Journal of Stochastic Analysis

Volume 1 | Number 4 Article 14

December 2020

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Recommended Citation

Navas, Eduard; Pineda, Ebner; and Urbina, Wilfredo O. (2020) "The Boundedness of General Alternative Singular Integrals with Respect to the Gaussian Measure," *Journal of Stochastic Analysis*: Vol. 1 : No. 4 , Article 14.
DOI: 10.31390/josa.1.4.14
Available at: https://digitalcommons.lsu.edu/josa/vol1/iss4/14

Journal of Stochastic Analysis
Vol. 1, No. 4 (2020) Article 14 (26 pages)
DOI: 10.31390/josa.1.4.14

THE BOUNDEDNESS OF GENERAL ALTERNATIVE SINGULAR INTEGRALS WITH RESPECT TO THE GAUSSIAN MEASURE

EDUARD NAVAS, EBNER PINEDA, AND WILFREDO O. URBINA*

Dedicated to Professor Leonard Gross on the occasion of his 88th birthday

Abstract. In this paper we introduce a new class of Gaussian singular integrals, the general alternative Gaussian singular integrals and study the boundedness of them in $L^p(\gamma_d)$, $1 < p < \infty$ and its weak $(1,1)$ boundedness with respect to the Gaussian measure following [7] and [1], respectively.

1. Introduction and Preliminaries

Singular integrals are some of the most important operators in classical harmonic analysis. They first appear naturally in the proof of the $L^p(\mathbb{T})$ convergence of Fourier series, $1 < p < \infty$; where the notion of the *conjugated function* is needed[1]

$$\tilde{f}(x) = \text{p.v.}\frac{1}{\pi}\int_{-\pi}^{\pi} \frac{f(x-y)}{2\tan\frac{y}{2}}dy = \lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{\pi>|y|>\varepsilon} \frac{f(x-y)}{2\tan\frac{y}{2}}dy.$$

This notion was extended to the non-periodic case with the definition of the *Hilbert transform,*

$$Hf(x) = \text{p.v.}\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(x-y)}{y}dy = \lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{|y|>\varepsilon} \frac{f(x-y)}{y}dy;$$

and then to $\mathbb{R}^d$, with the notion of *Riesz transform*; see E. Stein [9, Chap III, §1],

$$\begin{aligned} R_j f(x) &= \text{p.v. } C_d \int_{\mathbb{R}^d} \frac{y_i}{|y|^{d+1}} f(x-y)dy \\ &= \lim_{\varepsilon\to 0} C_d \int_{|y|>\varepsilon} \frac{y_j}{|y|^{d+1}} f(x-y)dy, \qquad (1.1) \end{aligned}$$

Received 2019-12-2; Accepted 2020-6-22; Communicated by A. Stan et al.
2010 *Mathematics Subject Classification.* Primary 42B25, 42B35 ; Secondary 46E30, 47G10.
Key words and phrases. Gaussian harmonic analysis, Gaussian Lebesgue spaces, Ornstein-Uhlenbeck semigroup, Gaussian singular integrals.
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[1]For a detailed study of this problem see for instance E. Stein [9, Chapter II, III], J. Duoandikoetxea [3, Chapter 4, 5], L. Grafakos [5, Chapter 4] or A. Torchinski [10, Chapter XI].

for $j = 1, \cdots, d$, $f \in L^p(\mathbb{R}^d)$ with $C_d = \frac{\Gamma(\frac{d+1}{2})}{\pi^{(d+1)/2}}$. Taking Fourier transform, we get

$$\widehat{(R_j f)}(\zeta) = i\frac{\zeta_j}{|\zeta|}\hat{f}(\zeta),$$

and thus $R_j f$ is a classical multiplier operator, with multiplier $m(y) = C_d\ i\,\frac{y_j}{|y|}$, and hence

$$R_j = \frac{\partial}{\partial x_j}(-\Delta)^{-1/2}, \tag{1.2}$$

where $\Delta = \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}$ is the Laplacian operator and $(-\Delta)^{-1/2}$ is the (classical) *Riesz potential* of order 1/2.

This was later generalized to the famous *Calderón-Zygmund class* of singular integrals:

Definition 1.1. We will say that a C^1 function $K(x,y)$, defined off the diagonal of $\mathbb{R}^d \times \mathbb{R}^d$, i.e. $x \neq y$ is a *Calderón-Zygmund kernel* provided that the following conditions are satisfied:

i) $|K(x,y)| \leq \frac{C}{|x-y|^d}$,

ii) $|\partial_y K(x,y)| \leq \frac{C}{|x-y|^{d+1}}$.

associated with K we define the operator T by means of the formula

$$Tf(x) = p.v. \int_{\mathbb{R}^d} K(x,y)f(y)d = \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} K(x,y)f(y)dy.$$

with $f \in C_0^\infty(\mathbb{R}^d)$. We say that T is a Calderón-Zygmund operator if T admits a continuous extension to $L^2(\mathbb{R}^d)$.

For more details on this see, E. Stein [9], [3] or [5].

The *Ornstein-Uhlenbeck operator* in $\mathbb{R}^d$ is a second order differential operator defined as

$$L = \frac{1}{2}\Delta_x - \langle x, \nabla_x \rangle = \sum_{i=1}^d \Big[\frac{1}{2}\frac{\partial^2}{\partial x_i^2} - x_i\frac{\partial}{\partial x_i}\Big], \tag{1.3}$$

where $\nabla_x = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_d})$ is the gradient, and Δ_x is the Laplace operator defined on the space of test functions $C_0^\infty(\mathbb{R}^d)$ of smooth functions with compact support on $\mathbb{R}^d$.

The Hermite polynomials in d-variables, $\{\vec{H}_\nu\}_\nu$ are eigenfunctions of L with corresponding eigenvalues $\lambda_\nu = -|\nu| = -\sum_{i=1}^d \nu_i$, i.e.

$$L\vec{H}_\nu = \lambda_\nu \vec{H}_\nu = -|\nu|\vec{H}_\nu. \tag{1.4}$$

The operator L has a self-adjoint extension to $L^2(\gamma_d)$, that will be also denoted as L, that is,

$$\int_{\mathbb{R}^d} Lf(x)g(x)\gamma_d(dx) = \int_{\mathbb{R}^d} f(x)Lg(x)\gamma_d(dx), \tag{1.5}$$

so L is the natural *"symmetric" Laplacian* in the Gaussian context.

For $i = 1, 2, \cdots, d$ let us consider the differential operators

$$\partial_\gamma^i = \frac{1}{\sqrt{2}}\frac{\partial}{\partial x_i}. \tag{1.6}$$

∂_γ^i is not symmetric nor antisymmetric in $L^2(\gamma_d)$. In fact, its formal $L^2(\gamma_d)$-adjoint[2] is,

$$(\partial_\gamma^i)^* = -\frac{1}{\sqrt{2}} e^{x_i^2} \frac{\partial}{\partial x_i}(e^{-x_i^2} I) = \sqrt{2} x_i I - \frac{1}{\sqrt{2}} \frac{\partial}{\partial x_i}, \tag{1.7}$$

where I is the identity, which can obtained simply by integration by parts. Observe that $(\partial_\gamma^i)^*$ can be written as

$$(\partial_\gamma^i)^* = -e^{|x|^2}(\partial_\gamma^i e^{-|x|^2} I). \tag{1.8}$$

Moreover, it is easy to see that

$$(-L) = \sum_{i=1}^{d} (\partial_\gamma^i)^* \partial_\gamma^i. \tag{1.9}$$

In analogy with the classical case (1.2), the Gaussian Riesz transforms in $\mathbb{R}^d$ are defined spectrally, for $1 \leq i \leq d$, as

$$\mathcal{R}_i = \partial_\gamma^i (-L)^{-1/2}, \tag{1.10}$$

where $(-L)^{-1/2}$ the Gaussian Riesz potential of order $1/2$. The meaning of this is that for any multi-index ν such that $|\nu| > 0$, its action on the Hermite polynomial $\vec{H}_\nu$ is

$$\mathcal{R}_i \vec{H}_\nu = \sqrt{\frac{2}{|\nu|}} \nu_j \vec{H}_{\nu - \vec{e_i}} \tag{1.11}$$

where $\vec{e_i}$ is the unitary vector with zeros in all coordinates except for the i-th coordinate that is one if $\nu_i > 0$, and zero otherwise.

It can be proved, for details see [12], that the kernel of $\mathcal{R}_i$ is given by

$$\mathcal{K}_i(x,y) = \frac{1}{\pi^{d/2}\Gamma(1/2)} \int_0^1 \left(\frac{1-r^2}{-\log r}\right)^{1/2} \frac{y_i - r x_i}{(1-r^2)^{\frac{(d+3)}{2}}} e^{-\frac{|y-rx|^2}{1-r^2}} dr, \tag{1.12}$$

and therefore, we get the integral representation of $\mathcal{R}_i$,

$$\begin{aligned} \mathcal{R}_i f(x) &= \text{p.v.} \int_{\mathbb{R}^d} \mathcal{K}_i(x,y) f(y) dy & (1.13) \\ &= \text{p.v.} \frac{1}{\pi^{d/2}\Gamma(1/2)} \int_{\mathbb{R}^d} \Big(\int_0^1 \left(\frac{1-r^2}{-\log r}\right)^{1/2} \frac{y_i - r x_i}{(1-r^2)^{\frac{(d+3)}{2}}} e^{-\frac{|y-rx|^2}{1-r^2}} \Big) \\ &\qquad \times dr f(y) dy. & (1.14) \end{aligned}$$

In the Gaussian case, the *higher order Gaussian Riesz transforms* are defined directly,

Definition 1.2. For $\beta = (\beta_1, \beta_2, \cdots, \beta_d) \in \mathbb{N}_0^d$, the *higher order Riesz transforms* are defined spectrally as

$$\mathcal{R}_\beta = \partial_\gamma^\beta (-L)^{-|\beta|/2}, \tag{1.15}$$

[2]In $L^2(\mathbb{R}^d)$, $\frac{\partial}{\partial x_i}$ is antisymmetric, by integration by parts.

where $|\beta| = \sum_{j=1}^{d} \beta_j$ and $\partial_\gamma^\beta = \frac{1}{2^{|\beta|/2}} \partial_{x_1}^{\beta_1} \cdots \partial_{x_d}^{\beta_d}$. The meaning of this is that for any multi-index ν such that $|\nu| > 0$, its action on the Hermite polynomial $\vec{H}_\nu$ is

$$\mathcal{R}_\beta \vec{H}_\nu = \Big(\frac{2}{|\nu|}\Big)^{|\beta|/2} \Big[\prod_{i=1}^{d} \nu_i(\nu_i - 1)\cdots(\nu_i - \beta_i + 1)\Big] \vec{H}_{\nu-\beta} \tag{1.16}$$

if $\beta_i \leq \nu_i$ for all $i = 1, 2, \cdots, d$, and zero otherwise.

Observe that (1.16) follows directly from the definition of $\mathcal{R}_\beta$ since $\vec{H}_\nu$ is eigenfunction of the Ornstein-Uhlenbeck operator $-L$, with eigenvalue $|\nu|$, and therefore, $(-L)^{-|\beta|/2} \vec{H}_\nu = \frac{1}{|\nu|^{|\beta|/2}} \vec{H}_\nu$.

The higher order Gaussian Riesz transforms have kernel given by

$$\mathcal{K}_\beta(x, y) = \frac{1}{\pi^{d/2}\Gamma(|\beta|/2)} \int_0^1 \Big(\frac{-\log r}{1-r^2}\Big)^{\frac{|\beta|-2}{2}} r^{|\beta|} \vec{H}_\beta\Big(\frac{y - rx}{\sqrt{1-r^2}}\Big) \frac{e^{-\frac{|y-rx|^2}{1-r^2}}}{(1-r^2)^{d/2+1}} \frac{dr}{r},$$

for details see [12]. Therefore,

$$\begin{aligned} \mathcal{R}_\beta f(x) &= \text{p.v.} \int_{\mathbb{R}^d} \mathcal{K}_\beta(x, y) f(y) dy \\ &= \text{p.v.} \frac{1}{\pi^{d/2}\Gamma(|\beta|/2)} \int_{\mathbb{R}^d} \int_0^1 \Big(\frac{-\log r}{1-r^2}\Big)^{\frac{|\beta|-2}{2}} r^{|\beta|} \vec{H}_\beta\Big(\frac{y - rx}{\sqrt{1-r^2}}\Big) \\ &\qquad \times \frac{e^{-\frac{|y-rx|^2}{1-r^2}}}{(1-r^2)^{d/2+1}} \frac{dr}{r} f(y) dy. \end{aligned}$$

The *general Gaussian singular integrals*, are generalizations of the Gaussian higher order Riesz transform. The first formulation of general Gaussian singular integrals was given initially by W. Urbina in [11]. Later, S. Pérez [7] extend it.

Definition 1.3. Given a C^1-function F, satisfying the orthogonality condition

$$\int_{\mathbb{R}^d} F(x) \gamma_d(dx) = 0, \tag{1.17}$$

and such that for every $\varepsilon > 0$, there exist constants, C_ε and C'_ε such that

$$|F(x)| \leq C_\epsilon e^{\epsilon|x|^2} \quad \text{and} \quad |\nabla F(x)| \leq C'_\epsilon e^{\epsilon|x|^2}. \tag{1.18}$$

Then, for each $m \in \mathbb{N}$ the *generalized Gaussian singular integral* is defined as

$$T_{F,m} f(x) = \int_{\mathbb{R}^d} \int_0^1 \Big(\frac{-\log r}{1-r^2}\Big)^{\frac{m-2}{2}} r^m F\Big(\frac{y - rx}{\sqrt{1-r^2}}\Big) \frac{e^{-\frac{|y-rx|^2}{1-r^2}}}{(1-r^2)^{d/2+1}} \frac{dr}{r} f(y) dy. \tag{1.19}$$

$T_{F,m}$ can be written as

$$T_{F,m} f(x) = \int_{\mathbb{R}^d} \mathcal{K}_{F,m}(x, y) f(y) dy,$$

denoting,

$$\begin{aligned}\mathcal{K}_{F,m}(x,y) &= \int_0^1 \left(\frac{-\log r}{1-r^2}\right)^{\frac{m-2}{2}} r^{m-1} F\Big(\frac{y-rx}{\sqrt{1-r^2}}\Big) \frac{e^{-\frac{|y-rx|^2}{1-r^2}}}{(1-r^2)^{d/2+1}}\,dr \\ &= \int_0^1 \varphi_m(r) F\Big(\frac{y-rx}{\sqrt{1-r^2}}\Big) \frac{e^{-\frac{|y-rx|^2}{1-r^2}}}{(1-r^2)^{d/2+1}}\,dr \qquad (1.20)\\ &= \frac{1}{2}\int_0^1 \psi_m(t) F\Big(\frac{y-\sqrt{1-t}\,x}{\sqrt{t}}\Big) \frac{e^{-u(t)}}{t^{d/2+1}}\,dt,\end{aligned}$$

with $\varphi_m(r) = \left(\frac{-\log r}{1-r^2}\right)^{\frac{m-2}{2}} r^{m-1}$; and taking the change of variables $t = 1-r^2$, with $\psi_m(t) = \varphi_m(\sqrt{1-t})/\sqrt{1-t}$, and $u(t) = \frac{|\sqrt{1-t}x-y|^2}{t}$.

In [7] S. Pérez proved that the operator $T_{F,m}$ is a bounded operator in $L^p(\gamma_d)$, $1 < p < \infty$

Theorem 1.1. *The operators $T_{F,m}$ are $L^p(\gamma_d)$ bounded for $1 < p < \infty$, that is to say there exists $C > 0$, depending only on p and dimension such that*

$$\|T_{F,m} f\|_{p,\gamma} \leq C\|f\|_{p,\gamma}, \qquad (1.21)$$

for any $f \in L^p(\gamma_d)$.

Now, reversing the order in (1.9), one gets another second order differential operator, that will be denoted as $\overline{L}$,

$$(-\overline{L}) = \sum_{i=1}^{d} \partial_\gamma^i (\partial_\gamma^i)^* = (-L) + d\,I = -\frac{1}{2}\Delta_x + \langle x, \nabla_x\rangle + d\,I, \qquad (1.22)$$

and therefore,

$$\overline{L} = L - d\,I = \frac{1}{2}\Delta_x - \langle x, \nabla_x\rangle - d\,I. \qquad (1.23)$$

We will call $\overline{L}$ the *alternative Ornstein-Uhlenbeck* operator. The Hermite polynomials $\{\vec{H}_\nu\}_\nu$ are also eigenfunctions of $\overline{L}$, with eigenvalues $\overline{\lambda}_\nu = -(|\nu| + d)$, i. e.

$$\overline{L}\vec{H}_\nu = (\lambda_\nu - d)\vec{H}_\nu = -(|\nu| + d)\vec{H}_\nu. \qquad (1.24)$$

In [1], H. Aimar, L. Forzani and R. Scotto considered the following *alternative Riesz transforms*, by taking the derivatives $(\partial_\gamma^i)^*$ and Riesz potentials of the operator $(-\overline{L})$,

$$\overline{\mathcal{R}}_i = (\partial_\gamma^i)^* (-\overline{L})^{-1/2}. \qquad (1.25)$$

They also considered alternative higher order Gaussian Riesz transforms, that is, for a multi-index β, $|\beta| \geq 1$ taking the representation of the gradient (1.8),

$$(\partial_\gamma^\beta)^* = \frac{(-1)^{|\beta|}}{2^{|\beta|/2}} e^{|x|^2} (\partial^\beta e^{-|x|^2}\, I)$$

and the Riesz potentials associated with $\overline{L}$, these new singular integral operators are defined as follows:

Definition 1.4. The *alternative Gaussian Riesz transform* $\overline{\mathcal{R}}_\beta$ for $|\beta| \geq 1$ is defined spectrally as

$$\overline{\mathcal{R}}_\beta f(x) = (\partial_\gamma^\beta)^*(-\overline{L})^{-|\beta|/2} f(x).$$

Thus, the action of $\overline{\mathcal{R}}_\beta$ over the Hermite polynomial $\vec{H}_\nu$ is given by

$$\overline{\mathcal{R}}_\beta \vec{H}_\nu = \frac{1}{2^{|\beta|/2}(|\nu|+d)^{|\beta|/2}} \vec{H}_{\nu+\beta}, \tag{1.26}$$

using the fact that the Hermite polynomials $\{\vec{H}_\nu\}$ are eigenfunctions of $\overline{L}$,

$$(-\overline{L})^{-|\beta|/2} H_\nu = \frac{1}{(|\nu|+d)^{|\beta|/2}} \vec{H}_\nu,$$

and Rodrigues' formula for the Hermite polynomials, and therefore,

$$\overline{\mathcal{R}}_\beta \vec{h}_\nu(x) = \frac{1}{(|\nu|+d)^{|\beta|/2}} \Big[\prod_{i=1}^d (\nu_i+\beta_i)(\nu_i+\beta_i-1)\cdots(\nu_i+d) \Big]^{1/2} \vec{h}_{\nu+\beta}(x). \tag{1.27}$$

It can be proved that the alternative higher order Gaussian Riesz transforms have then the following integral representation,

$$\overline{\mathcal{R}}_\beta f(x) = \text{p.v.} C_\beta e^{|x|^2} \int_{\mathbb{R}^d} \overline{\mathcal{K}}_\beta(x,y) f(y)\, \gamma_d(dy)$$

where

$$\overline{\mathcal{K}}_\beta(x,y) = C_\beta \int_0^1 \left(\frac{-\log r}{1-r^2} \right)^{\frac{|\beta|-2}{2}} r^{d-1} \vec{H}_\beta \left(\frac{x-ry}{\sqrt{1-r^2}} \right) \frac{e^{-\frac{|x-ry|^2}{1-r^2}}}{(1-r^2)^{\frac{d}{2}+1}}\, dr,$$

for details see [12, Chapter 9].

Now, if

$$\begin{aligned} K(x,y) &= e^{|x|^2} \overline{\mathcal{K}}_\beta(x,y) e^{-|y|^2} \\ &= C_\beta \int_0^1 \left(\frac{-\log r}{1-r^2} \right)^{\frac{|\beta|-2}{2}} r^{d-1} H_\beta \left(\frac{x-ry}{\sqrt{1-r^2}} \right) \frac{e^{\frac{-|y-rx|^2}{1-r^2}}}{(1-r^2)^{\frac{d}{2}+1}} dr, \end{aligned}$$

then, $\overline{\mathcal{R}}_\beta$ can be written as

$$\begin{aligned} \overline{\mathcal{R}}_\beta f(x) &= \text{p.v.} C_\beta \int_{\mathbb{R}^d} \int_0^1 \left(\frac{-\log r}{1-r^2} \right)^{\frac{|\beta|-2}{2}} r^{d-1} H_\beta \left(\frac{x-ry}{\sqrt{1-r^2}} \right) \\ &\qquad \times \frac{e^{\frac{-|y-rx|^2}{1-r^2}}}{(1-r^2)^{\frac{n}{2}+1}} dr\, f(y)\, dy. \end{aligned} \tag{1.28}$$

Following the same idea to define general Gaussian singular integrals we now introduce a new class of Gaussian singular integrals, the *general alternative Gaussian singular integrals* as,

Definition 1.5. Given a C^1-function F, satisfying the orthogonality condition

$$\int_{\mathbb{R}^d} F(x)\gamma_d(dx) = 0, \tag{1.29}$$

and such that for every $\varepsilon > 0$, there exist constants, C_ε and C'_ε such that

$$|F(x)| \leq C_\epsilon e^{\epsilon|x|^2} \quad \text{and} \quad |\nabla F(x)| \leq C'_\epsilon e^{\epsilon|x|^2}. \tag{1.30}$$

Then, for each $m \in \mathbb{N}$ the *generalized alternative Gaussian singular integral* is defined as

$$\overline{T}_{F,m} f(x) = \int_{\mathbb{R}^d} \int_0^1 \left(\frac{-\log r}{1-r^2}\right)^{\frac{m-2}{2}} r^{d-1} F\Big(\frac{x - ry}{\sqrt{1-r^2}}\Big) \frac{e^{-\frac{|y-rx|^2}{1-r^2}}}{(1-r^2)^{d/2+1}} dr f(y) dy. \tag{1.31}$$

Thus, $\overline{T}_{F,m}$ can be written as

$$\overline{T}_{F,m} f(x) = \int_{\mathbb{R}^d} \overline{\mathcal{K}}_{F,m}(x,y) f(y) dy,$$

where,

$$\begin{aligned} \overline{\mathcal{K}}_{F,m}(x,y) &= \int_0^1 \left(\frac{-\log r}{1-r^2}\right)^{\frac{m-2}{2}} r^{d-1} F\Big(\frac{x - ry}{\sqrt{1-r^2}}\Big) \frac{e^{-\frac{|y-rx|^2}{1-r^2}}}{(1-r^2)^{d/2+1}} dr \\ &= \int_0^1 \varphi_m(r) F\Big(\frac{x - ry}{\sqrt{1-r^2}}\Big) \frac{e^{-\frac{|y-rx|^2}{1-r^2}}}{(1-r^2)^{d/2+1}} dr \\ &= \frac{1}{2}\int_0^1 \psi_m(t) F\Big(\frac{x - \sqrt{1-t}\, y}{\sqrt{t}}\Big) \frac{e^{-u(t)}}{t^{d/2+1}} dt, \end{aligned} \tag{1.32}$$

with $\varphi_m(r) = \left(\frac{-\log r}{1-r^2}\right)^{\frac{m-2}{2}} r^{d-1}$; and after making the change of variables $t = 1 - r^2$, $\psi_m(t) = \varphi_m(\sqrt{1-t})/\sqrt{1-t} = \left(\frac{-\log\sqrt{1-t}}{t}\right)^{\frac{m-2}{2}} (\sqrt{1-t})^{d-2}$, and $u(t) = \frac{|y-\sqrt{1-t}x|^2}{t}$.

Observe that the hypothesis on F for the general Gaussian singular integrals, (1.18) and the conditions on F for the general alternative Gaussian singular integrals, (1.30) are the same. We will prove the boundedness of $\overline{T}_{F,m}$ on $L^p(\gamma_d)$, $1 < p < \infty$ following [7], for $d > 1$.

Theorem 1.2. *The operators $\overline{T}_{F,m}$ are $L^p(\gamma_d)$ bounded for $1 < p < \infty$, for $d > 1$; that is to say there exists $C > 0$, depending only on p and dimension such that*

$$\|\overline{T}_{F,m} f\|_{p,\gamma} \leq C\|f\|_{p,\gamma}, \tag{1.33}$$

for any $f \in L^p(\gamma_d)$.

In [1], H. Aimar, L. Forzani and R. Scotto obtained a surprising result: the alternative Riesz transforms $\overline{\mathcal{R}}_\beta$ are weak type $(1,1)$ for all multi-index β, i. e. independently of their orders which is a contrasting fact with respect to the anomalous behavior of the higher order Riesz transforms $\mathcal{R}_\beta$. We prove that the general alternative Gaussian singular integrals $\overline{T}_{F,m}$ are also weak $(1,1)$ with respect to the Gaussian measure.

Theorem 1.3. *For $d>1$, there exists a constant C depending only on d and m such that for all $\lambda>0$ and $f\in L^1(\gamma_d)$, we have*

$$\gamma_d\Big(\Big\{x\in\mathbb{R}^d:\ \overline{T}_{F,m}(x)>\lambda\Big\}\Big)\le\frac{C}{\lambda}\int_{\mathbb{R}^d}|f(y)|\gamma_d(dy).$$

As usual in what follows C represents a constant that is not necessarily the same in each occurrence.

2. Proofs of the Main Results.

In what follows we need the following technical results.

Lemma 2.1. *For the function $\psi_m(t)=\varphi_m(\sqrt{1-t})/\sqrt{1-t}$, considered in the Definition 1.5 we have*

i) *There exists a constant $C>0$ such that*

$$|\psi_m(t)|\le\frac{C}{\sqrt{1-t}},\tag{2.1}$$

for $0\le t<1$ and $d>1$

ii) *There exists a constant $C>0$ such that*

$$|\psi_m(t)-\psi_m(0)|\le C\frac{t}{\sqrt{1-t}},\tag{2.2}$$

for $0\le t<1$, and $d>1$ where $\psi_m(0)=\psi_m(0^+)=2^{-(m-2)/2}$.

Proof. ii) It is clear, by L'Hopital's rule that

$$\lim_{t\to0^+}\frac{-\log\sqrt{1-t}}{t}=\lim_{t\to0^+}\frac{1}{2(1-t)}=1/2,$$

and therefore

$$\psi_m(0^+)=\lim_{t\to0^+}\psi_m(t)=\lim_{t\to0^+}=\frac{\left(\frac{-\log\sqrt{1-t}}{t}\right)^{\frac{m-2}{2}}}{\sqrt{1-t}}(\sqrt{1-t})^{d-1}=2^{-(m-2)/2}.$$

Now,

$$\begin{aligned}|\psi_m(t)-\psi_m(0)| &= \left|\frac{\left(\frac{-\log\sqrt{1-t}}{t}\right)^{\frac{m-2}{2}}}{\sqrt{1-t}}(\sqrt{1-t})^{d-1}-(1/2)^{(m-2)/2}\right|\\ &= \frac{t}{\sqrt{1-t}}\left|B(t)\right|,\end{aligned}$$

where

$$B(t)=\frac{\left(\frac{-\log\sqrt{1-t}}{t}\right)^{\frac{m-2}{2}}(\sqrt{1-t})^{d-1}-(1/2)^{(m-2)/2}\sqrt{1-t}}{t}.$$

Clearly the function B is continuous on $(0,1)$. Thus it is enough to prove that $\lim_{t\to 0^+} B(t)$ and $\lim_{t\to 1^-} B(t)$ exist, since then B is continuous on $[0,1]$ and therefore it is bounded there.

Let us consider fist the limit $\lim_{t\to 1^-} B(t)$. Observe that using L'Hopital's rule, can be proved that

$$\lim_{t\to 1^-}(-\log\sqrt{1-t})\sqrt{1-t} = 0. \tag{2.3}$$

If $m=1$ or $m=2$, $\lim_{t\to 1^-}(-\log\sqrt{1-t})^{(m-2)/2}(\sqrt{1-t})^{d-1} = 0$, since $d>1$, and therefore

$$\lim_{t\to 1^-} B(t) = 0.$$

On the other hand, if $m>2$, and $m/2 \le d$, then

$$\begin{aligned}
&\lim_{t\to 1^-}(-\log\sqrt{1-t})^{(m-2)/2}(\sqrt{1-t})^{d-1}\\
&\qquad = \lim_{t\to 1^-}\left(-\log\left(\sqrt{1-t}\right)\sqrt{1-t}\right)^{(m/2-1)}(\sqrt{1-t})^{d-m/2} = 0.
\end{aligned}$$

Now, if $m>2$, and $m/2>d$, taking n such that $n\in\mathbb{N} : n\le m/2 < n+1$ then, using L'Hopital's rule n times

$$\begin{aligned}
&\lim_{t\to 1^-}(-\log\sqrt{1-t})^{(m-2)/2}(\sqrt{1-t})^{d-1}\\
&= \frac{(\frac{m}{2}-1)}{(d-1)}\lim_{t\to 1^-}\left(-\log\sqrt{1-t}\right)^{m/2-2}(\sqrt{1-t})^{d-1}\\
&= \frac{(\frac{m}{2}-1)(\frac{m}{2}-2)}{(d-1)^2}\lim_{t\to 1^-}\left(-\log\sqrt{1-t}\right)^{m/2-3}(\sqrt{1-t})^{d-1}\\
&\vdots\\
&\cdots\\
&\vdots\\
&= \frac{(\frac{m}{2}-1)(\frac{m}{2}-2)\cdots(\frac{m}{2}-n)}{(d-1)^n}\lim_{t\to 1^-}\left(-\log\sqrt{1-t}\right)^{m/2-(n+1)}(\sqrt{1-t})^{d-1}\\
&= 0,
\end{aligned}$$

as $m/2-(n+1)<0$. Hence,

$$\lim_{t\to 1^-} B(t) = 0.$$

Now, we consider the limit $\lim_{t\to 0^+} B(t)$. Using again L'Hospital's rule,

$$\lim_{t\to 0^+} B(t) = \lim_{t\to 0^+} \frac{\left(\frac{-\log\sqrt{1-t}}{t}\right)^{\frac{m-2}{2}} (\sqrt{1-t})^{d-1} - (1/2)^{(m-2)/2}\sqrt{1-t}}{t}$$

$$= \lim_{t\to 0^+} (\frac{m}{2}-1)\left(\frac{-\log\sqrt{1-t}}{t}\right)^{\frac{m}{2}-2}\left(\frac{\frac{t}{2(1-t)}+\frac{1}{2}\log(1-t)}{t^2}\right)(\sqrt{1-t})^{d-1}$$

$$-\left(\frac{-\log\sqrt{1-t}}{t}\right)^{\frac{m}{2}-1}(d-1)(\sqrt{1-t})^{d-2}\frac{1}{2\sqrt{1-t}} + (1/2)^{m/2-1}\frac{1}{2\sqrt{1-t}}.$$

Observe that using L'Hopital rule twice, we have that

$$\begin{aligned}
\lim_{t\to 0^+}\left(\frac{\frac{t}{2(1-t)}+\frac{1}{2}\log(1-t)}{t^2}\right) &= \lim_{t\to 0^+}\left(\frac{\frac{1}{2(1-t)^2}-\frac{1}{2(1-t)}}{2t}\right)\\
&= \frac{1}{4}\lim_{t\to 0^+}\frac{1}{1-t}.\lim_{t\to 0^+}\frac{\frac{1}{1-t}-1}{t}\\
&= \frac{1}{4}\lim_{t\to 0^+}\frac{1}{1-t}.\lim_{t\to 0^+}\frac{1}{(1-t)^2} = \frac{1}{4}.
\end{aligned}$$

Therefore,

$$\lim_{t\to 0^+} B(t)$$

$$= \lim_{t\to 0^+} (\frac{m}{2}-1)\left(\frac{-\log\sqrt{1-t}}{t}\right)^{\frac{m}{2}-2}\left(\frac{\frac{t}{2(1-t)}+\frac{1}{2}\log(1-t)}{t^2}\right)(\sqrt{1-t})^{d-1}$$

$$-\left(\frac{-\log\sqrt{1-t}}{t}\right)^{m/2-1}(d-1)(\sqrt{1-t})^{d-2}\frac{1}{2\sqrt{1-t}} + (1/2)^{m/2-1}\frac{1}{2\sqrt{1-t}}$$

$$= \left(\frac{m}{2}-1\right)(1/2)^{m/2} - (d-1)(1/2)^{m/2} + (1/2)^{m/2} = \left(\frac{m}{2}+1-d\right)(1/2)^{m/2}$$

i) It is enough to prove that φ_m is bounded, i.e. there exist a constant $C > 0$ such that $|\varphi_m(r)| \leq C$, for all $r \in [0,1]$ and $d > 1$.

Since $\varphi_m(r)$ is continuous on $(0,1)$ it is enough to see that $\lim_{r\to 0^+} \varphi_m(r)$ and $\lim_{r\to 1^-} \varphi_m(r)$ exist and therefore $\varphi_m(r)$ is a continuous function on $[0,1]$ and then bounded. Now, from computations done in ii), we have

$$\lim_{r\to 1^-} \varphi_m(r) = \lim_{r\to 1^-}\left(\frac{-\log r}{1-r^2}\right)^{\frac{m-2}{2}} r^{d-1} = 2^{-(m-2)/2},$$

and

$$\lim_{r\to 0^+} \varphi_m(r) = \lim_{r\to 0^+}\left(\frac{-\log r}{1-r^2}\right)^{\frac{m-2}{2}} r^{d-1} = 0.$$

□

In what follows we use the same notation as Proposition 4.23 [12], see also [8],

$$a = a(x,y) = |x|^2 + |y|^2,\ b = b(x,y) := 2\langle x, y\rangle,$$

$$u(t) = u(t;x,y) := \frac{|y-\sqrt{1-t}x|^2}{t} = \frac{a}{t} - \frac{\sqrt{1-t}}{t}b - |x|^2,$$

$$t_0 := \frac{2\sqrt{a^2-b^2}}{a+\sqrt{a^2-b^2}} \sim \frac{\sqrt{a^2-b^2}}{a} \sim \frac{\sqrt{a-b}}{\sqrt{a+b}} = \frac{|x-y|}{|x+y|},$$

and

$$u_0 := u(t_0) = \frac{\sqrt{a^2-b^2}}{2} + \frac{a}{2} - |x|^2 = \frac{|y|^2-|x|^2}{2} + \frac{\sqrt{a^2-b^2}}{2}.$$

For the proof of Theorem 1.2 we will need Lemma 4.36 [12], see also [8],

Lemma 2.2. *For every* $0 \leq \eta \leq 1$ *and* $\nu > 0$, *there exists a constant* C *such that if* $\langle x, y\rangle > 0$ *and* $|x-y| > C_d m(x)$, *we have,*

$$\int_0^1 (u(t))^{\eta/2} e^{-\nu u(t)} \frac{dt}{t^{3/2}\sqrt{1-t}} \leq C \frac{e^{-\nu u_0}}{t_0^{1/2}}. \tag{2.4}$$

Now, we are ready to prove Theorem 1.2.

Proof. The proof follows the same scheme as the one of S. Pérez for Theorem 1.1, see [7] (or Theorem 9.17 of [12]). As usual, we split these operators into a local and a global part,

$$\begin{aligned} \overline{T}_{F,m}f(x) &= C_d \int\limits_{|x-y|<d\,m(x)} \overline{\mathcal{K}}_{F,m}(x,y)f(y)dy \\ &\qquad + C_d \int\limits_{|x-y|\geq d\,m(x)} \overline{\mathcal{K}}_{F,m}(x,y)f(y)dy \\ &= \overline{T}_{F,m,L}f(x) + \overline{T}_{F,m,G,}f(x), \end{aligned}$$

where

$$\overline{T}_{F,m,L}f(x) = \overline{T}_{F,m}(f\chi_{B_h(\cdot)})(x)$$

is the *local part* and

$$\overline{T}_{F,m,G,}f(x) = \overline{T}_{F,m}(f\chi_{B_h^c(\cdot)})(x)$$

is the *global part* of $\overline{T}_{F,m}$, and

$$B_h = B(x, C_d\, m(x)) = \{y \in \mathbb{R}^d : |y-x| < C_d\, m(x)\}$$

is an admissible ball for the Gaussian measure.

i) For the local part $\overline{T}_{F,m,L}$, we will prove that it is always of weak type $(1,1)$. The needed estimates follow from an idea that the local part differs from a Calderón-Zygmund singular integral by an operator that is $L^1(\gamma_d)$-bounded; in other words, the operator defined by the difference of $\overline{T}_{F,m}$ and an appropriated approximation of it (which is an operator defined as the convolution with a Calderón-Zygmund kernel) is $L^1(\mathbb{R}^d)$-bounded.
   - First, observe that if F satisfies the orthogonality condition (1.29) and (1.30), setting

$$\mathcal{K}(x) = \int_0^\infty F\Big(-\frac{x}{t^{1/2}}\Big) e^{-|x|^2/t} \frac{dt}{t^{d/2+1}},$$

then, $\mathcal{K}$ is a Calderón-Zygmund kernel of convolution type (see [3], [9] or [5]), as the integral is absolutely convergent when $x \neq 0$. Making the change of variables $s = |x|/t^{1/2}$ we get

$$\mathcal{K}(x) := \frac{2\int_0^\infty F\Big(-\frac{x}{|x|}s\Big)e^{-s^2}s^{d-1}ds}{|x|^d} = \frac{\Omega(x)}{|x|^d},$$

with Ω homogeneous of degree zero, and therefore K is homogeneous of degree $-d$. Moreover, Ω is C^1 with mean zero on S^{d-1}, since

$$\begin{aligned}\int_{S^{d-1}} \Omega(x')d\sigma(x') &= 2\int_0^\infty \int_{S^{d-1}} F(-x's)d\sigma(x')e^{-s^2}s^{d-1}ds \\ &= 2\int_{\mathbb{R}^d} F(-y)e^{-|y|^2}dy = 0.\end{aligned}$$

Therefore, by the classical Calderón-Zygmund theory, the convolution operator defined using convolution with the kernel $\mathcal{K}$, is continuous in $L^p(\mathbb{R}^d)$, $1 < p < \infty$ and weak type $(1,1)$, with respect to the Lebesgue measure. Therefore, by Theorem 4.32 of [12], see also Proposition 4.3 of [7], its local part is bounded in $L^p(\gamma_d)$, $1 < p < \infty$ and of weak type $(1,1)$ with respect to γ_d.

- Second, we need to get rid of the function ψ_m. Using Lemma 2.1, we can write

$$\begin{aligned}\overline{\mathcal{K}}_{F,m}(x,y) &= \frac{1}{2}\psi_m(0)\int_0^1 F\left(\frac{x-\sqrt{1-t}y}{\sqrt{t}}\right)\frac{e^{-u(t)}}{t^{\frac{d}{2}+1}}dt \\ &\quad +\frac{1}{2}\int_0^1 (\psi_m(t)-\psi_m(0))\,F\left(\frac{x-\sqrt{1-t}y}{\sqrt{t}}\right)\frac{e^{-u(t)}}{t^{\frac{d}{2}+1}}dt\end{aligned}$$

Set

$$\mathcal{K}_1(x,y) := \int_0^1 F\left(\frac{x-\sqrt{1-t}y}{\sqrt{t}}\right)\frac{e^{-u(t)}}{t^{\frac{d}{2}+1}}dt.$$

Now, about the local part we know that $u(t) \geq |y-x|^2/t - 2d$, then, using condition (1.30), we get

$$\begin{aligned}&\left|\int_0^1 (\psi_m(t)-\psi_m(0))\,F\left(\frac{x-\sqrt{1-t}y}{\sqrt{t}}\right)\frac{e^{-u(t)}}{t^{\frac{d}{2}+1}}dt\right| \\ &\qquad \leq \int_0^1 |\psi_m(t)-\psi_m(0)|\left|F\left(\frac{x-\sqrt{1-t}y}{\sqrt{t}}\right)\right|\frac{e^{-u(t)}}{t^{\frac{d}{2}+1}}dt \\ &\qquad \leq C\int_0^1 \frac{t}{\sqrt{1-t}}e^{\epsilon\frac{|x-\sqrt{1-t}y|^2}{t}}\frac{e^{-u(t)}}{t^{\frac{d}{2}+1}}dt \\ &\qquad = C\int_0^1 \frac{e^{\epsilon v(t)-u(t)}}{t^{\frac{d}{2}}}\frac{dt}{\sqrt{1-t}},\end{aligned}$$

where $v(t) = \frac{|x-\sqrt{1-t}y|^2}{t}$. Observe that

$$\epsilon v(t) - u(t) = \epsilon(v(t)-u(t)) - (1-\epsilon)u(t) = \epsilon(|x|^2-|y|^2) - (1-\epsilon)u(t).$$

Then,

$$\left|\int_0^1 (\psi_m(t)-\psi_m(0))\, F\left(\frac{x-\sqrt{1-t}y}{\sqrt{t}}\right)\frac{e^{-u(t)}}{t^{\frac{d}{2}+1}}dt\right|$$
$$\leq Ce^{\epsilon(|x|^2-|y|^2)}\int_0^1 \frac{e^{-(1-\epsilon)u(t)}}{t^{\frac{d}{2}}}\frac{dt}{\sqrt{1-t}} \leq CC_\epsilon \int_0^1 \frac{e^{-\delta\frac{|x-y|^2}{t}}}{t^{\frac{d}{2}}}\frac{dt}{\sqrt{1-t}}$$

Set

$$\mathcal{K}_2(x) := \int_0^1 \frac{e^{-\frac{\delta|x|^2}{t}}}{t^{\frac{d}{2}}}\frac{dt}{\sqrt{1-t}}$$

- Third, we need to control the difference between $\mathcal{K}_1$ and the Calderón-Zygmund kernel $\mathcal{K}$.

Claim

$$|\mathcal{K}_1(x,y)-\mathcal{K}(x-y)| \leq C\frac{1+|x|^{1/2}}{|x-y|^{d-1/2}}$$

Proof of the claim We need to estimate,

$$|\mathcal{K}_1(x,y)-\mathcal{K}(x-y)|$$
$$= \left|\int_0^1 F\left(\frac{x-\sqrt{1-t}y}{\sqrt{t}}\right)\frac{e^{-u(t)}}{t^{\frac{d}{2}+1}}dt - \int_0^\infty F\left(\frac{y-x}{\sqrt{t}}\right)\frac{e^{-\frac{|x-y|^2}{t}}}{t^{\frac{d}{2}+1}}dt\right|$$
$$\leq \left|\int_{t_0}^1 F\left(\frac{x-\sqrt{1-t}y}{\sqrt{t}}\right)\frac{e^{-u(t)}}{t^{\frac{d}{2}+1}}dt - \int_{t_0}^\infty F\left(\frac{y-x}{\sqrt{t}}\right)\frac{e^{-\frac{|x-y|^2}{t}}}{t^{\frac{d}{2}+1}}dt\right|$$
$$+\left|\int_0^{t_0} F\left(\frac{x-\sqrt{1-t}y}{\sqrt{t}}\right)\frac{e^{-u(t)}}{t^{\frac{d}{2}+1}}dt - \int_0^{t_0} F\left(\frac{y-x}{\sqrt{t}}\right)\frac{e^{-\frac{|x-y|^2}{t}}}{t^{\frac{d}{2}+1}}dt\right|$$
$$= \text{(I)}+\text{(II)}.$$

Using again the notation of Proposition 4.23 of [12], and the fact that on the local part $u(t) \geq |y-x|^2/t - 2d$, there is a $\delta > 0$ such that,

$$\begin{aligned}
\text{(I)} &\leq \int_{t_0}^1 \left|F\left(\frac{x-\sqrt{1-t}y}{\sqrt{t}}\right)\right|\frac{e^{-u(t)}}{t^{\frac{d}{2}+1}}dt + \int_{t_0}^\infty \left|F\left(\frac{y-x}{\sqrt{t}}\right)\right|\frac{e^{-\frac{|x-y|^2}{t}}}{t^{\frac{d}{2}+1}}dt \\
&\leq C_1\int_{t_0}^1 \frac{e^{\epsilon v(t)-u(t)}}{t^{\frac{(d-1)}{2}}}\frac{dt}{t^{\frac{3}{2}}} + C_2\int_{t_0}^\infty e^{(\epsilon-1)\frac{|x-y|^2}{t}}\frac{dt}{t^{\frac{d}{2}+1}} \\
&\leq C_1 e^{\epsilon(|x|^2-|y|^2)}\int_{t_0}^1 \frac{e^{-(1-\epsilon)u(t)}}{t^{\frac{(d-1)}{2}}}\frac{dt}{t^{\frac{3}{2}}} + C_2\int_{t_0}^\infty e^{(\epsilon-1)\frac{|x-y|^2}{t}}\frac{dt}{t^{\frac{d}{2}+1}} \\
&\leq C_\epsilon\int_{t_0}^1 \frac{e^{-(1-\epsilon)\frac{|x-y|^2}{t}}}{t^{\frac{(d-1)}{2}}}\frac{dt}{t^{\frac{3}{2}}} + C_2\int_{t_0}^\infty e^{(\epsilon-1)\frac{|x-y|^2}{t}}\frac{dt}{t^{\frac{d}{2}+1}} \\
&\leq 2C\int_{t_0}^\infty \frac{e^{-\delta\frac{|x-y|^2}{t}}}{t^{\frac{(d-1)}{2}}}\frac{dt}{t^{\frac{3}{2}}} \\
&\leq C\frac{1}{|x-y|^{d-1}}\cdot\frac{1}{t_0^{1/2}} \leq C\frac{1+|x|^{1/2}}{|x-y|^{d-\frac{1}{2}}}.
\end{aligned}$$

Now, we need to bound (II).
Set $w(s) = x - \sqrt{1-s}y$, $z(s) = y - \sqrt{1-s}x$, then $w'(s) = \frac{y}{2\sqrt{1-s}}$ and $z'(s) = \frac{x}{2\sqrt{1-s}}$.
Then,

$$
\begin{aligned}
&\left| F\left(\frac{x-\sqrt{1-t}y}{\sqrt{t}}\right) e^{-u(t)} - F\left(\frac{x-y}{\sqrt{t}}\right) e^{-\frac{|x-y|^2}{t}} \right| \\
&= \left| F\left(\frac{w(t)}{\sqrt{t}}\right) e^{\frac{-|z(t)|^2}{t}} - F\left(\frac{w(0)}{\sqrt{t}}\right) e^{-\frac{|z(0)|^2}{t}} \right| \\
&= \left| \int_0^t \frac{\partial}{\partial s}\left(F\left(\frac{w(s)}{\sqrt{t}}\right) e^{-\frac{|z(s)|^2}{t}} \right) ds \right| \\
&= \Big| \int_0^t \left\langle \frac{w'(s)}{\sqrt{t}}, \nabla F\left(\frac{w(s)}{\sqrt{t}}\right) \right\rangle e^{-\frac{|z(s)|^2}{t}} \\
&\qquad\qquad -2\left\langle z'(s), \frac{z(s)}{t} \right\rangle F\left(\frac{w(s)}{\sqrt{t}}\right) e^{-\frac{|z(s)|^2}{t}} ds \Big| \\
&\le \int_0^t \left|\frac{w'(s)}{\sqrt{t}}\right| \left|\nabla F\left(\frac{w(s)}{\sqrt{t}}\right)\right| e^{-\frac{|z(s)|^2}{t}} ds \\
&\qquad\qquad +2\int_0^t \left|\frac{z'(s)}{\sqrt{t}}\right| \left|\frac{z(s)}{\sqrt{t}}\right| \left|F\left(\frac{w(s)}{\sqrt{t}}\right)\right| e^{-\frac{|z(s)|^2}{t}} ds \\
&\le \int_0^t \frac{|y|}{\sqrt{t}2\sqrt{1-s}} e^{\epsilon' \frac{|w(s)|^2}{t} - \frac{|z(s)|^2}{t}} ds \\
&\qquad\qquad +2\int_0^t \frac{|x|}{\sqrt{t}2\sqrt{1-s}} \left|\frac{z(s)}{\sqrt{t}}\right| e^{\epsilon' \frac{|w(s)|^2}{t} - \frac{|z(s)|^2}{t}} ds \\
&\le \frac{|y|}{2\sqrt{t}} \int_0^t \frac{1}{\sqrt{1-s}} e^{\epsilon' \frac{s}{t}(|x|^2-|y|^2) - (1-\epsilon')\frac{|z(s)|^2}{t}} ds \\
&\qquad\qquad + \frac{|x|}{\sqrt{t}} \int_0^t \frac{1}{\sqrt{1-s}} \left|\frac{z(s)}{\sqrt{t}}\right| e^{\epsilon' \frac{s}{t}(|x|^2-|y|^2) - (1-\epsilon')\frac{|z(s)|^2}{t}} ds \\
&\le \frac{C|y|}{2\sqrt{t}} \int_0^t \frac{1}{\sqrt{1-s}} e^{-(1-\epsilon')\frac{|z(s)|^2}{t}} ds \\
&\qquad\qquad + \frac{C|x|}{\sqrt{t}} \int_0^t \frac{1}{\sqrt{1-s}} \left|\frac{z(s)}{\sqrt{t}}\right| e^{-\frac{(1-\epsilon')}{2}\frac{|z(s)|^2}{t}} e^{-\frac{(1-\epsilon')}{2}\frac{|z(s)|^2}{t}} ds \\
&\le \frac{C|y|}{2\sqrt{t}} \int_0^t \frac{1}{\sqrt{1-s}} e^{-(1-\epsilon')\frac{|z(s)|^2}{t}} ds + \frac{C|x|}{\sqrt{t}} \int_0^t \frac{1}{\sqrt{1-s}} e^{-\frac{(1-\epsilon')}{2}\frac{|z(s)|^2}{t}} ds.
\end{aligned}
$$

On the other hand, in the local part, we have

$$
\begin{aligned}
\frac{|z(s)|^2}{t} &= \frac{|y-\sqrt{1-s}x|^2}{t} = \frac{|(y-x)-(\sqrt{1-s}-1)x|^2}{t} \\
&\ge \frac{\left(|(y-x)| - |(\sqrt{1-s}-1)x|\right)^2}{t} = \frac{\left(|(y-x)| - |(1-\sqrt{1-s})x|\right)^2}{t}.
\end{aligned}
$$

Since $0 \le s \le t \le 1$, we have:

$$\frac{1-\sqrt{1-s}}{t} \quad \le \quad 1.$$

$$\begin{aligned}
\frac{\left(|(y-x)| - |(\sqrt{1-s}-1)x|\right)^2}{t} &\ge \frac{|(y-x)|^2}{t} - 2\frac{|y-x||x|(1-\sqrt{1-s})}{t} \\
&\ge \frac{|(y-x)|^2}{t} - 2|y-x||x| \\
&\ge \frac{|(y-x)|^2}{t} - 2C.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\left| F\left(\frac{x-\sqrt{1-t}y}{\sqrt{t}}\right) e^{-u(t)} - F\left(\frac{x-y}{\sqrt{t}}\right) e^{-\frac{|x-y|^2}{t}} \right| \\
&\le \frac{C|y|}{2\sqrt{t}} \int_0^t \frac{1}{\sqrt{1-s}} e^{-(1-\epsilon')\frac{|y-x|^2}{t}} ds + \frac{C|x|}{\sqrt{t}} \int_0^t \frac{1}{\sqrt{1-s}} e^{-\frac{(1-\epsilon')}{2}\frac{|y-x|^2}{t}} ds \\
&\le \frac{C(|y|+|x|)}{\sqrt{t}} \int_0^t \frac{1}{\sqrt{1-s}} e^{-\frac{(1-\epsilon')}{2}\frac{|y-x|^2}{t}} ds \\
&= \frac{C(|y|+|x|)}{\sqrt{t}} e^{-\delta\frac{|y-x|^2}{t}} \int_0^t \frac{1}{\sqrt{1-s}} ds \\
&= \frac{C(|y|+|x|)}{\sqrt{t}} e^{-\delta\frac{|y-x|^2}{t}} 2(1-\sqrt{1-t}),
\end{aligned}$$

where $\delta = \frac{(1-\epsilon')}{2} > 0$. Hence,

$$\begin{aligned}
(\mathrm{II}) &= \int_0^{t_0} \left| F\left(\frac{x-\sqrt{1-t}y}{\sqrt{t}}\right) e^{-u(t)} - F\left(\frac{x-y}{\sqrt{t}}\right) e^{-\frac{|x-y|^2}{t}} \right| \frac{dt}{t^{d/2+1}} \\
&\le 2C(|y|+|x|) \int_0^{t_0} \frac{e^{-\delta\frac{|x-y|^2}{t}}}{t^{d/2}} \frac{(1-\sqrt{1-t})}{t} \frac{dt}{\sqrt{t}} \\
&\le 2C(|y|+|x|) \int_0^{t_0} \frac{e^{-\delta\frac{|x-y|^2}{t}}}{t^{d/2}} \frac{dt}{\sqrt{t}} \le \frac{2C(|y|+|x|)}{|x-y|^d} \int_0^{t_0} \frac{dt}{\sqrt{t}} \\
&= \frac{2C(|y|+|x|)}{|x-y|^d} t_0^{1/2}.
\end{aligned}$$

Now, using that

$$t_0 \quad = \quad \frac{2\sqrt{a^2-b^2}}{a+\sqrt{a^2-b^2}} \le \frac{2\sqrt{a^2-b^2}}{a} = \frac{2|x+y||x-y|}{|x|^2+|y|^2}$$

we get,

$$\begin{aligned}
\frac{C(|y|+|x|)}{|x-y|^{d}} t_0^{1/2} &\leq \frac{C(|y|+|x|)}{|x-y|^{d}} \left(\frac{2|x+y||x-y|}{|x|^2+|y|^2}\right)^{1/2} \\
&= \frac{C(|y|+|x|)}{|x-y|^{d-\frac{1}{2}}} \left(\frac{|x+y|^{1/2}}{(|x|^2+|y|^2)^{1/2}}\right) \\
&= \frac{C|y|}{|x-y|^{d-\frac{1}{2}}} \left(\frac{|x+y|^{1/2}}{(|x|^2+|y|^2)^{1/2}}\right) \\
&\qquad + \frac{C|x|}{|x-y|^{d-\frac{1}{2}}} \left(\frac{|x+y|^{1/2}}{(|x|^2+|y|^2)^{1/2}}\right) \\
&\leq \frac{C|y|}{|x-y|^{d-\frac{1}{2}}} \left(\frac{|x+y|^{1/2}}{(|y|^2)^{1/2}}\right) + \frac{C|x|}{|x-y|^{d-\frac{1}{2}}} \left(\frac{|x+y|^{1/2}}{(|x|^2)^{1/2}}\right) \\
&= \frac{2C}{|x-y|^{d-\frac{1}{2}}} \left(|x+y|^{1/2}\right) \\
&\leq \frac{2C}{|x-y|^{d-\frac{1}{2}}} \left(|x|^{1/2}+|y|^{1/2}\right).
\end{aligned}$$

Also in the local part, we have

$$||x|-|y|| \leq |x-y| \leq C_1, \text{ i.e. } |y| \leq C_1 + |x|.$$

Then,

$$\begin{aligned}
\frac{C(|y|+|x|)}{|x-y|^{d}} t_0^{1/2} &\leq \frac{C}{|x-y|^{d-\frac{1}{2}}} \left(|x|^{1/2} + (C_1^{1/2}+|x|^{1/2})\right) \\
&\leq \frac{C|x|^{1/2}+C}{|x-y|^{d-\frac{1}{2}}} \\
&= \frac{C\left(1+|x|^{1/2}\right)}{|x-y|^{d-\frac{1}{2}}},
\end{aligned}$$

and therefore,

$$(\mathrm{II}) \leq \frac{C\left(1+|x|^{1/2}\right)}{|x-y|^{d-\frac{1}{2}}}.$$

Set

$$\mathcal{K}_3(x,y) := \frac{1+|x|^{\frac{1}{2}}}{|x-y|^{d-\frac{1}{2}}}.$$

Observe that $K_3(x,y)$ defines a function in the variable x which is $L^1(\mathbb{R}^d)$, uniformly in the variable y.

Hence, writing $\overline{\mathcal{K}}_{F,m}(x,y)$ as

$$\begin{aligned}
\overline{\mathcal{K}}_{F,m}(x,y) &= \frac{1}{2}\int_0^1 \psi_m(t) F\Big(\frac{y-\sqrt{1-t}\,x}{\sqrt{t}}\Big)\frac{e^{-u(t)}}{t^{d/2+1}}dt,\\
&= \frac{1}{2}\psi_m(0)\int_0^1 F\Big(\frac{y-\sqrt{1-t}\,x}{\sqrt{t}}\Big)\frac{e^{-u(t)}}{t^{d/2+1}}dt\\
&\qquad +\frac{1}{2}\int_0^1 (\psi_m(t)-\psi_m(0)) F\Big(\frac{y-\sqrt{1-t}\,x}{\sqrt{t}}\Big)\frac{e^{-u(t)}}{t^{d/2+1}}dt\\
&= \frac{1}{2}\psi_m(0)\Big[\int_0^1 F\Big(\frac{y-\sqrt{1-t}\,x}{\sqrt{t}}\Big)\frac{e^{-u(t)}}{t^{d/2+1}}dt\\
&\qquad\qquad -\int_0^\infty F\Big(\frac{y-x}{\sqrt{t}}\Big)\frac{e^{-\frac{|x-y|^2}{t}}}{t^{d/2+1}}dt\Big]\\
&\quad +\frac{1}{2}\psi_m(0)\int_0^\infty F\Big(\frac{y-x}{\sqrt{t}}\Big)\frac{e^{-\frac{|x-y|^2}{t}}}{t^{d/2+1}}dt\\
&\qquad +\frac{1}{2}\int_0^1 (\psi_m(t)-\psi_m(0)) F\Big(\frac{y-\sqrt{1-t}\,x}{\sqrt{t}}\Big)\frac{e^{-u(t)}}{t^{d/2+1}}dt.
\end{aligned}$$

Using the estimates above, we conclude that the local part $T_{F,m,L}$ can be bounded as

$$\begin{aligned}
|\overline{T}_{F,m,L}f(x)| &= |\overline{T}_{F,m}f(\chi_{B_h(x)})(x)| = \Big|\int_{B_h(x)} \overline{\mathcal{K}}_{F,m}(x,y)f(y)\,dy\Big|\\
&\le C\int_{B_h(x)} \mathcal{K}_3(x,y)|f(y)|\,dy + C\Big|p.v.\int_{B_h(x)} \mathcal{K}(x-y)f(y)\,dy\Big|\\
&\qquad + C\int_{B_h(x)} \mathcal{K}_2(x-y)|f(y)|\,dy\\
&= (I)+(II)+(III).
\end{aligned}$$

By Theorem 4.32 of [12], (II) is bounded in $L^p(\gamma_d)$, $1<p<\infty$ and of weak type $(1,1)$ with respect to γ_d. So it remains to prove that (I) and (III) are also bounded. In order to do that, we need to use a covering lemma, Lemma 4.3 of [12]; taking a countable family of admissible balls $\mathcal{F}$.

Given $B\in\mathcal{F}$, if $x\in B$ then $B_h(x)\subset\hat{B}$, and therefore,

$$\begin{aligned}
(I) &= (1+|x|^{1/2})\sum_{k=0}^{\infty}\int_{2^{-(k+1)}C_d m(x)<|x-y|<2^{-k}C_d m(x)} \frac{|f(y)|\chi_{\hat{B}}}{|x-y|^{d-1/2}}dy\\
&\le C_d 2^d \mathcal{M}(f\chi_{\hat{B}})(x)(1+|x|^2)m(x)^{1/2}\sum_{k=0}^{\infty}2^{-(k+1)/2}\\
&\le C\mathcal{M}(f\chi_{\hat{B}})(x)(\chi_{B_h(\cdot)})(x),
\end{aligned}$$

where $\mathcal{M}(g)$ is the classical Hardy-Littlewood maximal function of the function g.

On the other hand, let us consider $\varphi(y) = C_\delta e^{-\delta|y|^2}$, where C_δ is a constant such that $\int_{\mathbb{R}^d} \varphi(y)dy = 1$. φ is a non-increasing radial function, and given $t > 0$, we rescale this function as $\varphi_{\sqrt{t}}(y) = t^{-d/2}\phi(y/\sqrt{t})$, and, since $0 \leq \varphi \in L^1(\mathbb{R}^d)$, $\{\varphi_{\sqrt{t}}\}_{t>0}$ is a classical approximation of the identity in $\mathbb{R}^d$. Then, since $\int_0^1 (1/\sqrt{1-t})dt < \infty$,

$$\begin{aligned}(III) &= \int_{B_h(x)} \mathcal{K}_2(x-y)|f(y)|\, dy = \int_{B_h(x)} \Big(\int_0^1 \varphi_{\sqrt{t}}(x-y)\frac{dt}{\sqrt{1-t}}\Big)|f(y)|dy\\ &\leq \int_{B_h(x)} \Big(\sup_{t>0} \varphi_{\sqrt{t}}(x-y)\Big)\Big(\int_0^1 \frac{dt}{\sqrt{1-t}}\Big)|f(y)|dy\\ &\leq C\int_{B_h(x)} \Big(\sup_{t>0} \varphi_{\sqrt{t}}(x-y)\Big)|f(y)|dy.\end{aligned}$$

Again, using the family $\mathcal{F}$ if $x \in B$ then $B_h(x) \subset \hat{B}$, then, by a similar argument as before,

$$(III) = \int_{B_h(x)} \mathcal{K}_2(x-y)|f(y)|\, dy \leq C\int_{\mathbb{R}^d} \Big(\sup_{t>0} \varphi_{\sqrt{t}}(x-y)\Big)|f(y)|\chi_{\hat{B}}(y)dy$$

which yields, using Theorem 4 in Stein's book [9, Chapter II Sec. 4.], we get

$$\begin{aligned}(III) &= \int_{B_h(x)} \mathcal{K}_2(x-y)|f(y)|\, dy \leq \sum_{B\in\mathcal{F}} \sup_{t>0} \Big|(\varphi_{\sqrt{t}} * |f\chi_{\hat{B}}|)(x)\Big|\chi_B(x)\\ &\leq \sum_{B\in\mathcal{F}} \mathcal{M}(f\chi_{\hat{B}})(x)\chi_B(x).\end{aligned}$$

Therefore, the local part $T_{F,m,L}$ is bounded in $L^p(\gamma_d)$, $1 < p < \infty$ and of weak type $(1,1)$ with respect to γ_d.

ii) Now, for the global part $\overline{T}_{F,m,G}$, we will prove that it is $L^p(\gamma_d)$-bounded for all $1 < p < \infty$. The idea will be to exploit the size of the kernel and treat $\overline{T}_{F,m,G}$ as a positive operator.

Observe that, from Lemma 2.1,

$$|\psi_m(t)| \leq \frac{C}{\sqrt{1-t}}.$$

Hence, using (1.30) and $v(t) = \frac{|x-\sqrt{1-t}y|^2}{t}$, we get

$$\begin{aligned}\overline{\mathcal{K}}_{F,m}(x,y)| &\leq C\int_0^1 \Big|F\Big(\frac{x-\sqrt{1-t}\,y}{\sqrt{t}}\Big)\Big|\frac{e^{-u(t)}}{t^{d/2+1}}\frac{dt}{\sqrt{1-t}}\\ &\leq C_\epsilon \int_0^1 e^{\epsilon v(t)}\frac{e^{-u(t)}}{t^{d/2+1}}\frac{dt}{\sqrt{1-t}},\end{aligned}$$

for some $\epsilon > 0$ to be determined.

Let us take $E_x = \{y : \langle x, y\rangle > 0\}$ and consider two cases:

- Case #1: $b = 2\langle x, y\rangle \leq 0$. Now, as $v(t) = \frac{a}{t} - \frac{\sqrt{1-t}b}{t} - |y|^2$,

$$\frac{a}{t} - |y|^2 \leq v(t) = \frac{a}{t} - \frac{\sqrt{1-t}}{t}b - |y|^2 \leq \frac{2a}{t},$$

and so, the change of variables $s = a(\frac{1}{t} - 1)$ gives

$$\begin{aligned}
\int_0^1 \frac{e^{\epsilon v(t)-u(t)}}{t^{\frac{d}{2}+1}} \frac{dt}{\sqrt{1-t}} \\
&= e^{|x|^2-|y|^2} \int_0^1 \frac{e^{-(1-\epsilon)v(t)}}{t^{\frac{d}{2}+1}} \frac{dt}{\sqrt{1-t}} \\
&\leq e^{|x|^2-|y|^2} \int_0^1 \frac{e^{-(1-\epsilon)(\frac{a}{t}-|y|^2)}}{t^{\frac{d}{2}+1}} \frac{dt}{\sqrt{1-t}} \\
&= e^{|x|^2-|y|^2+|y|^2-\epsilon|y|^2} \int_0^1 \frac{e^{-(1-\epsilon)(\frac{a}{t})}}{t^{\frac{d}{2}+1}} \frac{dt}{\sqrt{1-t}} \\
&= \frac{e^{|x|^2-|y|^2+|y|^2-\epsilon|y|^2}}{a^{\frac{d}{2}}} \int_0^\infty e^{-(1-\epsilon)(s+a)} (s+a)^{\frac{d}{2}-\frac{1}{2}} \frac{ds}{s^{\frac{1}{2}}} \\
&= \frac{e^{-|y|^2+\epsilon|x|^2}}{a^{\frac{d}{2}}} \int_0^\infty e^{-(1-\epsilon)s} (s+a)^{\frac{d}{2}-\frac{1}{2}} \frac{ds}{s^{\frac{1}{2}}} \\
&\leq \frac{e^{-|y|^2+\epsilon|x|^2}}{a^{\frac{d}{2}}} C \left(\int_0^\infty e^{-(1-\epsilon)s} s^{\frac{d}{2}-1} ds + a^{\frac{d}{2}-\frac{1}{2}} \int_0^\infty e^{-(1-\epsilon)s} s^{\frac{1}{2}-1} ds \right) \\
&= C_\epsilon \frac{e^{-|y|^2+\epsilon|x|^2}}{a^{\frac{d}{2}}} \left(\Gamma(\frac{d}{2}) + a^{\frac{d}{2}-\frac{1}{2}} \Gamma(\frac{1}{2}) \right) = C_\epsilon e^{-|y|^2+\epsilon|x|^2} \left(\frac{1}{a^{\frac{d}{2}}} + \frac{1}{a^{\frac{1}{2}}} \right) \\
&\leq C_\epsilon e^{-|y|^2+\epsilon|x|^2},
\end{aligned}$$

as $a > \frac{1}{2}$ over the global region. Thus,

$$\begin{aligned}
\left|\overline{\mathcal{K}}_{F,m}(x,y)\right| &\leq \int_0^1 \left| F\left(\frac{x-\sqrt{1-t}y}{\sqrt{t}} \right) \right| \frac{e^{-u(t)}}{t^{\frac{d}{2}+1}} \frac{dt}{\sqrt{1-t}} \\
&\leq C_\epsilon \int_0^1 \frac{e^{\epsilon v(t)-u(t)}}{t^{\frac{d}{2}+1}} \frac{dt}{\sqrt{1-t}} \leq C_\epsilon e^{-|y|^2+\epsilon|x|^2}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_{\mathbb{R}^d} \left(\int_{B_h^c(x)\cap E_x^c} \left|\overline{\mathcal{K}}_{F,m}(x,y)\right| |f(y)|\, dy \right)^p e^{-|x|^2} dx \\
&\leq C_\epsilon \int_{\mathbb{R}^d} \left(\int_{B_h^c(x)\cap E_x^c} e^{-|y|^2+\epsilon|x|^2} |f(y)|\, dy \right)^p e^{-|x|^2} dx \\
&\leq C_\epsilon \int_{\mathbb{R}^d} \left(\int_{B_h^c(x)} |f(y)|\, e^{-|y|^2} dy \right)^p e^{\epsilon|x|^2 p-|x|^2} dx \\
&\leq C_\epsilon \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |f(y)|^p \gamma(dy) \right) e^{(\epsilon p-1)|x|^2} dx \\
&= C_\epsilon \|f\|_{p,\gamma}^p \int_{\mathbb{R}^d} e^{(\epsilon p-1)|x|^2} dx = C_\epsilon \|f\|_{p,\gamma}^p,
\end{aligned}$$

for $\epsilon < 1/p$.

- Case #2: $b = 2\langle x, y\rangle > 0$. Consider again

$$u_0 = u(t_0) = \frac{|y|^2 - |x|^2}{2} + \frac{\sqrt{a^2 - b^2}}{2} \leq (a^2 - b^2)^{1/2}.$$

Let $y \in B_h^c(x)$,

$$\begin{aligned}
|\overline{K}_{F,m}(x,y)| &\leq C\int_0^1 \left|F\left(\frac{x-\sqrt{1-t}y}{\sqrt{t}}\right)\right| \frac{e^{-u(t)}}{t^{\frac{d}{2}+1}} \frac{dt}{\sqrt{1-t}} \\
&\leq C\int_0^1 \frac{e^{\frac{\epsilon|x-\sqrt{1-t}y|^2}{t} - u(t)}}{t^{\frac{d}{2}+1}} \frac{dt}{\sqrt{1-t}} = C\int_0^1 \frac{e^{\epsilon v(t) - u(t)}}{t^{\frac{d}{2}+1}} \frac{dt}{\sqrt{1-t}} \\
&= Ce^{\epsilon(|x|^2-|y|^2)} \int_0^1 \frac{e^{-(1-\epsilon)u(t)}}{t^{\frac{d-1}{2}}} \frac{dt}{t^{\frac{3}{2}}\sqrt{1-t}} \\
&= Ce^{\epsilon(|x|^2-|y|^2)} \int_0^1 \frac{e^{-(\frac{d-1}{d})u(t)}}{t^{\frac{d-1}{2}}} \frac{e^{\epsilon u(t) - \frac{u(t)}{d}} dt}{t^{\frac{3}{2}}\sqrt{1-t}}
\end{aligned}$$

Now, we know that

$$\frac{e^{-(\frac{d-1}{d})u(t)}}{t^{\frac{d-1}{2}}} = \left(\frac{e^{-u(t)}}{t^{\frac{d}{2}}}\right)^{\frac{d-1}{d}} \leq \left(\frac{e^{-u(t_0)}}{t_0^{\frac{d}{2}}}\right)^{\frac{d-1}{d}} = \frac{e^{-(\frac{d-1}{d})u(t_0)}}{t_0^{\frac{d-1}{2}}}$$

Then, by Lemma 2.2, taking $v = \frac{1}{d} - \epsilon > 0$

$$\begin{aligned}
|\overline{K}_{F,m}(x,y)| &\leq Ce^{\epsilon(|x|^2-|y|^2)} \frac{e^{-(\frac{d-1}{d})u(t_0)}}{t_0^{\frac{d-1}{2}}} \int_0^1 \frac{e^{(\epsilon-\frac{1}{d})u(t)}dt}{t^{\frac{3}{2}}\sqrt{1-t}} \\
&\leq Ce^{\epsilon(|x|^2-|y|^2)} \frac{e^{-(\frac{d-1}{d})u(t_0)}}{t_0^{\frac{d-1}{2}}} \frac{e^{(\epsilon-\frac{1}{d})u(t_0)}}{t_0^{\frac{1}{2}}} \\
&= Ce^{\epsilon(|x|^2-|y|^2)} \frac{e^{-(1-\epsilon)u(t_0)}}{t_0^{\frac{d}{2}}}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\int_{\mathbb{R}^d} \left(\int_{B_h^c(x)\cap E_x} |\overline{K}_{F,m}(x,y)f(y)|dy\right)^p e^{-|x|^2}dx \\
&\leq C\int_{\mathbb{R}^d} \left(\int_{B_h^c(x)\cap E_x} e^{\epsilon(|x|^2-|y|^2)} \frac{e^{-(1-\epsilon)u(t_0)}}{t_0^{\frac{d}{2}}} |f(y)|dy\right)^p e^{-|x|^2}dx \\
&= C\int_{\mathbb{R}^d} \left(\int_{B_h^c(x)\cap E_x} e^{(\epsilon-\frac{1}{p})(|x|^2-|y|^2)} \frac{e^{-(1-\epsilon)u(t_0)}}{t_0^{\frac{d}{2}}} |f(y)|e^{-\frac{|y|^2}{p}}dy\right)^p dx.
\end{aligned}$$

Therefore, it is enough to check that the operator defined using the kernel,

$$\widetilde{K}(x,y) = e^{(\epsilon-\frac{1}{p})(|x|^2-|y|^2)} \frac{e^{-(1-\epsilon)u(t_0)}}{t_0^{\frac{d}{2}}} \chi_{B_h^c(x)}(y),$$

is of strong type p with respect to the Lebesgue measure. Using the inequality $||y|^2 - |x|^2| \leq |x+y||x-y|$, definition of t_0 and that, as $b > 0$, then on the global region, $|x+y||x-y| \geq d$, we conclude that

$$\begin{aligned} e^{(\epsilon-\frac{1}{p})(|x|^2-|y|^2)}\frac{e^{-(1-\epsilon)u(t_0)}}{t_0^{\frac{d}{2}}} &= \frac{1}{t_0^{\frac{d}{2}}}e^{\left[(\frac{1}{p}-\epsilon)-\frac{(1-\epsilon)}{2}\right](|y|^2-|x|^2)}e^{-\frac{(1-\epsilon)}{2}|x+y||x-y|} \\ &\leq \frac{1}{t_0^{\frac{d}{2}}}e^{\left|(\frac{1}{p}-\epsilon)-\frac{(1-\epsilon)}{2}\right|\left||y|^2-|x|^2\right|}e^{-\frac{(1-\epsilon)}{2}|x+y||x-y|} \\ &\leq \frac{1}{t_0^{\frac{d}{2}}}e^{\left|(\frac{1}{p}-\epsilon)-\frac{(1-\epsilon)}{2}\right||x+y||x-y|-\frac{(1-\epsilon)}{2}|x+y||x-y|} \\ &= \frac{1}{t_0^{\frac{d}{2}}}e^{-\alpha_p|x+y||x-y|} \leq C|x+y|^d e^{-\alpha_p|x+y||x-y|}, \end{aligned}$$

where

$$\alpha_p = \frac{(1-\epsilon)}{2} - |(\frac{1}{p}-\epsilon) - \frac{(1-\epsilon)}{2}|.$$

Now, as $p > 1$, taking $\epsilon < \frac{1}{p}$ we get that que $\alpha_p > 0$.
Observe that the last expression is symmetric in x and y and, therefore, it suffices to prove its integrability with respect to one of them

$$\begin{aligned} \int_{\mathbb{R}^d}|x+y|^d e^{-\alpha_p|x+y||x-y|}dy &\leq C + C\int_{|x-y|<1}|x|^d e^{-\alpha_p|x||x-y|}dy \\ &\quad + C\int_{|x-y|<1}|x+y|^d e^{-\alpha_p|x+y|}dy \\ &\leq C\int_{\mathbb{R}^d}e^{\alpha_p|v|}dv + C_d\int_0^\infty r^{2d-1}e^{-\alpha_p r}dr \leq C. \end{aligned}$$

Observe that, once $p > 1$ is chosen, then the operator defined using the kernel $\widetilde{K}(x,y)$ is in fact $L^q(\mathbb{R}^d)$-bounded for $1 \leq q \leq \infty$, but for the proof of the theorem it is enough the case $p = q$.

□

Now we will prove Theorem 1.3, following the proof of Theorem 1.2 in [1], (see also Theorem 9.17 of [12])

Proof. As usual, for each $x \in \mathbb{R}^d$, we write this operator as the sum of two operators which are obtained by splitting $\mathbb{R}^d$ into a local region,

$$B_h(x) = \{y \in \mathbb{R}^d : |y-x| < C_d m(x)\},$$

an admissible ball and its complement $B_h^c(x)$ called the global region. Thus,

$$\begin{aligned}
\overline{T}_{F,m}f(x) &= C_d \int\limits_{|x-y|<d\, m(x)} \overline{\mathcal{K}}_{F,m}(x,y)f(y)dy \\
&\qquad + C_d \int\limits_{|x-y|\geq d\, m(x)} \overline{\mathcal{K}}_{F,m}(x,y)|f(y)|dy \\
&= \overline{T}_{F,m,L}f(x) + \overline{T}_{F,m,G}f(x),
\end{aligned}$$

where as before

$$\overline{T}_{F,m,L}f(x) = \overline{T}_{F,m}(f\chi_{B_h(\cdot)})(x)$$

is the *local part* and

$$\overline{T}_{F,m,G}f(x) = \overline{T}_{F,m}(f\chi_{B_h^c(\cdot)})(x)$$

is the *global part* of $\overline{T}_{F,m}$.

We will prove that these two operators are γ_d-weak type $(1,1)$ and so will be $\overline{T}_{F,m}$. In order to prove that $\overline{T}_{F,m,L}f(x)$ is γ-weak type $(1,1)$, we will apply Theorem 4.30 of [12]. In our case,

$$Tf(x) = p.v. \int_{\mathbb{R}^d} \mathcal{K}(x,y)f(y)dy$$

with

$$\begin{aligned}
\mathcal{K}(x,y) &= e^{|x|^2}\overline{\mathcal{K}}_{F,m}(x,y)e^{-|y|^2} \\
&= C\int_0^1 \left(\frac{-\log r}{1-r^2}\right)^{\frac{m-2}{2}} r^{d-1}F\left(\frac{x-ry}{\sqrt{1-r^2}}\right)\frac{e^{-\frac{|y-rx|^2}{1-r^2}}}{(1-r^2)^{\frac{d}{2}+1}}\,dr
\end{aligned}$$

and, therefore,

$$\begin{aligned}
\frac{\partial \mathcal{K}}{\partial y_j}(x,y) &= 2C\int_0^1 \left(\frac{-\log r}{1-r^2}\right)^{\frac{|m-2}{2}} r^{d-1} \\
&\quad \times\left[\frac{-r}{\sqrt{1-r^2}}\frac{\partial F}{\partial y_j}\left(\frac{x-ry}{\sqrt{1-r^2}}\right) + F\left(\frac{x-ry}{\sqrt{1-r^2}}\right)\frac{y_j-rx_j}{1-r^2}\right]\frac{e^{-\frac{|y-rx|^2}{1-r^2}}}{(1-r^2)^{\frac{d}{2}+1}}\,dr.
\end{aligned}$$

Now, we show that the hypotheses of Theorem 4.30 of [12] are fulfilled for this operator. Thus, we prove that, in the local region $B_h(x)$, we have,

$$|\mathcal{K}(x,y)| \leq \frac{C}{|x-y|^d}$$

and

$$\left|\frac{\partial \mathcal{K}}{\partial y_j}(x,y)\right| \leq \frac{C}{|x-y|^{d+1}}.$$

There exists a constant $C>0$ such that for every $y \in B_h(x)$

$$C^{-1} \leq e^{|y|^2-|x|^2} \leq C,$$

then

$$|\mathcal{K}(x,y)| \leq C|e^{-|x|^2+|y|^2}\mathcal{K}(x,y)| = C|\overline{\mathcal{K}}_{F,m}(x,y)|$$

and

$$\left|\frac{\partial \mathcal{K}}{\partial y_j}(x,y)\right| \leq C \left| e^{-|x|^2+|y|^2} \frac{\partial \overline{\mathcal{K}}_{F,m}}{\partial y_j}(x,y)\right|.$$

On the other hand, on $B_h(x)$, we have

$$e^{-c\frac{|y-rx|^2}{1-r^2}} = e^{-c\frac{|x-y|^2}{1-r^2}} e^{-c\frac{1-r}{1+r}|x|^2} e^{-c\frac{(x-y)\cdot x}{1-r}} \leq Ce^{-c\frac{|x-y|^2}{1-r}},$$

thus by this inequality and using the hypothesis on F, (1.30), we have

$$\left|F\left(\frac{x-ry}{\sqrt{1-r^2}}\right)\right| e^{-\frac{|y-rx|^2}{1-r^2}} \leq C_\epsilon \, e^{\epsilon\frac{|x-ry|^2}{1-r^2}} e^{-\frac{|y-rx|^2}{1-r^2}} \leq Ce^{-c\frac{|x-y|^2}{1-r}}$$

and

$$\left|\nabla F\left(\frac{x-ry}{\sqrt{1-r^2}}\right)\right| e^{-\frac{|y-rx|^2}{1-r^2}} \leq C'_\epsilon \, e^{\epsilon\frac{|x-ry|^2}{1-r^2}} e^{-\frac{|y-rx|^2}{1-r^2}} \leq Ce^{-c\frac{|x-y|^2}{1-r}}$$

$$\begin{aligned}
|\mathcal{K}(x,y)| &\leq C\int_0^1 \left(\frac{-\log r}{1-r^2}\right)^{\frac{m-2}{2}} \frac{e^{-c\frac{|x-y|^2}{1-r}}}{(1-r)^{\frac{d}{2}+1}}\,dr \\
&\leq C\left[\int_0^{\frac12} (-\log r)^{\frac{m-2}{2}}\,dr + \int_{\frac12}^1 \frac{e^{-c\frac{|x-y|^2}{1-r}}}{(1-r)^{\frac{d}{2}+1}}\,dr\right] \\
&\leq C\left(1+\frac{1}{|x-y|^d}\right) \leq \frac{C}{|x-y|^d}
\end{aligned}$$

and

$$\begin{aligned}
\left|\frac{\partial \mathcal{K}}{\partial y_j}(x,y)\right| &\leq C\int_0^1 \left(\frac{-\log r}{1-r^2}\right)^{\frac{m-2}{2}} \frac{e^{-c\frac{|x-y|^2}{1-r}}}{(1-r)^{\frac{n+3}{2}}}\,dr \\
&\leq C\left[\int_0^{\frac12} (-\log r)^{\frac{|m-2}{2}}\,dr + \int_{\frac12}^1 \frac{e^{-c\frac{|x-y|^2}{1-r}}}{(1-r)^{\frac{n+3}{2}}}\,dr\right] \\
&\leq C\left(1+\frac{1}{|x-y|^{d+1}}\right) \\
&\leq \frac{C}{|x-y|^{d+1}}.
\end{aligned}$$

From Theorem 1.2 we know that the operator $\overline{T}_{F,m,}$ is bounded on $L^p(\gamma_d)$ for any $p>1$. Thus, γ_d-weak type $(1,1)$ of $\overline{T}_{F,m,L}$ follows, using Theorem 4.30 of [12].

In order to prove that $\overline{T}_{F,m,G}$ is also γ_d-weak type $(1,1)$ we use Forzani's generalized Gaussian maximal function,

$$\mathcal{M}_\Phi f(x) = \sup_{0<r<1} \frac{1}{\gamma_d\left((1+\delta)B\left(\frac{x}{r}, \frac{|x|}{r}(1-r)\right)\right)} \int_{\mathbb{R}^d} \Phi\left(\frac{|x-ry|}{\sqrt{1-r^2}}\right) |f(y)|\,\gamma_d(dy), \tag{2.5}$$

where $\delta = \delta_{r,x} = \frac{r}{|x|(1-r)} \min\left\{\frac{1}{|x|}, \sqrt{1-r}\right\}$, see Definition 4.17 of [12], and prove that on $\mathbb{R}^d \setminus B_h(x)$,

$$|\overline{T}_{F,m,G} f(x)| \leq C\mathcal{M}_\Phi f(x), \tag{2.6}$$

with $\Phi(t) = e^{-ct^2}$.

$$\begin{aligned}
|\overline{\mathcal{K}}_{F,m}(x,y)| &= \left| \int_0^1 \left(\frac{-\log r}{1-r^2}\right)^{\frac{m-2}{2}} r^{d-1} F\left(\frac{x-ry}{\sqrt{1-r^2}}\right) \frac{e^{-\frac{|x-ry|^2}{1-r^2}}}{(1-r^2)^{\frac{n}{2}+1}}\, dr \right| \\
&\leq C\int_0^{\frac{3}{4}} (-\log r)^{\frac{m-2}{2}} \frac{e^{-\frac{|x-ry|^2}{2(1-r^2)}}}{(1-r^2)^{\frac{n}{2}}}\, dr \\
&\quad + C\int_{\frac{3}{4}}^{1-\zeta/|x|^2} \frac{e^{-\frac{|x-ry|^2}{2(1-r^2)}}}{(1-r^2)^{\frac{n-1}{2}}} \left(|x| \vee (1-r^2)^{-\frac{1}{2}}\right) \frac{dr}{|x|(1-r^2)^{3/2}} \\
&\quad + C\int_{1-\zeta/|x|^2}^{1} \frac{e^{-c\frac{|x-ry|^2}{1-r^2}}}{(1-r^2)^{\frac{n-1}{2}}} \left(|x| \vee (1-r^2)^{-\frac{1}{2}}\right) \frac{e^{-\bar{c}\frac{|x-y|^2}{1-r}}}{1-r}\, dr.
\end{aligned}$$

Hence,

$$|\overline{\mathcal{K}}_{F,m}(x,y)| = C\left(\overline{\mathcal{K}}^1_{F,m}(x,y) + \overline{\mathcal{K}}^2_{F,m}(x,y) + \overline{\mathcal{K}}^3_{F,m}(x,y)\right),$$

where the inequality is obtained by annihilating the Hermite polynomial with part of the exponential, then splitting the unit interval of the integral into three subintervals $[0, 3/4]$, $[3/4, 1-\zeta/|x|^2]$, and $[1-\zeta/|x|^2, 1]$ and taking into account that on the second one $|x| \vee (1-r^2)^{-1/2} \geq |x|$, on the third one $|x| \vee (1-r^2)^{-1/2} \geq (1-r^2)^{-1/2}$ and $|x-ry| \geq \bar{c}|x-y|$, and on the last two intervals the function $-\log r/(1-r^2)$ is bounded by a constant.

Thus, by using the definition of kernels $\overline{\mathcal{K}}^j_{F,m}$, $j = 1,2,3$; using Fubini's theorem to interchange the order of integration on each operator $\overline{T}^j_{F,m,G}$, $j = 1,2,3$, using the inequality

$$\gamma_d\left(B\left(\frac{x}{r}, \frac{|x|}{r}\, s\right)\right) \leq C\, s^{(d-1)/2} \exp\left(-\frac{|x|^2}{r^2}(1-s)^2\right)\frac{1}{|x|}. \tag{2.7}$$

see Proposition 1.7 of [12], and using the definition of $\mathcal{M}_\Phi f$ with $\Phi(t) = e^{-ct^2}$, we get,

$$\begin{aligned}
\overline{T}^1_{F,m,G} f(x) &= e^{|x|^2} \int_{\mathbb{R}^d} \int_0^{\frac{3}{4}} (-\log r)^{\frac{m-2}{2}} \frac{e^{-\frac{|x-ry|^2}{2(1-r^2)}}}{(1-r^2)^{\frac{n}{2}}}\, dr\ |f(y)|\ \gamma_d(dy) \\
&= \int_0^{\frac{3}{4}} (-\log r)^{\frac{m-2}{2}} e^{|x|^2} \int_{\mathbb{R}^d} \frac{e^{-\frac{|x-ry|^2}{2(1-r^2)}}}{(1-r^2)^{\frac{n}{2}}} |f(y)| \gamma_d(dy)\, dr \\
&\leq C\int_0^{\frac{3}{4}} (-\log r)^{\frac{m|-2}{2}} dr \quad \mathcal{M}_\Phi f(x) \leq C\ \mathcal{M}_\Phi f(x),
\end{aligned}$$

$$\begin{aligned}
&\overline{T}^2_{F,m,G}f(x)\\
&= e^{|x|^2}\int_{\mathbb{R}^d}\int_{\frac{3}{4}}^{1-\zeta/|x|^2} \frac{e^{-\frac{|x-ry|^2}{2(1-r^2)}}}{(1-r^2)^{\frac{n-1}{2}}}\ (|x|\vee(1-r^2)^{-\frac{1}{2}})\ \frac{dr}{|x|(1-r^2)^{3/2}}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times|f(y)|\ \gamma_d(dy)\\
&= \int_{3/4}^{1-\zeta/|x|^2} e^{|x|^2}\int_{\mathbb{R}^d}\frac{e^{-c\frac{|x-ry|^2}{(1-r^2)}}}{(1-r^2)^{(n-1)/2}}(|x|\vee(1-r^2)^{-1/2})\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times|f(y)|\gamma_d(dy)\frac{dr}{|x|(1-r^2)^{3/2}}\\
&\leq C\frac{1}{|x|}\int_{3/4}^{1-\zeta/|x|^2}\frac{dr}{(1-r)^{3/2}}\ \ \mathcal{M}_\Phi f(x)\leq C\mathcal{M}_\Phi f(x),
\end{aligned}$$

and, finally,

$$\begin{aligned}
&\overline{T}^3_{F,m,G}f(x)\\
&= e^{|x|^2}\int_{\mathbb{R}^d}\int_{1-\zeta/|x|^2}^{1}\frac{e^{-c\frac{|x-ry|^2}{1-r^2}}}{(1-r^2)^{\frac{n-1}{2}}}(|x|\vee(1-r^2)^{-\frac{1}{2}})\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times\frac{e^{-\bar{c}\frac{|x-y|^2}{1-r}}}{1-r}\ dr\ |f(y)|\ \gamma_d(dy)\\
&= \int_{1-\zeta/|x|^2}^{1} e^{|x|^2}\int_{\mathbb{R}^d}\frac{e^{-c\frac{|x-ry|^2}{(1-r^2)}}}{(1-r^2)^{(n-1)/2}}(|x|\vee(1-r^2)^{-1/2})\\
&\quad \frac{e^{-\bar{c}\frac{|x-y|^2}{1-r}}}{1-r}\ |f(y)|\ \gamma_d(dy)\ dr\\
&\leq \int_{1-\zeta/|x|^2}^{1} e^{|x|^2}\int_{\mathbb{R}^d}\frac{e^{-c\frac{|x-ry|^2}{(1-r^2)}}}{(1-r^2)^{(n-1)/2}}(|x|\vee(1-r^2)^{-1/2})\frac{1}{|x-y|^2}\ |f(y)|\gamma_d(dy)\ dr\\
&\leq C|x|^2\int_{1-\zeta/|x|^2}^{1} dr\ \ \mathcal{M}_\Phi f(x)\leq C\mathcal{M}_\Phi f(x).
\end{aligned}$$

So, since

$$|\overline{T}_{F,m,G}f(x)|\leq C\sum_{j=1}^{3}\overline{T}^j_{F,m,G}f(x),$$

(2.6) follows. Then, using Theorem 4.18 of [12] (see also Theorem 1.1 of [1]) we get the γ_d-weak type $(1,1)$ inequality for $\overline{T}_{F,m,G}$. □

In an forthcoming paper [6], following [2], we prove that the general alternative Gaussian singular integrals $\overline{T}_{F,m}$ are also continuous on Gaussian variable Lebesgue spaces under a condition of regularity on $p(\cdot)$.

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