Communications on Stochastic Analysis

Volume 2 | Number 2 Article 4

8-1-2008

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Recommended Citation

Balan, Raluca and Kim, Doyoon (2008) "The stochastic heat equation driven by a Gaussian noise: germ Markov property," *Communications on Stochastic Analysis*: Vol. 2 : No. 2 , Article 4.
DOI: 10.31390/cosa.2.2.04
Available at: https://digitalcommons.lsu.edu/cosa/vol2/iss2/4

Communications on Stochastic Analysis
Vol. 2, No. 2 (2008) 229-249

Serials Publications
www.serialspublications.com

THE STOCHASTIC HEAT EQUATION DRIVEN BY A GAUSSIAN NOISE: GERM MARKOV PROPERTY

RALUCA BALAN* AND DOYOON KIM

Abstract. Let $u = \{u(t,x); t \in [0,T], x \in \mathbb{R}^d\}$ be the process solution of the stochastic heat equation $u_t = \Delta u + \dot{F}, u(0,\cdot) = 0$ driven by a Gaussian noise $\dot{F}$, which is white in time and has spatial covariance induced by the kernel f. In this paper we prove that the process u is locally germ Markov, if f is the Bessel kernel of order $\alpha = 2k, k \in \mathbb{N}_+$, or f is the Riesz kernel of order $\alpha = 4k, k \in \mathbb{N}_+$.

1. Introduction

This article is based on the theory of stochastic partial differential equations (s.p.d.e.'s) initiated by John Walsh in 1986 [24]. This theory relies on the construction of a stochastic integral with respect to the so-called "worthy martingale measures" and focuses mainly on equations driven by a space-time white noise.

Recently, there has been a considerable amount of interest in s.p.d.e.'s driven by a noise term which is white in time, but is "colored" in space, in the sense that it has a spatial covariance structure induced by a kernel f, which is the Fourier transform of a tempered distribution μ. This line of research was initiated in [15], [4] for the stochastic wave equation in the case $d = 2$, and then generalized in [3], [6] to a larger class of s.p.d.e.'s in arbitrary spatial dimensions. The new theory requires an extension of Walsh's stochastic integral, to include the case when the integrand is a (Schwartz) distribution; this type of extension is needed for instance for the stochastic wave equation with $d \geq 3$. Under a certain integrability condition imposed on the measure μ, one can prove that any second-order s.p.d.e. with constant coefficients has a process solution, i.e. a solution which can be identified with a multiparameter stochastic process u. As far as we know, the literature to date does not contain any study of the germ Markov property of this process solution. Such a study is of great importance because it allows us to conclude that the behavior of the process in any time-space region A is independent of its behavior outside the region, given the values of the process in a thin area around the boundary ∂A of the region.

2000 *Mathematics Subject Classification.* Primary 60H15; Secondary 60G60.
Key words and phrases. Gaussian noise, stochastic integral, stochastic heat equation, Reproducing Kernel Hilbert Space, germ Markov property.
The first author was supported by a grant from the Natural Sciences and Engineering Research Council of Canada. The second author was supported by a post-doctoral fellowship at University of Ottawa.

There is an immense amount of literature dedicated to the study of different types of Markov properties for multiparameter processes (see [2] and the references therein). In particular, the germ Markov property of multiparameter Gaussian processes is an area which received a lot of attention in the 1970's, which allows for the use of various tools imported from Hilbert space analysis, like the method of the Reproducing Kernel Hilbert Space (RKHS). An excellent all-time reference is the monograph [20]; earlier references include [18], [14], [17].

In the 1990's, the problem of the germ Markov property for multi-dimensional Gaussian processes appears again in the literature, this time for processes which arise as solutions of s.p.d.e.'s driven by a white noise: the case of elliptic equations was thoroughly treated in [7], while the systematic study of this problem for the quasi-linear parabolic equations (in the case $d = 1$) is found in [16]. The case of hyperbolic equations turns out to be the most difficult one: the only reference here seems to be [5], in which the authors investigate very carefully the structure of the germ σ-fields induced by the process solution of the stochastic wave equation (in the case $d = 2$) driven by a Lévy noise without Gaussian component. (This type of noise induces a very particular form for the process solution, which is used in a fundamental way for deriving its germ Markov property.) Finally, the more recent work [19] contains a detailed analysis of the relationship between the germ and sharp σ-fields (which are used for defining the sharp and germ Markov property, respectively), in the case of the process solution associated to the Bessel equation driven by a white noise (in the case $d = 2$).

The goal of the present paper is to investigate the structure of the RKHS associated to the process solution u of the stochastic heat equation driven by a spatial covariance kernel f, and to identify some examples of functions f for which the process u possesses the germ Markov property. As far as we know, this is the first attempt to tackle a problem of this type in the literature. Our two main results (Theorem 4.9 and Theorem 4.15) state that the process solution u has the germ Markov property, if the function f is either the Bessel kernel of order $\alpha = 2k, k \in \mathbf{N}$ or the Riesz kernel of order $\alpha = 4k, k \in \mathbf{N}$. In order to prove these results, we use some results from the L_p-theory of parabolic equations with mixed norms, due to [12], [13]. The germ Markov property of the process solution u in the case of the Bessel or Riesz kernels with $\alpha > 0$ arbitrary (or in the case of other kernel functions) remains an open problem.

The paper is organized as follows. In Section 2, we introduce the framework for the study of s.p.d.e.'s, including the construction of the extended stochastic integral due to Dalang (as in [3]). In Section 3, we include all the ingredients which are necessary to formulate the result about the existence of the process solution u, and we examine the structure of the Gaussian space and the RKHS associated to this process. In Section 4, we give the definition of the germ Markov property and we prove our two main results, by applying a fundamental result of [14]. The two appendices contain some technical proofs.

2. The Framework

2.1. Basic Notation. We denote by $\mathcal{D}(U)$ the space of all infinitely differentiable functions on $\mathbb{R}^n$ whose support is compact and contained in the open set U, and by

$\mathcal{D}'(U)$ the space of *distributions*, i.e. continuous linear functionals on $\mathcal{D}(U)$. We denote by $\mathcal{S}(\mathbb{R}^n)$ the Schwartz space of all rapidly decreasing functions on $\mathbb{R}^n$, and by $\mathcal{S}'(\mathbb{R}^n)$ the space of *tempered distributions*, i.e. continuous linear functionals on $\mathcal{S}(\mathbb{R}^n)$. The space $\mathcal{S}'(\mathbb{R}^n)$ can be viewed as a subspace of $\mathcal{D}'(\mathbb{R}^n)$. An important subspace of $\mathcal{S}'(\mathbb{R}^n)$ is the space $\mathcal{O}'_C(\mathbb{R}^n)$ of all distributions with *rapid decrease* (see Chapter VII, section 5, [22]).

For an arbitrary function ϕ on $\mathbb{R}^n$, the translation by x is denoted with ϕ_x, i.e. $\phi_x(y) = \phi(x+y)$, and the reflection by 0 is denoted with $\tilde{\phi}$, i.e. $\tilde{\phi}(x) = \phi(-x)$. These notions have obvious extensions to distributions. The Fourier transform of $\phi \in L_1(\mathbb{R}^n)$ is defined by $\mathcal{F}\phi(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \exp(-i\xi \cdot x)\phi(x)dx$. The map $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ can be extended to $L_2(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$. For every subset $U \subset \mathbb{R}^n$, we denote

$$\langle \varphi, \psi \rangle_{L_2(U)} = \int_U \varphi(x)\overline{\psi(x)}dx,$$

whenever the integral is defined.

2.2. The Gaussian Noise. Let $T > 0$ and f be a locally integrable function on $\mathbb{R}^d$. As in [3], for every $\varphi, \psi \in \mathcal{D}((0,T) \times \mathbb{R}^d)$ we define

$$J_f(\varphi, \psi) = \langle \varphi, \psi \rangle_0 = \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(t,x) f(x-y) \psi(t,y) dy dx dt. \tag{2.1}$$

The Gaussian noise mentioned in the Introduction will be a zero-mean Gaussian process with covariance J_f. The existence of this process is based on the following Bochner–Schwartz type result (see Theorem 2, p. 157, [11]):

Lemma 2.1. *The bi-functional J_f is nonnegative-definite if and only if there exists a tempered measure μ on $\mathbb{R}^d$ such that*

$$f(x) = \int_{\mathbb{R}^d} e^{-i\xi \cdot x} \mu(d\xi), \quad \forall x \in R^d.$$

In this case, for every $\varphi, \psi \in \mathcal{D}((0,T) \times \mathbb{R}^d)$

$$\langle \varphi, \psi \rangle_0 = \int_0^T \int_{\mathbb{R}^d} \mathcal{F}\varphi(t,\xi) \overline{\mathcal{F}\psi(t,\xi)} \mu(d\xi) dt,$$

where $\mathcal{F}\varphi(t,\cdot)$ denotes the Fourier transform of $\varphi(t,\cdot)$.

The basic example of kernel functions is the white noise kernel: $f(x) = \delta(x)$, $\mu(d\xi) = d\xi$.

More interesting examples of kernel functions f arise when $\mu(d\xi) = g(|\xi|^2)d\xi$, for a certain function g on $(0,\infty)$. In this case, f becomes the inverse Fourier transform of the tempered distribution $\xi \mapsto g(|\xi|^2)$ and we can define the operator $g(-\Delta) : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ by $\mathcal{F}[g(-\Delta)\phi](\xi) = g(|\xi|^2)\mathcal{F}\phi(\xi)$, or equivalently

$$g(-\Delta)\phi = \phi * f.$$

(see p. 149-152, [10], or p. 117-132, [23]). Here are some typical examples:

Example 2.2. If $\mu(d\xi) = |\xi|^{-\alpha} d\xi$, then f is the Riesz kernel of order α:

$$f(x) = R_\alpha(x) := \gamma_\alpha |x|^{-d+\alpha}, \quad 0 < \alpha < d,$$

where γ_α is an appropriate constant. In this case, for every $\phi \in \mathcal{S}(\mathbb{R}^d)$ we have $(-\Delta)^{-\alpha/2}\phi = \phi * R_\alpha$.

Example 2.3. If $\mu(d\xi) = (1+|\xi|^2)^{-\alpha/2}d\xi$, then f is the Bessel kernel of order α:

$$f(x) = B_\alpha(x) := c_\alpha \int_0^\infty \tau^{(\alpha-d)/2-1} e^{-\tau-|x|^2/(4\tau)} d\tau, \quad \alpha > 0,$$

where c_α is an appropriate constant. In this case, for every $\phi \in \mathcal{S}(\mathbb{R}^d)$ we have $(1-\Delta)^{-\alpha/2}\phi = \phi * B_\alpha$.

Example 2.4. If $\mu(d\xi) = e^{-4\pi^2\alpha|\xi|^2}$, then f is the heat kernel

$$f(x) = G_\alpha(x) := (4\pi\alpha)^{-d/2} e^{-|x|^2/(4\alpha)}, \quad \alpha > 0.$$

In this case, for every $\phi \in \mathcal{S}(\mathbb{R}^d)$ we have $e^{\alpha\Delta}\phi = \phi * G_\alpha$.

In what follows, we let $F = \{F(\varphi); \varphi \in \mathcal{D}((0,T)\times\mathbb{R}^d)\}$ be a zero-mean Gaussian process with covariance J_f, i.e. $\forall \varphi, \psi \in \mathcal{D}((0,T)\times\mathbb{R}^d)$

$$E(F(\varphi)F(\psi)) = \langle \varphi, \psi \rangle_0. \tag{2.2}$$

The Gaussian space H^F of the process F is defined as the closed linear subspace of $L_2(\Omega)$, generated by the variables $\{F(\varphi), \varphi \in \mathcal{D}((0,T)\times\mathbb{R}^d)\}$.

2.3. The Stochastic Integral. In this subsection, we summarize the construction of the generalized stochastic integral $M(\varphi) = \int_0^T \int_{\mathbb{R}^d} \varphi(t,x) M(dt,dx)$ with respect to the martingale measure M induced by the noise F, due to Dalang (see [3]). For our purposes, it is enough to consider only the case of deterministic integrands. Here is the construction procedure:

Step 0. For each $\varphi \in \mathcal{D}((0,T)\times R^d)$, we set $M(\varphi) = F(\varphi) \in H^F$.

Step 1. Let $\mathcal{B}_b(\mathbb{R}^d)$ be the class of all bounded Borel subsets of $\mathbb{R}^d$ and $\mathcal{E}^{(d)}$ be the class of all linear combinations of functions $1_{[0,t]\times A}$, $t \in [0,T], A \in \mathcal{B}_b(\mathbb{R}^d)$. We endow $\mathcal{E}^{(d)}$ with the inner product $\langle \cdot, \cdot \rangle_0$ given by formula (2.1) and we denote by $\|\cdot\|_0$ the corresponding norm.

For each $t \in (0,T), A \in \mathcal{B}_b(\mathbb{R}^d)$, there exists a sequence $\{\varphi_n\}_n \subset \mathcal{D}((0,T)\times\mathbb{R}^d)$ such that $\varphi_n \to 1_{[0,t]\times A}$ and $\text{supp}\varphi_n \subseteq K$ for all n (see [4], p. 190). By the bounded convergence theorem, $\|\varphi_n - 1_{[0,t]\times A}\|_+ \to 0$, where $\|\cdot\|_+$ is the norm defined in Step 2 below. Hence $\|\varphi_n - 1_{[0,t]\times A}\|_0 \to 0$ and $\mathbb{E}(M(\varphi_m) - M(\varphi_n))^2 = \|\varphi_m - \varphi_n\|_0 \to 0$ as $m,n\to\infty$, i.e. the sequence $\{M(\varphi_n)\}_n$ is Cauchy in $L_2(\Omega)$. A standard argument shows that its limit does not depend on $\{\varphi_n\}_n$. We set $M_t(A) = M(1_{[0,t]\times A}) =_{L_2(\Omega)} \lim_n M(\varphi_n) \in H^F$. (Note that the process $M = \{M_t(A); t \in [0,T], A \in \mathcal{B}(\mathbb{R}^d)\}$ is a worthy martingale measure, in the sense of [24].) We extend M by linearity to $\mathcal{E}^{(d)}$. A limiting argument and relation (2.2) shows that, for every $\varphi, \psi \in \mathcal{E}^{(d)}$

$$\mathbb{E}(M(\varphi)M(\psi)) = \langle \varphi, \psi \rangle_0. \tag{2.3}$$

Step 2. Let $\mathcal{P}_+^{(d)} = \{\varphi : [0,T]\times\mathbb{R}^d \to \mathbb{R} \text{ measurable}; \|\varphi\|_+ < \infty\}$, where $\|\varphi\|_+^2 := \int_0^T \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\varphi(t,x)| f(x-y) |\varphi(t,y)| dydxdt$. We endow $\mathcal{P}_+^{(d)}$ with the inner product $\langle \cdot, \cdot \rangle_0$ given by formula (2.1) and we denote by $\|\cdot\|_0$ the corresponding norm; note that $\|\cdot\|_0 \leq \|\cdot\|_+$. An argument similar to that used in the proof of Proposition 2.3, [24], shows that $\mathcal{E}^{(d)}$ is dense in $\mathcal{P}_+^{(d)}$ with respect to $\|\cdot\|_+$, and hence for each $\varphi \in \mathcal{P}_+^{(d)}$, there exists a sequence $\{\varphi_n\}_n \subset \mathcal{E}^{(d)}$ such that $\|\varphi_n - \varphi\|_+ \to 0$. As in Step 1, we set $M(\varphi) =_{L_2(\Omega)} \lim_n M(\varphi_n) \in H^F$. Note that relation (2.3) holds for every $\varphi, \psi \in \mathcal{P}_+^{(d)}$.

Step 3. Let $\mathcal{P}_0^{(d)}$ be the completion of $\mathcal{E}^{(d)}$ with respect to $< \cdot, \cdot >_0$. The space $\mathcal{P}_0^{(d)}$ is the largest space of integrands φ for which we can define the stochastic integral $M(\varphi)$. According to [3],p. 9, the space $\mathcal{P}_0^{(d)}$ has the following alternative definition.

Let $\overline{\mathcal{P}}^{(d)} = \{\varphi : [0,T] \to \mathcal{S}'(\mathbb{R}^d);\ \mathcal{F}\varphi(t,\cdot) \text{ function} \forall t \in [0,T],\ (t,\xi) \mapsto \mathcal{F}\varphi(t,\xi)$ measurable, $\|\varphi\|_0 < \infty\}$, where $\|\varphi\|_0^2 = \int_0^T \int_{\mathbb{R}^d} |\mathcal{F}\varphi(t,\xi)|^2 \mu(d\xi)dt$. Let $\mathcal{E}_0^{(d)} = \{\varphi \in \mathcal{P}_+^{(d)}; \varphi(t,\cdot) \in \mathcal{S}(\mathbb{R}^d),\ \forall t \in [0,T]\}$ and $\mathcal{P}_0^{(d)}$ be the closure of $\mathcal{E}_0^{(d)}$ in $\overline{\mathcal{P}}^{(d)}$ with respect to $\|\cdot\|_0$. Note that

$$\langle \varphi, \psi \rangle_0 = \int_0^T \int_{\mathbb{R}^d} \mathcal{F}\varphi(t,\xi)\overline{\mathcal{F}\psi(t,\xi)}\mu(d\xi)dt, \quad \varphi, \psi \in \mathcal{P}_0^{(d)}.$$

For each $\varphi \in \mathcal{P}_0^{(d)}$, there exists a sequence $\{\varphi_n\}_n \subset \mathcal{E}_0^{(d)}$ such that $\|\varphi_n - \varphi\|_0 \to 0$. As in Step 1, we set $M(\varphi) =_{L_2(\Omega)} \lim_n M(\varphi_n) \in H^F$. Note that relation (2.3) holds for every $\varphi, \psi \in \mathcal{P}_0^{(d)}$, and hence

$$\varphi \mapsto M(\varphi) \text{ is an isometry between } \mathcal{P}_0^{(d)} \text{ and } H^F. \tag{2.4}$$

In summary, the previous construction is based on the diagram

$$\begin{array}{ccccc} \mathcal{D}((0,T)\times\mathbb{R}^d) \subset \mathcal{E}_0^{(d)} \subset & \mathcal{P}_+^{(d)} & \subset \mathcal{P}_0^{(d)} \subset \overline{\mathcal{P}}^{(d)} \\ & \cup & \\ & \mathcal{E}^{(d)} & \end{array}$$

and the following 3 approximation techniques:

- indicator functions can be approximated by functions in $\mathcal{D}((0,T)\times\mathbb{R}^d)$;
- functions in $\mathcal{P}_+^{(d)}$ can be approximated by indicator functions;
- *distributions* in $\mathcal{P}_0^{(d)}$ can be approximated by “smooth” functions in $\mathcal{P}_+^{(d)}$.

In particular, we conclude that $\mathcal{D}((0,T)\times\mathbb{R}^d)$ is dense in $\mathcal{P}_0^{(d)}$ with respect to $\|\cdot\|_0$.

2.4. Alternative Characterization of the Space $\mathcal{P}_0^{(d)}$. As in [9], for every $\varphi, \psi \in \mathcal{D}(\mathbb{R}^d)$, define

$$\langle \varphi, \psi \rangle_{0,x} := \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \varphi(x)f(x-y)\psi(y)dydx = \int_{\mathbb{R}^d} \mathcal{F}\varphi(\xi)\overline{\mathcal{F}\psi(\xi)}\,\mu(d\xi),$$

and let $\mathcal{P}_{0,x}^{(d)}$ be the completion of $\mathcal{D}(\mathbb{R}^d)$ with respect to $\langle\cdot,\cdot\rangle_{0,x}$. Note that

$$\langle \varphi, \psi \rangle_0 = \int_0^T \langle \varphi(t,\cdot), \psi(t,\cdot)\rangle_{0,x} dt, \quad \forall \varphi, \psi \in \mathcal{P}_0^{(d)}. \tag{2.5}$$

Remark 2.5. Note that for every $\varphi \in \mathcal{P}_0^{(d)}$, $\varphi(t,\cdot) \in \mathcal{P}_{0,x}^{(d)}$ for a.e. $t \in [0,T]$. Since $\mathcal{E}^{(d)}$ is dense in $\mathcal{P}^{(d)}$, one can prove that the map $t \mapsto \varphi(t,\cdot)$ is strongly measurable from $[0,T]$ to $\mathcal{P}_{0,x}^{(d)}$ (in the sense of Definition on p.649, [8]). Using (2.5), we conclude that $\mathcal{P}_0^{(d)} \subset L_2((0,T),\mathcal{P}_{0,x}^{(d)})$ and $\|\varphi\|_0 = \|\varphi\|_{L_2((0,T),\mathcal{P}_{0,x}^{(d)})}$, $\forall \varphi \in \mathcal{P}_0^{(d)}$.

3. The Process Solution

3.1. The Equation and its Solution. We consider the stochastic heat equation with vanishing initial conditions, written *formally* as:

$$u_t - \Delta u = \dot{F}, \quad \text{in } (0,T)\times\mathbb{R}^d, \quad u(0,\cdot)=0. \tag{3.1}$$

The solution of this equation is defined formally as follows. Suppose for the moment that $\dot{F}$ is a random variable with values in $\mathcal{S}'(\mathbb{R}^{d+1})$. For every fixed $\omega \in \Omega$, let $\{u(\varphi); \varphi \in \mathcal{D}((0,T)\times\mathbb{R}^d)\}$ be the distribution solution of (3.1). It is known that

$$u(\varphi) = (G * \dot{F})(\varphi) = \dot{F}(\varphi * \tilde{G}) \tag{3.2}$$

where G is the fundamental solution of the heat equation:

$$G(t,x) = \begin{cases} (4\pi t)^{-d/2}\exp\left(-\frac{|x|^2}{4t}\right) & \text{if } t>0, x\in R^d \\ 0 & \text{if } t\le 0, x \in R^d \end{cases}$$

(Since $G \in \mathcal{O}'_C(\mathbb{R}^{d+1})$, we have $\varphi * \tilde{G} \in \mathcal{S}(\mathbb{R}^{d+1})$ for every $\varphi \in \mathcal{D}(\mathbb{R}^{d+1})$ and the convolution $G * \dot{F}$ is well defined; see Theorem XI, Chapter VII, [22].)

Going back to our framework, we have the following lemma.

Lemma 3.1. *If*

$$\int_{R^d} \frac{1}{1+|\xi|^2}\mu(d\xi) < \infty, \tag{3.3}$$

then: (a) $(G_{tx})\tilde{} \in \mathcal{P}_+^{(d)}$ *for every* $(t,x) \in [0,T]\times\mathbb{R}^d$*; (b)* $\varphi * \tilde{G} \in \mathcal{P}_0^{(d)}$ *for every* $\varphi \in \mathcal{D}((0,T)\times\mathbb{R}^d)$.

Proof. (a) Set $g_{t,x} = (G_{tx})\tilde{}$. Note that g_{tx} is measurable and $\|g_{tx}\|_+ = \|g_{tx}\|_0$, since $g_{tx} \ge 0$. We have $\mathcal{F}g_{t,x}(s,\xi) = c_d 1_{t>s}\exp(-\mathrm{i}\xi x - (t-s)|\xi|^2)$, where c_d is an appropriate constant. Hence

$$\|g_{tx}\|_0^2 = \int_0^T \int_{\mathbb{R}^d} |\mathcal{F}g_{t,x}(s,\xi)|^2\,\mu(d\xi)\,ds \le N(T)\int_{\mathbb{R}^d}\frac{1}{1+|\xi|^2}\,\mu(d\xi) < \infty,$$

where $N(T)$ is a constant depending on T.

(b) We apply Remark 4, [3] to the function $\psi = \varphi * \tilde{G}$. For this we need to check that: (i) $t \mapsto \mathcal{F}\psi(t,\xi)$ is continuous $\forall \xi \in R^d$; (ii) there exists a nonnegative function $k(t,\xi)$ which is square-integrable with respect to $dt \times \mu(d\xi)$ such that $|\mathcal{F}\psi(t,\xi)| \le k(t,\xi)$, $\forall t \in [0,T], \xi \in R^d$.

(i) is clearly satisfied, and (ii) will follow from (3.3) once we prove that

$$|\mathcal{F}\psi(t,\xi)| \le \frac{N}{1+|\xi|^2} := k(t,\xi), \quad \forall t\in[0,T],\ \forall \xi \in \mathbb{R}^d, \tag{3.4}$$

where N is a constant. Since $(1+|\xi|^2)|\mathcal{F}\psi(t,x)| = |\mathcal{F}\phi(t,\xi)| \leq \|\phi(t,\cdot)\|_{L_1(\mathbb{R}^d)}$ where $\phi = (1-\Delta)\psi$, relation (3.4) follows if we prove that $\|\phi(t,\cdot)\|_{L_1(\mathbb{R}^d)} \leq N$. Note that

$$-\phi_t - \Delta\phi = (1-\Delta)\varphi \quad \text{in} \quad (0,T)\times\mathbb{R}^d, \quad \phi(T,x)=0,$$

since ψ is the unique solution of: $-\psi_t - \Delta\psi = \varphi$ in $(0,T)\times\mathbb{R}^d$, $\psi(T,x)=0$. Thus $\phi(s,y) = \int_s^T \int_{\mathbb{R}^d} G(t-s,x-y)(1-\Delta)\varphi(t,x)\,dx\,dt$. From this we calculate $\|\phi(s,\cdot)\|_{L_1(\mathbb{R}^d)}$, which turns out to be bounded (note that $(1-\Delta)\varphi$ has a compact support in $(0,T)\times\mathbb{R}^d$). □

A consequence of Lemma 3.1.(b) is the fact that $M(\varphi * \tilde{G})$ is well-defined for every $\varphi \in \mathcal{D}((0,T)\times\mathbb{R}^d)$. By analogy with (3.2) (and using a slight abuse of terminology), we introduce the following definition:

Definition 3.2. The process $\{u(\varphi); \varphi \in \mathcal{D}((0,T)\times\mathbb{R}^d)\}$ defined by

$$u(\varphi) := M(\varphi * \tilde{G}) = \int_0^T \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}_+} \int_{\mathbb{R}^d} \varphi(t+s,x+y)G(s,y)dyds \right) M(dt,dx)$$

is called the *distribution-valued solution* of the stochastic heat equation (3.1), with vanishing initial conditions.

Theorem 3.3 ([3], [6]). *Let $\{u(\varphi); \varphi \in \mathcal{D}((0,T)\times\mathbb{R}^d)\}$ be the distribution-valued solution of the stochastic heat equation (3.1). In order that there exists a jointly measurable process $X = \{X(t,x); t\in[0,T], x\in\mathbb{R}^d\}$ such that*

$$u(\varphi) = \int_0^T \int_{\mathbb{R}^d} X(t,x)\varphi(t,x)dxdt \quad \forall \varphi \in \mathcal{D}((0,T)\times\mathbb{R}^d) \quad a.s.$$

it is necessary and sufficient that (3.3) holds. In this case, X is a modification of the process $u = \{u(t,x); t\in[0,T], x\in\mathbb{R}^d\}$ defined by

$$u(t,x) := M((G_{tx})^{\tilde{}}) = \int_0^T \int_{\mathbb{R}^d} G(t-s,x-y)M(ds,dy). \tag{3.5}$$

Definition 3.4. The process $u = \{u(t,x); t\in[0,T], x\in\mathbb{R}^d\}$ defined by (3.5) is called the *process solution* of the stochastic heat equation (3.1), with vanishing initial conditions.

Note that the process u is a zero-mean Gaussian process. In the present paper, we are interested in examining the germ Markov property of the process u.

3.2. The Gaussian space. The Gaussian space H^u of the process u is defined as the closed linear subspace of $L_2(\Omega)$, generated by the variables $\{u(t,x), t \in [0,T], x\in\mathbb{R}^d\}$. The next result shows that this space coincides with the space H^F.

Lemma 3.5. *We have $H^u = H^F$.*

Proof. a) First, we prove that $H^u \subseteq H^F$. Recall that $u(t,x) = M((G_{tx})^{\tilde{}})$ and $(G_{tx})^{\tilde{}} \in \mathcal{P}_+^{(d)} \subset \mathcal{P}_0^{(d)}$, by Lemma 3.1.(a). Since $\mathcal{D}((0,T)\times\mathbb{R}^d)$ is dense in $\mathcal{P}_0^{(d)}$ with respect to $\|\cdot\|_0$, there exists a sequence $\{\varphi_n\}_{n\geq1} \subseteq \mathcal{D}((0,T)\times\mathbb{R}^d)$ such that

$\|\varphi_n - (G_{tx})\tilde{}\,\|_0 \to 0$. Hence $\mathbb{E}(M(\varphi_n) - u(t,x))^2 \to 0$. But $M(\varphi_n) \in H^F$ for all n and therefore $u(t,x) \in H^F$.

b) Let H^u_* be the closed linear subspace of $L_2(\Omega)$, generated by the variables $\{u(\eta), \eta \in \mathcal{D}((0,T)\times\mathbb{R}^d)\}$. To prove that $H^F \subseteq H^u_*$, let $\varphi \in \mathcal{D}((0,T)\times\mathbb{R}^d)$ be arbitrary and $\eta = -\varphi_t - \Delta\varphi \in \mathcal{D}((0,T)\times\mathbb{R}^d)$. Then $\varphi = \eta * \tilde{G}$ and $F(\varphi) = M(\varphi) = M(\eta * \tilde{G}) = u(\eta) \in H^u_*$.

c) Finally, we prove that $H^u_* \subseteq H^u$. Let $\eta \in \mathcal{D}((0,T)\times\mathbb{R}^d)$ be arbitrary. Note that $u(\eta) = M(\varphi)$ where $\varphi = \eta * \tilde{G}$. Let K be a compact set such that $\operatorname{supp}\eta \subset [0,T]\times K$.

For each $n \geq 1$, let $\{Q^{(n)}_m, m \in \mathbb{Z}\}$ be a partition of $(0,T)\times\mathbb{R}^d$ such that each $Q^{(n)}_m = R^{(n)}_m \times S^{(n)}_m$, where $R^{(n)}_m$ is an interval in $[0,T]$ and $S^{(n)}_m$ is a cube in $\mathbb{R}^d$. Suppose that $\max_{m\in\mathbb{Z}} |Q^{(n)}_m| \to 0$ as $n \to \infty$. For each $Q^{(n)}_m$, we choose $(t^{(n)}_m, x^{(n)}_m) \in Q^{(n)}_m$ such that $t^{(n)}_m \geq t$ for all $t \in R^{(n)}_m$. We consider the Riemann sum:

$$\varphi_n(s,y) = \sum_{m\in I_n} |Q^{(n)}_m| G(t^{(n)}_m - s, x^{(n)}_m - y)\eta(t^{(n)}_m, x^{(n)}_m),$$

where $I_n = \{m \in \mathbb{Z}; t^{(n)}_m > s, x^{(n)}_m \in K\}$. Clearly $\varphi_n(s,y) \to \varphi(s,y)$ for every (s,y). We claim that

$$\|\varphi_n - \varphi\|_0 \to 0. \tag{3.6}$$

From here, it follows that $\mathbb{E}(M(\varphi_n)-M(\varphi))^2 \to 0$. This concludes that proof, since $\varphi_n = \sum_{m\in I_n} a^{(n)}_m (G_{t^{(n)}_m, x^{(n)}_m})\tilde{}$ and hence $M(\varphi_n) = \sum_{m\in I_n} a^{(n)}_m u(t^{(n)}_m, x^{(n)}_m) \in H^u$, where $a^{(n)}_m = |Q^{(n)}_m|\ \eta(t^{(n)}_m, x^{(n)}_m)$. The proof of (3.6) is given in Appendix A. □

3.3. The RKHS. Let $\mathcal{H}^u = \{h_Y; h_Y(t,x) = \mathbb{E}(Yu(t,x)), Y \in H^u\}$ be the RKHS of the process u, endowed with the inner product $\langle h_Y, h_Z\rangle_{\mathcal{H}^u} := \mathbb{E}(YZ)$, $Y, Z \in H^u$. Note that any function $h \in \mathcal{H}^u$ is continuous on $[0,T]\times\mathbb{R}^d$ and satisfies $h(0,\cdot) = 0$. Moreover, $\mathcal{H}^u$ is the closure of $\{R((t,x),\cdot\,); (t,x) \in [0,T]\times\mathbb{R}^d\}$ with respect to $\|\cdot\|_{\mathcal{H}^u}$, where $R((t,x),(s,y)) = \mathbb{E}(u(t,x)u(s,y))$.

By Lemma 3.5 and (2.4), it follows that

$$\mathcal{H}^u = \{h(t,x) = \mathbb{E}(M(\varphi)u(t,x)); \varphi \in \mathcal{P}^{(d)}_0\}.$$

Moreover, if $h(t,x) = \mathbb{E}(M(\varphi)u(t,x))$, $g(t,x) = \mathbb{E}(M(\eta)u(t,x))$ with $\varphi, \eta \in \mathcal{P}^{(d)}_0$, then $\langle h, g\rangle_{\mathcal{H}^u} = \langle \varphi, \eta\rangle_0$.

Let $g_{t,x} = (G_{tx})\tilde{}$. Using the fact that $u(t,x) = M(g_{tx})$ and (2.5), we get:

$$h(t,x) = \mathbb{E}(M(\varphi)u(t,x)) = \langle \varphi, g_{tx}\rangle_0 = \int_0^t \int_{\mathbb{R}^d} \mathcal{F}\varphi(s,\xi)\overline{\mathcal{F}g_{t,x}(s,\xi)}\,\mu(d\xi)\,ds. \tag{3.7}$$

Lemma 3.6. *Let $h(t,x) = \mathbb{E}(M(\varphi)u(t,x)), \varphi \in \mathcal{P}^{(d)}_0$. For $\eta \in \mathcal{D}((0,T)\times\mathbb{R}^d)$, let ϕ be a solution to the equation $-\phi_t - \Delta\phi = \eta$ in $(0,T)\times\mathbb{R}^d$, $\phi(T,\cdot) = 0$. Then*

$$\langle h, \eta\rangle_{L_2((0,T)\times\mathbb{R}^d)} = \langle \varphi, \phi\rangle_0.$$

In particular, for any $\phi \in \mathcal{D}((0,T)\times\mathbb{R}^d)$, we have

$$\langle h, -\phi_t - \Delta\phi\rangle_{L_2((0,T)\times\mathbb{R}^d)} = \langle \varphi, \phi\rangle_0.$$

Proof. Using (3.7) and applying Fubini's theorem (since $\varphi \in \mathcal{P}_0^{(d)}$, $g_{tx} \in \mathcal{P}_+^{(d)}$, and η has compact support), we get

$$\langle h, \eta \rangle_{L_2((0,T)\times\mathbb{R}^d)} = \int_0^T \langle h(s,\cdot), \eta(s,\cdot)\rangle_{L_2(\mathbb{R}^d)}\, ds$$

$$= \int_0^T \int_{\mathbb{R}^d} \mathcal{F}\varphi(r,\xi) \int_r^T \int_{\mathbb{R}^d} \overline{\mathcal{F}g_{s,x}(r,\xi)\, \eta(s,x)}\, dx\, ds\, \mu(d\xi)\, dr. \quad (3.8)$$

Note that for every $r \in (0,T)$ and $y \in \mathbb{R}^d$

$$\phi(r,y) = \int_r^T \int_{\mathbb{R}^d} G(s-r, x-y)\eta(s,x)\, dx\, ds.$$

Hence

$$\mathcal{F}\phi(r,\xi) = \int_r^T \int_{\mathbb{R}^d} \mathcal{F}g_{s,x}(r,\xi)\eta(s,x)\, dx\, ds \quad (3.9)$$

From (3.8) and (3.9), we conclude that

$$\langle h, \eta \rangle_{L_2((0,T)\times\mathbb{R}^d)} = \langle \varphi, \phi \rangle_0$$

(recall that $\phi \in \mathcal{P}_0^{(d)}$ by Lemma 3.1). This finishes the proof. □

4. The Germ Markov Property

4.1. The Definition. Let $S \subseteq [0,T] \times \mathbb{R}^d$ be an arbitrary set. Let $\mathcal{F}_S^u$ be the σ-field generated by the variables $\{u(t,x); (t,x) \in S\}$, K_S^u be the closed linear subspace of $L_2(\Omega)$ generated by the variables $\{u(t,x); (t,x) \in S\}$, and $\mathcal{K}_S^u$ be the closed linear subspace of $\mathcal{H}^u$ generated by the functions $\{R((t,x),\cdot\,); (t,x) \in S\}$. Let

$$\mathcal{G}_S^u = \bigcap_{O \text{ open};\ O \supset S} \mathcal{F}_O^u, \quad H_S^u = \bigcap_{O \text{ open};\ O \supset S} K_O^u, \quad \mathcal{H}_S^u = \bigcap_{O \text{ open};\ O \supset S} \mathcal{K}_O^u.$$

Definition 4.1. The process $u = \{u(t,x); (t,x) \in [0,T] \times \mathbb{R}^d\}$ is *locally germ Markov* if for every relatively compact open set $A \subset [0,T] \times \mathbb{R}^d$, $\mathcal{G}_{\overline{A}}^u$ and $\mathcal{G}_{\overline{A^c}}^u$ are conditionally independent given $\mathcal{G}_{\partial A}^u$, where $\partial A = \overline{A} \cap \overline{A^c}$.

Based on the fact that $\mathcal{G}_S^u = \sigma(H_S^u)$, one can prove that u is locally germ Markov if and only if for every open set A, $H^u = H_{\overline{A}}^u \oplus (H_{\overline{A^c}}^u \ominus H_{\partial A}^u)$ (see Lemma 1.3, [14]), or equivalently, using the isometry between H^u and $\mathcal{H}^u$

$$\mathcal{H}^u = \mathcal{H}_{\overline{A}}^u \oplus (\mathcal{H}_{\overline{A^c}}^u \ominus \mathcal{H}_{\partial A}^u). \quad (4.1)$$

(Here $\oplus, \ominus$ denote the usual operations on Hilbert subspaces: if H is a Hilbert space, S is a closed subspace and $S^\perp$ is its orthogonal complement, then we write $H = S \oplus S^\perp$ and $S^\perp = H \ominus S$.)

On the other hand, $h(t,x) = \langle h, R((t,x),\cdot\,)\rangle_{\mathcal{H}^u}$ for every $h \in \mathcal{H}^u$. Hence

$$\text{supp}\, h \subseteq B^c \quad \text{if and only if} \quad h \in (\mathcal{H}_B^u)^\perp. \quad (4.2)$$

Based on (4.1) and (4.2), we have the following fundamental result.

Theorem 4.2 (Theorem 5.1, [14])**.** *The Gaussian process u is locally germ Markov if and only if the following two conditions hold:*

(i) If $h, g \in \mathcal{H}^u$ are such that (supp h) $\cap$ (supp g) $= \emptyset$ *and* supp h *is compact, then* $< h, g >_{\mathcal{H}^u} = 0$.

(ii) If $\zeta = h + g \in \mathcal{H}^u$, where h, g are such that (supp h) $\cap$ (supp g) $= \emptyset$ *and* supp h *is compact, then $h, g \in \mathcal{H}^u$.*

In the next two subsections, we will suppose that f is the Bessel kernel of order $\alpha > 0$ (Example 2.3), respectively the Riesz kernel of order $0 < \alpha < d$ (Example 2.2). In both cases we must have $\alpha > d - 2$, in order to have condition (3.3) satisfied. Our goal is to prove that conditions (i) and (ii) of Theorem 4.2 hold. For this, we will assume that $\alpha = 2k, k \in \mathbb{N}_+$ in the case of the the Bessel kernel, respectively $\alpha = 4k, k \in \mathbb{N}_+$ in the case of the Riesz kernel (and $d = 4k + 1$, since $d - 2 < \alpha < d$).

4.2. The Case of the Bessel Kernel. In this subsection we will assume that f is the Bessel kernel, i.e. $f = B_\alpha$ with $\alpha > \max\{0, d-2\}$.

In this case, $\mathcal{P}_{0,x}^{(d)} = H_2^{-\alpha/2}(\mathbb{R}^d)$, where

$$H_2^\gamma(\mathbb{R}^d) = \{\varphi \in \mathcal{S}'(\mathbb{R}^d); \mathcal{F}\varphi \text{ is a function}, \|\varphi\|_{\gamma,2}^2 = \int_{\mathbb{R}^d} |\mathcal{F}\varphi(\xi)|^2 (1+|\xi|^2)^\gamma d\xi < \infty\}$$

denotes the fractional Sobolev space of index $\gamma \in \mathbb{R}$ (see e.g. [10], p.191-192).

By Remark 2.5, $\mathcal{P}_0^{(d)} \subset L_2((0,T), H_2^{-\alpha/2}(\mathbb{R}^d))$. On the other hand, we have the denseness of $\mathcal{E}_0^{(d)}$ in $L_2((0,T), H_2^{-\alpha/2}(\mathbb{R}^d))$ (for a proof, one may see the proof of Theorem 3.10 in [12]). Thus $\mathcal{P}_0^{(d)} = L_2((0,T), H_2^{-\alpha/2}(\mathbb{R}^d))$ and

$$\|\varphi\|_0 = \|\varphi\|_{L_2((0,T), H_2^{-\alpha/2}(\mathbb{R}^d))}, \quad \forall \varphi \in \mathcal{P}_0^{(d)}.$$

For each $t \in [0,T]$, let $\varphi_1(t,\cdot) := (1-\Delta)^{-\alpha/2}\varphi(t,\cdot)$ and note that the map $\varphi \mapsto \varphi_1$ is an isometry between $L_2((0,T), H_2^{-\alpha/2}(\mathbb{R}^d))$ and $L_2((0,T), H_2^{\alpha/2}(\mathbb{R}^d))$.

Since $\mathcal{F}\varphi_1(t,\xi) = (1+|\xi|^2)^{-\alpha/2}\mathcal{F}\varphi(t,\xi)$, it is not difficult to see that: $\forall \varphi \in L_2((0,T), H_2^{-\alpha/2}(\mathbb{R}^d))$

$$\langle \varphi_1, \phi \rangle_{L_2((0,T)\times\mathbb{R}^d)} = \langle \varphi, \phi \rangle_0, \quad \forall \phi \in \mathcal{D}((0,T)\times\mathbb{R}^d). \tag{4.3}$$

To investigate the relationship between h and φ_1, we need some general results from the L_p-theory of parabolic equations.

Recall that $H_2^0(\mathbb{R}^d) = L_2(\mathbb{R}^d)$, and $H_2^\gamma(\mathbb{R}^d) \subset H_2^{\gamma'}(\mathbb{R}^d)$ if $\gamma > \gamma'$. For every $\gamma, \beta \in \mathbb{R}$, $(1-\Delta)^{\gamma/2}$ is a unitary isomorphism between $H_2^\beta(\mathbb{R}^d)$ and $H_2^{\gamma-\beta}(\mathbb{R}^d)$. In particular, $(1-\Delta)^{\gamma/2}$ is a unitary isomorphism between $H_2^\gamma(\mathbb{R}^d)$ and $L_2(\mathbb{R}^d)$. For every $v \in H_2^\gamma(\mathbb{R}^d)$, $\phi \in \mathcal{D}(\mathbb{R}^d)$, we denote

$$(v, \phi) = \int_{\mathbb{R}^d} [(1-\Delta)^{\gamma/2} v](x) \overline{[(1-\Delta)^{-\gamma/2}\phi](x)} dx.$$

Note that, if $v \in H_2^\gamma(\mathbb{R}^d)$ and $\gamma \geq 0$, then $(v, \phi) = \langle v, \phi \rangle_{L_2(\mathbb{R}^d)}$ for all $\phi \in \mathcal{D}(\mathbb{R}^d)$.

Definition 4.3. If $t \mapsto v(t,\cdot)$ is a function from $[0,T]$ to $H_2^\gamma(\mathbb{R}^d)$, with $\gamma \in \mathbb{R}$, we say that v is a *solution* of

$$dv = (\Delta v + g)dt \quad \text{in} \quad (0,T)\times\mathbb{R}^d, \quad v(0,\cdot)=0$$

if for any $t\in(0,T)$ and for any $\psi \in \mathcal{D}(\mathbb{R}^d)$, we have

$$(v(t,\cdot),\psi) = \int_0^t (v(s,\cdot),\Delta\psi)ds + \int_0^t (g(s,\cdot),\psi)ds. \tag{4.4}$$

We write $v \in \mathcal{H}_{2,0}^\gamma(T)$ if $v_{xx} \in L_2((0,T),H_2^{\gamma-2}(\mathbb{R}^d))$, $v(0,\cdot)=0$, and there exists $g \in L_2((0,T),H_2^{\gamma-2}(\mathbb{R}^d))$ satisfying (4.4). By $\|v\|_{\mathcal{H}_{2,0}^\gamma(T)}$ we mean

$$\|v\|_{\mathcal{H}_{2,0}^\gamma(T)} = \|v_{xx}\|_{L_2((0,T),H_2^{\gamma-2}(\mathbb{R}^d))} + \|g\|_{L_2((0,T),H_2^{\gamma-2}(\mathbb{R}^d))}.$$

Remark 4.4. It is known that (4.4) implies that: $\forall t\in(0,T)$, $\forall \phi \in \mathcal{D}((0,T)\times\mathbb{R}^d)$

$$(v(t,\cdot),\phi(t,\cdot)) = \int_0^t (v(s,\cdot),(\partial/\partial t+\Delta)\phi(s,\cdot))ds + \int_0^t (g(s,\cdot),\phi(s,\cdot))ds$$

(see e.g. Proposition 10, [9] for a stochastic version of this result). In particular, (4.4) implies that: $\forall \phi \in \mathcal{D}((0,T)\times\mathbb{R}^d)$

$$\int_0^T (v(s,\cdot),(-\partial/\partial t-\Delta)\phi(s,\cdot))ds = \int_0^T (g(s,\cdot),\phi(s,\cdot))ds \tag{4.5}$$

(by taking $t=T$ and using the fact that $\phi(T,\cdot)=0$).

Theorem 4.5 ([12], [13])**.** *Given $g \in L_2((0,T),H_2^\gamma(\mathbb{R}^d))$, $\gamma\in\mathbb{R}$, there exists a unique solution $v \in \mathcal{H}_{2,0}^{\gamma+2}(T)$ to the equation*

$$dv = (\Delta v + g)dt \quad \textit{in} \quad (0,T)\times\mathbb{R}^d, \quad v(0,\cdot)=0.$$

Moreover, there exists a constant N (independent of v) such that

$$\|v\|_{\mathcal{H}_{2,0}^{\gamma+2}(T)} \le N\|g\|_{L_2((0,T),L_2(\mathbb{R}^d))}.$$

We now return to our framework.

Theorem 4.6. *Let $h(t,x) = \mathbb{E}(M(\varphi)u(t,x))$, where u is defined in* (3.5) *and $\varphi \in \mathcal{P}_0^{(d)}$. Set $\varphi_1 = (1-\Delta)^{-\alpha/2}\varphi$. Then h is the unique solution in $\mathcal{H}_{2,0}^{\alpha/2+2}(T)$ to the equation*

$$dh = (\Delta h + \varphi_1)\,dt \quad \textit{in } (0,T)\times\mathbb{R}^d, \qquad h(0,\cdot)=0. \tag{4.6}$$

Proof. Recall that $\varphi_1 \in L_2((0,T),H_2^{\alpha/2}(\mathbb{R}^d))$. Thus by Theorem 4.5 there exists a unique solution $v \in \mathcal{H}_{2,0}^{\alpha/2+2}(T)$ to the equation (4.6).

We are now proving that $h=v$. This will follow once we prove that

$$\langle h,\eta\rangle_{L_2((0,T)\times\mathbb{R}^d)} = \langle v,\eta\rangle_{L_2((0,T)\times\mathbb{R}^d)}, \quad \forall \eta\in\mathcal{D}((0,T)\times\mathbb{R}^d).$$

Let $\eta \in \mathcal{D}((0,T)\times\mathbb{R}^d)$ be arbitrary and ϕ be the unique solution of

$$-\phi_t - \Delta\phi = \eta \quad \text{in } (0,T)\times\mathbb{R}^d, \quad \phi(T,\cdot)=0.$$

By Lemma 3.6 and (4.3)

$$\langle h,\eta\rangle_{L_2((0,T)\times\mathbb{R}^d)} = \langle\varphi,\phi\rangle_0 = \langle\varphi_1,\phi\rangle_{L_2((0,T)\times\mathbb{R}^d)}.$$

On the other hand, by (4.5) we have

$$\langle v, \eta \rangle_{L_2((0,T)\times\mathbb{R}^d)} = \langle \varphi_1, \phi \rangle_{L_2((0,T)\times\mathbb{R}^d)}$$

This concludes the proof. □

Corollary 4.7. *If f is the Bessel kernel of order α, then*

$$\mathcal{H}^u = \mathcal{H}^{\alpha/2+2}_{2,0}(T)$$

and the norms in the two spaces are equivalent.

Proof. From the argument at the beginning of subsection 3.3, for every $h \in \mathcal{H}^u$, we have $h(t,x) = \mathbb{E}M(\varphi)u(t,x)$, where u is defined in (3.5) and $\varphi \in \mathcal{P}_0^{(d)}$. Then by Theorem 4.6, $h \in \mathcal{H}^{\alpha/2+2}_{2,0}(T)$.

For $v \in \mathcal{H}^{\alpha/2+2}_{2,0}(T)$, there is a $\varphi_1 \in L_2((0,T), H_2^{\alpha/2}(\mathbb{R}^d))$ satisfying (4.4) with φ_1 in place of g. Then

$$\varphi := (1-\Delta)^{\alpha/2}\varphi_1 \in L_2((0,T), H_2^{-\alpha/2}(\mathbb{R}^d)) = \mathcal{P}_0^{(d)}.$$

Set $h(t,x) = \mathbb{E}M(\varphi)u(t,x) \in \mathcal{H}^u$, then by Theorem 4.6 it follows that $v = h$, so $v \in \mathcal{H}^u$.

Now notice that

$$\begin{aligned}\|h\|_{\mathcal{H}^u} = \|\varphi\|_0 = \|\varphi\|_{L_2((0,T),H_2^{-\alpha/2}(\mathbb{R}^d))} &= \|(1-\Delta)^{-\alpha/2}\varphi\|_{L_2((0,T),H_2^{\alpha/2}(\mathbb{R}^d))} \\ &= \|\varphi_1\|_{L_2((0,T),H_2^{\alpha/2}(\mathbb{R}^d))} \simeq \|h\|_{\mathcal{H}^{\alpha/2+2}_{2,0}(T)},\end{aligned}$$

where $\simeq$ indicates the equivalence of the norms, which follows from the definition of the norm of $\mathcal{H}^{\alpha/2+2}_{2,0}(T)$ and the estimate in Theorem 4.5. This finishes the proof. □

Remark 4.8. Under the conditions of Theorem 4.6, we can also say that h is the unique solution in $W_2^{1,2}((0,T)\times\mathbb{R}^d)$ of:

$$h_t = \Delta h + \varphi_1 \quad \text{in} \quad (0,T)\times\mathbb{R}^d, \quad h(0,\cdot) = 0.$$

Here $W_2^{1,2}((0,T)\times\mathbb{R}^d)$ is the space of all measurable functions $v : (0,T)\times\mathbb{R}^d \to \mathbb{R}$, such that the weak derivatives v_t, v_{x_i}, $v_{x_ix_j}$ exist and are in $L_2((0,T)\times\mathbb{R}^d)$.

We are now ready to prove the main result of this subsection.

Theorem 4.9. *Suppose that f is the Bessel kernel of order $\alpha = 2k, k \in \mathbb{N}_+$ such that $\alpha > d-2$. Then the process solution u of the stochastic heat equation (3.1) with vanishing initial conditions is locally germ Markov.*

Proof. We need to verify conditions (i) and (ii) of Theorem 4.2.

We first verify condition (i). Let $h, g \in \mathcal{H}^u$ are such that $(\text{supp } h)\cap(\text{supp } g) = \emptyset$ and supp h is compact. We have to prove that $\langle h, g\rangle_{\mathcal{H}^u} = 0$. We know that there exist $\varphi, \eta \in \mathcal{P}_0^{(d)}$ such that $h(t,x) = \mathbb{E}(M(\varphi)u(t,x))$ and $g(t,x) = \mathbb{E}(M(\eta)u(t,x))$. Then by Theorem 4.6 and Corollary 4.7 (also recall the definition of the norm of $\mathcal{H}^{\gamma}_{2,0}(T)$),

$$\langle h, g\rangle_{\mathcal{H}^u} = \langle h, g\rangle_{\mathcal{H}^{k+2}_{2,0}(T)} = \langle h_{xx}, g_{xx}\rangle_{L_2((0,T),H_2^k(\mathbb{R}^d))} + \langle \varphi_1, \eta_1\rangle_{L_2((0,T),H_2^k(\mathbb{R}^d))},$$

where $\varphi_1 = (1-\Delta)^k\varphi$ and $\eta_1 = (1-\Delta)^k\eta$. Note that (see also Remark 4.8)

$$\operatorname{supp} h_{xx} \subset \operatorname{supp} h, \quad \operatorname{supp} \varphi_1 \subset \operatorname{supp} h.$$

Similar inclusions hold for g and η_1. Thus it is clear that

$$\langle h_{xx}, g_{xx}\rangle_{L_2((0,T),H_2^k(\mathbb{R}^d))} = \langle \varphi_1, \eta_1\rangle_{L_2((0,T),H_2^k(\mathbb{R}^d))} = 0,$$

from which we arrive at $\langle h, g\rangle_{\mathcal{H}^u} = 0$. This finishes the proof of the condition (i).

To prove the condition (ii), let $\zeta = h + g \in \mathcal{H}^u$, where h and g are such that $(\operatorname{supp} h) \cap (\operatorname{supp} g) = \emptyset$ and $\operatorname{supp} h$ is compact. We have to prove that $h \in \mathcal{H}^u$.

Let χ be an infinitely differentiable function such that $\chi = 1$ on $\operatorname{supp} h$ and $\chi = 0$ on an open set containing $\operatorname{supp} g$. By Corollary 4.7, it follows that $\zeta \in \mathcal{H}_{2,0}^{k+2}(T)$, and hence $h = \chi\zeta \in \mathcal{H}_{2,0}^{k+2}(T) = \mathcal{H}^u$. The theorem is proved. □

4.3. The Case of the Riesz Kernel. In this subsection, we will assume that f is the Riesz kernel, i.e. $f = R_\alpha$ with $\max\{d-2, 0\} < \alpha = 4k < d$, $k \in \mathbb{N}_+$.

According to [21], for any $0 < \beta < d/2$ we can define the Riesz potential

$$I^\beta\varphi(y) := \varphi * R_\beta(y) = \frac{1}{\gamma_n(\beta)} \int_{\mathbb{R}^d} \frac{\varphi(y)}{|x-y|^{d-\beta}} dy \quad \text{for all } \varphi \in L_2(\mathbb{R}^d).$$

Let $q = q_{d,\beta} > 2$ be such that $1/q = 1/2 - \beta/d$.

The space $I^\beta(L_2(R^d))$ of all Riesz potentials has the following properties (see [21]):

(1) $I^\beta(L_2(R^d)) \subset L_q(\mathbb{R}^d)$ and there exists a constant $N > 0$ such that

$$\|I^\beta\varphi\|_{L_q(\mathbb{R}^d)} \le N\|\varphi\|_{L_2(\mathbb{R}^d)}, \quad \text{for all } \varphi \in L_2(\mathbb{R}^d).$$

(2) For every $f \in I^\beta(L_2(R^d))$, we define the Riesz derivative $\mathbb{D}^\beta f$ as $\lim_{\varepsilon\to 0} \mathbb{D}_\varepsilon^\beta f$ in $L_2(\mathbb{R}^d)$, where

$$(\mathbb{D}_\varepsilon^\beta f)(x) = \frac{1}{c_{d,l}(\beta)} \int_{|y|>\varepsilon} \frac{(\Delta_y^l f)(x)}{|y|^{d+\beta}} dy \quad \text{(is independent of } l),$$

$\Delta_y^l f = (I - \tau_y)^l f$ and $(\tau_y f)(x) = f(x-y)$. Then

$$\mathcal{F}(\mathbb{D}^\beta f)(\xi) = \mathcal{F}f(\xi)|\xi|^\beta, \quad \forall f \in \mathcal{S}(\mathbb{R}^d).$$

(3) $I^\beta(L_2(R^d)) = \{f \in L_q(\mathbb{R}^d); \mathbb{D}^\beta f \text{ exists and is in } L_2(\mathbb{R}^d)\}$.

(4) $\mathbb{D}^\beta$ is the left inverse of I^β in $L_2(\mathbb{R}^d)$, i.e. $\mathbb{D}^\beta(I^\beta\varphi) = \varphi$ for all $\varphi \in L_2(\mathbb{R}^d)$.

(5) $\mathcal{D}(\mathbb{R}^d)$ is dense in $I^\beta(L_2(R^d))$ with respect to the norm $\|f\|_{L_q(\mathbb{R}^d)} + \|\mathbb{D}^\beta f\|_{L_2(\mathbb{R}^d)}$.

In what follows, we will use these properties with $\beta = \alpha/2 = 2k$, where $d-2 < 4k < d$. We let $q = q_{d,4k} > 2$ be such that $1/q = 1/2 - 2k/d$.

Proposition 4.10. *Let $2 < d/(2k)$ and $1/q = 1/2 - 2k/d$. If $f \in I^{2k}(L_2(\mathbb{R}^d))$, where $k \in \mathbb{N}_+$, then*

$$\mathcal{F}\left(\mathbb{D}^{2k} f\right) = |\xi|^{2k}\mathcal{F}f,$$

where $\mathcal{F}f$ is an element in $\mathcal{S}'(\mathbb{R}^d)$.

Proof. First of all, $|\xi|^{2k}\mathcal{F}f$ is well-defined as an element of $\mathcal{S}'$ since $|\xi|^{2k} = (\xi_1^2 + \cdots + \xi_d^2)^k$ is infinitely differentiable.

Due to the fact that $\mathcal{D}(\mathbb{R}^d)$ is dense in $I^{2k}(L_2(\mathbb{R}^d))$, there exists a sequence $\{f_n\} \subset \mathcal{D}(\mathbb{R}^d)$ such that

$$\|f_n - f\|_{L_q(\mathbb{R}^d)} + \|\mathbb{D}^{2k} f_n - \mathbb{D}^{2k} f\|_{L_2(\mathbb{R}^d)} \to 0.$$

Note that $\mathcal{F}\left(\mathbb{D}^{2k} f_n\right) = |\xi|^{2k}\mathcal{F}f_n \in \mathcal{S}(\mathbb{R}^d)$. Also note that

$$\begin{aligned}\left(\mathcal{F}\left(\mathbb{D}^{2k} f_n\right), \phi\right) &= \left(|\xi|^{2k}\mathcal{F}f_n, \phi\right) = \left(\mathcal{F}f_n, |\xi|^{2k}\phi\right)\\ &= \left(f_n, \mathcal{F}^{-1}\left(|\xi|^{2k}\phi\right)\right) \to \left(f, \mathcal{F}^{-1}\left(|\xi|^{2k}\phi\right)\right) = \left(|\xi|^{2k}\mathcal{F}f, \phi\right),\end{aligned}$$

where the convergence is possible because $f_n \to f$ in $L_q(\mathbb{R}^d)$. This along with the fact that $\mathbb{D}^{2k} f_n \to \mathbb{D}^{2k} f$ in $L_2(\mathbb{R}^d)$ implies that

$$\mathcal{F}\left(\mathbb{D}^{2k} f\right) = |\xi|^{2k}\mathcal{F}f.$$

□

Corollary 4.11. *Let $2 < d/(2k)$ and $1/q = 1/2 - 2k/d$. If $f \in I^{2k}(L_2(\mathbb{R}^d))$, where $k \in \mathbb{N}_+$, then*

$$\mathbb{D}^{2k} f = (-\Delta)^k f.$$

Proof. Note that

$$\left(\mathcal{F}\left(\mathbb{D}^{2k} f\right), \phi\right) = \left(|\xi|^{2k}\mathcal{F}f, \phi\right) = \left(\mathcal{F}f, |\xi|^{2k}\phi\right).$$

On the other hand,

$$\begin{aligned}\left(\mathcal{F}\left((-\Delta)^k f\right), \phi\right) &= \left((-\Delta)^k f, \mathcal{F}^{-1}\phi\right) = \left(f, (-\Delta)^k \mathcal{F}^{-1}\phi\right)\\ &= \left(f, \mathcal{F}^{-1}\left(|\xi|^{2k}\phi\right)\right) = \left(\mathcal{F}f, |\xi|^{2k}\phi\right).\end{aligned}$$

□

Lemma 4.12. *Let $2 < d/2k$. There exists a linear operator $J : \mathcal{P}_0^{(d)} \to L_2((0,T)\times\mathbb{R}^d)$ such that J is one-to-one and onto, and satisfies:*

$$J\varphi(t,x) = I^{2k}\left(\varphi(t,\cdot)\right)(x)$$

for any $\varphi \in \mathcal{D}((0,T)\times\mathbb{R}^d)$.

Proof. For all $\varphi \in \mathcal{D}((0,T)\times\mathbb{R}^d)$, set

$$J(\varphi) = I^{2k}\left(\varphi(t,\cdot)\right)(x) = c_{d,k}\int_{\mathbb{R}^d} \frac{\varphi(t,y)}{|x-y|^{d-2k}}\,dy,$$

where $c_{d,k}$ is an appropriate constant. Denote $\varphi_0 := J(\varphi)$. We see that $\varphi_0(t,x)$ is measurable in $(t,x) \in (0,T)\times\mathbb{R}^d$. Since $\varphi(t,\cdot) \in \mathcal{S}(\mathbb{R}^d)$ for each fixed $t \in (0,T)$, we have

$$\mathcal{F}\varphi_0(t,\cdot)(\xi) = \mathcal{F}\varphi(t,\xi)|\xi|^{-2k}$$

in the sense that

$$\int_{\mathbb{R}^d} \varphi_0(t,x)\overline{\phi(x)}\,dx = \int_{\mathbb{R}^d} \mathcal{F}\varphi(t,\xi)|\xi|^{-2k}\overline{\mathcal{F}\phi(\xi)}\,d\xi,$$

where $\phi \in \mathcal{S}(\mathbb{R}^d)$. Also notice that, for a.e. $t \in (0,T)$,

$$\mathcal{F}\varphi(t,\xi)|\xi|^{-2k} \in L_2(\mathbb{R}^d)$$

because $\varphi \in \mathcal{P}_0^{(d)}$ (recall that $\alpha = 4k$), i.e.,

$$\int_0^T \int_{\mathbb{R}^d} |\mathcal{F}\varphi(t,\xi)|\xi|^{-2k}|^2 \, d\xi \, dt < \infty.$$

Thus, for a.e. $t \in (0,T)$, as a function of $x \in \mathbb{R}^d$, $\varphi_0(t,x) \in L_2(\mathbb{R}^d)$ and

$$\mathcal{F}\varphi_0(t,\cdot)(\xi) = \mathcal{F}\varphi(t,\xi)|\xi|^{-2k}.$$

Thus

$$\begin{aligned}\|J(\varphi)\|^2_{L_2((0,T)\times\mathbb{R}^d)} &= \int_0^T \int_{\mathbb{R}^d} |\varphi_0(t,x)|^2 \, dx \, dt \\ &= \int_0^T \int_{\mathbb{R}^d} |\mathcal{F}\varphi(t,\xi)|^2 |\xi|^{-4k} \, d\xi \, dt = \|\varphi\|_0^2.\end{aligned}$$

From this and the fact that $\mathcal{D}((0,T)\times\mathbb{R}^d)$ is dense in $\mathcal{P}_0^{(d)}$, we extend J to all elements in $\mathcal{P}_0^{(d)}$.

It is easy to see that J is linear as well as one-to-one. To prove the fact that J is onto, take $\varphi_0 \in L_2((0,T)\times\mathbb{R}^d)$. Also take a sequence $\{\varphi_{0n}\} \subset \mathcal{D}((0,T)\times\mathbb{R}^d)$ such that $\varphi_{0n} \to \varphi_0$ in $L_2((0,T)\times\mathbb{R}^d)$. Let $\varphi_n = (-\Delta)^k \varphi_{0n}$. Then

$$\mathcal{F}\varphi_n(t,\xi) = |\xi|^{2k}\mathcal{F}\varphi_{0n}(t,\xi)$$

and

$$\int_0^T \int_{\mathbb{R}^d} |\mathcal{F}\varphi_n(t,\xi)|^2 |\xi|^{-4k} \, d\xi \, dt = \int_0^T \int_{\mathbb{R}^d} |\mathcal{F}\varphi_{0n}|^2 \, d\xi \, dt.$$

Thus $\{\varphi_n\}$ is a Cauchy sequence in $\mathcal{P}_0^{(d)}$, so there is a $\varphi \in \mathcal{P}_0^{(d)}$ such that $\varphi_n \to \varphi$ in $\mathcal{P}_0^{(d)}$. To prove $J(\varphi) = \varphi_0$, we only need to prove that

$$I^{2k}(\varphi_n) = \varphi_{0n},$$

which follows from

$$\mathcal{F}\left(I^{2k}(\varphi_n)\right) = |\xi|^{-2k}\mathcal{F}\varphi_n.$$

□

From the above result it follows that:

Corollary 4.13. *For φ, $\eta \in \mathcal{P}_0^{(d)}$, let $\varphi_0 := J(\varphi)$ and $\eta_0 := J(\eta)$. Then*

$$\langle \varphi, \eta \rangle_0 = \langle \varphi_0, \eta_0 \rangle_{L_2((0,T)\times\mathbb{R}^d)}.$$

Now we set

$$\varphi_1(t,x) = I^{2k}\left(\varphi_0(t,\cdot)\right)(x) = c_{d,k} \int_{\mathbb{R}^d} \frac{\varphi_0(t,y)}{|x-y|^{d-2k}} \, dy,$$

where $\varphi_0 \in L_2((0,T)\times\mathbb{R}^d)$. Notice that φ_1 is a measurable function of $(t,x) \in (0,T)\times\mathbb{R}^d$. Since $\varphi_0(t,x) \in L_2(\mathbb{R}^d)$ for a.e. $t \in (0,T)$,

$$\varphi_1(t,\cdot) \in L_q(\mathbb{R}^d), \qquad \|\varphi_1(t,\cdot)\|_{L_q(\mathbb{R}^d)} \le N\|\varphi_0(t,\cdot)\|_{L_2(\mathbb{R}^d)},$$

for a.e. $t \in (0,T)$, where $1/q = 1/2 - 2k/d$. This implies that

$$\varphi_1 \in L_{2,q}((0,T)\times\mathbb{R}^d),$$

$$\|\varphi_1\|_{L_{2,q}((0,T)\times\mathbb{R}^d)} = \left(\int_0^T \|\varphi_1(t,\cdot)\|^2_{L_q(\mathbb{R}^d)}\,dt\right)^{1/2} \le N\|\varphi_0\|_{L_2((0,T)\times\mathbb{R}^d)}.$$

From the L_p-theory of parabolic equations with mixed norms, there exists a unique function $w \in W^{1,2}_{2,q}((0,T)\times\mathbb{R}^d)$ satisfying

$$w_t - \Delta w = \varphi_1, \qquad w(0,\cdot) = 0. \tag{4.7}$$

Lemma 4.14. *Let $h(t,x) = \mathbb{E}M(\varphi)u(t,x)$, where u is defined in* (3.5). *Let $\varphi_0 = J(\varphi)$, φ_1 be a function defined as above, and w be the solution to* (4.7). *Then*

$$w = h.$$

Proof. For $\eta \in \mathcal{D}((0,T)\times\mathbb{R}^d)$, find a function ϕ, a unique solution to

$$-\phi_t - \Delta\phi = \eta, \qquad \phi(T,\cdot) = 0.$$

Then by Lemma 3.6

$$\langle h,\eta\rangle_{L_2((0,T)\times\mathbb{R}^d)} = \langle\varphi,\phi\rangle_0$$

and

$$\langle w,\eta\rangle_{L_2((0,T)\times\mathbb{R}^d)} = \langle w_t - \Delta w,\phi\rangle_{L_2((0,T)\times\mathbb{R}^d)} = \langle\varphi_1,\phi\rangle_{L_2((0,T)\times\mathbb{R}^d)}.$$

Find a sequence $\{\varphi_{0n}\} \subset \mathcal{D}((0,T)\times\mathbb{R}^d)$ such that $\varphi_{0n} \to \varphi_0$ in $L_2((0,T)\times\mathbb{R}^d)$. Let $\psi_n = I^{2k}(\varphi_{0n})$ and w_n be the solution to $(\partial/\partial t - \Delta)w_n = \psi_n$ and $w_n(0,\cdot) = 0$. Then

$$\langle w,\eta\rangle_{L_2((0,T)\times\mathbb{R}^d)} = \lim_{n\to\infty}\langle w_n,\eta\rangle_{L_2((0,T)\times\mathbb{R}^d)} = \lim_{n\to\infty}\langle\psi_n,\phi\rangle_{L_2((0,T)\times\mathbb{R}^d)}$$

$$= \lim_{n\to\infty}\langle I^{2k}(\varphi_{0n}),\phi\rangle_{L_2((0,T)\times\mathbb{R}^d)} = \lim_{n\to\infty}\int_0^T\int_{\mathbb{R}^d}\mathcal{F}\varphi_{0n}(t,\xi)\overline{\mathcal{F}\phi(t,\xi)}|\xi|^{-2k}\,d\xi\,dt.$$

On the other hand,

$$\langle h,\eta\rangle_{L_2((0,T)\times\mathbb{R}^d)} = \langle\varphi,\phi\rangle_0 = \lim_{n\to\infty}\int_0^T\int_{\mathbb{R}^d}\mathcal{F}\varphi_n(t,\xi)\overline{\mathcal{F}\phi(t,\xi)}|\xi|^{-4k}\,d\xi\,dt$$

$$= \lim_{n\to\infty}\int_0^T\int_{\mathbb{R}^d}\mathcal{F}\varphi_{0n}(t,\xi)\overline{\mathcal{F}\phi(t,\xi)}|\xi|^{-2k}\,d\xi\,dt,$$

where $\varphi_n = (-\Delta)^{2k}\varphi_{0n}$. This finishes the proof. □

Now we prove the main result of this subsection.

Theorem 4.15. *Suppose that f is the Riesz kernel of order $\alpha = 4k, k \in \mathbb{N}_+$ and $d = 4k+1$. Then the process solution u of the stochastic heat equation (3.1) with vanishing initial conditions is locally germ Markov.*

Proof. Again we need to verify conditions (i) and (ii) in Theorem 4.2.

We first verify condition (i).

Let $h(t,x) = \mathbb{E}M(\varphi)u(t,x)$ and $g(t,x) = \mathbb{E}M(\eta)u(t,x)$, where $\varphi,\eta \in \mathcal{P}_0^{(d)}$, such that $\operatorname{supp} h \cap \operatorname{supp} g = \emptyset$. We need to show that $\langle\varphi,\eta\rangle_0 = 0$. Let $\varphi_0 = J(\varphi)$ and $\eta_0 = J(\eta)$, where J is the operator defined in Lemma 4.12. Also let $\varphi_1 = I^{2k}(\varphi_0(t,\cdot))$ and $\eta_1 = I^{2k}(\eta_0(t,\cdot))$. For a.e. $t \in (0,T)$, we have

$$\varphi_0(t,x) = \mathbb{D}^{2k}\varphi_1(t,x) = (-\Delta)^k\varphi_1,$$

where the second equality follows from Corollary 4.11. Let $\phi(t,x) \in \mathcal{D}((0,T)\times\mathbb{R}^d)$ such that $\phi(t,x) = 0$ on $\operatorname{supp}\varphi_1$. Then for a.e. $t \in (0,T)$,

$$\int_{\mathbb{R}^d} \varphi_0(t,x)\phi(t,x)\,dx = \int_{\mathbb{R}^d} \varphi_1(t,x)(-\Delta)^k\phi(t,x)\,dx.$$

Thus

$$\int_0^T \int_{\mathbb{R}^d} \varphi_0(t,x)\phi(t,x)\,dx\,dt = \int_0^T \int_{\mathbb{R}^d} \varphi_1(t,x)(-\Delta)^k\phi(t,x)\,dx\,dt = 0.$$

This shows that $\operatorname{supp}\varphi_0 \subset \operatorname{supp}\varphi_1$. Similarly, we have $\operatorname{supp}\eta_0 \subset \operatorname{supp}\eta_1$. Hence we have $\langle \varphi_0, \eta_0\rangle_{L_2((0,T)\times\mathbb{R}^d)} = 0$. This and Corollary 4.13 prove that $\langle \varphi, \eta\rangle_0 = 0$.

We now verify condition (ii).

Assume that $\zeta = h + g$, where $\zeta(t,x) = \mathbb{E}M(\nu)u(t,x)$, $\operatorname{supp} h \cap \operatorname{supp} g = \emptyset$, and $\operatorname{supp} h$ is compact. We have to prove that $h \in \mathcal{H}^u$.

Let χ be an infinitely differentiable function defined on $[0,T]\times\mathbb{R}^d$ such that $0 \le \chi \le 1$, $\chi = 1$ on $\operatorname{supp}\mu$, and $\chi = 0$ in an (relative) open set containing $\operatorname{supp}\nu$. Then $\chi\zeta = h$ and $h \in W^{1,2}_{2,q}((0,T)\times\mathbb{R}^d)$. Let $(\partial/\partial t - \Delta)\zeta = \nu_1$, where $\nu_1 = I^{2k}(\nu_0)$ and $\nu_0 = J(\nu)$. Now we set

$$\varphi_1 := \chi\nu_1, \quad \eta_1 := (1-\chi)\nu_1, \quad \varphi_0 := \chi\nu_0.$$

Since $\varphi_0 \in L_2((0,T)\times\mathbb{R}^d)$ there exists a sequence $\{\varphi_{0n}\} \subset \mathcal{D}((0,T)\times\mathbb{R}^d)$ such that $\varphi_{0n} \to \varphi_0$ in $L_2((0,T)\times\mathbb{R}^d)$. Let $\varphi_n = (-\Delta)^k\varphi_{0n}$. Then $\varphi_n \in \mathcal{D}((0,T)\times\mathbb{R}^d)$ and $\{\varphi_n\}$ is a Cauchy sequence in $\mathcal{P}_0^{(d)}$ due to the fact that

$$\mathcal{F}\varphi_n(t,\xi) = |\xi|^{2k}\mathcal{F}\varphi_{0n}(t,\xi).$$

Let φ be the limit in $\mathcal{P}_0^{(d)}$ of φ_n. Then $J(\varphi) = \varphi_0$. Now let $\hat\varphi_1 = I^{2k}(\varphi_0)$ and $\hat h(t,x) = \mathbb{E}M(\varphi)u(t,x)$.

We now prove that $h = \hat h$, which will imply that $h \in \mathcal{H}^u$. For this, it suffices to prove that $\varphi_1 = \hat\varphi_1$. By Lemma 4.14, we have $(\partial/\partial t - \Delta)\hat h = \hat\varphi_1$. Notice that $\nu_1 = \varphi_1 + \eta_1$ and $\operatorname{supp}\varphi_1 \cap \operatorname{supp}\eta_1 = \emptyset$. Thus for a.e. $t \in (0,T)$, we have

$$\operatorname{supp}_x \varphi_1(t,\cdot) \cap \operatorname{supp}_x \eta_1(t,\cdot) = \emptyset.$$

Thus by Lemma B.1 it follows that, for a.e. $t \in (0,T)$,

$$\mathbb{D}^{2k}\varphi_1(t,\cdot) = \varphi_0(t,\cdot).$$

We also have

$$\mathbb{D}^{2k}\hat\varphi_1(t,\cdot) = \varphi_0(t,\cdot)$$

for a.e. $t \in (0,T)$. Since $\varphi_1(t,\cdot) - \hat\varphi_1(t,\cdot) \in I^{2k}(L_2(\mathbb{R}^d))$ for a.e. $t \in (0,T)$, there exists a function $f_t(x)$ such that

$$\varphi_1(t,\cdot) - \hat\varphi_1(t,\cdot) = I^{2k}(f_t)$$

for a.e. $t \in (0,T)$. From the fact that $\mathbb{D}^{2k}$ is the left inverse of the I^{2k}, we know that

$$0 = \mathbb{D}^{2k}\left(\varphi_1(t,\cdot) - \hat\varphi_1(t,\cdot)\right) = \mathbb{D}^{2k}I^{2k}(f_t) = f_t.$$

We also know that $I^{2k}(0) = 0$. Hence $\varphi_1(t,\cdot) = \hat\varphi_1(t,\cdot)$ for a.e. $t \in (0,T)$. Therefore, $\varphi_1 = \hat\varphi_1$ as elements in $L_{2,q}((0,T)\times\mathbb{R}^d)$. The theorem is proved. □

Appendix A. Proof of Relation (3.6)

We need to prove that

$$\int_0^T \int_{\mathbb{R}^d} |\mathcal{F}\varphi_n(s,\xi) - \mathcal{F}\varphi(s,\xi)|^2 \mu(d\xi) ds \to 0, \quad \text{as } n \to \infty.$$

Note that $\mathcal{F}\varphi_n(s,\xi) = \sum_{m \in I_n} c_d |Q_m^{(n)}| \exp\{-\mathrm{i}\xi \cdot x_m^{(n)} - (t_m^{(n)} - s)|\xi|^2\} \eta(t_m^{(n)}, x_m^{(n)})$ converges pointwise to

$$\mathcal{F}\varphi(s,\xi) = \int_s^T \int_{\mathbb{R}^d} c_d \exp\{-\mathrm{i}\xi \cdot x - (t-s)|\xi|^2\} \eta(t,x)\, dx\, dt.$$

In order to apply the dominated convergence theorem, we need to prove that there exists a function Ψ, which is square-integrable with respect to $ds \times \mu(d\xi)$ such that $|\mathcal{F}\varphi_n(s,\xi)| \le \Psi(s,\xi)$. Let N be a constant such that $|\eta| \le N$. Then

$$|\mathcal{F}\varphi_n(s,\xi)| \le \sum_{m \in I_n} c_d |Q_m^n| |\eta(t_m^{(n)}, x_m^{(n)})| \exp\{-(t_m^{(n)} - s)|\xi|^2\}$$

$$\le \sum_{m \in I_n} c_d N |Q_m^{(n)}| e^{-(t_m^{(n)} - s)|\xi|^2} \le \int_s^T \int_{K'} c_d N e^{-(t-s)|\xi|^2}\, dx\, dt$$

$$\le c_d N |K'| (T-s) 1_{|\xi|<1} + c_d N |K'| \frac{1 - e^{(s-T)|\xi|^2}}{|\xi|^2} 1_{|\xi| \ge 1} := \Psi(s,\xi),$$

where K' is a compact set containing K, and the third inequality above is due to the fact that $1_{Q_m^{(n)}}(t,x) e^{-(t_m^{(n)}-s)|\xi|^2} \le 1_{Q_m^{(n)}}(t,x) e^{-(t-s)|\xi|^2}$. Clearly Ψ is square-integrable with respect to $ds \times \mu(d\xi)$. This concludes the proof of (3.6).

Appendix B. An Auxiliary Lemma

The following technical result was used in the proof of Theorem 4.15 for the verification of condition (ii).

Lemma B.1. *Let $2 < d/2k$ and $1/q = 1/2 - 2k/d$. Assume that $\kappa \in I^{2k}(L_2(\mathbb{R}^d))$ and $\kappa = \mu + \nu$, where $\mu, \nu \in L_q(\mathbb{R}^d)$, $\mathrm{supp}\, \mu \cap \mathrm{supp}\, \nu = \emptyset$, and $\mathrm{supp}\, \mu$ is compact. Then*

$$\mu \in I^{2k}(L_2(\mathbb{R}^d)), \qquad \mathbb{D}^{2k}\mu = \chi \mathbb{D}^{2k}\kappa,$$

where χ is an infinitely differentiable function such that $0 \le \chi \le 1$, $\chi = 1$ on $\mathrm{supp}\, \mu$, and $\chi = 0$ in an open set containing $\mathrm{supp}\, \nu$.

Proof. Since $\kappa \in I^{2k}(L_2(\mathbb{R}^d))$, we have

$$\mathbb{D}_\varepsilon^{2k}\kappa := \frac{1}{c(d,l,2k)} \int_{|z|>\varepsilon} \frac{\left(\Delta_z^l \kappa\right)(x)}{|z|^{d+2k}}\, dz \to \mathbb{D}^{2k}\kappa \quad \text{in} \quad L_2(\mathbb{R}^d).$$

Note that $\mathbb{D}_\varepsilon^{2k}\kappa \in L_2(\mathbb{R}^d)$ (see Proposition 2.4 in [1]). Let

$$\Lambda_\varepsilon^{2k}\kappa := \frac{1}{c(d,l,2k)} \int_{\varepsilon<|z|\le\varepsilon_0} \frac{\left(\Delta_z^l \kappa\right)(x)}{|z|^{d+2k}}\, dz, \quad \varepsilon < \varepsilon_0.$$

Since $\Lambda_\varepsilon^{2k}\kappa = \mathbb{D}_\varepsilon^{2k}\kappa - \mathbb{D}_{\varepsilon_0}^{2k}\kappa$, we have $\Lambda_\varepsilon^{2k}\kappa \in L_2(\mathbb{R}^d)$. Moreover,

$$\Lambda_{\varepsilon_1}^{2k}\kappa - \Lambda_{\varepsilon_2}^{2k}\kappa = \int_{\varepsilon_1<|z|\le\varepsilon_2} \frac{\left(\Delta_z^l \kappa\right)(x)}{|z|^{d+2k}}\,dz = \mathbb{D}_{\varepsilon_1}^{2k}\kappa - \mathbb{D}_{\varepsilon_2}^{2k}\kappa.$$

Form this and the fact that $\mathbb{D}_\varepsilon^{2k}\kappa$ is Cauchy in $L_2(\mathbb{R}^d)$, it follows that $\{\Lambda_\varepsilon^{2k}\kappa\}$ is a Cauchy sequence in $L_2(\mathbb{R}^d)$. Consider $\mathbb{D}_\varepsilon^{2k}(\chi\kappa)$, which is well-defined because $\chi\kappa \in L_q(\mathbb{R}^d)$. Note that

$$\mathbb{D}_\varepsilon^{2k}(\chi\kappa) = \Lambda_\varepsilon^{2k}(\chi\kappa) + \int_{|z|\ge\varepsilon_0} \frac{\left(\Delta_z^l \chi\kappa\right)(x)}{|z|^{d+2k}}\,dz$$

and

$$\Lambda_\varepsilon^{2k}(\chi\kappa) = \chi\Lambda_\varepsilon^{2k}\kappa \tag{B.1}$$

for a sufficiently small $\varepsilon_0 > 0$. The proof of (B.1) is stated below. Also note that

$$\Lambda_\varepsilon^{2k}(\chi\kappa) \in L_2(\mathbb{R}^d), \quad \int_{|z|\ge\varepsilon_0} \frac{\left(\Delta_z^l \chi\kappa\right)(x)}{|z|^{d+2k}}\,dz \in L_2(\mathbb{R}^d),$$

where the latter follows from the fact that $\chi\kappa \in L_2(\mathbb{R}^d)$ (χ has a compact support and $2 \le q$). In addition, $\Lambda_\varepsilon^{2k}\kappa$ approaches a function in $L_2(\mathbb{R}^d)$ as $\varepsilon \searrow 0$. Thus $\mathbb{D}_\varepsilon^{2k}(\chi\kappa)$ converges a function in $L_2(\mathbb{R}^d)$. This shows that $\mu = \chi\kappa \in I^{2k}(L_2(\mathbb{R}^d))$. To prove $\mathbb{D}^{2k}\mu = \chi\mathbb{D}^{2k}\kappa$, we can just use Corollary 4.11.

Let us prove (B.1) as follows. For each $x \in \operatorname{supp}\mu$, find $\delta_x > 0$ such that $B(x,\delta_x) \cap \operatorname{supp}\nu = \emptyset$. Consider

$$\{B(x,\delta_x/5) : x \in \operatorname{supp}\mu\}.$$

Since $\operatorname{supp}\mu$ is compact, we have a finite subset $\{x_i : i \in I\} \subset \operatorname{supp}\mu$ such that

$$\bigcup_{i\in I} B(x_i,\delta_{x_i}/5) \supset \operatorname{supp}\mu.$$

Let

$$\mathcal{O}_1 := \bigcup_{i\in I} B(x_i,\delta_{x_i}/4), \quad \mathcal{O}_2 := \bigcup_{i\in I} B(x_i,\delta_{x_i}/3), \quad \mathcal{O}_3 := \bigcup_{i\in I} B(x_i,\delta_{x_i}/2).$$

Then

$$\mathcal{O}_i \supset \operatorname{supp}\mu, \quad i = 1,2,3$$

and

$$\mathcal{O}_i \cap \operatorname{supp}\nu = \emptyset, \quad i = 1,2,3.$$

We may assume that $\chi = 1$ on $\mathcal{O}_2$ and $\chi = 0$ on $\mathcal{O}_3^c$. We also assume that $\varepsilon_0 < \delta/(20l)$, where $\delta = \min_{i\in I}\delta_{x_i}$. By definition we have

$$\Lambda_\varepsilon^{2k}(\chi\kappa)(x) = \frac{1}{c}\int_{\varepsilon<|z|\le\varepsilon_0} \frac{\left(\Delta_z^l(\chi\kappa)\right)(x)}{|z|^{d+2k}}\,dz$$

$$= \frac{1}{c}\int_{\varepsilon<|z|\le\varepsilon_0} \frac{\sum_{k=0}^{l}(-1)^k c_l^k \chi(x-kz)\kappa(x-kz)}{|z|^{d+2k}}\,dz$$

Let $x \in \mathcal{O}_1$, i.e. $x \in B(x_i,\delta_{x_i}/4)$. Then

$$|x_i - (x-kz)| \le |x_i - x| + |kz| \le \delta_{x_i}/4 + l\delta/(20l)$$

$$\leq \delta_{x_i}/4 + \delta_{x_i}/12 = \delta_{x_i}/3.$$

Thus $x - kz \in \mathcal{O}_2$. This shows that $\chi(x - kz) = 1$ and

$$\Lambda_\varepsilon^{2k}(\chi\kappa)(x) = \chi(x)\Lambda_\varepsilon^{2k}\kappa(x), \quad x \in \mathcal{O}_1.$$

Now let $x \in \mathcal{O}_1^c$. Then for every x_i, $i \in I$,

$$|x_i - (x - kz)| \geq |x_i - x| - l|z|$$

$$\geq \delta_{x_i}/4 - l\delta/(20l) \geq \delta_{x_i}/4 - \delta_{x_i}/20 = \delta_{x_i}/5.$$

This implies that $x - kz \notin \operatorname{supp}\mu$. In that case we have

$$\chi(x - kz)\kappa(x - kz) = 0$$

because $\kappa(x - kz) = 0$ in case $\chi(x - kz) > 0$. Thus $\Lambda_\varepsilon^{2k}(\chi\kappa)(x) = 0$. In addition, $\chi(x)\Lambda_\varepsilon^{2k}\kappa(x) = 0$ (recall that $x \in \mathcal{O}_1^c$ and $x - kz \notin \operatorname{supp}\mu$) because $\kappa(x - kz) = 0$ in case $\chi(x) > 0$. Indeed, if $x \in \mathcal{O}_3 \setminus \mathcal{O}_1$, then there exists x_j, $j \in I$, such that $x \in B(x_j, \delta_{x_j}/2)$. Then

$$|x_j - (x - kz)| \leq |x_j - x| + l|z| < \delta_{x_j}.$$

Thus by the choice of δ_{x_j} it follows that $x - kz \notin \operatorname{supp}\nu$, that is, $\kappa(x - kz) = 0$. □

Acknowledgment. The authors are grateful to an anonymous referee who carefully read the article and made some suggestions for future research.

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