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MARTINGALES AND COCYCLES IN QUANTUM PROBABILITY

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Dedicated to Professor Leonard Gross on the occasion of his 88th birthday

Abstract. A formulation of the "martingale problem" in Quantum Probability is proposed and it is shown that the property of being a quantum stochastic cocycle with respect to the Brownian shift in the Fock space is equivalent to an "additive cocycle property" of the martingale-candidate which in its turn implies the martingale property.

1. Introduction

In the theory of (classical) stochastic processes, the standard Brownian motion (SBM for short, see [5], [11] for an introduction) represents a diffusion caused only by fluctuations and one attempts to construct other diffusions, either by using the "fundamental solutions" of a class of elliptic partial differential equations to construct the associated Markov process, or by solving the associated stochastic differential equation (driven by the same partial differential operator) using Ito's ideas of stochastic integrals. For a brief introduction to these ideas, the reader is referred to [12], and here we restrict ourselves to stating a couple of typical results. In the following, $\{\omega(t)\}_{\{t\geq 0\}}$ denotes the $\mathbb{R}^d$-valued SBM with the probability space $(\Omega \equiv C_0(\mathbb{R}_+, \mathbb{R}^d), \mathcal{B}, \mathbb{P})$, $\mathbb{P}$ being the Wiener measure and L is an elliptic second-order (time-independent) partial differential operator given as:

$$(L\varphi)(x) = \frac{1}{2}\sum_{j,k=1}^{d} a_{jk}(x)\frac{\partial^2\varphi}{\partial x_j x_k}(x) + \sum_{j=1}^{d} b_j(x)\frac{\partial\varphi}{\partial x_j}. \tag{1.1}$$

For simplicity of presentation, we also consider two functions $A : \mathbb{R}^d \to d \times d$ real positive matrices, and $b : \mathbb{R}^d \to \mathbb{R}^d$ along with their respective point-wise norms $|A(x)|$ and $|b(x)|$.

Proposition 1.1. *Let A, b, L be as given above, satisfying furthermore:*

$$\begin{aligned} &|A(x) - A(x')| + |b(x) - b(x')| \leq C_1|x - x'|, \\ &|A(x)| + |b(x)| \leq C_2 \text{ and} \\ &\langle\theta, A(x)\theta\rangle \geq C_3|\theta|^2 \text{ for all } x, x', \theta \in \mathbb{R}^d, \end{aligned} \tag{1.2}$$

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where $<,>$ denotes the Euclidean inner product. Then there exists a unique positive "fundamental" solution $p(t,y;0,x)$ for $t \geq 0,\ x,y \in \mathbb{R}^d$ of the partial differential equation:

$$\frac{\partial p}{\partial t}(t,y;0,x) - L_{(y)}\ p(t,y;0,x) = 0,\ \textit{for } t > 0$$
$$\textit{and } \lim_{t \to 0+} p(t,y;0,x) = \delta(x-y) \tag{1.3}$$

where $\delta(\cdot)$ is the Dirac-delta distribution, or equivalently the solution of the initial value problem:

$$\frac{\partial u}{\partial t}(t,y) - (Lu)(t,y) = 0,\ u(0,x) = f(x) \tag{1.4}$$

is given by

$$u(t,y) = \int_{\mathbb{R}^d} p(t,y;0,x)\ f(x)dx. \tag{1.5}$$

Moreover, p satisfies the Chapman-Kolmogorov (semigroup) relation, viz. for $0 \leq s \leq t,\ x,y \in \mathbb{R}^d$,

$$\begin{aligned} p(t,y;0,x) &= \int_{\mathbb{R}^d} p(t,y;s,z)\ p(s,z;0,x)\ dz \\ &= \int_{\mathbb{R}^d} p(t-s,y;0,z)\ p(s,z;0,x)\ dz. \end{aligned} \tag{1.6}$$

The function p can be interpreted as a "transition probability density function", leading to a standard construction of an associated Markov process or Markov semigroup (see e.g. [5], [11]). The other classical method involves solving an associated stochastic differential equation (using Ito-integrals), showing the existence of a Brownian filtration $\{\mathcal{B}_t\}_{t\geq 0}$-adapted continuous process $\{Z(t)\}_{t \geq 0}$ with initial value $Z(0) = x \in \mathbb{R}^d$, satisfying

$$Z(t) = x + \int_0^t b(Z(\tau))d\tau + \int_0^t \sigma(Z(\tau))d\omega(\tau), \tag{1.7}$$

such that $\mathbb{E}(|Z(t)|^2) < \infty$ and that $\{Z(t)\}$ is a Markov process, under appropriate conditions on σ and b. The two approaches are related by the identification: $A = \sigma\sigma^*$, via an application of Ito's formula, so that for a bounded C^2-function $\varphi : \mathbb{R}^d \to \mathbb{R}$ one has:

$$\varphi(Z(t)) - \varphi(x) - \int_0^t (L\varphi)(Z(\tau))d\tau = \int_0^t \sum_{j,k=1}^d \frac{\partial \varphi}{\partial x_j}(Z(\tau))\ \sigma_{jk}\ d\omega_k(\tau). \tag{1.8}$$

We note that in (1.8)

$$\mathcal{M}_t \equiv \varphi(Z(t)) - \varphi(x) - \int_0^t (L\varphi)(Z(\tau))d\tau \tag{1.9}$$

is a $\mathcal{B}_t$-martingale, i.e. for $0 \le s \le t$

$$\mathbb{E}_s(\mathcal{M}_t) \equiv \mathbb{E}(\mathcal{M}_t \mid \mathcal{B}_s) = \mathcal{M}_s. \tag{1.10}$$

In some situations, the hypotheses on a, b (or in the second method on σ, b) may not be satisfied so that one does not have these solutions available. However, in many cases with less conditions on a, b appearing in the elliptic operator L, Stroock and Varadhan ([12], [15]) formulated a new point of view, viz. solving the associated "martingale problem". A typical result (in the time-homogenous case) is given next.

Proposition 1.2. *Let $A : \mathbb{R}^d \to d \times d$ positive matrices be bounded continuous and let $b : \mathbb{R}^d \to \mathbb{R}^d$ be bounded measurable, and consider the elliptic operator L given in (1.1). Assume furthermore that A is strictly elliptic, i.e. $\exists\ C > 0$ such that $\langle \theta, A(x)\theta\rangle \le C|\theta|^2,\ \forall\ x, \theta \in \mathbb{R}^d$. Then the "martingale problem" for L has a unique solution, i.e. there exists a probability measure P_x on $\{\Omega, \mathcal{B}_t\}$ and a stochastic process $\{X(t)\}$ starting at $x \in \mathbb{R}^d$ at $t = 0$ such that the family $\{\mathcal{M}_t\}_{t\ge 0}$ given in (1.9), with $X(t)$ replacing $Z(t)$, is a P_x-martingale for every bounded $C^2(\mathbb{R}^d)$-function φ.*

In the theory of quantum stochastic processes, path-wise description has to be abandoned and instead one has "fluctuation trajectories" in a suitable Hilbert space or "maps in a Hilbert-space, fluctuating in time". One way to represent these ideas is to put the "fluctuations" in their "quantum receptacle", viz. the Fock space, then study the description of operator-processes in them which is precisely the aim of section 2 here. The main results containing the intimate connections between operator - and map - cocycles with "martingale-property" constitute the section 3 while we conclude with some remarks in section 4.

2. Fock Space and Classical Stochastic Processes in it.

Let $\mathfrak{H}$ be a (separable) Hilbert space (over $\mathbb{C}$), and we define the (symmetric)-Fock space $\mathfrak{F}$ over the base-space $\mathfrak{H}$ as $\mathfrak{F} \equiv \Gamma_{\text{sym}}(\mathfrak{H}) \equiv \bigoplus_{n=0}^{\infty} \mathfrak{H}^{\text{sn}}$, where $\mathfrak{H}^{\text{s0}} = \mathbb{C}$ and $\mathfrak{H}^{\text{sn}} = n$th symmetric tensor power of $\mathfrak{H}$, for $n = 1, 2, \ldots$; and the inner product of $\mathfrak{F}$ is the natural inner-product of infinite direct sum of Hilbert spaces. It is useful, for the sake of concreteness and for most of our needs here, to think of $\mathfrak{H}$ as $L^2(\mathbb{R}_+, k)$, the Lebesgue space of square-integrable k-valued functions on $\mathbb{R}_+$ with k as an auxilliary separable Hilbert space (often called **multilicity** or **noise** space).

It is convenient to have a **distinguished total set** of vectors (called **exponential** or coherent vectors) which is useful to make computations on:

$$e(f) = 1 \oplus f \oplus \frac{f^{\otimes 2}}{\sqrt{2!}} \oplus \cdots \oplus \frac{f^{\otimes n}}{\sqrt{n!}} \oplus \cdots , \tag{2.1}$$

for $f \in \mathfrak{H}$, so that

$$\langle e(f), e(g)\rangle_{\mathfrak{F}} = \exp\left(\langle f, g\rangle_{\mathfrak{H}}\right). \tag{2.2}$$

In (2.1) $f^{\otimes n}$ is the n-fold tensor product of the vector $f \in \mathfrak{H}$. The functional relation (2.2) explains the name exponential vector for $e(f)$. Furthermore, when $\mathfrak{H} = L^2(\mathbb{R}_+)$, the map $\mathfrak{F} \leftrightarrow L^2(\mathbb{P})$ (where $\mathbb{P}$ is the Wiener measure on the Borel space $C_0(\mathbb{R}_+)$, the Frechet space of continuous functions on $\mathbb{R}_+$, with initial value 0 at $0 \in \mathbb{R}_+$) given by:

$$e(f) \leftrightarrow \exp\left\{ \int_0^\infty f(t)d\omega(t) - \frac{1}{2}\int_0^\infty f(t)^2 dt \right\}$$

is an isomorphism (so called Wiener-Ito-Segal isomorphism), where $\omega(\cdot)$ is the 1-dimensional SBM.

Another (exponential)-functorial property of exponential vectors is the following: if $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ so that $f_j (j = 1, 2)$ is the component of the vector $f \in \mathfrak{H}$ in the subspace $\mathfrak{H}_j$, then there is a unitary isomorphism between $\mathfrak{F}$ (on the base $\mathfrak{H}_1 \oplus \mathfrak{H}_2$) and $\mathfrak{F}$ (on base $\mathfrak{H}_1$) $\otimes\mathfrak{F}$ (on base $\mathfrak{H}_2$) given by the map; $e(f) \leftrightarrow e(f_1) \otimes e(f_2)$. In the concrete case of $\mathfrak{H} = L^2(\mathbb{R}_+, k) \simeq L^2([0, t], k) \oplus L^2([t, \infty), k)$, this translates to:

$$e(f) \simeq e(f_t) \otimes e(f^t), \text{ with } t \geq 0 \text{ and} \tag{2.3}$$

$f_t = \mathcal{X}_{[0,t]} f, f^t = \mathcal{X}_{[t,\infty)} f$. This property lifts unitarily to the whole Fock space, is called the continuous time (tensor)-decomposition property of the Fock space and is represented as $\mathfrak{F} \simeq \mathfrak{F}_t \otimes \mathfrak{F}^t$ for $\forall\ t \geq 0$, where $\mathfrak{F}_t$ and $\mathfrak{F}^t$ are spanned by vectors of the type $e(f_t)$ and $e(f^t)$ respectively for $f \in L^2(\mathbb{R}_+, k)$.

Next we define the shift-isometry in the base space $\mathfrak{H} = L^2(\mathbb{R}_+)$ by setting for $\forall\ s \geq 0$;

$$(\Theta_s f)(t) = \begin{cases} 0 \text{ if } t < s \\ f(t-s) \text{ if } t \geq s, \ldots \end{cases} \tag{2.4}$$

and it is easy to check that $\{\Theta_s\}_{s \geq 0}$ is a C_0-semigroup of isometries in $L^2(\mathbb{R}_+)$. This family can be lifted canonically to the Fock space as:

$$\Gamma_s(e(f)) = e(\Theta_s f) \text{ for } \forall\ s \geq 0, f \in l^2(\mathbb{R}_+)$$

and extended linearly to a C_0-semigroup of isometries on $\mathfrak{F}$. In terms of the SBM, this can be connected canonically with the so-called **Brownian shift** $\hat{\Theta}_s$ by:

$$\int_0^\infty f(\tau)(\hat{\Theta}_s \omega)(\tau)d\tau = \int_0^\infty (\Theta_s f)(\tau)\omega(dz),$$

to conclude that

$$(\hat{\Theta}_s \omega)(\tau) = \omega(\tau + s) - \omega(s), \tag{2.5}$$

and that

$$\Gamma_s(e(f)) = e(\Theta_s f) \leftrightarrow \exp\left\{ \int_0^\infty f(\tau)(\hat{\Theta}_s \omega)(d\tau) - \frac{1}{2}\int_0^\infty f(\tau)^2 d\tau \right\}. \tag{2.6}$$

We can now define the basic operator processes, in terms of which many of the fundamental objects in Quantum Stochastic Calculus (QSC) are described:

$$\begin{aligned} &\text{annihilation operator} &:& \quad a(f)e(g) = \langle f, g\rangle e(g), \\ &\text{creation operator} &:& \quad a^+(f)e(g) = s - \tfrac{d}{d\epsilon} e(g+\epsilon f)\Big|_{\epsilon=0}, \\ &\text{conservation operator} &:& \quad \Lambda(T)e(g) = s - \tfrac{d}{d\epsilon}(e(\exp(\epsilon T)g))\Big|_{\epsilon=0}, \end{aligned} \tag{2.7}$$

and linearly extended to the dense (exponential)-manifold $\mathcal{E} \subseteq \mathfrak{F}$, spanned by the exponential vectors, where $f, g \in \mathfrak{H}$ and $T \in \mathcal{B}(\mathfrak{H})$, and $s - \frac{d}{d\epsilon}(\cdot)\Big|_{\epsilon=0}$ means the derivative of $(\cdot)$ at zero in the strong topology of $\mathfrak{H}$, $\{\exp(\epsilon T)\}_{\epsilon\geq 0}$ is the semigroup generated by T.

One can convince oneself that these three operators are densely defined unbounded operators and that $a^+(f)$ is the adjoint of $a(f)$ on $\mathcal{E}$.

If we now specialise to the case $\mathfrak{H} = L^2(\mathbb{R}_+)$, and set $f = \mathcal{X}_{[0,t]}$ and $T =$ the operator of multiplication $M_{\mathcal{X}_{[o,t]}}$ by $\mathcal{X}_{[0,t]}$ in $\mathfrak{H}$, then we get

$$\begin{aligned} &\text{the } \textbf{annihilation process} &:& \quad \left\{A(t) = a(\mathcal{X}_{[0,t]})\right\}_{t\geq 0} \\ &\text{the } \textbf{creation process} &:& \quad \left\{A^+(t) = a^+(\mathcal{X}_{[0,t]})\right\}_{t\geq 0} \text{ and} \\ &\text{the } \textbf{conservation process} &:& \quad \left\{\Lambda(t) = \Lambda(M_{\mathcal{X}_{[0,t]}})\right\}_{t\geq 0}. \end{aligned} \tag{2.8}$$

These three operator families or the fundamental quantum processes constitute the building blocks of Q.S.C., and if we set

$$Q(t) = 2^{-1/2}\left[A(t) + A^+(t)\right], P(t) = 2^{-1/2} i \left[A(t) - A^+(t)\right], \tag{2.9}$$

then one can verify that on $\mathcal{E}$,

$$[Q(s), P(t)] = i \min(t, s), \text{ for } t, s \geq 0. \tag{2.10}$$

Furthermore, $Q(t)$ and $P(t)$ are unitarily equivalent under the (unitary) Wiener Transform, defined by the linear extension of the map $e(f) \mapsto e(if)$, and the pullback of $Q(t)$ and $P(t)$ onto $L^2(\mathbb{P})$ by the Wiener-Ito-Segal transform gives a pair of non-commuting versions (see the relation (2.10)) of the SBM, looked upon as an operator of multiplication by the SBM in $L^2(\mathbb{P})$. In a similar vein, for $\lambda > 0$ and each $t \geq 0$, we set

$$\pi_\lambda(t) = \Lambda(t) + \sqrt{\lambda} Q(t) + \lambda t \tag{2.11}$$

to discover that $\pi_\lambda(t)$ describes a Poisson process in Fock spaces $\mathfrak{F}$, with intensity λ. Thus in the Fock space description, the classical Brownian and Poisson processes (σ-algebras) are "rolled" into one, viz. the increasing family $\{\mathcal{B}(\mathfrak{F}_t) \otimes I^t\}_{t\geq 0}$ of $*$-subalgebras of $\mathcal{B}(\mathfrak{F})$, the von Neumann algebra of all linear bounded operators on $\mathfrak{F}$, constituting the quantum filtration.

To describe Q.S.P's and quantum martingales in a Fock space, we need to bring in another Hilbert space h, the initial Hilbert space and look at the structure.

$$\begin{aligned} \widetilde{\mathfrak{H}} &= h \otimes \mathfrak{F} \simeq (h \otimes \mathfrak{F}_t) \otimes \mathfrak{F}^t \\ &= \widetilde{\mathfrak{H}}_t \otimes \mathfrak{F}^t \text{ for each } t \geq 0, \end{aligned} \tag{2.12}$$

where $\widetilde{\mathfrak{H}}_0$ is clearly h. In the language of Quantum Theory, one can say that h is the Hilbert space of the quantum system under observation, $\mathfrak{F}$ is the Hilbert space of "noise" or "heat-bath" and $\widetilde{\mathfrak{H}}$ is the Hilbert space of the composite system.

An operator family $\{X(t)\}_{t\geq 0}$ in $\widetilde{\mathfrak{H}}$ is said to be **adapted** if

(i) $\mathcal{D}\underline{\otimes}\mathcal{E} \subseteq \text{Dom}(X(t))\ \forall\ t \geq 0$, with $\mathcal{D}$ dense in h and $\underline{\otimes}$ is the algebraic tensor-product of subspaces:

(ii) $X(t) = X_0(t) \otimes I_{\mathfrak{F}^t}$ with respect to the Hilbert space decomposition (2.12) and $X_0(t)$ is an operator defined on $\mathcal{D} \otimes \mathcal{E}_t$;

(iii) the map $t \mapsto X(t)ue(f)$ is strongly continuous for each $u \in \mathcal{D}$ and $f \in L^2(\mathbb{R}_+)$.

Every such adapted families of operators $\{X(t)\}_{t\geq 0}$ in $\widetilde{\mathfrak{H}}$ is called a Quantum Stochastic Process (QSP) and it can be verified that the operator families $\{A(t), A^+(t), \Lambda(t)\}$ are each adapted.

The distinguished vector $\Omega \equiv e(0)$, called the vacuum vector, induces the (vacuum)-**expectation** map $\mathbb{E}$: Operator-families in $\widetilde{\mathfrak{H}} \mapsto$ operators in h by $\langle u, \mathbb{E}(X(t))v\rangle = \langle u \otimes \Omega, X(t)v \otimes \Omega\rangle$; $u \in h$ and $v \in \mathcal{D}$. More generally, using the continuous time-decomposition property (2.3) of the Fock-space over $L^2(\mathbb{R}_+)$, one defines the **conditional expectation** map for $0 \leq s \leq t$, $v \in \mathcal{D}$ such that $v \otimes e(f) \in \text{Dom}(X(t))$,

$$\langle u \otimes e(g_s), \mathbb{E}_s(X(t))v \otimes e(f_s)\rangle = \langle u \otimes e(g_s) \otimes e(0^s), X(t)v \otimes e(f_s) \otimes e(0^s)\rangle$$

which defines an adapted operator family $\mathbb{E}_s(X(t))$ in $\widetilde{\mathfrak{H}}_s$. Note that $\mathbb{E}_0 = \mathbb{E}$.

An adapted operator family (or QSP) $\{X(t)\}_{t\geq 0}$ is a **martingale** if for $0 \leq s \leq t$,

$$\mathbb{E}_s(X(t)) = X(s). \tag{2.13}$$

Then it is also easy to verify that the triple $\{A(t), A^+(t), \Lambda(t)\}_{t\geq 0}$ is a martingale.

2.1. Operator - and Map - cocycles. Next we look at operator-cocycles in $\widetilde{\mathfrak{H}}$. For this, let $V \equiv \{V_t\}_{t\geq 0}$ be an adapted family of bounded operators in $\widetilde{\mathfrak{H}}$ and V is said to be a **cocycle** with respect to the shift-isometry Γ_s introduced in (2.4)-(2.5) if for $0 \leq s \leq t$,

$$V_t = V_s \sigma_s(V_{t-s}), \text{ where} \tag{2.14}$$

$\sigma_s : \mathcal{B}(\widetilde{\mathfrak{H}}) \mapsto \mathcal{B}(\widetilde{\mathfrak{H}})$ is given by (for $Y \in \mathcal{B}(\widetilde{\mathfrak{H}})$)

$$\sigma_s(Y) = (I_h \otimes \Gamma_s)Y(I_h \otimes \Gamma_s^*). \tag{2.15}$$

Remark 2.1. (i) If we define a map $i_\tau : \widetilde{\mathfrak{H}} \simeq$ as in $h \otimes \mathfrak{F}_\tau \otimes \mathfrak{F}^\tau \mapsto \mathfrak{F}_\tau \otimes (h \otimes \mathfrak{F}^\tau)$ by setting $i_\tau(u \otimes e(f)) = e(f_\tau) \otimes (u \otimes e(f^\tau))$ and extending the same linearly, we can easily verify that for each $\tau \geq 0$, i_τ extends to a unitary flip-isomorphism of Hilbert spaces. Then it is clear that

$$i_s \cdot \sigma_s(Y) = [I_{\mathfrak{F}_s} \otimes (\Gamma_s Y \Gamma_s^*)] \cdot i_s \text{ on } \widetilde{\mathfrak{H}}, \tag{2.16}$$

where $(\Gamma_s Y \Gamma_s^*)$ acts in $h \otimes \mathfrak{F}^s$ and we shall identify $\sigma_s(Y)$ with $I_{\mathfrak{F}_s} \otimes (\Gamma_s Y \Gamma_s^*)$.

(ii) It is useful to note that for every $s \geq 0$

$$\mathbb{E}_s \circ \sigma_s = \mathbb{E}_0 = \mathbb{E}. \tag{2.17}$$

(iii) In the sequel, we shall assume furthermore that the map $\mathbb{R}_+ \ni t \mapsto V_t \in \mathcal{B}(\widetilde{\mathfrak{H}})$ is strongly continuous and that $V_0 = I_{\widetilde{\mathfrak{H}}}$.

The next proposition sums up some of the important properties of a cocycle.

Proposition 2.2. *Let V be a strongly continuous (see Remark 2.1 (iii)) cocycle in $\widetilde{\mathfrak{H}}$. Then $\{P_t \equiv \mathbb{E}(V_t)\}_{t \geq 0}$ is a C_0-semigroup of bounded operators in h.*

(**Sketch of proof**): Using the definitions of cocycle ((2.14)) and of σ_s ((2.15)), one gets for $0 \leq s \leq t$ that,

$$\begin{aligned} \mathbb{E}_0(V_t) &= \mathbb{E}_0(\mathbb{E}_s(V_t)) = \mathbb{E}_0(\mathbb{E}_s(V_s \cdot \sigma_s(V_{t-s}))) \\ &= \mathbb{E}_0(V_s(\mathbb{E}_s \circ \sigma_s(V_{t-s}))) = \mathbb{E}_0(V_s)\mathbb{E}_0(V_{t-s}) \\ &= P_s P_{t-s}, \end{aligned}$$

where we have also used the "projection" property of $\{\mathbb{E}_s\}$, i.e. $\mathbb{E}_{s_1} \circ \mathbb{E}_{s_2} = \mathbb{E}_{\min(s_1,s_2)}$ and the Remark 2.1 (ii). The strong continuity of P_t in h follows from the strong continuity V_t in $\widetilde{\mathfrak{H}}$. □

Remark 2.3. Often such cocycles are constructed as solutions of Quantum Stochastic Differential equations (QSDE) of the Hudson-Parthasarathy type:

$$dV_t = V_t \left[L dA^+ + K dA + G dt\right], V_0 = I_{\widetilde{\mathfrak{H}}}, \tag{2.18}$$

where L, K, G are closed (closable) operators in h, looked upon as operators in $\widetilde{\mathfrak{H}}$. If $L, K, G \in \mathcal{B}(h)$, then it can be shown ([9], [10], [13]) that the solution V_t exists and is a cocycle. Furthermore, if $K = -L^*, G = -1/2L^*L + iH$ (with H selfadjoint), then the solution V_t is an adapted family of unitary cocyles in $\widetilde{\mathfrak{H}}$. When L, K, G's are not bounded, one needs some (sufficient) conditions on them to ensure the existence of an associated cocycle V_t ([3], [8], [13]) and for more recent advances in this direction, the reader is referred to [7].

To make realistic contact with various theories of classical stochastic processes, one needs to go one step further, viz. study and construct cocyles of maps on $\mathcal{B}(h)$. Let $V \equiv \{V_t\}_{t \geq 0}$ be a family of contractive cocyles in $\widetilde{\mathfrak{H}}$ as described above and let $x \in \mathcal{B}(h)$. Set for $t \geq 0$,

$$j_t(x) = V_t(x \otimes I_{\mathfrak{F}})V_t^* \in \mathcal{B}(\widetilde{\mathfrak{H}}), \tag{2.19}$$

and observe that (i) $j_t : \mathcal{B}(h) \mapsto \mathcal{B}(\widetilde{\mathfrak{H}})$ extends as a bounded $*$-map i.e. $\|j_t(x)\| \leq \|x\|$ and $j_t(x^*) = (j_t(x))^*$; (ii) j_t is unital if V is coisometric, is a homomorphism if V is isometric. Note furthermore that very significantly,

(iii) if V is a unitary cocycle in $\widetilde{\mathfrak{H}}$, then j_t is a unital $*$-homomorphism of $\mathcal{B}(h)$-valued cocycle called Quantum Stochastic flow (QSF) (see [1], [4], [13]). The appropriate statement is in the next proposition.

Proposition 2.4. *Let V be a strongly continuous unitary cocycle in $\widetilde{\mathfrak{H}}$ and let j_i be defined by (2.19). Then*

(i) j_t is a bounded $*$-homomorphic cocycle on $\mathcal{B}(h)$ with respect to the shift isometry, i.e.

$$\|j_t(x)\| \leq \|x\| \text{ and } j_t(x) = \hat{j}_s \circ \sigma_s \circ j_{t-s}(x), \tag{2.20}$$

where $x \in \mathcal{B}(h)$ and $\hat{j}_s$ is the "lift" of the map j_s from $\mathcal{B}(h)$ to $\mathcal{B}(h \otimes \mathfrak{F}_{[s,t]})$, which is the range of $\sigma_s \cdot j_{t-s}(x)$. Here by $\mathfrak{F}_{[s,t]}$ we mean the natural extension of (2.3) to $\Gamma(L^2([s,t]))$;

(ii) if furthermore $t \mapsto V_t^*$ is also strongly continuous on $\widetilde{\mathfrak{H}}$, then the map: $\mathbb{R}_+ \times \mathcal{B}(h) \ni (t,x) \mapsto j_t(x) \in \mathcal{B}(h \otimes \mathfrak{F})$ is jointly strongly continuous with respect to strong operator topology of $\mathcal{B}(h)$.

(iii) Set $\mathcal{J}_t(x) = \mathbb{E}(j_t(x))$ for $x \in \mathcal{B}(h)$. Then $\{\mathcal{J}_t\}_{t\geq 0}$ is C_0-semigroup of $*$-preserving contractive, completely positive maps on $\mathcal{B}(h)$.

(Sketch of proof): (i) The map-cocycle property follows from the defining property (2.14) of V as:

$$\begin{aligned} j_t(x) &= V_t(x \otimes I_{\mathfrak{F}})V_t^* = V_s\sigma_s(V_{t-s})(x \otimes I)\sigma_s(V_{t-s}^*)V_s^* \\ &= V_s\sigma_s(V_{t-s}(x \otimes I)V_{t-s}^*)V_s^* = \hat{j}_s \circ \sigma_s(j_{t-s}(x)), \end{aligned}$$

remembering the explanation above of the notation $\hat{j}_s(x)$ in the statement of this proposition.

(ii) From (2.19) it follows that for $0 \leq s \leq t$, and $x, y \in \mathcal{B}(h)$,

$$\begin{aligned} j_t(x) - j_s(y) &= \hat{j}_s \circ \sigma_{s\circ}(j_{t-s}(x)) - \hat{j}_s \circ \sigma_s \circ (y \otimes I_{\mathfrak{F}}) \\ &= \hat{j}_s \circ \sigma_s \circ (j_{t-s}(x) - x \otimes I_{\mathfrak{F}}) + \hat{j}_s \circ \sigma_s \circ ((x-y) \otimes I_{\mathfrak{F}}) \end{aligned} \tag{2.21}$$

The second term in (2.21) has an easy estimate: for $u \in h, f \in L^2(\mathbb{R}_+)$,

$$\left[\hat{j}_s \circ \sigma_s((x-y) \otimes I_{\mathfrak{F}})\right] ue(f) = j_s(x-y)(ue(f)) = V_s((x-y) \otimes I)V_s^* ue(f),$$

so that the norm of the left hand side $= \|((x-y) \otimes I)V_s^* ue(f)\|$ leading to the strong continuity of the map $\mathcal{B}(h) \ni x \mapsto j_s(x)ue(f)$ for fixed s, u and f, with respect to the strong topology of $\mathcal{B}(h)$. The extension of strong continuity of the same map on the whole of $h \otimes \mathfrak{F}$ follows from the totality of vectors of the form $\{ue(f) \mid u \in h, f \in L^2(\mathbb{R}_+)\}$ in $\widetilde{\mathfrak{H}}$ and the fact $\|j_s(x)\| \leq \|x\|$. For the first term in (2.21), we note that for $\Psi \in \widetilde{\mathfrak{H}}$

$$\begin{aligned} &(j_{t-s}(x) - x \otimes I_{\mathfrak{F}})\Psi = (V_{t-s}(x \otimes I_{\mathfrak{F}})V_{t-s}^* - x \otimes I_{\mathfrak{F}})\Psi \\ &= V_{t-s}(x \otimes I_{\mathfrak{F}})(V_{t-s}^* - I_{\widetilde{\mathfrak{H}}})\Psi + (V_{t-s} - I_{\widetilde{\mathfrak{H}}})(x \otimes I_{\mathfrak{F}})\Psi \end{aligned}$$

and the norm of the right hand side goes to zero as $(t-s) \to 0$ since V is unitary and since both V and V^* are assumed to be strongly continuous on $\widetilde{\mathfrak{H}}$.

(iii) That $\{\mathcal{J}_t\}_{t\geq 0}$ is a semigroup follows from the cocycle property: for $x \in \mathcal{B}(h)$

$$\begin{aligned} \mathcal{J}_t(x) &= \mathbb{E}(j_t(x)) = \mathbb{E} \circ (\mathbb{E}_s(\hat{j}_s \circ \sigma_s(j_{t-s}(x)))) \\ &= \mathbb{E} \circ \hat{j}_s \circ ((\mathbb{E}_s \circ \sigma_s)(j_{t-s}(x)) = \mathcal{J}_s(\mathcal{J}_{t-s}(x)), \end{aligned}$$

where we have used the property (2.17). Furthermore, since $\langle u, \mathcal{J}_t(x)v\rangle = \langle V_t^*(u \otimes e(0)), (x \otimes I_{\mathfrak{F}})V_t^*(v \otimes e(0))\rangle$, the continuity of the map $t \mapsto \mathcal{J}_t(x)$ in weak*-topology of $\mathcal{B}(h)$ follows and for a semigroup of maps, this implies the strong-continuity of the predual semigroup $\mathcal{J}_{t,*}$-on $\mathcal{B}_1(h)$, the predual of $\mathcal{B}(h)$. That $\mathcal{J}_t$ is a $*$-preserving completely positive map for every fixed $t \geq 0$ follows from the facts that j_t is a $*$-homomorphic map on $\mathcal{B}(h)$. □

Remark 2.5. (i) More abstractly, the family $\{j_t(\cdot)\}_{t\geq 0}$, the QSF's are constructed as solutions of $\mathcal{B}(h)-$valued QSDE's of the type (similar to that in Remark 2.3)

$$dj_t(x) = j_t(\delta(x)dA^+(t) - j_t(\delta^*(x))dA(t) + j_t(\mathcal{L}(x))dt, \tag{2.22}$$

with $j_0(x) = x$ for every $x \in \mathcal{B}(h)$, and where δ is a derivation on $\mathcal{B}(h)$, δ^* is defined by $\delta^*(x) = \delta(x)^*$ and $\mathcal{L}$ is the generator of the semigroup $\mathcal{J}_t$ (for further discussions, see [4], [13]). In cases where δ and $\mathcal{L}$ are bounded maps, the above QSDE can be solved easily, but in a more general case this is a difficult problem.

(ii) A more tractable way of generating a solution of (2.22) is when j_t is obtained in terms of V_t as in (2.19) since solving for V_t, either as a solution of QSDE (2.18) in $\widetilde{\mathfrak{H}}$ or as a cocycle by other means, is easier. For the second method (following [2], [3], [6], [7], [14]), the "minimal" semigroup is constructed, associated with the following:

(a) G the generator of a C_0-contraction semigroup $\{P_t\}_{t\geq 0}$ on h, (b) L a closed operator in h satisfying

$$\langle Lu, Lv\rangle + \langle Gu, v\rangle + \langle u, Gv\rangle = 0, \tag{2.23}$$

for all $u, v \in D(G)$ which is contained in $D(L)$. With an additional assumption of holomorphy of the semigroup $\{P_t\}$ in h, a unique cocycle $\{V_t\}$ has been constructed in [7]. However, here we shall not go into those details, but shall assume that we have such a cocyle $\{V_t\}_{t\geq 0}$ in $\widetilde{\mathfrak{H}}$ or a map-cocycle $\{j_t\}_{t\geq 0}$ and study the consequences.

3. Cocycles and Martingales

In the theory of classical stochastic processes, the "cocycles" are often generated by solving classical stochastic differential equations [5]. However, as we have described in the Introduction, there are many situations, where this does not happen and one, following Stroock and Varadhan [15], looks for solving the associated martingale problem. In this, the generator of the semigroup, driving the process is known and one looks at the difference of the process and the pure time-integral part as in (1.9) and attempts to show that this difference is a suitable martingale. Under further restrictive conditions on the process, the martingale may admit a representation as an appropriate stochastic integral, in which case one has effectively solved the associated stochastic differential equation.

To put the above in perspective, let us describe the simple classical case of SBM-flow in the Fock-space language described earlier. For this, we set the initial Hilbert space $h = L^2(\mathbb{R})$ and for $\varphi \in L^\infty(\mathbb{R})$, define

$$j_t(M_\varphi) = M_{\varphi(\cdot+\omega(t))}, \tag{3.1}$$

where M_φ is the operator of multiplication by φ in h, and $M_{\varphi(\cdot+\omega(t))}$ is the operator of multiplication by the translated function $\varphi(\cdot + \omega(t))$ in $\widetilde{\mathfrak{H}} = h \otimes \mathfrak{F}$. It is not difficult to make the following observations:

(i) If we set $V_t = T_{\omega(t)}$ in $h \otimes \mathfrak{F} \simeq h \otimes L^2(\mathbb{P})$, where T is the translation operator in h given by $(T_x f)(y) = f(y+x)$ for $f \in h; x, y \in \mathbb{R}$, then

$$\begin{aligned} j_t(M_\varphi) &= V_t(M_\varphi \otimes I_{\mathfrak{F}})V_t^*, \text{ for } t \geq 0 \\ &= M_{\varphi(\cdot+\omega(t))}. \end{aligned} \tag{3.2}$$

(ii) Since V_t is unitary, j_t has all the properties as stated in Proposition 2.4, viz. it is a family of unital $*$-homomorphic maps from $L^\infty(\mathbb{R}) \subseteq \mathcal{B}(h)$ into a (commutative) unital $*$-subalgebra of $\mathcal{B}(h \otimes \mathfrak{F})$.

(iii) One can compute the expectation semigroups: $P_t = \mathbb{E}(V_t)$ is the Heat semigroup with generator Δ, the Laplacian acting in h, while $\mathcal{J}_t(M_\varphi) = \mathbb{E}(j_t(M_\varphi))$ is also the Heat semigroup acting on $L^\infty(\mathbb{R}) \subseteq \mathcal{B}(h)$, with ω^*-generator $\mathcal{L}$, the Laplacian Δ acting on $L^\infty(\mathbb{R})$.

(iv) That $\{j_t(\cdot)\}_{t\geq 0}$ is a cocycle of maps can be seen as follows. By (2.15) and (2.5),

$$\begin{aligned} \sigma_s \circ (j_{t-s}(M_\varphi)) &= (I_h \otimes \Gamma_s) M_{\varphi(\cdot+\omega(t-s))} I_h \otimes \Gamma_s^* \\ &= I_{\mathfrak{F}_s} \otimes \Gamma_s M_{\varphi(\cdot+\omega(t-s))} \Gamma_s^* = M_{\varphi(\cdot+(\omega(t)-\omega(s)))}, \end{aligned}$$

and therefore, $\hat{j}_s \circ \sigma_s \circ (j_{t-s}(M_\varphi)) = j_s(M_{\varphi(\cdot+(\omega(t)-\omega(s)))}) = M_{\varphi(\cdot+\omega(t))} = j_t(M_\varphi)$.

(v) Finally, for $\varphi \in BC^2(\mathbb{R})$ (twice continuously differentiable functions with bounded derivatives), if we set

$$\begin{aligned} Y_t(\varphi) &\equiv j_t(M_\varphi) - M_\varphi - \int_0^t j_\tau(M_{\varphi''}) d\tau \\ &= M_{\varphi(\cdot+\omega(t))} - M_\varphi - \int_0^t M_{\varphi''(\cdot+\omega(\tau))} d\tau, \end{aligned}$$

then $Y_t(\varphi)$ is a martingale, i.e. $\mathbb{E}_s(Y_t(\varphi)) = Y_s(\varphi)$ for $0 \leq s \leq t$. In this special case, one can say furthermore, (since the problem can be set in $h \otimes L^2(\mathbb{P})$, the relevant filtration is the one coming from the S.B.M), that $Y_t(\varphi)$ admits the representation:

$$\begin{aligned} Y_t(\varphi) &= \int_0^t j_\tau(M_{\varphi'}) d\omega(\tau) \\ &= \int_0^t M_{\varphi'(\cdot+\omega(\tau))} d\omega(\tau), \end{aligned} \tag{3.3}$$

the Ito-stochastic integral with respect to S.B.M.

Now we are ready to state our main theorem, which in effect says that in the Fock-space description as given above, the properties of any Q.S.P. being a cocycle and of it being a martingale are essentially equivalent.

Theorem 3.1. *(i) The following are equivalent:*
(a) $\{V_t\}_{t\geq 0}$ *is a strongly continuous family of contractive cocycles in* $h \otimes \mathfrak{F}$*,*

(b) set

$$\mathcal{M}_t\xi \equiv V_t\xi - \xi - \int_0^t ds\ V_s(G \otimes I_{\mathfrak{F}})\xi, \tag{3.4}$$

with $\xi \in$ *dense* $\mathcal{D} \subseteq D(G) \otimes_{alg} \mathcal{E}$ *where* G *is the generator of the expectation semigroup* $\{P_t\}_{t\geq 0}$*, then for* $0 \leq s \leq t$

$$\mathcal{M}_t\xi = \mathcal{M}_s\xi + V_s \circ \sigma_s(M_{t-s})\xi; \tag{3.5}$$

and in such a case $\{\mathcal{M}_t\}_{t\geq 0}$ *is a martingale i.e.* $\mathbb{E}_s(\mathcal{M}_t) = \mathcal{M}_s$.

(ii) The following are equivalent.

(a) $j_t : \mathcal{B}(h) \to \mathcal{B}(h \otimes \mathfrak{F})$ *is a family (for* $t \geq 0$*) of map-cocycles, satisfying (2.20),*
(b) setting

$$Y_t(x) = j_t(x) - x \otimes I - \int_0^t ds j_s(\mathcal{L}(x)), \tag{3.6}$$

for $x \in D(\mathcal{L}) \subseteq \mathcal{B}(h)$ *with* $\mathcal{L}$*, the generator of the expectation semigroup* $\{\mathcal{J}_t\}_{t\geq 0}$*, one has for* $0 \leq s \leq t$*,*

$$Y_t(x) = Y_s(x) + \hat{j}_s \circ \sigma_s(Y_{t-s}(x)), \tag{3.7}$$

and in such a case $\{Y_t(x)\}_{t\geq 0}$ *is a martingale, i.e.*

$$\mathbb{E}_s(Y_t(x)) = Y_s(x)\ \forall\ x \in D(\mathcal{L}).$$

Proof. (ia) $\Rightarrow$ (ib): For $0 \leq s \leq t, \xi \in \mathcal{D}$, one has

$$\begin{aligned}
(\mathcal{M}_t - \mathcal{M}_s)\xi &= V_t\xi - V_s\xi - \int_s^t d\tau V_\tau(G \otimes I_{\mathfrak{F}})\xi \\
&= \left\{ V_s(\sigma_s \circ (V_{t-s} - I)) - \int_s^t d\tau [V_s \circ \sigma_s \circ V_{\tau-s}(G \otimes I)] \right\} \xi \\
&= V_s \circ \sigma_s \circ \left\{ V_{t-s} - I - \int_0^{t-s} d\tau V_\tau(G \otimes I) \right\} \xi,
\end{aligned}$$

where we have used the cocycle property (2.14) of V_t and interchanged the integral and the composition of the operator V_s and map σ_s, using their boundedness. This gives us the relation (3.5) and by taking conditional expectation given $\mathcal{B}(\mathfrak{F}_s)$, on both the sides of (3.5) we have that

$$\begin{aligned}
\mathbb{E}_s(\mathcal{M}_t) &= \mathcal{M}_s + V_s(\mathbb{E}_s \circ \sigma_s(\mathcal{M}_{t-s})) \\
&= \mathcal{M}_s + V_s\mathbb{E}_0(\mathcal{M}_{t-s}),
\end{aligned}$$

using the property (2.17). On the other hand by taking vacuum-expectation of both sides of the definition (3.4) we get for $v \in h, u \in D(G)$ that

$$\begin{aligned}\langle v, \mathbb{E}(\mathcal{M}_t)u\rangle &= \langle v\otimes e(0), \mathcal{M}_t(u\otimes e(0))\rangle \\ &= \left\langle v\otimes e(0), \left\{V_t u\otimes e(0) - u\otimes e(0) - \int_0^t ds V_s(Gu\otimes e(0))\right\}\right\rangle \\ &= \left\langle v, P_t u - u - \int_0^t ds\ P_s G u\right\rangle = 0,\end{aligned}$$

since G is the generator of the semigroup $\{P_t\}_{t\geq 0}$ and we have the required result, viz. $\mathbb{E}_s(\mathcal{M}_t) = \mathcal{M}_s$.

(ib) $\Rightarrow$ (ia): Set $D_{s,t} = V_t - V_s \cdot \sigma_s \cdot (V_{t-s})$ and on using (3.4) and (3.5) we observe that for ξ as before,

$$\begin{aligned}V_t\xi - V_s\xi &= (\mathcal{M}_t - \mathcal{M}_s)\xi + \int_s^t V_\tau(G\otimes I_{\mathfrak{F}})\xi d\tau \\ &= (V_s\circ\sigma_s(\mathcal{M}_{t-s}))\xi + \int_s^t V_\tau(G\otimes I_{\mathfrak{F}})\xi d\tau \\ &= V_s\cdot\sigma_s\left\{V_{t-s}\xi - \xi - \int_0^{t-s} d\tau V_\tau(G\otimes I_{\mathfrak{F}})\xi\right\} + \int_s^t V_\tau(G\otimes I_{\mathfrak{F}})\xi d\tau.\end{aligned}$$

Therefore, we get that

$$\begin{aligned}D_{s,t}\xi &= \int_s^t d\tau V_\tau(G\otimes I_{\mathfrak{F}})\xi - V_s\cdot\sigma_s\left(\int_0^{t-s} d\tau V_\tau(G\otimes I_{\mathfrak{F}})\xi\right), \\ &= \int_s^t d\tau V_\tau(G\otimes I_{\mathfrak{F}})\xi - V_s\circ\sigma_s\left[\int_s^t d\tau V_{\tau-s}(G\otimes I_{\mathfrak{F}})\xi\right] \\ &= \int_s^t D_{s,\tau}(G\otimes I_{\mathfrak{F}})\xi d\tau.\end{aligned}$$

This shows that $D_{s,t}\xi$ is strongly differentiable for all $\xi \in \mathcal{D}$, in particular for $\xi = u\otimes e(f)$ with $u \in D(G)$ and $f \in L^2(\mathbb{R}_+)$. Furthermore, for $s \leq t$,

$$\frac{d}{dt}D_{s,t}(u\otimes e(f)) = D_{s,t}(Gu\otimes e(f))$$

and since the semigroup P_t leaves $D(G)$ invariant, the map:

$$[0,t) \ni \tau \mapsto D_{s,\tau}((P_{t-\tau}u)\otimes e(f))$$

is strongly differentiable and for $0 \leq s \leq \tau \leq t$,

$$\begin{aligned}\frac{d}{d\tau} D_{s,\tau}((P_{t-\tau}u) \otimes e(f)) &= D_{s,\tau}((GP_{t-\tau}u) \otimes e(f)) \\ -D_{s,\tau}((P_{t-\tau}Gu) \otimes e(f)) &= 0.\end{aligned}$$

Thus the above-mentioned map is independent of τ for $\tau \in [s,t]$ and therefore

$$D_{s,s}((P_{t-s}u) \otimes e(f)) = D_{s,t}((P_0 u) \otimes e(f))$$

or $D_{s,t}(u \otimes e(f)) = 0$ since $D_{s,s} = 0$. Finally we note that the family $D_{s,t} \in \mathcal{B}(h \otimes \mathfrak{F})$ and hence $D_{s,t} = 0$ for $0 \leq s \leq t$ or $V_t = V_s \cdot \sigma_s(V_{t-s})$.

(iia) $\Rightarrow$ (iib): The idea of the proof is very similar to that given above. From the definition (3.6), we get that

$$\begin{aligned}Y_t(x) - Y_s(x) &= j_t(x) - j_s(x) - \int_s^t d\tau j_\tau(\mathcal{L}(x)) \\ &= \hat{j}_s \cdot \sigma_s \cdot \left[j_{t-s}(x) - (x \otimes I_{\mathfrak{F}}) - \int_0^{t-s} d\tau j_\tau(\mathcal{L}(x)) d\tau \right],\end{aligned}$$

for $0 \leq s \leq t$ and for $x \in D(\mathcal{L}) \subseteq \mathcal{B}(h)$, leading to (3.7).

Taking conditional expectation given 's' on both sides of (3.6), we have that

$$\begin{aligned}\mathbb{E}_s(Y_t(x)) &= Y_s(x) + j_s \circ (\mathbb{E}_s \circ \sigma_s) \left(j_{t-s}(x) - x \otimes I - \int_0^{t-s} d\tau j_\tau(\mathcal{L}(x)) \right) \\ &= Y_s(x) + j_s \circ \left\{ \mathcal{J}_{t-s}(x) - x - \int_0^{t-s} d\tau \mathcal{J}_\tau(\mathcal{L}(x)) \right\} = Y_s(x)\end{aligned}$$

since the expression in the parenthesis $\{\cdot\} = 0$, $\mathcal{L}$ being the generator of the semigroup $\{\mathcal{J}_t\}_{t \geq 0}$ on $\mathcal{B}(h)$ and $x \in D(\mathcal{L})$.

(iib) $\Rightarrow$ (iia): Set $\Xi_{s,t}(x) = j_t(x) - \hat{j}_s \circ \sigma_s \circ (j_{t-s}(x))$ for $x \in D(\mathcal{L})$ and $0 \leq s \leq t$ and note that $\Xi_{s,t}$ is a bounded linear map from $\mathcal{B}(h)$ into $\mathcal{B}(h \otimes \mathfrak{F})$. Furthermore, a simple calculation using (3.7) yields that

$$j_t(x) - j_s(x) = Y_t(x) - Y_s(x) + \int_s^t d\tau j_\tau(\mathcal{L}(x))$$

$$= \hat{j}_s \circ \sigma_s \circ \left(j_{t-s}(x) - x - \int_0^{t-s} d\tau j_\tau(\mathcal{L}(x)) \right) + \int_s^t d\tau j_\tau(\mathcal{L}(x))$$

$$\text{or } j_t(x) - \hat{j}_s \circ \sigma_s \circ (j_{t-s}(x)) = \int_s^t d\tau \left[j_\tau(\mathcal{L}(x)) - \hat{j}_s \circ \hat{\sigma}_s(j_{\tau-s}(\mathcal{L}(x))) \right]$$

or equivalently, that $\Xi_{s,t}$ satisfies: $\Xi_{s,t}(x) = \int_s^t d\tau \Xi_{s,\tau}(\mathcal{L}(x))$. This implies that the map $t \mapsto \Xi_{s,t}(x)$ is ω^*-differentiable on $\mathcal{B}(h)$ for $x \in D(\mathcal{L})$ and

$$\frac{d}{dt}\Xi_{s,t}(x) = \Xi_{s,t}(\mathcal{L}(x)).$$

Noting that the semigroup $\mathcal{J}_t$ leaves the $D(\mathcal{L})$ invariant we compute for $0 \leq s \leq \tau \leq t$ and for $x \in D(\mathcal{L})$,

$$\frac{d}{d\tau}\Xi_{s,\tau} \circ \mathcal{J}_{t-\tau}(x) = \Xi_{s,\tau}(\mathcal{L}\mathcal{J}_{t-\tau}(x)) - \Xi_{s,\tau}(\mathcal{J}_{t-\tau}(\mathcal{L}(x))) = 0$$

and therefore $(\Xi_{s,\tau} \circ \mathcal{J}_{t-\tau})(x)$ is independent of τ for $0 \leq s \leq \tau \leq t$. This leads to the conclusion that $\Xi_{s,s} \circ \mathcal{J}_{t-s}(x) = \Xi_{s,t}(\mathcal{J}_0(x)) = 0$, since $\Xi_{s,s} = 0$. $\square$

Remark 3.2. It needs to be emphasized that the stronger properties (3.5) and (3.7) are equivalent to the cocycle properties of V_t and of j_t respectively and not just the property of $\mathcal{M}_t$ and Y_t being martingales in the respective cases. For example, in the case of $\{V_t\}_{t\geq 0}$, if one starts with the weaker assumption that $\{\mathcal{M}_t\xi\}$ is a martingale, then one has the following result.

Theorem 3.3. *Recall the definitions of $\mathcal{M}_t$ and Y_t as in (3.4) and (3.6) respectively. Then the martingale properties: $\mathbb{E}_s(\mathcal{M}_t) = \mathcal{M}_s$ and $\mathbb{E}_s(Y_t(x)) = Y_s(x)$ for $x \in D(\mathcal{L})$ implies the Q.S.P.'s $\{V_t\}_{t\geq 0}$ and $\{j_t(x)\}_{t\geq 0}$ are both Markov processes (relative to the driving semigroups $\{P_t\}_{t\geq 0}$ and $\{\mathcal{J}_t\}_{t\geq 0}$ respectively); i.e. $\mathbb{E}_s(V_t) = V_s \circ P_{t-s}$ and $\mathbb{E}_s(j_t(x)) = j_s(\mathcal{J}_{t-s}(x))$ for $0 \leq s \leq t$ respectively.*

Proof. (only for $\{V_t\}$, that of $\{j_t(x)\}$ is very similar): The martingale condition for $\mathcal{M}_t$ implies that for $\xi \in \mathcal{D}$,

$$\mathbb{E}_s\left(V_t\xi - \xi - \int_0^t d\tau V_\tau (G \otimes I_{\mathfrak{F}})\xi\right) = V_s\xi - \xi - \int_0^s d\tau V_\tau(G \otimes I_{\mathfrak{F}})\xi$$

or, $\mathbb{E}_s(V_t\xi) = V_s\xi + \int_s^t d\tau \mathbb{E}_s(V_\tau(G \otimes I_{\mathfrak{F}})\xi)$. Thus

$$\begin{aligned}
\mathbb{E}_s(V_t\xi - V_s \circ \sigma_s(V_{t-s})\xi) &= V_s\xi + \int_s^t d\tau \mathbb{E}_s(V_\tau(G \otimes I_{\mathfrak{F}})\xi) - V_s\mathbb{E}(V_{t-s}\xi) \\
&= V_s\xi + \int_s^t d\tau \mathbb{E}_s(V_\tau(G \otimes I_{\mathfrak{F}})\xi) - V_s P_{t-s}\xi \\
&= \int_s^t d\tau \mathbb{E}_s(V_\tau - V_s \cdot \sigma_s(V_{\tau-s}))(G \otimes I_{\mathfrak{F}})\xi, \qquad (3.8)
\end{aligned}$$

where we have used the relation

$$V_s(P_{t-s}\xi - \xi) = V_s \int_0^{t-s} d\tau P_\tau (G \otimes I_{\mathfrak{F}})\xi = \mathbb{E}_s\left(\int_s^t d_\tau V_s \sigma_s(V_{\tau-s})(G \otimes I_{\mathfrak{F}})\xi\right).$$

If we set $D_{s,t} = \mathbb{E}_s\{V_t - V_s \cdot \sigma_s(V_{t-s})\}$ for $0 \le s \le t$, then (3.8) leads to the equation

$$\begin{aligned} D_{s,t}\xi &= \int_s^t D_{s,\tau}(G \otimes I)\xi d\tau, \\ \text{or } \frac{d}{dt}D_{s,t}\xi &= D_{s,t}(G \otimes I)\xi. \end{aligned} \tag{3.9}$$

Finally, we compute for $0 \le s \le \tau \le t$

$$\frac{d}{d\tau}D_{s,\tau}P_{t-\tau}\xi = D_{s,\tau}(G \otimes I)P_{t-\tau}\xi - D_{s,\tau}P_{t-\tau}(G \otimes I)\xi = 0$$

which yields the relation:

$$D_{s,t}P_0\xi = D_{s,s}P_{t-s}\xi = 0$$

and since $D_{s,t}$ is a family of bounded operators in $h \otimes \mathfrak{F}$, we have that $D_{s,t} = E_s(V_t) - V_sP_{t-s} = 0$ for $0 \le s \le t$, which is the Markov property. □

Thus we have seen that the cocycle property of V_t (or j_t respectively) is **equivalent** to the additive cocycle property of $\mathcal{M}_t$ (or Y_t respectively) which in its turn implies the martingale property of V_t (or j_t respectively). In the converse direction, on the other hand, the martingale properties of the processes V_t (or j_t respectively) implies only the Markov property in the respective cases, and does **not necessarily** lead to the cocycle property.

(ii) It is worth mentioning that there is a variant of the Theorem 3.1 in which the cocyle property of $\{V_t\}_{t\ge0}$ to be replaced by the property of evolution of 2-parameter families, viz. for $0 \le r \le s \le t$, $V_{r,t} = V_{r,s}V_{s,t}$, along with corresponding strong continuities with respect to both variables. However, in this case, the expectation of $V_{r,t}$ is also an evolution $P_{r,t}$ in contrast to being a semigroup and this makes discussions a little more complicated. We have the following result, which we state without proof since it is mostly an adaptation of that of Theorem 3.1. First we need the following definition.

Definition 3.4. Let $\{P_{r,t}\}_{0\le r\le t}$ be a family of contractive evolutions in h with (both the right-and the left-) strong derivative $\{G_s\}_{s\ge0}$, such that

$$[0,\infty) \ni s \mapsto P_{r,s}G_s\xi \text{ and } [0,\infty) \ni s \mapsto G_sP_{s,t}\xi \text{ are}$$

both strongly continuous for $\xi \in \bigcap\limits_{s\ge0} D(G_s)$, which is assumed to be dense and such that:

$$\begin{aligned} P_{r,t}\xi &= \xi + \int_r^t dsP_{r,s}G_s\xi \text{ and} \\ P_{r,t}\xi &= \xi - \int_r^t dsG_sP_{s,t}\xi. \end{aligned} \tag{3.10}$$

Unlike the theory of semigroups, the theory of evolutions is not so well-known and for some information on this, the reader is referred to [16].

Theorem 3.5. *Let $\{V_{r,t}\}_{0\leq r\leq t}$ be a family of contractive Q.S.P.'s on $\widetilde{\mathfrak{H}}$ such that its expectation evolution $\{P_{r,t}\}$ on h has strong derivative $\{G_s\}_{0\leq s}$ in the sense of the definition 3.4 above. Set for $0 \leq r \leq t$,*

$$\mathcal{M}_{r,t}\xi = V_{r,t}\xi - \xi - \int_r^t ds\ V_{r,s}(G_s \otimes I_{\mathfrak{F}})\xi \tag{3.11}$$

for $\xi \in \bigcap_{s\geq 0} D(G_s) \otimes \mathcal{E}$. Then the following are equivalent:
(i) $\{V_{r,t}\}_{0\leq r\leq t}$ is an evolution,
(ii) for $0 \leq r \leq s \leq t; \mathcal{M}_{r,t} = \mathcal{M}_{r,s} + V_{r,s} \circ \mathcal{M}_{s,t}$. Furthermore, (ii) implies that $\{\mathcal{M}_{r,t}\}$ is a martingale, i.e.

$$\mathbb{E}_s(\mathcal{M}_{r,t}) = \mathcal{M}_{r,s}.$$

4. Concluding Remarks:

As described in section 1, Stroock and Varadhan, in their book ([15], see also [12]), introduced a concept, "the Martingale formulation" for the construction of a stochastic process, driven by a certain class of second order partial differential operators (in some cases with time-dependent coefficients) acting on smooth functions. For simplicity of presentation and to connect with our main theorem, Theorem 3.1, we shall restrict ourselves to the case with time-independent coefficients.

In contrast to the approach in (classical) stochastic processes, we look at the corresponding "Heat semigroup" $\{\mathcal{J}_t\}_{t\geq 0}$ generated by $\mathcal{L}$ on the $*$-algebra $\mathcal{B}(L^2(\mathbb{R}^d))$. As we have mentioned in the Introduction, one can associate with the fundamental solution $p(t, x; 0, y)$ of (1.3), a one-parameter semigroup $\mathcal{J}_t$ on $\Omega = C(\mathbb{R}_+, \mathbb{R}^d)$, satisfying:

$$\begin{aligned}(\mathcal{J}_t f)(x) &\equiv \int_{\mathbb{R}^d} p(t, x; 0, y) f(y) dy \\ &= f(x) + \int_0^t d\tau \int_{\mathbb{R}^d} p(\tau, x; 0, y)(\mathcal{L}f)(y) dy,\end{aligned} \tag{4.1}$$

where the generator $\mathcal{L}$, in this case, is the differential operator L of (1.1), and $f \in \mathrm{Dom}(\mathcal{L})$. Also one can construct canonically a classical stochastic process $\{x(t)\}$ with which is associated the quantum stochastic map (flow) $\{j_t\}_{t\geq 0}$ by setting $j_t(f)(x) = f(x(t))$. Then

$$(j_t f)(x) - f(x) - \int_0^t j_\tau(\mathcal{L}(f))(x) d\tau = f(x(t)) - f(x) - \int_0^t L(f)(x(\tau)) d\tau. \tag{4.2}$$

Stroock and Varadhan construct $x(t)$ such that the second expression in (4.2) is a martingale. What we have shown in Theorem 3.1 (and stated without proof for the time-dependent case in Theorem 3.5) is that the martingale formulation of Stroock-Varadhan is equivalent to the cocycle property of the map $\{j_t\}_{t\geq 0}$ as in (2.20), except that the theory presented here is on the whole $\mathcal{B}(h)$ (with $h = L^2(\mathbb{R}^d)$ in this case) while the classical processes are represented as maps on an appropriate commutative $*$-subalgebra of $\mathcal{B}(\mathfrak{H})$. It has also been observed in Remark 2.5 that the problem of constructing a cocycle of maps j_t on suitable subalgebras of $\mathcal{B}(h)$ can often be achieved by constructing isometric cocycles V_t on $h \otimes \mathfrak{F}$, and this has been achieved in many cases (see [3], [7]).

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