac{1}{2}\mathbb{E}\int_s^{t+h}\|\hat{Z}_r^h\|^2dr.$$

Combining Gronwall lemma and Burkholder-Davis-Gundy inequality to obtain

$$\mathbb{E}\sup_{t\leq s\leq t+h}|\hat{Y}_s^h|^2+\frac{1}{h}\mathbb{E}\int_s^{t+h}\|\hat{Z}_r^h\|^2dr\leq Ch. \tag{4.9}$$

Now, we suppose that

$$\partial_t\varphi(t,x)+\mathcal{L}^{\nu}\varphi(t,x)+f(t,x,u(t,x),(\varphi_k^1(t,x))_{k=1}^{\infty})<0.$$

Then, there exists $h_0>0$ and $\delta>0$ such that

$$\varepsilon_h=\frac{1}{h}\int_t^{t+h}\left[\left(\partial_t\varphi+\mathcal{L}^{\nu}\varphi\right)(r,X_r^{t,x})+f(r,\Lambda_r^{t,x})\right]dr\leq-\delta,\quad\forall\ 0<h<h_0,$$

where $\Lambda_r^{t,x} = \big(r, X_r^{t,x}, \varphi(r, X_r^{t,x}), (\varphi_k^1(r, X_r^{t,x}))_{k=1}^{\infty}\big)$. So, using (4.7), we have

$$\hat{Y}_t^h = \bar{Y}_t^h - \varphi(t,x) = \bar{Y}_t^h - u(t,x) = \bar{Y}_t^h - Y_t^{t,x} \geq 0.$$

Therefore, we deduce by using (4.9) that

$$\begin{aligned}
\delta &\leq \mathbb{E}|\frac{1}{h}\hat{Y}_t^h - \varepsilon_h| \leq \frac{1}{h}\mathbb{E}\int_t^{t+h} \Big(|\hat{Y}_r^h| + \|\hat{Z}_r^h\|\Big)\, dr \\
&\leq C\left(\mathbb{E}\sup_{t\leq s\leq t+h}|\hat{Y}_s^h|^2\right)^{1/2} + C\left(\frac{1}{h}\mathbb{E}\int_t^{t+h}\|\hat{Z}_r\|^2 dr\right)^{1/2} \\
&\leq Ch^{1/2}, \quad \forall\ 0 < h \leq h_0.
\end{aligned}$$

That is impossible. Consequently

$$-\partial_t\varphi(t,x) - \mathcal{L}^{\nu}\varphi(t,x) - f(t,x,u(t,x),(\varphi_k^1(t,x))_{k=1}^{\infty}) \leq 0.$$

By the same argument, we prove that u is a viscosity supersolution. From where u is a viscosity solution of the system (4.6). $\square$

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