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A NONSTANDARD PROOF OF DE FINETTI'S THEOREM FOR BERNOULLI RANDOM VARIABLES

IRFAN ALAM*

Dedicated to Professor Leonard Gross on the occasion of his 88th birthday

ABSTRACT. We give a nonstandard analytic proof of de Finetti's theorem for an exchangeable sequence of Bernoulli random variables. The theorem postulates that such a sequence is uniquely representable as a mixture of iid sequences of Bernoulli random variables. We use combinatorial arguments to show that this probability distribution is induced by a hyperfinite sample mean. A brief introduction to nonstandard analytic concepts is also provided.

1. de Finetti's Theorem and its Proof

This paper presents a nonstandard analytic treatment of de Finetti's theorem for a sequence of exchangeable Bernoulli random variables. We begin with the definition of exchangeability.

Definition 1.1. A finite collection X_1, \dots, X_n of random variables is said to be *exchangeable* if for any permutation $\sigma \in S_n$, the random vectors (X_1, \dots, X_n) and $(X_{\sigma(1)}, \dots, X_{\sigma(n)})$ have the same distribution. An infinite sequence X_1, X_2, \dots of random variables is said to be *exchangeable* if any finite subcollection of the X_i is exchangeable in the above sense.

Throughout the paper, all Bernoulli random variables take values in $\{0, 1\}$. A well-known result of de Finetti says that a sequence of exchangeable Bernoulli random variables is conditionally independent given the value of a random parameter in $[0, 1]$, the parameter being sampled through a unique probability measure on $[0, 1]$. More precisely, we may write de Finetti's theorem in the following form.

Theorem 1.2 (de Finetti). *Let X_1, X_2, \dots be a sequence of exchangeable Bernoulli random variables. There exists a unique measure μ on the interval $[0, 1]$ such that the following holds:*

$$\mathbb{P}(X_1 = e_1, \dots, X_k = e_k) = \int_{[0,1]} p^{\sum_{j=1}^k e_j} (1-p)^{k-\sum_{j=1}^k e_j} d\mu(p) \quad (1.1)$$

for any $k \in \mathbb{N}$ and $e_1, \dots, e_k \in \{0, 1\}$.

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The integrand on the right side is the probability that k iid Bernoulli(p) random variables have the outcomes e_1, \dots, e_k . In this sense, de Finetti's theorem expresses an exchangeable sequence of Bernoulli random variables as a *mixture* of iid sequences of Bernoulli random variables.

See de Finetti [8, 9] for the original formulations of this theorem. Aldous [3] and Kingman [11] are good resources for an introduction to exchangeability and related topics. See Kirsch [12] for a recent elementary proof of de Finetti's theorem.

We will give a nonstandard proof of Theorem 1.2. In nonstandard analytic language, the idea is that the measure μ will be shown to be induced by a hyperfinite sample mean $\frac{X_1 + \dots + X_N}{N}$. A very brief introduction to nonstandard methods is provided in Section 2. We refer the reader to books such as [2] and [10] for more details.

For the rest of this section, we fix an exchangeable sequence X_1, X_2, \dots of Bernoulli random variables. We also fix $k \in \mathbb{N}$ and $e_1, \dots, e_k \in \{0, 1\}$. Taking $\alpha = \sum_{j=1}^k e_j$ and writing the integral in (1.1) as an expectation in terms of a random variable $Y \sim \mu$, de Finetti's theorem may be restated as follows:

$$\mathbb{P}(X_1 = e_1, \dots, X_k = e_k) = \mathbb{E}_\mu(Y^\alpha(1 - Y)^{k-\alpha}). \quad (1.2)$$

Written this way, it is clear that any measure satisfying the conclusion of de Finetti's theorem must be unique. Indeed, taking $\alpha = k$ and varying k through \mathbb{N} in (1.2) shows that such a measure has a unique sequence of moments, which implies that they agree on expected values of continuous functions on $[0, 1]$ (using the Weierstrass approximation theorem).

Hence, it is enough to prove the existence of a probability measure on $[0, 1]$ satisfying the conclusion of de Finetti's theorem. Toward that end, we will verify equation (1.2) for a standard measure μ that is naturally induced by an appropriate Loeb measure. Fix $N > \mathbb{N}$ and define:

$$Y_N = \frac{X_1 + \dots + X_N}{N}. \quad (1.3)$$

Note that we are abusing notation by using (X_i) to denote both the standard sequence $(X_i)_{i \in \mathbb{N}}$ of random variables and the nonstandard extension of this sequence, with the usage being clear from context. More precisely, if $\mathcal{X}: \Omega \times \mathbb{N} \rightarrow S$ is defined by $\mathcal{X}(\omega, i) := X_i(\omega)$ for all $\omega \in \Omega$ and $n \in \mathbb{N}$, then for any $i \in {}^*\mathbb{N}$, the internal random variable $X_i: {}^*\Omega \rightarrow {}^*S$ is defined as follows:

$$X_i(\omega) = {}^*\mathcal{X}(\omega, i) \text{ for all } \omega \in {}^*\Omega \text{ and } i \in {}^*\mathbb{N}.$$

Let ${}^*\mathbb{P}: {}^*\mathcal{F} \rightarrow {}^*[0, 1]$ denote the nonstandard extension of \mathbb{P} . Then ${}^*\mathbb{P}$ is an internal probability measure. Note that Y_N takes values in $\left\{0, \frac{1}{N}, \dots, \frac{N-1}{N}, \frac{N}{N} = 1\right\}$.

Naively conditioning on the value of Y_N , we obtain the following:

$$\begin{aligned} & \mathbb{P}(X_1 = e_1, \dots, X_k = e_k) \\ &= \sum_{i=0}^N {}^*\mathbb{P}\left(X_1 = e_1, \dots, X_k = e_k \mid Y_N = \frac{i}{N}\right) {}^*\mathbb{P}\left(Y_N = \frac{i}{N}\right). \end{aligned} \quad (1.4)$$

Note that we could have started the sum in (1.4) at $i = \alpha$ since the conditional probabilities in this sum are zero for all $i < \alpha$.

The random variable Y_N induces an internal finitely additive internal probability measure \mathbb{P}_N on ${}^*[0, 1]$, which is supported on $\left\{0, \frac{1}{N}, \dots, \frac{N-1}{N}, \frac{N}{N} = 1\right\}$, in the following way:

$$\mathbb{P}_N(B) = {}^*\mathbb{P}(Y_N \in B) \text{ for all } {}^*\text{-Borel sets } B \subseteq {}^*[0, 1]. \quad (1.5)$$

Consider the associated Loeb measure $L\mathbb{P}_N$. With $\mathcal{B}([0, 1])$ denoting the Borel sigma algebra of $[0, 1]$, define $\mu: \mathcal{B}([0, 1]) \rightarrow [0, 1]$ by:

$$\mu(A) := L\mathbb{P}_N(\text{st}^{-1}(A)) \text{ for all Borel subsets } A \subseteq [0, 1]. \quad (1.6)$$

By Theorem 2.14, μ is a well-defined Radon probability measure on $[0, 1]$ such that the following holds:

$${}^*\mathbb{E}_{\mathbb{P}_N}({}^*f) \approx \mathbb{E}_\mu(f) \text{ for all bounded nonnegative } f: [0, 1] \rightarrow \mathbb{R}_{\geq 0}. \quad (1.7)$$

Consider the function $f: [0, 1] \rightarrow \mathbb{R}_{\geq 0}$ defined by

$$f(p) = p^\alpha(1-p)^{k-\alpha} \text{ for all } p \in [0, 1]. \quad (1.8)$$

Noting the form of the right side in (1.2), and using (1.4) and (1.7), it is clear that we need the following to be true:

Theorem 1.3. *We have*

$$\begin{aligned} & \sum_{i=0}^N {}^*\mathbb{P}\left(X_1 = e_1, \dots, X_k = e_k \mid Y_N = \frac{i}{N}\right) {}^*\mathbb{P}\left(Y_N = \frac{i}{N}\right) \\ & \approx \sum_{i=0}^N \left(\frac{i}{N}\right)^\alpha \left(1 - \frac{i}{N}\right)^{k-\alpha} {}^*\mathbb{P}\left(Y_N = \frac{i}{N}\right). \end{aligned} \quad (1.9)$$

The rest of this paper will build toward a proof of Theorem 1.3. The strategy is to use the following simple fact from nonstandard analysis:

Lemma 1.4. *If $\alpha_j, \beta_j \in {}^*\mathbb{R}_{\geq 0}$ (where $j \in H$ for some hyperfinite set H) and $\frac{\alpha_j}{\beta_j} \approx 1$ for all $j \in H$, then*

$$\frac{\sum_{j \in H} \alpha_j}{\sum_{j \in H} \beta_j} \approx 1. \quad (1.10)$$

Proof. Let H , α_j , and β_j be as in the statement of the lemma. Note that α_j, β_j must all be strictly positive. For any real number $\epsilon \in \mathbb{R}_{>0}$, the condition that $\frac{\alpha_j}{\beta_j} \approx 1$ for all $j \in H$ implies that

$$1 - \epsilon < \frac{\alpha_j}{\beta_j} < 1 + \epsilon \text{ for all } j \in H.$$

Multiplying all sides of the above inequality by β_j , we have:

$$\beta_j(1 - \epsilon) < \alpha_j < \beta_j(1 + \epsilon) \text{ for all } j \in H.$$

Summing as j varies over the hyperfinite set (in this step, we are also using transfer of a similar inequality for finite sums), we get:

$$(1 - \epsilon) \sum_{j \in H} \beta_j < \sum_{j \in H} \alpha_j < (1 + \epsilon) \sum_{j \in H} \beta_j. \quad (1.11)$$

Dividing all sides of (1.11) by $\sum_{j \in H} \beta_j$ and noting that $\epsilon \in \mathbb{R}_{>0}$ was arbitrarily chosen completes the proof. \square

For brevity in future computations, we define

$$a_i = {}^*\mathbb{P} \left(X_1 = e_1, \dots, X_k = e_k \mid Y_N = \frac{i}{N} \right) \quad (1.12)$$

$$\text{and } b_i = \left(\frac{i}{N} \right)^\alpha \left(1 - \frac{i}{N} \right)^{k-\alpha} \text{ for all } i \in \{0, 1, 2, \dots, N\}. \quad (1.13)$$

Let us first try to understand the conditional probabilities a_i . As explained earlier, the a_i are zero for $i < \alpha$. By summing over all possible cases, we have:

$$\begin{aligned} a_i &= {}^*\mathbb{P} \left(X_1 = e_1, \dots, X_k = e_k \mid Y_N = \frac{i}{N} \right) \\ &= \sum_{(u_1, \dots, u_N) \in \mathcal{G}} {}^*\mathbb{P} \left(X_1 = u_1, \dots, X_N = u_N \mid X_1 + \dots + X_N = i \right), \end{aligned} \quad (1.14)$$

where

$$\mathcal{G} := \left\{ (u_1, \dots, u_N) \in \{0, 1\}^N : u_j = e_j \text{ for all } j \in \{1, \dots, k\} \text{ and } \sum_{j=1}^N u_j = i \right\}.$$

It is clear that the internal cardinality of \mathcal{G} is the number of ways of choosing $u_{k+1}, \dots, u_N \in \{0, 1\}$ such that $\sum_{j=k+1}^N u_j = i - \alpha$. By a simple counting argument, this yields:

$$\#(\mathcal{G}) = \binom{N-k}{i-\alpha}. \quad (1.15)$$

Also, by the transfer of exchangeability of the X_i , it is clear that:

$$\begin{aligned} & {}^*\mathbb{P}\left(X_1 = u_1, \dots, X_N = u_N \mid X_1 + \dots + X_N = i\right) \\ &= \frac{1}{\text{Number of ways of writing } i \text{ as a sum of } N \text{ zeroes and ones}} \end{aligned} \quad (1.16)$$

for all $(u_1, \dots, u_N) \in \mathcal{G}$.

To see (1.16), first define \mathcal{G}' as the set of those (u_1, \dots, u_N) such that $\sum_{j=1}^N u_j = i$.

Then exchangeability implies that

$$\begin{aligned} & {}^*\mathbb{P}((X_1, \dots, X_N) = \vec{u} \mid X_1 + \dots + X_N = i) \\ &= {}^*\mathbb{P}((X_1, \dots, X_N) = \vec{u}' \mid X_1 + \dots + X_N = i) \text{ for all } \vec{u}, \vec{u}' \in \mathcal{G}'. \end{aligned}$$

Since the sum of ${}^*\mathbb{P}((X_1, \dots, X_N) = \vec{u} \mid X_1 + \dots + X_N = i)$ as \vec{u} varies over \mathcal{G}' is equal to one, it must be the case that

$${}^*\mathbb{P}((X_1, \dots, X_N) = \vec{u} \mid X_1 + \dots + X_N = i) = \frac{1}{\#(\mathcal{G}')} \text{ for all } \vec{u} \in \mathcal{G}'. \quad (1.17)$$

In particular, since $\mathcal{G} \subseteq \mathcal{G}'$, equation (1.17) explains (1.16). Now, another simple counting argument shows that $\#(\mathcal{G}') = \binom{N}{i}$. Thus, (1.16) becomes:

$${}^*\mathbb{P}\left(X_1 = u_1, \dots, X_N = u_N \mid X_1 + \dots + X_N = i\right) = \frac{1}{\binom{N}{i}} \quad (1.18)$$

for all $(u_1, \dots, u_N) \in \mathcal{G}$.

Using (1.18) and (1.15) in (1.14), we obtain:

$$a_i = \frac{\binom{N-k}{i-\alpha}}{\binom{N}{i}} \text{ for all } i \in \{1, \dots, N\}, \quad (1.19)$$

where $\binom{N-k}{i-\alpha}$ is understood to be zero when $i < \alpha$.

Using (1.19), we first prove Theorem 1.3 in a pathological case of zero probability (see Lemma 1.5) that we will avoid afterward. Note that the conclusion of de Finetti's theorem implies that this pathological case can never happen, unless all the random variables X_i are zero almost surely. However, since we are proving de Finetti's theorem, we have to take care of this case in a non-circular way, without using de Finetti's theorem.

Lemma 1.5. *Suppose $\mathbb{P}(X_1 = e_1, \dots, X_k = e_k) = 0$. Then, (1.9) holds.*

Proof. Suppose $\mathbb{P}(X_1 = e_1, \dots, X_k = e_k) = 0$. Suppose $i \geq \alpha$ and consider the event $\left\{Y_N = \frac{i}{N}\right\}$, which is the same as the event $\{X_1 + \dots + X_N = i\}$.

If the sum of N zero-one random variables is $i \geq \alpha$ then some subcollection of k such random variables must have had exactly α ones. Therefore, if \mathcal{C} denotes the

collection of all k tuples of distinct indices from $\{1, \dots, N\}$ (so that the internal cardinality $\#(\mathcal{C})$ is $\binom{N}{k}$), then we have

$$\{X_1 + \dots + X_N = i\} \subseteq \bigcup_{(j_1, \dots, j_k) \in \mathcal{C}} \{X_{j_1} = e_1, \dots, X_{j_k} = e_k\}.$$

By exchangeability, all events in the union on the right have the same probability as the event $\{X_1 = e_1, \dots, X_k = e_k\}$, which is assumed to have probability zero. Since ${}^*\mathbb{P}$ is hyperfinitely subadditive, this implies that ${}^*\mathbb{P}(X_1 + \dots + X_N = i) = 0$ whenever $i \geq \alpha$. Thus (using (1.19)), proving (1.9) is equivalent to proving the following:

$$\begin{aligned} & \sum_{i=0}^{\alpha-1} \frac{\binom{N-k}{i-\alpha}}{\binom{N}{i}} \mathbb{P}\left(Y_N = \frac{i}{N}\right) \\ & \approx \sum_{i=0}^{\alpha-1} \left(\frac{i}{N}\right)^\alpha \left(1 - \frac{i}{N}\right)^{k-\alpha} \mathbb{P}\left(Y_N = \frac{i}{N}\right). \end{aligned} \quad (1.20)$$

But the left side of (1.20) is zero (as $\binom{N-k}{i-\alpha} = 0$ for $i < \alpha$), while the right side is an infinitesimal (being a finite sum of infinitesimals). This completes the proof. \square

Also using (1.19), we obtain the following result about the ratio of a_i and b_i :

Lemma 1.6. *There exists a constant $r \approx 1$, such that for each $i \in {}^*\mathbb{N}_{>k}$, we have*

$$\frac{a_i}{b_i} = \frac{i!}{(i-\alpha)!i^\alpha} \left(1 - \frac{1}{N-i}\right) \dots \left(1 - \frac{k-\alpha-1}{N-i}\right) r \leq r. \quad (1.21)$$

Proof. From (1.19) and (1.13), we obtain:

$$\begin{aligned} \frac{a_i}{b_i} &= \frac{\frac{(N-k)(N-k-1)\dots(N-k-(i-\alpha-1))}{(i-\alpha)!}}{\frac{N(N-1)\dots(N-(i-1))}{i!} \left(\frac{i}{N}\right)^\alpha \left(1 - \frac{i}{N}\right)^{k-\alpha}} \\ &= \frac{i!}{(i-\alpha)!i^\alpha} \frac{N^k}{(N-i)^{k-\alpha}} \frac{(N-k)(N-k-1)\dots(N-k-(i-\alpha-1))}{N(N-1)\dots(N-(i-1))} \\ &= \frac{i!}{(i-\alpha)!i^\alpha} \frac{N^k(N-i)(N-(i+1))\dots N-(i+k-\alpha-1)}{N(N-1)\dots(N-(k-1))(N-i)^{k-\alpha}}. \end{aligned}$$

Let

$$r := \frac{N^k}{N(N-1)\dots(N-(k-1))} = \frac{1}{1\left(1 - \frac{1}{N}\right)\dots\left(1 - \frac{k-1}{N}\right)} \approx 1. \quad (1.22)$$

Thus the proof is complete because:

$$\begin{aligned} & \frac{(N-i)(N-(i+1))\dots(N-(i+k-\alpha-1))}{(N-i)^{k-\alpha}} \\ & = 1 \left(1 - \frac{1}{N-i}\right) \dots \left(1 - \frac{k-\alpha-1}{N-i}\right). \end{aligned}$$

\square

Lemma 1.7. *Suppose $\alpha \geq 1$. There is an $M_1 > \mathbb{N}$ such that $M_1 < N - \sqrt{N}$ and*

$$\sum_{i=0}^{M_1} a_i {}^* \mathbb{P} \left(Y_N = \frac{i}{N} \right) \approx 0 \text{ and } \sum_{i=0}^{M_1} b_i {}^* \mathbb{P} \left(Y_N = \frac{i}{N} \right) \approx 0.$$

Proof. Fix any $M_1 > \mathbb{N}$ such that $M_1 < \min\{N^{\frac{1}{3}}, N - \sqrt{N}\}$.

Note that $\sum_{i=0}^k a_i$ is an infinitesimal. Hence, by (1.21), it suffices to show that

$\sum_{i=0}^{M_1} b_i$ is an infinitesimal. Now,

$$\sum_{i=0}^{M_1} b_i = \sum_{i=0}^{M_1} \left(\frac{i}{N} \right)^\alpha \left(1 - \frac{i}{N} \right)^{k-\alpha} \leq \frac{M_1^{1+\alpha}}{N^\alpha} < \frac{N^{\frac{1+\alpha}{3}}}{N^\alpha} = \frac{1}{N^{\frac{2\alpha-1}{3}}}.$$

But the right side is an infinitesimal because $2\alpha > 1$ (as $\alpha \geq 1$ is assumed in the statement of the lemma). This completes the proof. \square

For the rest of this paper, let

$$M_2 := [N - \sqrt{N}] + 1, \tag{1.23}$$

where $[\cdot]$ is the greatest integer function.

Corollary 1.8. *For $i \in {}^* \mathbb{N}$ with $\mathbb{N} < i \leq M_2$, we have $\frac{a_i}{b_i} \approx 1$.*

Proof. Note that $\frac{i!}{(i-\alpha)!i^\alpha} = 1$ when $\alpha = 0, 1$. And for $\alpha \geq 2$, we have

$$\frac{i!}{(i-\alpha)!i^\alpha} = \left(1 - \frac{1}{i} \right) \dots \left(1 - \frac{\alpha-1}{i} \right) \approx 1 \text{ if } i > \mathbb{N}.$$

Thus, we have:

$$\frac{i!}{(i-\alpha)!i^\alpha} \approx 1 \text{ for all } i > \mathbb{N}. \tag{1.24}$$

Now let i be as in the statement of the corollary, i.e., $\mathbb{N} < i \leq M_2$. Then, $N - i \geq N - M_2 \geq \sqrt{N}$. Then,

$$\left(1 - \frac{1}{N-i} \right) \dots \left(1 - \frac{k-\alpha-1}{N-i} \right) \approx 1 \text{ as well.} \tag{1.25}$$

Using (1.24) and (1.25) in (1.21) completes the proof. \square

Lemma 1.9. *Suppose $\alpha \leq (k-1)$. Then*

$$\sum_{i=M_2+1}^N a_i {}^* \mathbb{P} \left(Y_N = \frac{i}{N} \right) \approx 0 \text{ and } \sum_{i=M_2+1}^N b_i {}^* \mathbb{P} \left(Y_N = \frac{i}{N} \right) \approx 0. \tag{1.26}$$

Proof. By (1.21), it suffices to show that the second sum is an infinitesimal. Since the b_i are all positive, we have the following estimate for the second term:

$$\begin{aligned}
\sum_{i=M_2+1}^N b_i {}^* \mathbb{P} \left(Y_N = \frac{i}{N} \right) &\leq \left(\max_{M_2+1 \leq i \leq N} b_i \right) \sum_{i=M_2+1}^N {}^* \mathbb{P} \left(Y_N = \frac{i}{N} \right) \\
&\leq \max_{M_2+1 \leq i \leq N} \left(\frac{i}{N} \right)^\alpha \left(1 - \frac{i}{N} \right)^{k-\alpha} \\
&\leq 1 \cdot \left(1 - \frac{N - \sqrt{N}}{N} \right)^{k-\alpha} \\
&= \left(\frac{1}{\sqrt{N}} \right)^{k-\alpha},
\end{aligned}$$

where the last term is infinitesimal since $k - \alpha \geq 1$. \square

We are now in a position to prove Theorem 1.3.

Proof. The case when $\alpha = 0$ is verified directly by plugging in $\alpha = 0$ to the formulae for a_i and b_i and using Lemma 1.4.

In the case when $\alpha = k$, using (1.9) and (1.13), we get:

$$\frac{a_i}{b_i} = \frac{\binom{N-k}{i-k}}{\binom{N}{i} \frac{i^k}{N^k}} = \frac{i!}{(i-k)! i^k} \frac{(N-k)! N^k}{N!}.$$

This expression is infinitesimally close to 1 whenever $i > N$. Thus, Lemma 1.7 and Lemma 1.4 complete the proof in this case.

By Lemma 1.5, we may also assume that

$$\mathbb{P}(X_1 = e_1, \dots, X_k = e_k) \neq 0.$$

Then using (1.4), we obtain

$$\sum_{i=0}^N {}^* \mathbb{P} \left(X_1 = e_1, \dots, X_k = e_k \mid Y_N = \frac{i}{N} \right) {}^* \mathbb{P} \left(Y_N = \frac{i}{N} \right) \not\approx 0.$$

Thus, by Lemmas 1.7 and 1.9, we obtain:

$$\begin{aligned}
&\sum_{i=0}^N {}^* \mathbb{P} \left(X_1 = e_1, \dots, X_k = e_k \mid Y_N = \frac{i}{N} \right) {}^* \mathbb{P} \left(Y_N = \frac{i}{N} \right) \\
&\approx \sum_{i=M_1+1}^{M_2} a_i {}^* \mathbb{P} \left(Y_N = \frac{i}{N} \right),
\end{aligned}$$

and

$$\sum_{i=0}^N \left(\frac{i}{N} \right)^\alpha \left(1 - \frac{i}{N} \right)^{k-\alpha} {}^* \mathbb{P} \left(Y_N = \frac{i}{N} \right) \approx \sum_{i=M_1+1}^{M_2} b_i {}^* \mathbb{P} \left(Y_N = \frac{i}{N} \right).$$

Corollary 1.8 together with Lemma 1.4 now complete the proof in this case. \square

As e_1, \dots, e_k was an arbitrarily fixed finite sequence of zeros and ones, this proves de Finetti's Theorem 1.2 using Theorem 2.14.

We finish this section with a combinatorial-probabilistic interpretation of the proof. A main ingredient in the proof was Corollary 1.8. It shows that when i is large (in the sense that it is hyperfinite) but not too large (in the sense that it is less than $M_2 = \lceil N - \sqrt{N} \rceil + 1$), then $\frac{a_i}{b_i}$ is infinitesimally close to 1. Looking at the expressions (1.19) and (1.13) for a_i and b_i respectively, we can express the ratio as follows:

$$\frac{a_i}{b_i} = \frac{\binom{N-k}{i-\alpha} \binom{k}{\alpha}}{\binom{N}{i}} \cdot \frac{1}{\binom{k}{\alpha} \left(\frac{i}{N}\right)^\alpha \left(1 - \frac{i}{N}\right)^{k-\alpha}}.$$

The first term on the right is an expression related to a certain hypergeometric random variable, while the second term is related to a certain binomial random variable. We can thus interpret Corollary 1.8 as a statement about asymptotically approximating a hypergeometric random variable with a binomial random variable. More explicitly, Corollary 1.8 says that as long as i is neither too small nor too large, then the probabilities P_1 and P_2 described by the following are very close to each other in the sense that $\frac{P_1}{P_2} \approx 1$:

- (1) Uniformly choose a random subset of size i (here $i \geq \alpha$) from $\{1, \dots, N\}$ —thus all the $\binom{N}{i}$ subsets are equally likely to be chosen. Then P_1 is the probability that exactly α elements of $\{1, \dots, k\}$ appear in this random subset of size i .
- (2) Take a coin with a probability of Heads being $\frac{i}{N}$. Then P_2 is the probability that exactly α Heads appear in k independent tosses of this coin.

2. A Brief Overview of Nonstandard Analysis

This section provides an introduction to nonstandard methods. A significant part of this discussion is borrowed from a similar introduction in the arXiv version of Alam [1]. There are many approaches to nonstandard analysis, eight of which were described in Benci–Forti–di Nasso [5]. We follow the superstructure approach, as done in Albeverio et al. [2]. Very roughly, a nonstandard extension of a set S (consisting of atoms or urelements; that is, we view each element of X as an “individual” without any structure, set-theoretic or otherwise) is a superset *S that preserves the “first-order” properties of S . That is, a property which is expressible using finitely many symbols without quantifying over any collections of subsets of S is true if and only if the same property is true of *S . This is called the **transfer principle** (or just *transfer* for brevity). The set *S should contain, as a subset, *T for each $T \subseteq S$.

Like subsets, other mathematical objects defined on S also have extensions. So, a function $f: S \rightarrow T$ extends to a map ${}^*f: {}^*S \rightarrow {}^*T$, and relations on S extend to relations on *S . Hence there is a binary relation ${}^* <$ on ${}^*\mathbb{R}$, which we still denote by $<$ (an abuse of notation that we frequently make), and which is the same as the usual order when restricted to \mathbb{R} . Thus, ${}^*\mathbb{R}$ is an ordered field of characteristic 0.

Indeed all (the infinitely many) axioms for ordered fields of characteristic 0 hold for it by transfer. The symbols in a sentence such as “ $\forall x > 0 \exists y(x = y^2)$ ” (which is expressing the proposition that each positive number has a square root) have new meanings in the nonstandard universe: by “ $<$ ”, we are now interpreting the extension of the order on \mathbb{R} . Yet the sentence is true in ${}^*\mathbb{R}$ by transfer!

A non-first-order property may not transfer. For instance, we shall see (cf. Proposition 2.2) that any “non-trivial” extension of \mathbb{R} contains *infinite* elements (that is, those that are larger than all real numbers in absolute value), as well as *infinitesimal* elements (that is, those that are smaller than all positive real numbers in absolute value). Thus, the Archimedean property of \mathbb{R} does not transfer. The set of finite nonstandard real numbers, denoted by ${}^*\mathbb{R}_{\text{fin}}$, is a subring of the non-Archimedean field ${}^*\mathbb{R}$. To see what went wrong, note that the following sentences formally express the Archimedean property for \mathbb{R} and its transfer, respectively:

$$\forall x \in \mathbb{R} \quad \exists n \in \mathbb{N} (n > x). \quad (2.1)$$

$$\forall x \in {}^*\mathbb{R} \quad \exists n \in {}^*\mathbb{N} (n > x). \quad (2.2)$$

The transferred sentence (2.2) no longer expresses the Archimedean property (though it still expresses an interesting fact about ${}^*\mathbb{R}$). The issue is that we are only able to quantify over ${}^*\mathbb{N}$ (and not over \mathbb{N}) after transfer. To keep quantifying over \mathbb{N} , we would have to transfer an “infinite statement” (saying that for every x , either $1 > x$, or $2 > x$, or $3 > x$, or \dots), which is not a valid first-order sentence.

Another non-example is the least upper bound principle—the set \mathbb{N} , viewed as a subset of ${}^*\mathbb{R}$, is bounded (by any positive infinite element), yet has no least upper bound (as any upper bound minus one is also an upper bound). The issue here is that the least upper bound property for \mathbb{R} is expressed via the second-order statement:

$$\begin{aligned} & \forall A \subseteq \mathbb{R} \\ & \langle \exists x \in \mathbb{R} (\forall y \in \mathbb{R} \{ (y \in A) \rightarrow (y \leq x) \}) \rangle \rightarrow \\ & \exists z \in \mathbb{R} \\ & \{ (\forall y \in \mathbb{R} [(y \in A) \rightarrow (y \leq z)]) \\ & \quad \wedge [\forall w \in \mathbb{R} (\forall y \in \mathbb{R} \{ [(y \in A) \rightarrow (y \leq w)] \rightarrow (z \leq w) \})] \} \}. \end{aligned}$$

One way to express this as a first-order statement is to quantify over the powerset, $\mathcal{P}(\mathbb{R})$, of \mathbb{R} . If our nonstandard map * was able to extend sets of subsets of X as well, then the above would transfer to the following **-least upper bound property*:

$$\begin{aligned} & \forall A \in {}^*\mathcal{P}(\mathbb{R}) \\ & \langle \exists x \in {}^*\mathbb{R} (\forall y \in {}^*\mathbb{R} \{ (y \in A) \rightarrow (y \leq x) \}) \rangle \rightarrow \\ & \exists z \in {}^*\mathbb{R} \\ & \{ (\forall y \in {}^*\mathbb{R} [(y \in A) \rightarrow (y \leq z)]) \\ & \quad \wedge [\forall w \in {}^*\mathbb{R} (\forall y \in {}^*\mathbb{R} \{ [(y \in A) \rightarrow (y \leq w)] \rightarrow (z \leq w) \})] \} \}. \end{aligned}$$

Notice that any quantification over a standard set was “transferred” to a quantification over the corresponding nonstandard extension of that set. The non-quantified occurrences of \in in the original sentence were as relation symbols (that is, ‘ $a \in b$ ’ is true just in case a is an element of b). Strictly speaking, an occurrence of a relation (or function) symbol must be transferred to the nonstandard extension of that relation (function) symbol. Thus, the second line of the transferred sentence must technically be ‘ $[\exists x \in {}^*\mathbb{R}(\forall y \in {}^*\mathbb{R}\{(y^* \in A) \rightarrow (y^* \leq x)\})]$ ’. However, as before, we suppress the $*$ on the transferred relation symbols for better readability.

In practice, we often write informal logic sentences as long as it is clear that they can be made formal. For instance, instead of writing

$$(\forall y \in \mathbb{R}\{(y \in A) \rightarrow (y \leq x)\}),$$

one would often write ‘ $\forall y \in A(y \leq x)$ ’.

The above discussion implies that \mathbb{N} would not be an element of ${}^*\mathcal{P}(\mathbb{R})$ (as it does not satisfy the $*$ -least upper bound property), whatever the latter object is. By the transfer of the sentence “ $\forall A \in \mathcal{P}(\mathbb{R}) \forall x \in A (x \in \mathbb{R})$ ”, the object ${}^*\mathcal{P}(\mathbb{R})$ would in fact be a subset of $\mathcal{P}({}^*\mathbb{R})$. The previous example leads to the observation that ${}^*\mathcal{P}(\mathbb{R})$ is not a superset of $\mathcal{P}(\mathbb{R})$ in the literal sense. It does, however, contain as an element the extension *A for any $A \in \mathcal{P}(\mathbb{R})$.

In general, we fix a set S consisting of atoms (that is, we view each element of S as an “individual” without any structure, set-theoretic or otherwise), and extend what is called the superstructure $V(S)$ of S , which is defined inductively as follows (here, for any set A , the set $\mathcal{P}(A)$ denotes the power set of A):

$$\begin{aligned} V_0(S) &:= S, \\ V_n(S) &:= \mathcal{P}(V_{n-1}(S)) \text{ for all } n \in \mathbb{N}, \\ V(S) &:= \bigcup_{n \in \mathbb{N} \cup \{0\}} V_n(S). \end{aligned} \tag{2.3}$$

Choosing S suitably, the superstructure $V(S)$ can be made to contain all mathematical objects relevant for a given theory. For example, if $\mathbb{R} \subseteq S$, then all collections of subsets of \mathbb{R} live as objects in $V_2(S) \subseteq V(S)$. For a finite subset consisting of k objects from $V_m(S)$, the ordered k -tuple of those objects is an element of $V_n(S)$ for some larger n ; and hence the set of all k -tuples of objects in $V_m(S)$ lies as an object in $V_{n+1}(S)$. For example, if $x, y \in V_m(S)$, then the ordered pair (x, y) is just the set $\{\{x\}, \{x, y\}\} \in V_{m+2}(S)$. Identifying functions and relations with their graphs, $V(S)$ also contains, if $\mathbb{R} \subseteq S$, all functions from \mathbb{R}^n to \mathbb{R} , all relations on \mathbb{R}^n , etc., for all $n \in \mathbb{N}$.

We extend the superstructure $V(S)$ via a *nonstandard map*,

$$*: V(S) \rightarrow V({}^*S),$$

which, by definition, is any map satisfying the following axioms:

- (NS1) The transfer principle holds.
- (NS2) ${}^*\alpha = \alpha$ for all $\alpha \in S$.
- (NS3) $\{{}^*a : a \in A\} \subsetneq {}^*A$ for any infinite set $A \in V(S)$.

A nonstandard map may not be unique. In practice, however, we fix a standard universe $V(S)$ and a nonstandard map $*$. The reader is referred to Chang–Keisler [6, Theorem 4.4.5, p. 268] or Albeverio et al. [2, Chapter 1] for a proof of the existence of a nonstandard map.

An object that belongs to $*A$ for some $A \in V(S)$ is called *internal*. A useful way to understand this concept is to think that internal objects are those that inherit properties from their standard counterparts by transfer. For instance, the internal subsets of $*S$ are precisely the elements of $*\mathcal{P}(S)$ —a (reasonable) property satisfied by all elements of $\mathcal{P}(S)$ (that is, by all subsets of S) will thus transfer to all internal sets. As a consequence, the class of internal sets is closed under Boolean operations such as finite unions, finite intersections, etc.

Definition 2.1. For a cardinal number κ , a nonstandard extension is called κ -*saturated* if any collection of internal sets that has cardinality less than κ and that has the finite intersection property has a non-empty intersection.

We will henceforth assume that the nonstandard extension we work with is sufficiently saturated (cf. Chang and Keisler [6, Lemma 5.1.4, p. 294 and Exercise 5.1.21, p. 305]). The next proposition shows that infinite (and infinitesimal) elements do exist in any sufficiently saturated nonstandard extension.

Proposition 2.2. *$*\mathbb{R}$ contains infinite as well as infinitesimal elements.*

Proof. By saturation, the set $\bigcap_{n \in \mathbb{N}} \{x \in *\mathbb{R} : x > n\}$ is non-empty. It is clear that any element in this set must be infinite. The multiplicative inverse of any infinite element is infinitesimal. \square

The next result says that all legitimately nonstandard natural numbers (that is, those elements of $*\mathbb{N}$ that are not elements of \mathbb{N}) are infinite.

Proposition 2.3. *Any $N \in *\mathbb{N} \setminus \mathbb{N}$ is infinite. We express this by writing $N > \mathbb{N}$.*

Proof. Let $N \in *\mathbb{N} \setminus \mathbb{N}$. Suppose, if possible, that N is finite. In particular, there exist elements of \mathbb{N} that are larger than N . Thus the set $\{n \in \mathbb{N} : n > N\}$ is non-empty and hence has a smallest element, say n_0 . By transfer of the fact that elements in \mathbb{N} are at least one unit apart, we know that $n_0 - N \geq 1$. If $n_0 - N = 1$, then $N = n_0 - 1 \in \mathbb{N}$, a contradiction. Hence, we must have $n_0 - N \geq 2$ (by transfer of the fact that if the distance between two natural numbers is larger than one, then it is at least two). But then $n_0 - 1 \geq N + 1$ and $n_0 - 1 \in \mathbb{N}$, contradicting the minimality of n_0 . \square

As discussed earlier, the existence of an infinite element in $*\mathbb{R}$ implies that the set \mathbb{N} has an upper bound in $*\mathbb{R}$, but it does not have a least upper bound. Since all bounded internal subsets of $*\mathbb{R}$ have a least upper bound in $*\mathbb{R}$ by transfer, it follows that the set \mathbb{N} is not internal. The following useful result is a consequence of this fact. See also Albeverio et al. [2, Proposition 1.2.7, p.21].

Proposition 2.4. *Let A be an internal set.*

(i) [**Overflow**] *If $\mathbb{N} \subseteq A$, then there is an $N > \mathbb{N}$ such that*

$$\{n \in *\mathbb{N} : n \leq N\} \subseteq A.$$

(ii) [**Underflow**] *If A contains all hyperfinite natural numbers, then there is an $n_0 \in \mathbb{N}$ such that ${}^*\mathbb{N}_{\geq n_0} := \{n \in {}^*\mathbb{N} : n \geq n_0\} \subseteq A$.*

We have seen several examples of internal sets and functions— ${}^*\mathbb{N}$, ${}^*\mathbb{R}$, *f (for any standard function f), etc. Unlike these examples, (NS3) guarantees the existence of internal objects that are not ${}^*\alpha$ for any $\alpha \in V(S)$. For instance, for any $N > \mathbb{N}$, the set $\{1, \dots, N\}$ of the “first N nonstandard natural numbers” is internal, yet it does not equal the nonstandard extension of any standard set. This set is rigorously defined as the initial segment of N in ${}^*\mathbb{N}$. The fact that it is internal follows from the transfer of the following sentence:

$$\forall n \in \mathbb{N} \exists! A \in \mathcal{P}(\mathbb{N}) [\forall x \in \mathbb{N} (x \in A \leftrightarrow x \leq n)].$$

For a standard set A , let $\mathcal{P}_{\text{fin}}(A)$ denote the collection of finite subsets of A . There is a function $\# : \mathcal{P}_{\text{fin}}(A) \rightarrow \mathbb{N} \cup \{0\}$ that counts the number of elements in each finite subset. By transfer, we have a corresponding counting function ${}^*\# : {}^*\mathcal{P}_{\text{fin}}(A) \rightarrow {}^*\mathbb{N} \cup \{0\}$ (which we often still denote by $\#$ by an abuse of notation) that satisfies the same first order properties as the usual counting function (for example, it satisfies the inclusion-exclusion principle). The elements of ${}^*\mathcal{P}_{\text{fin}}(A)$ are called the hyperfinite subsets of *A . Hyperfinite sets behave like finite sets even though they are not finite in the standard sense. For instance, an internal set H is hyperfinite if and only if there is an $N \in {}^*\mathbb{N}$ and an internal bijection $f : H \rightarrow \{1, \dots, N\}$.

There is a “sum function” that takes any finite set of real numbers as an input and produces the sum of those real numbers. By transfer, we can thus abstractly make sense of “hyperfinite sums” (that is, the sum of hyperfinitely many nonstandard real numbers). For nonstandard real numbers a_i , this is the sense in which

we interpret objects such as $\sum_{i=1}^N a_i$ where $N \in {}^*\mathbb{N}$ (or in general, $\sum_{i \in H} a_i$, where H

is a hyperfinite set).

The next result says that one can think of a finite nonstandard real number z as having a real part, and an infinitesimal part (in fact, this real part is just $\sup\{y \in \mathbb{R} : y \leq z\}$). See Cutland [7, Theorem 2.10, p. 55] for a proof.

Proposition 2.5. *For all $z \in {}^*\mathbb{R}_{\text{fin}}$, there is a unique $x \in \mathbb{R}$ (called the standard part of z) such that $(z - x)$ is infinitesimal. We write $\text{st}(z) = x$ or $z \approx x$.*

The next result gives a nice characterization of limit points of sequences in terms of standard parts of terms with hyperfinite indices (see Cutland [7, Theorems 3.1 and 3.3] for proofs of the two statements):

Proposition 2.6. *For a sequence of real numbers $\{a_n\}_{n \in \mathbb{N}}$, there is an extended sequence $\{a_n\}_{n \in {}^*\mathbb{N}}$ (by viewing the original sequence as a function on \mathbb{N}). A real number L is an accumulation point of the sequence $\{a_n\}_{n \in \mathbb{N}} \iff$ there is an $N > \mathbb{N}$ such that $\text{st}(a_N) = L$. Thus $\lim a_n = L \iff \text{st}(a_N) = L$ for all $N > \mathbb{N}$.*

Note that, more generally, one can define the notion of standard parts for elements in the nonstandard extension of any Hausdorff space. In general, we will need a point to be *nearstandard*, instead of finite, for it to have a standard part.

Definition 2.7. For a topological space T and a point $x \in {}^*T$, we say that z is nearstandard to x if $z \in {}^*O$ for any open neighborhood O of x .

If T is a Hausdorff space, then a point $z \in {}^*T$ can be nearstandard to at most one point $x \in T$. In such a case, we write $x = \mathbf{st}(z)$ (also, $z \in \mathbf{st}^{-1}(x)$). Using this notation, we have the following useful characterization of continuity (see, for example, Albeverio et al. [2, Proposition 1.3.3, p. 27] for the one-dimensional case, with the higher dimensional case following a similar argument):

Proposition 2.8. *Let S and T be Hausdorff spaces, and let $f: S \rightarrow T$ be a function. Then f is continuous at $x \in S$ if and only if ${}^*f(\mathbf{st}^{-1}(x)) \subseteq \mathbf{st}^{-1}(f(x))$.*

We shall also need the following characterization of compact spaces (see [2, Proposition 2.1.6]).

Proposition 2.9. *A topological space T is compact if and only all points in *T are nearstandard.*

The following consequence of saturation will be useful in the sequel (see also [2, Lemma 3.1.1, p. 64]).

Proposition 2.10. *A countable union of disjoint internal sets is internal if and only if all but finitely many of them are empty.*

Proof. Suppose $\{A_i\}_{i \in \mathbb{N}}$ is a countable collection of disjoint internal sets. Let $A = \cup_{i \in \mathbb{N}} A_i$. If all but finitely many of the A_i are empty, then A being a finite union of internal sets is also internal due to transfer.

Conversely, if A is internal, then $A \setminus A_i$ is internal for each $i \in \mathbb{N}$ by transfer. In that case, if all but finitely many of the A_i are not empty, then the collection $\{A \setminus A_i\}_{i \in \mathbb{N}}$ would satisfy the finite intersection property. By saturation, this would lead to $\cap_{i \in \mathbb{N}} (A \setminus A_i) \neq \emptyset$, which is absurd. This completes the proof by contradiction. \square

We now describe the concept of Loeb measures. Let \mathfrak{X} be an internal set in a nonstandard universe ${}^*V(S)$. Let \mathcal{A} be an *internal algebra* on \mathfrak{X} , i.e., an internal set consisting of (internal) subsets of \mathfrak{X} that is closed under complements and finite unions. Given an internal probability measure ν (that is, the internal map $\nu: \mathcal{A} \rightarrow {}^*\mathbb{R}_{\geq 0}$ satisfies $\nu(\mathfrak{X}) = 1$, and $\nu(A \cup B) = \nu(A) + \nu(B)$ whenever $A \cap B = \emptyset$), the map $\mathbf{st}(\nu): \mathcal{A} \rightarrow \mathbb{R}_{\geq 0}$ is an ordinary finitely additive probability measure. By Proposition 2.10, it follows that $\mathbf{st}(\nu)$ satisfies the premises of Carathéodory Extension Theorem. By that theorem, it extends to a unique probability measure on $\sigma(\mathcal{A})$ (the smallest sigma algebra containing \mathcal{A}), whose completion is called the **Loeb measure** of ν . The corresponding complete measure space $(\mathfrak{X}, L(\mathcal{A}), L\nu)$ is called the **Loeb space** of $(\mathfrak{X}, \mathcal{A}, \nu)$. Note that this construction could have been done with any finite internal measure ν .

It is in general difficult to visualize Loeb measurable sets that are not in the original internal algebra. The following approximation result helps us to approximate any Loeb measurable sets by sets in the original internal algebra. See Albeverio et al. [2, Theorem 3.1.2, p. 64] for a proof.

Proposition 2.11. *Let $(\mathfrak{X}, L(\mathcal{A}), L\nu)$ be the Loeb probability space of $(\mathfrak{X}, \mathcal{A}, \nu)$.*

- (i) For each $A \in \sigma(\mathcal{A})$, there is a set $B \in \mathcal{A}$ such that $L\nu(A \Delta B) = 0$.
(ii) For each $A \in \sigma(\mathcal{A})$ and $\epsilon \in \mathbb{R}_{>0}$, there are sets $B, D \in \mathcal{A}$ such that $B \subseteq A \subseteq D$ and

$$L\mathbb{P}(D) - \epsilon \leq L\mathbb{P}(A) \leq L\mathbb{P}(B) + \epsilon.$$

We will use the following simplification of Ross [13, Theorem 5.1, p. 105] extensively:

Proposition 2.12. *Let $(\mathfrak{X}, L(\mathcal{A}), L\nu)$ be the Loeb probability space of $(\mathfrak{X}, \mathcal{A}, \nu)$. Suppose $F: \mathfrak{X} \rightarrow {}^*\mathbb{R}$ is an internal function that is measurable in the sense that $F^{-1}(B) \in \mathcal{A}$ for all $B \in {}^*\mathcal{B}(\mathbb{R})$ (where $\mathcal{B}(\mathbb{R})$ is the Borel σ -algebra on \mathbb{R}). If $F(x) \in {}^*\mathbb{R}_{fn}$ for $L\nu$ -almost all $x \in \mathfrak{X}$, then $\mathbf{st}(F)$ is Loeb measurable (i.e., measurable as a map from $(\mathfrak{X}, L(\mathcal{A}))$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$).*

For any probability measure ν , there is an *integral operator* \int that takes certain functions (those in the space $L^1(\nu)$ of integrable real-valued functions on the underlying sample space of ν) to their integrals with respect to ν . By transfer, for any internal probability space $(\mathfrak{X}, \mathcal{A}, \nu)$, we also have the associated space ${}^*L^1(\mathfrak{X}, \nu)$ of * -integrable functions, and a corresponding * -integral operator ${}^*\int$. For any * -integrable $F: \mathfrak{X} \rightarrow {}^*\mathbb{R}$, one then has ${}^*\int_{\mathfrak{X}} F d\nu \in {}^*\mathbb{R}$, which we call the ** -integral* of F over $(\mathfrak{X}, \mathcal{A}, \nu)$.

The * -integral on ${}^*L^1(\mathfrak{X})$ inherits many properties (an important one being linearity) from the ordinary integral by transfer. If F is finite almost surely with respect to the corresponding Loeb measure, then $\mathbf{st}(F)$ is Loeb measurable by Proposition 2.12. In that case, it is interesting to study the relation between the * -integral of F and the Loeb integral of $\mathbf{st}(F)$. The following result covers this for a useful class of functions (see Ross [13, Theorem 6.2, p.110] for a proof):

Theorem 2.13. *Suppose $(\mathfrak{X}, \mathcal{A}, \nu)$ is an internal probability space and $F \in {}^*L^1(\mathfrak{X}, \nu)$ is such that $L\nu(F \in {}^*\mathbb{R}_{fn}) = 1$. Then the following are equivalent:*

- (1) ${}^*\int_{\mathfrak{X}} |F| d\nu \in {}^*\mathbb{R}_{fn}$, and
- $$\mathbf{st}\left({}^*\int_{\mathfrak{X}} |F| d\nu\right) = \lim_{m \rightarrow \infty} \mathbf{st}\left({}^*\int_{\mathfrak{X}} |F| \mathbb{1}_{\{|F| \leq m\}} d\nu\right).$$
- (2) For every $M > \mathbb{N}$, we have $\mathbf{st}\left({}^*\int_{\mathfrak{X}} |F| \mathbb{1}_{\{|F| > M\}} d\nu\right) = 0$.
- (3) ${}^*\int_{\mathfrak{X}} |F| d\nu \in {}^*\mathbb{R}_{fn}$; and for any $A \in \mathcal{A}$ we have:

$$\nu(A) \approx 0 \Rightarrow {}^*\int_{\mathfrak{X}} |F| \mathbb{1}_A d\nu \approx 0.$$

- (4) $\mathbf{st}(F)$ is Loeb integrable, and $\mathbf{st}\left({}^*\int_{\mathfrak{X}} |F| d\nu\right) = \int_{\mathfrak{X}} |\mathbf{st}(F)| dL\nu$.

A function satisfying the conditions in Theorem 2.13 is called S -integrable on $(\mathfrak{X}, \mathcal{A}, \nu)$. The notion of S -integrability, first developed by Anderson [4], is one of the most ubiquitous concepts in nonstandard measure theory. Using this concept, we obtain the main result that was needed in this paper. The following result is applicable to more general situations (refer to the settings in Sections 3.4 and 3.5 of Albeverio et al. [2]). However, we restrict to compact Hausdorff spaces and real-valued functions on them for convenience.

Theorem 2.14. *Let S be a compact Hausdorff space. Suppose ${}^*\mathcal{B}(S)$ is the internal algebra of * -Borel subsets of S . Let ν be an internal (finitely additive) probability measure on $({}^*S, {}^*\mathcal{B}(S))$. Let $L\nu$ be the associated Loeb measure. Define a map $\mu: \mathcal{B}(S) \rightarrow [0, 1]$ by:*

$$\mu(B) := L\nu(\mathbf{st}^{-1}(B)) \text{ for all } B \in \mathcal{B}(S). \quad (2.4)$$

Then, we have:

- (i) μ is a Radon probability measure.
- (ii) For any nonnegative continuous function $f: S \rightarrow \mathbb{R}_{\geq 0}$, we have:

$$\int_{{}^*S} {}^*f d\nu \approx \int_S f d\mu. \quad (2.5)$$

Proof. Note that since S is a compact space, we have $\mathbf{st}^{-1}(S) = {}^*S$. That μ is well-defined (that is, $\mathbf{st}^{-1}(B)$ is Loeb measurable for each $B \in \mathcal{B}(S)$) and is a Radon measure then follow from Proposition 3.4.5 and Corollary 3.4.3 in Albeverio et al. [2, pp. 88-89].

To see (ii), let $f: S \rightarrow \mathbb{R}_{\geq 0}$ be a nonnegative function (which is automatically bounded, as the domain is a compact space). Since f is bounded, it follows that $\mathbf{st}({}^*f)$ is Loeb measurable, satisfying the following (see Proposition 2.12 and (2) \Rightarrow (4) of Theorem 2.13):

$$\int_{{}^*S} {}^*f d\nu \approx \int_{{}^*S} \mathbf{st}({}^*f) dL\nu. \quad (2.6)$$

Also, with λ denoting the one-dimensional Lebesgue measure, we have (since $\mathbf{st}({}^*f)$ is nonnegative):

$$\begin{aligned} \int_{{}^*S} \mathbf{st}({}^*f) dL\nu &= \int_{(0, \infty)} L\nu \{x \in {}^*S : \mathbf{st}({}^*f(x)) > y\} d\lambda(y) \\ &= \int_{(0, \infty)} L\nu \{x \in {}^*S : f(\mathbf{st}(x)) > y\} d\lambda(y). \end{aligned} \quad (2.7)$$

We used the nonstandard characterization of continuity (i.e., that $\mathbf{st}({}^*f(x)) = {}^*f(\mathbf{st}(x))$ for all nearstandard points $x \in {}^*S$, which in our case includes all $x \in {}^*S$ since S is compact) to obtain (2.7) in the above.

For $y \in (0, \infty)$, let

$$\begin{aligned} A_y &:= \{x \in {}^*S : f(\mathbf{st}(x)) > y\} \\ \text{and } B_y &:= \{x \in S : f(x) > y\}. \end{aligned}$$

It is routine to verify that

$$A_y = \mathbf{st}^{-1}(B_y) \text{ for all } y \in (0, \infty). \quad (2.8)$$

Thus, (2.7) becomes:

$$\begin{aligned} \int_{*S} \mathbf{st}(*f) dL\nu &= \int_{(0, \infty)} L\nu(A_y) d\lambda(y) \\ &= \int_{(0, \infty)} L\nu(\mathbf{st}^{-1}(B_y)) d\lambda(y) \\ &= \int_{*S} \mathbf{st}(*f) dL\nu \\ &= \int_{(0, \infty)} \mu(B_y) d\lambda(y) \\ &= \int_S f d\mu. \end{aligned} \quad (2.9)$$

Equations (2.6) and (2.9) complete the proof. \square

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