Journal of Stochastic Analysis

Volume 1 | Number 4 Article 12

December 2020

Rényi Entropy on C*-Algebras

Farrukh Mukhamedov
United Arab Emirates University, 15551 Al-Ain, United Arab Emirates, far75m@yandex.ru

Kyouhei Ohmura
Tokyo University of Science, Noda, Chiba 278-8510, Japan, ohmura.kyouhei@gmail.com

Noboru Watanabe
Tokyo University of Science, Noda, Chiba 278-8510, Japan, watanabe@is.noda.tus.ac.jp

Recommended Citation

Mukhamedov, Farrukh; Ohmura, Kyouhei; and Watanabe, Noboru (2020) "Rényi Entropy on C*-Algebras," *Journal of Stochastic Analysis*: Vol. 1 : No. 4 , Article 12.
DOI: 10.31390/josa.1.4.12
Available at: https://digitalcommons.lsu.edu/josa/vol1/iss4/12

Journal of Stochastic Analysis
Vol. 1, No. 4 (2020) Article 12 (13 pages)
DOI: 10.31390/josa.1.4.12

RÉNYI ENTROPY ON C^*-ALGEBRAS

FARRUKH MUKHAMEDOV, KYOUHEI OHMURA*, AND NOBORU WATANABE

Dedicated to Professor Leonard Gross on the occasion of his 88th birthday

Abstract. In this paper, we formulate Rényi-type entropy depending on $\alpha \in [0,\infty)\backslash\{1\}$ and reference systems $\mathcal{S}$ on C^*-algebras, and prove that the introduced entropy corresponds to the quantum Rényi entropy defined by Petz and $\mathcal{S}$-mixing entropy given by Ohya under certain conditions. Moreover, using our entropy, we show that the complexities of the KMS state takes different values by choosing different reference systems $\mathcal{S}$ for any $\alpha \neq 1$.

1. Introduction

In classical information theory, the Shannon entropy gives the information amount of an information source by using the probability distribution $\{p_i\}$ [21]. However, in general, it is difficult to obtain the correct probability distribution for actual systems. The Rényi entropy is more widely used than Shannon's one since it can give the information amount for a system which $\{p_i^\alpha\}$ ($\alpha \in [0,\infty)\backslash\{1\}$) is found [19]. Moreover, since the Rényi entropy corresponds to the Shannon entropy when $\alpha \to 1$, the Rényi entropy is an extension of Shannon's one in the sense of the parameter α.

On the other hand, using the barycentric decomposition of states, the $\mathcal{S}$-mixing entropy was defined by Ohya as a Shannon-type entropy on the C^*-algebra $\mathcal{A}$ [11, 12, 22]. This entropy depends on the extreme points of $\mathcal{S}$ which decompose a state on $\mathcal{A}$. Therefore, one can investigate the complexity of a state in detail by choosing $\mathcal{S}$. The space $\mathcal{S}$ is called a reference system.

In [7], we formulated a Rényi-type entropy which depends on $\alpha \in [0,\infty)\backslash\{1\}$ and reference systems $\mathcal{S}$ based on the construction of the $\mathcal{S}$-mixing entropy. In this paper, we give the definition of the Rényi entropy and show several results by using the entropy following [7].

We organize the paper as follows: In section 2, we briefly recall the notion of barycentric decomposition. Section 3 is devoted to formulate the $\mathcal{S}$-mixing Rényi entropy. Moreover, we prove that the Rényi entropy corresponds to the $\mathcal{S}$-mixing entropy when $\alpha \to 1$. In section 4 we show that our Rényi entropy equals to the quantum Rényi entropy defined by Petz [16, 17] if $\mathcal{S}$ is the space of the set of all states and $\alpha > 1$. Besides, we prove that our entropy is less than or equal to the

Received 2019-12-19; Accepted 2020-12-5; Communicated by A. Stan et al.
2010 *Mathematics Subject Classification.* Primary 81P45; Secondary 46N50.
Key words and phrases. Quantum information theory, quantum entropy, $\mathcal{S}$-mixing entropy, Rényi entropy, quantum statistical mechanics, operator algebras.
* Corresponding author.

quantum Rényi entropy, if $0 \leq \alpha < 1$. This result implies if $0 \leq \alpha < 1$ and the quantum Rényi entropy takes the value $+\infty$ then one can measures the complexity of states by using the $\mathcal{S}$-mixing Rényi entropy. Section 5 is devoted to computing our entropy for general states in several reference systems.

2. Decomposition of States

2.1. States on C^*-algebras. We firstly recall the following famous theorem.

Theorem 2.1. (Banach-Alaoglu theorem) *Let X be a normed space and X^* be the set of all bounded linear functionals on X. Then*

$$(X^*)_1 := \{f \in X^* \ ; \ \|f\| \leq 1\} \qquad (\|f\| = \sup_{\|x\|=1} |f(x)|, \ x \in X)$$

is weakly* compact.

Let $\mathcal{A}$ be a unital C^*-algebra (i.e. with the identity $1_{\mathcal{A}}$) and $\mathfrak{S}$ be the set of all states on $\mathcal{A}$. $\mathfrak{S}$ is a closed set with respect to the weakly* topology. Due to Theorem 2.1, $\mathfrak{S}$ is weakly* compact. Moreover, obviously, $\mathfrak{S}$ is a convex set. Therefore, the following theorem holds.

Theorem 2.2. $\mathfrak{S}$ *is a weakly* compact convex set.*

Let $\theta(\mathbb{R})$ be the set of all strongly continuous 1-parameter automorphisms on $\mathcal{A}$. Similarly to Theorem 2.2, the set of all θ-invariant states $I(\theta)$ and (β, θ_t)-KMS states $K_\beta(\theta)$ (β is the inverse temperature) are weakly* compact convex subsets of $\mathfrak{S}$. Therefore, one can apply the decomposition theory introduced in next subsection to these state spaces.

2.2. Barycentric decomposition. In this subsection we briefly review the general decomposition theory of a compact convex set.

Let $\mathbf{S}$ be a locally convex space, $\mathbb{S}$ a nonempty compact convex subset of $\mathbf{S}$, $C(\mathbb{S})$ the space of real continuous functions on $\mathbb{S}$, and $M_+(\mathbb{S})$ the set of all positive Radon measures on $C(\mathbb{S})$.

The Riesz Theorem tells us that there exists a one-to-one correspondence between the positive linear functionals, L, on $C(\mathbf{S})$, and the Radon measures $\mu \in M_+(\mathbb{S})$ which satisfies $\mu(\mathbb{S}) = \|L\|$, such that:

$$L(f) = \int f(\omega) \mathrm{d}\mu(\omega), \qquad f \in C(\mathbb{S}).$$

We identify L with its corresponding μ, and write:

$$\mu(f) = \int f(\omega) \mathrm{d}\mu(\omega), \qquad f \in C(\mathbb{S}),$$

viewing the Radon measures as the positive linear functionals that they generate.

We now recall the notion of Baire measure.

Definition 2.3. The elements of the σ-algebra $\mathcal{B}_0$ generated by the closed G_δ-subsets of $\mathbb{S}$ are called the *Baire sets* of $\mathbb{S}$. A measure μ_0 on $\mathcal{B}_0$ is called a *Baire measure*.

The restriction of a Borel measure on $\mathcal{B}$ to $\mathcal{B}_0$ is a Baire measure. Therefore, a one-to-one correspondence between Radon measures μ, and Borel measures $\mathrm{d}\mu$, and Baire measures $\mathrm{d}\mu_0$ is established:

$$\mu(f) = \int f(\omega)\mathrm{d}\mu(\omega) = \int f(\omega)\mathrm{d}\mu_0(\omega), \qquad f \in C(\mathbb{S}).$$

In the sequel, we will use the term "measure μ" interchangeably as the above three concepts. Denote

$$M_1(\mathbb{S}) := \{\mu\ ;\ \mu \in M_+(\mathbb{S}),\ \mu(\mathbb{S}) = 1\}.$$

Now we introduce the notion of the barycenter of a measure $\mu \in M_1(\mathbb{S})$.

Definition 2.4. Under the above notations, the *barycenter* $b(\mu)$ of $\mu \in M_1(\mathbb{S})$ is given by

$$b(\mu) := \int \omega \mathrm{d}\mu(\omega). \tag{2.1}$$

We let $\psi \in \mathbb{S}$ be equal to the barycenter $b(\mu)$ of $\mu \in M_1(\mathbb{S})$, i.e.

$$\psi = \int \omega \mathrm{d}\mu(\omega). \tag{2.2}$$

In the above equation, we can consider that ψ is decomposed into simpler components of $\mathbb{S}$. This decomposition (2.2) is said to be the *barycentric decomposition* of ψ. We define

$$M_\psi(\mathbb{S}) := \{\mu\ ;\ \mu \in M_1(\mathbb{S}),\ b(\mu) = \psi\}.$$

$M_\psi(\mathbb{S})$ is the set of probability measures μ which represent ψ. In the next subsection, we will show a condition of the existence of measures which represent ψ.

Incidentally, if $\mu \in M_\psi(\mathbb{S})$ has countable support $\{\psi_1, \psi_2, \dots\}$ then μ is written as

$$\mu = \sum_j \lambda_j \delta_{\psi_j}, \qquad \lambda_j \geq 0,\ \sum_j \lambda_j = 1, \tag{2.3}$$

where δ_{ψ_j} is the Dirac measure with support $\{\psi_j\}$. Then the barycenter of μ is given by

$$b(\mu) = \sum_j \lambda_j \psi_j.$$

Under the above assumption, we denote the set of all probability measures with countable support whose barycenter is equal to $\psi \in \mathbb{S}$ by $D_\psi(\mathbb{S})$, i.e.

$$D_\psi(\mathbb{S}) = \{\mu\ ;\ b(\mu) = \sum_j \lambda_j \psi_j = \psi \text{ and } \psi_j \in \mathbb{S} \text{ for all } j\}.$$

2.3. Maximal measures. We first recall the theorem which ensures the existence of the extreme points of compact convex sets.

Theorem 2.5. (Krein-Milman theorem) *For a compact convex subset $\mathbb{S}$ of a locally convex topological vector space $\mathbf{S}$, one has*

(i): $\mathrm{ex}\mathbb{S} \neq \phi$,

(ii): $\mathbb{S} = \overline{\mathrm{co}}(\mathrm{ex}\mathbb{S})$,

where $\overline{\mathrm{co}}(\cdot)$ is the closure of the convex hull of $(\cdot)$.

Next, we state the relation between elements of $\mathbb{S}$ and measures.

Definition 2.6. Let μ and ν be Radon measures on $C(\mathbb{S})$. We write

$$\mu \prec \nu \quad \text{if} \quad \mu(f) \leq \nu(f) \quad \text{for all} \quad f \in C(\mathbb{S}),\ f \geq 0.$$

By Zorn's lemma, one has the following result [18].

Proposition 2.7. *For any $\mu \in M_1(\mathbb{S})$, there exists a maximal measure ν which satisfies*

$$\mu \prec \nu. \tag{2.4}$$

We define the set of all maximal measures (2.4) by $M_1^m(\mathbb{S})$. After these preliminaries we now introduce a condition of the existence of μ satisfying $\psi = b(\mu)$.

Theorem 2.8. *Let $\mathbb{S}$ be a compact convex set. For any $\psi \in \mathbb{S}$, there exists $\mu \in M_1^m(\mathbb{S})$ such that*

$$\psi = b(\mu)$$

and $\mu(Q) = 1$, where Q is a Baire set satisfying $\mathrm{ex}\mathbb{S} \subset Q \subset \mathbb{S}$.

A condition of the uniqueness of μ which satisfies the above condition is as follows.

Theorem 2.9. *There exists a unique maximal measure, for any $\psi \in \mathbb{S}$, if and only if, $\mathbb{S}$ is a Choquet simplex* [4, 7, 18].

Remark 2.10. Another noncommutative decomposition theory based on the Jacobi relation has been studied intensively in recent years [1, 3].

3. $\mathcal{S}$-mixing Rényi Entropy

3.1. $\mathcal{S}$-mixing entropy. Since the set of all states $\mathfrak{S}$ on a unital C^*-algebra $\mathcal{A}$ is a weakly* compact convex set (see theorem 2.2), one can apply the notion of barycentric decomposition to states $\varphi \in \mathfrak{S}$. Moreover, the set of all θ-invariant states (i.e. $\varphi \circ \theta_t = \varphi$) $I(\theta)$ and KMS states $K_\beta(\theta)$ are weakly* compact convex subsets of $\mathfrak{S}$. According to these facts, in the sequel, we will treat the weakly* compact convex set $\mathcal{S}$.

If $\varphi \in \mathcal{S}$ is represented by $\mu \in M_1(\mathcal{S})$ with countable support, then one has the coefficients $\lambda_j \geq 0$ ($\sum_j \lambda_j = 1$) (see (2.3)). From the result, Ohya defined the Shannon-type entropy on the unital C^*-algebra [12].

Definition 3.1. The *$\mathcal{S}$-mixing entropy* of $\varphi \in \mathcal{S}$ is defined as

$$S^{\mathcal{S}}(\varphi) := \begin{cases} \inf\{-\sum_j \lambda_j \log \lambda_j \ ; \ \mu = \sum_j \lambda_j \delta_{\varphi_j} \in D_\varphi(\mathcal{S})\} & (D_\varphi(\mathcal{S}) \neq \phi) \\ +\infty & (D_\varphi(\mathcal{S}) = \phi) \end{cases} \tag{3.1}$$

If φ can not be represented by countable support $\{\varphi_1, \varphi_2, \dots\}$, then the mixed state φ has uncountable support. Therefore, Ohya defined the complexity to be ∞ if $D_\varphi(\mathcal{S}) = \phi$ [13, 15]. This entropy is an extension of the von Neumann entropy [9] to the C^*-algebras.

Theorem 3.2. *Let $\mathcal{A}$ be the set of all bounded operators on a Hilbert space $\mathcal{H}$. If φ is given by the density operator ρ, i.e.*

$$\varphi(\cdot) = \mathrm{Tr}\rho(\cdot)$$

and $D_\varphi(\mathfrak{S}) \neq \phi$, the following equation holds:

$$S^{\mathfrak{S}}(\varphi) = -\mathrm{Tr}\rho \log \rho. \tag{3.2}$$

Remark 3.3. In [8], Mukhamedov and Watanabe formulated a new type of entropy of quantum channels based on the construction of the $\mathcal{S}$-mixing entropy. Using the entropy of channels, they showed several results for entangled states. Besides, the complexity of qubit channels and phase-damping channels were calculated.

3.2. $\mathcal{S}$-mixing Rényi entropy. In [7], we generalized the $\mathcal{S}$-mixing entropy (3.1) to the Rényi-type which depends on a positive real number $\alpha \neq 1$.

Definition 3.4. The $\mathcal{S}$*-mixing Rényi entropy* of order $\alpha \in [0,\infty)\backslash\{1\}$ of $\varphi \in \mathcal{S}$ is defined by

$$S_\alpha^{\mathcal{S}}(\varphi) := \begin{cases} \inf\{(1-\alpha)^{-1}\log\sum_j \lambda_j^\alpha\} \ ; \ \mu = \sum_j \lambda_j \delta_{\varphi_j} \in D_\varphi(\mathcal{S})\} & (D_\varphi(\mathcal{S}) \neq \phi) \\ +\infty & (D_\varphi(\mathcal{S}) = \phi) \end{cases} \tag{3.3}$$

Varying two parameters α and $\mathcal{S}$, we investigate the properties of the $\mathcal{S}$-mixing Rényi entropy. In the sequel, we will assume that $D_\varphi(\mathcal{S}) \neq \phi$.

Note 3.5. Unlike the case of finite-events, the classical Rényi entropy of countably infinite events $X = \{a_1, a_2, \dots\}$ may diverge for $0 \leq \alpha < 1$ [6]. Therefore, we have to carefully check the behavior of the $\mathcal{S}$-mixing Rényi entropy when $0 \leq \alpha < 1$. Hence we define a subset of $D_\varphi(\mathcal{S})$ as follows:

$$D'_\varphi(\mathcal{S}) := \left\{ \mu \in D_\varphi(\mathcal{S}) \ ; \ \exists\alpha \in [0,1),\ \frac{1}{1-\alpha}\log\sum_j \lambda_j^\alpha < \infty \right\}. \tag{3.4}$$

We first show the basic properties of the $\mathcal{S}$-mixing Rényi entropy.

Lemma 3.6. *Assume that*

$$\frac{1}{1-\alpha}\log\sum_j \lambda_j^\alpha < +\infty \qquad (0 \leq \alpha < 1),$$

where $\lambda_j \geq 0$ and $\sum_j \lambda_j = 1$. Then $(1-\alpha)^{-1}\log\sum_j \lambda_j^\alpha$ is a monotone decreasing function with respect to $\alpha \in [0,\infty)\backslash\{1\}$.

Proof. Let

$$\begin{aligned} f(\alpha) &:= \log\sum_j \lambda_j^\alpha \qquad (\alpha \in [0,\infty)\backslash\{1\}), \\ f(1) &= 0. \end{aligned}$$

Next let $0 < \theta < 1$ and $\alpha_1,\ \alpha_2 \neq 1$. Due to the Hölder's inequality, we have

$$\begin{aligned}\sum_j \lambda_j^{\theta\alpha_1+(1-\theta)\alpha_2} &= \sum_j (\lambda_j^{\alpha_1})^{\theta}(\lambda_j^{\alpha_2})^{1-\theta} \\ &\leq (\sum_j \lambda_j^{\alpha_1})^{\theta}(\sum_j \lambda_j^{\alpha_2})^{1-\theta}.\end{aligned}$$

Therefore,

$$f(\theta\alpha_1 + (1-\theta)\alpha_2) \leq \theta f(\alpha_1) + (1-\theta) f(\alpha_2).$$

This implies that f is a convex function. Hence the following different quotient:

$$\frac{f(\alpha) - f(1)}{\alpha - 1} = \frac{f(\alpha)}{\alpha - 1}$$

is a monotone increasing function w.r.t. $\alpha \neq 1$. Thus $(1-\alpha)^{-1} \log \sum_j \lambda_j^{\alpha}$ is monotone decreasing with respect to $\alpha \neq 1$. □

Theorem 3.7. *Let $D'_{\varphi}(\mathcal{S}) \neq \phi$. $S_{\alpha}^{\mathcal{S}}(\varphi)$ is a monotone decreasing function with respect to $\alpha \in [0, \infty)\backslash\{1\}$.*

Proof. Since $(1-\alpha)^{-1} \log \sum_j \lambda_j^{\alpha}$ $(\mu = \sum_j \lambda_j \delta_{\varphi_j} \in D'_{\varphi}(\mathcal{S}))$ is monotonically decreasing for $\alpha \neq 1$ (Lemma 3.6), we obtain the statement by taking the infimum over all $\mu \in D_{\varphi}(\mathcal{S})$. □

The following property does not require the assumption that $D'_{\varphi}(\mathcal{S}) \neq \phi$.

Proposition 3.8. *The $\mathcal{S}$-mixing Rényi entropy of $\varphi \in \mathcal{S}$ has the positivity:*

$$S_{\alpha}^{\mathcal{S}}(\varphi) \geq 0 \tag{3.5}$$

for any $\alpha \in [0, +\infty)\backslash\{1\}$.

Proof. Since

$$\sum_j \lambda_j^{\alpha} > 1 \qquad (0 \leq \alpha < 1),$$

we have

$$\log \sum_j \lambda_j^{\alpha} \geq 0 \qquad (0 \leq \alpha < 1).$$

Hence

$$\frac{1}{1-\alpha} \log \sum_j \lambda_j^{\alpha} \geq 0, \qquad \forall \alpha \in [0, 1). \tag{3.6}$$

On the other hand, if $\alpha \in (1, +\infty)$, then $\sum_j \lambda_j^{\alpha} \leq 1$. This implies that

$$\frac{1}{1-\alpha} \log \sum_j \lambda_j^{\alpha} > 0, \qquad \forall \alpha \in (1, +\infty). \tag{3.7}$$

(3.6) and (3.7) induces that

$$S_{\alpha}^{\mathcal{S}}(\varphi) = \inf_{\{\lambda_j\}} \frac{1}{1-\alpha} \log \sum_j \lambda_j^{\alpha} \geq 0, \qquad \forall \alpha \in [0, +\infty)\backslash\{1\}.$$

□

Recall that the classical Rényi entropy is an extension of the Shannon entropy by $\alpha \neq 1$ [19]. Now we study the relation between the $\mathcal{S}$-mixing Rényi entropy and the $\mathcal{S}$-mixing entropy when $\alpha \to 1$.

Theorem 3.9. *For $\varphi \in \mathcal{S}$ such that $D'_{\varphi}(\mathcal{S}) \neq \phi$,*

$$\lim_{\alpha \nearrow 1} S_{\alpha}^{\mathcal{S}}(\varphi) = S^{\mathcal{S}}(\varphi) \qquad (0 \leq \alpha < 1). \tag{3.8}$$

Proof. Since the $\mathcal{S}$-mixing Rényi entropy is a monotone decreasing function with respect to $\alpha \neq 1$ (Theorem 3.7), we can rewrite

$$\lim_{\alpha \nearrow 1} S_{\alpha}^{\mathcal{S}}(\varphi) = \inf_{0 \leq \alpha < 1} S_{\alpha}^{\mathcal{S}}(\varphi).$$

Therefore,

$$\begin{aligned} \lim_{\alpha \nearrow 1} S_{\alpha}^{\mathcal{S}}(\varphi) &= \inf_{0 \leq \alpha < 1} \inf_{\{\lambda_j\} \in D'_{\varphi}(\mathcal{S})} \frac{1}{1-\alpha} \log \sum_j \lambda_j^{\alpha} \\ &= \inf_{\{\lambda_j\} \in D'_{\varphi}(\mathcal{S})} \inf_{0 \leq \alpha < 1} \frac{1}{1-\alpha} \log \sum_j \lambda_j^{\alpha}. \end{aligned} \tag{3.9}$$

If $\frac{1}{1-\alpha} \log \sum_j \lambda_j^{\alpha} < \infty$ $(0 \leq \alpha < 1)$ then

$$\lim_{\alpha \nearrow 1} \frac{1}{1-\alpha} \log \sum_j \lambda_j^{\alpha} = -\sum_j \lambda_j \log \lambda_j$$

(see [6]). Hence (3.9) is

$$\inf_{\{\lambda_j\} \in D_{\varphi}(\mathcal{S})} -\sum_j \lambda_j \log \lambda_j.$$

Thus there holds

$$\lim_{\alpha \nearrow 1} S_{\alpha}^{\mathcal{S}}(\varphi) = S^{\mathcal{S}}(\varphi).$$

□

Therefore, the $\mathcal{S}$-mixing Rényi entropy is a general extension of the $\mathcal{S}$-mixing entropy by $0 \leq \alpha < 1$. On the other hand, we have the following inequality when $\alpha \searrow 1$.

Theorem 3.10. *For any $\varphi \in \mathcal{S}$,*

$$\lim_{\alpha \searrow 1} S_{\alpha}^{\mathcal{S}}(\varphi) \leq S^{\mathcal{S}}(\varphi) \qquad (\alpha > 1). \tag{3.10}$$

Proof.

$$\begin{aligned} \lim_{\alpha \searrow 1} S_{\alpha}^{\mathcal{S}}(\varphi) &= \sup_{\alpha > 1} S_{\alpha}^{\mathcal{S}}(\varphi) \\ &= \sup_{\alpha > 1} \inf_{\{\lambda_j\}} \frac{1}{1-\alpha} \log \sum_j \lambda_j^{\alpha} \\ &\leq \inf_{\{\lambda_j\}} \sup_{\alpha > 1} \frac{1}{1-\alpha} \log \sum_j \lambda_j^{\alpha} \\ &= S^{\mathcal{S}}(\varphi). \end{aligned}$$

□

4. Quantum Rényi Entropy

The noncommutative-extension of the Rényi entropy is done by Petz, and is called *quantum Rényi entropy* [14], [17]. The quantum Rényi entropy plays important roles in quantum information theory and quantum field theory. In the present section, we show the connections between the quantum Rényi entropy and our $\mathcal{S}$-mixing Rényi entropy.

We briefly review the definition of the quantum Rényi entropy. Let $\mathfrak{S}(\mathcal{H})$ be the set of all density operators on a Hilbert space $\mathcal{H}$.

Definition 4.1. The *quantum Rényi entropy* of order $\alpha \in [0,+\infty)\backslash\{1\}$ of $\rho \in \mathfrak{S}(\mathcal{H})$ is given by

$$S_\alpha(\rho) := \frac{1}{1-\alpha}\log \mathrm{Tr}\rho^\alpha. \tag{4.1}$$

Before proving the following Lemma, we state Karamata inequality [5].

Theorem 4.2. *Let $I \subset \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ a convex function. Let $N \subset \mathbb{N}\cup\{\infty\}$, and $\{x_k\}_{1\le k<N+1}$ and $\{y_k\}_{1\le k<N+1}$ be two sequences of numbers from I, such that:*

- $x_1 \ge x_2 \ge x_3 \ge \dots$ *and* $y_1 \ge y_2 \ge y_3 \ge \dots,$
- *For all* $1 \le k < N$, *we have* $\sum_{i=1}^k x_i \ge \sum_{i=1}^k y_i$,
- $\sum_{i=1}^N x_i = \sum_{k=1}^N y_i < \infty$.

Then we have:

$$\sum_{i=1}^N f(x_i) \ge \sum_{i=1}^N f(y_i). \tag{4.2}$$

If f is concave on I, the inequality (4.2) is reversed.

Lemma 4.3. *Let $\rho = \sum_j \lambda_j\rho_j$ be the decomposition of ρ into pure states. For any $\alpha > 1$, we have*

$$S_\alpha(\rho) \le \frac{1}{1-\alpha}\log\sum_j \lambda_j^\alpha. \tag{4.3}$$

Moreover, if $\rho_n \perp \rho_m$ $(n \ne m)$ then we obtain the equality.

Proof. Let $\rho = \sum_k p_k E_k$ denote a Schatten decomposition (i.e. 1-dimensional orthogonal decomposition) of ρ. We order the eigenvalues from largest to smallest:

$$p_1 \ge p_2 \ge \dots,$$
$$\lambda_1 \ge \lambda_2 \ge \dots.$$

Then

$$\sum_{k=1}^n p_k \ge \sum_{j=1}^n \lambda_j$$

holds for any $n \in \mathbb{N}$ [14]. Since $\alpha > 1$, the function $f(x) = x^\alpha$ is convex on the interval $[0,\infty)$. Thus, applying Karamata inequality, we obtain:

$$\sum_{k=1}^n p_k^\alpha \ge \sum_{j=1}^n \lambda_j^\alpha. \tag{4.4}$$

Applying first log to both sides of this inequality and then dividing both sides by the negative number $1-\alpha$ (since $\alpha > 1$), the inequality reverses, and we obtain inequality (4.3). If $0 < \alpha < 1$, the function $f(x) = x^\alpha$ is concave on $[0, \infty)$. Therefore, Karamata inequality reverses and so, inequality (4.3) is reversed. Applying log to both sides and dividing both sides by positive number $1 - \alpha$ (since $\alpha < 1$), the inequality remains the same, and we prove again inequality (4.3). □

Using this lemma, the $\mathcal{S}$-mixing Rényi entropy is equal to the quantum Rényi entropy in the following setting.

Theorem 4.4. *Assume that $\mathcal{A} = \mathbf{B}(\mathcal{H})$, $\mathfrak{S}$ is the set of all states on $\mathcal{A}$, and a state φ is defined by the density operator ρ on $\mathcal{H}$ as $\varphi = \mathrm{Tr}\rho$. Then we have*

$$S_\alpha^{\mathfrak{S}}(\varphi) = S_\alpha(\rho), \qquad \forall \alpha > 1. \tag{4.5}$$

Proof. Let $\rho = \sum_j \lambda_j \rho_j$ be the decomposition of ρ into pure states ρ_j and

$$\varphi_j(\cdot) := \mathrm{Tr}\rho_j \cdot \ .$$

Then one can see that $\varphi = \sum_j \lambda_j \varphi_j$ is the extremal decomposition of φ. Moreover, if $\varphi = \mathrm{Tr}\rho \in \mathrm{ex}\mathfrak{S}$, then the ρ is a pure state. Hence, by Lemma 4.3,

$$S_\alpha^{\mathfrak{S}}(\varphi) = \inf\{(1-\alpha)^{-1} \log \sum \lambda_j^\alpha\} = S_\alpha(\rho) \qquad (\forall \alpha > 1).$$

□

Therefore, the $\mathcal{S}$-mixing Rényi entropy is a general extension of the quantum Rényi entropy to C^*-algebras when $\alpha > 1$. On the other hand, if $0 \le \alpha < 1$, then we obtain a different result from above.

Theorem 4.5. *Let $D'_\varphi(\mathfrak{S}) \neq \phi$. Under the above assumptions and notations,*

$$S_\alpha^{\mathfrak{S}}(\varphi) \le S_\alpha(\rho), \qquad \forall \alpha \in [0, 1). \tag{4.6}$$

Proof. If $D'_\varphi(\mathfrak{S}) \neq \phi$ and $0 \le \alpha < 1$, then the reversed inequality (4.4) induces that

$$\frac{1}{1-\alpha} \log \sum_j \lambda_j^\alpha \le \frac{1}{1-\alpha} \log \sum_k p_k^\alpha.$$

Taking the infimum over all $\mu \in D_\varphi(\mathfrak{S})$, we obtain the desired statement. □

This inequality tells us that one can obtain the information amount of ρ by using the $\mathcal{S}$-mixing Rényi entropy if $S_\alpha(\rho) = +\infty$.

5. The Complexity of States

We have seen that the properties of the $\mathcal{S}$-mixing Rényi entropy by varying the parameter α. On the other hand, our entropy depends on the reference system $\mathcal{S}$. Therefore, varying the extreme points in $\mathrm{ex}\mathcal{S}$ which decompose φ, one can investigate the complexity of a state φ in detail. In this section, we study the complexity of states by choosing different reference systems $\mathcal{S}$.

First, we recall the theorems on the decomposition of KMS states [4], [14].

Theorem 5.1. *The set of all states $K_\beta(\theta)$ on a C^*-algebra is a Choquet simplex.*

Theorem 5.2. *There exists a unique maximal measure μ which decomposes $\varphi \in K_\beta(\theta)$ into $\mathrm{ex}K_\beta(\theta)$. Then μ is a central measure* [4].

Therefore, a KMS state $\varphi \in K_\beta(\theta)$ is uniquely decomposed into extreme KMS states $\varphi_j \in \mathrm{ex}K_\beta(\theta)$.

Theorem 5.3. *For any KMS states $\varphi \in K_\beta(\theta)$ and any $\alpha \in [0,+\infty)\backslash\{1\}$, the following inequalities hold:*

(1): $S_\alpha^{I(\theta)}(\varphi) \geq S_\alpha^{K(\theta)}(\varphi)$ *when* $D'_\varphi(I(\theta)) \neq \phi$.

(2): $S_\alpha^{\mathfrak{S}}(\varphi) \geq S_\alpha^{K(\theta)}(\varphi)$ *when* $D'_\varphi(\mathfrak{S}) \neq \phi$.

Proof. **(1)**
The decomposition of $\varphi \in K_\beta(\theta)$ into $\mathrm{ex}K_\beta(\theta)$ is unique. We write the (orthogonal) decomposition as

$$\varphi = \sum_n \lambda_n \varphi_n \ ; \ \varphi_n \in \mathrm{ex}K_\beta(\theta), \ \varphi_n \perp \varphi_m \ (n \neq m).$$

On the other hand, since $\mathrm{ex}K_\beta(\theta) \subset I(\theta)$, $\varphi_n \in \mathrm{ex}K_\beta(\theta)$ can be further decomposed into the elements of $\mathrm{ex}I(\theta)$. Let

$$\varphi_n = \sum_j \mu_k^{(n)} \psi_k^{(n)}, \qquad \psi_k^{(n)} \in \mathrm{ex}I(\theta)$$

be the ergodic decomposition. Therefore,

$$\varphi = \sum_{n,k} \lambda_n \mu_k^{(n)} \psi_k^{(n)}, \qquad \psi_k^{(n)} \in \mathrm{ex}I(\theta).$$

Thus

$$D_\varphi(K_\beta(\theta)) \ni \mu = \sum_n \lambda_n \delta_{\varphi_n},$$

$$D_\varphi(I(\theta)) \ni \mu' = \sum_{n,k} \lambda_n \mu_k^{(n)} \delta_{\psi_k^{(n)}}.$$

($\alpha > 1$):
When $\varphi \in K_\beta(\theta)$ is decomposed by $\mathrm{ex}I(\theta)$,

$$(1-\alpha)^{-1} \log \sum_{n,k} \left(\lambda_n \mu_k^{(n)}\right)^\alpha$$

$$= (1-\alpha)^{-1} \log \sum_n (\lambda_n)^\alpha \sum_j \left(\mu_k^{(n)}\right)^\alpha .$$

For all $\alpha > 1$, $\sum_j \left(\mu_k^{(n)}\right)^\alpha < 1$. This implies that

$$\log \sum_n \lambda_n^\alpha \sum_j \left(\mu_k^{(n)}\right)^\alpha \leq \log \sum_n \lambda_n^\alpha \cdot 1$$

$$\Rightarrow \frac{1}{1-\alpha} \log \sum_{n,k} \left(\lambda_n \mu_k^{(n)}\right)^\alpha \geq \frac{1}{1-\alpha} \log \sum_n \lambda_n^\alpha$$

$$\Rightarrow \frac{1}{1-\alpha}\log\sum_{n,k}\left(\lambda_n\mu_k^{(n)}\right)^{\alpha} \geq S_\alpha^{K(\theta)}(\varphi).$$

Taking the infimum over all

$$\mu = \sum_{n,k}\lambda_n\mu_k^{(n)}\delta_{\psi_k^{(n)}} \in D_\varphi(I(\theta)),$$

we obtain

$$S_\alpha^{I(\theta)}(\varphi) \geq S_\alpha^{K(\theta)}(\varphi) \qquad (\forall\alpha > 1). \tag{5.1}$$

($0 \leq \alpha < 1$):

Let $D'_\varphi(I(\theta)) \neq \phi$. From this assumption, take $\mu' = \sum_{n,k}\lambda_n\mu_k^{(n)}\delta_{\psi_k^{(n)}} \in D'_\varphi(I(\theta))$. For all $0 \leq \alpha < 1$,

$$\sum_{n,k}(\lambda_n)^\alpha\left(\mu_k^{(n)}\right)^\alpha \geq \sum_n \lambda_n^\alpha \cdot 1.$$

By the monotonicity of logarithm,

$$\log\sum_{n,k}\left(\lambda_n\mu_k^{(n)}\right)^\alpha \geq \log\sum_n\lambda_n^\alpha.$$

Hence

$$\frac{1}{1-\alpha}\log\sum_{n,k}\left(\lambda_n\mu_k^{(n)}\right)^\alpha \geq \frac{1}{1-\alpha}\log\sum_n\lambda_n^\alpha$$

$$\Rightarrow \frac{1}{1-\alpha}\log\sum_{n,k}\left(\lambda_n\mu_k^{(n)}\right)^\alpha \geq S_\alpha^{K(\theta)}(\varphi)$$

Taking the infimum over all $\mu \in D_\varphi(I(\theta)) \supset D'_\varphi(I(\theta))$, we obtain

$$S_\alpha^{I(\theta)}(\varphi) \geq S_\alpha^{K(\theta)}(\varphi) \qquad (0 \leq \forall\alpha < 1). \tag{5.2}$$

(2)

($\alpha > 1$):

Let

$$\varphi = \sum_n\lambda_n\varphi_n, \qquad \varphi_n \in \mathrm{ex}K_\beta(\theta).$$

Since $\mathrm{ex}K_\beta(\theta) \subset \mathfrak{S}$, φ_n can be decomposed by $\mathrm{ex}\mathfrak{S}$:

$$\varphi_n = \sum_l\tilde{\mu}_l^{(n)}\tilde{\psi}_l^{(n)}, \qquad \tilde{\psi}_l^{(n)} \in \mathrm{ex}\mathfrak{S}.$$

Therefore, for

$$\mu = \sum_{n,l}\lambda_n\tilde{\mu}_l^{(n)}\delta_{\tilde{\psi}_l^{(n)}} \in D_\varphi(\mathfrak{S}),$$

$$\frac{1}{1-\alpha}\log\sum_{n,l}\left(\lambda_n\tilde{\mu}_l^{(n)}\right)^\alpha \geq \frac{1}{1-\alpha}\log_n\lambda_n^\alpha$$

holds in the same way as (1). Taking the infimum over all $\mu \in D_\varphi(\mathfrak{S})$, we have

$$S_\alpha^{\mathfrak{S}}(\varphi) \geq S_\alpha^{K(\theta)}(\varphi) \qquad (\forall\alpha > 1).$$

$(0 \leq \alpha < 1)$:
From the assumption, we take $\mu = \sum_{n,l} \lambda_n \tilde{\mu}_l^{(n)} \in D'_\varphi(\mathfrak{S})$. For any $0 \leq \alpha < 1$,

$$\frac{1}{1-\alpha} \log \sum_{n,l} \left(\lambda_n \tilde{\mu}_l^{(n)} \right)^\alpha \geq S_\alpha^{K(\theta)}(\varphi).$$

Taking the infimum over all $\mu' \in D_\varphi(\mathfrak{S})$, the proof is completed.

□

Moreover, if the pair $(\mathcal{A}, \theta(\mathbb{R}))$ has $\mathbb{R}$-*abelianess*, we can discuss the inequality between $S_\alpha^{I(\theta)}(\varphi)$ and $S_\alpha(\varphi)$. Thus, we recall the definition below.

Let $(\mathcal{H}_\varphi, \pi_\varphi, x_\varphi)$ be the GNS-representation defined by $\varphi \in \mathfrak{S}(\mathcal{A})$ and $\{u_t^\varphi\ ;\ t \in \mathbb{R}\}$ be the strongly continuous unitary group on $\mathcal{H}_\varphi$.

Definition 5.4. Let E_φ be a projection from $\mathcal{H}_\varphi$ to the set of u_t^φ-invariant vectors. If $E_\varphi \pi_\varphi(\mathcal{A})'' E_\varphi$ is a commutative von Neumann algebra, $(\mathcal{A}, \theta(\mathbb{R}))$ is called $\mathbb{R}$-*abelian* for φ.

Furthermore, we mention the following theorem. Let $\mathcal{O}_\varphi(I(\theta))$ be the set of all orthogonal measures [4] whose barycenters are $\varphi \in I(\theta)$.

Theorem 5.5. *For $\varphi \in I(\theta)$, the followings are satisfied:*

- **(i):** *There exists $\mu \in \mathcal{O}_\varphi(I(\theta))$ whose pseudo-support* [4] *is* $\mathrm{ex}I(\theta)$.
- **(ii):** *If $(\mathcal{A}, \theta(\mathbb{R}))$ is $\mathbb{R}$-abelian, $I(\theta)$ is a Choquet simplex. Therefore, the above μ is a unique maximal measure.*

Now we prove the following inequalities.

Theorem 5.6. *Let $(\mathcal{A}, \theta(\mathbb{R}))$ be $\mathbb{R}$-abelian for $\varphi \in K_\beta(\theta)$,*

$$D'_\varphi(\mathfrak{S}) \neq \phi \quad \text{and} \quad D'_\varphi(I(\theta)) \neq \phi.$$

We have

$$S_\alpha^{\mathfrak{S}}(\varphi) \geq S_\alpha^{I(\theta)}(\varphi) \geq S_\alpha^{K(\theta)}(\varphi) \tag{5.3}$$

for any $\alpha \in [0, \infty) \backslash \{1\}$.

Proof. According to Theorem 5.5, the ergodic decomposition of φ is unique. Then φ has the unique maximal measure $\mu \in \mathcal{O}_\varphi(I(\theta))$ whose pseudo-support is $\mathrm{ex}I(\theta)$. Hence we can obtain the first inequality using the same way we proved Theorem 5.3. The second inequality holds from the results of Theorem 5.3 (1). □

Acknowledgments

We are heartily grateful to Professor Hiai whose support enabled us to build up a comprehensive treatment of the subject. We would also like to thank anonymous referees for their useful suggestions which allowed us to improve the content of the paper.

References

1. Accardi, L., Bożejko, M.: Interacting Fock Spaces and Gaussianization of Probability Measures, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **1** (1998), 4, 663-670.
2. Accardi, L., Frigerio, A.: Markovian cocycles, *Proc. Royal Irish Acad.* **83A** (1983), 2, 251-263.
3. Accardi, L., Kuo, H. H., Stan, A.: A Combinatorial Identity and Its Application to Gaussian Measures, in: *Quantum Probability and Infinite Dimensional Analysis: From Foundations to Applications* (Singapore: World Scientific), (2005) 1-11.
4. Bratteli, O., Robinson, D. W.: *Operator Algebras and Quantum Statistical Mechanics I*, Springer, New York, 1987.
5. Karamata, J.: Sur une inégalité relative aux fonctions convexes, *Publ. Math. Univ. Belgrade* **1** (1932), 145148.
6. Kovačevć, M., Stanojević, I., Šenk, V.: Some Properties of Rényi Entropy over Countably Infinite Alphabets, *Probl. Inf. Transm.* **49** (2013), 2, 99110.
7. Mukhamedov, F., Ohmura, K., Watanabe, N.: A formulation of Rényi entropy on C^*-algebras, *Quantum Information Processing* **18** (2019), 10, 318.
8. Mukhamedov, F., Watanabe, N.: On S-mixing entropy of quantum channels, *Quantum Inf. Process.* **17** (2018), 148-168.
9. von Neumann, J.: *Die Mathematischen Grundlagen der Quantenmechanik*, Springer, Berlin, 1932.
10. Ohmura, K., Watanabe, N.: Quantum Dynamical Mutual Entropy Based on AOW Entropy, *Open Systems & Information Dynamics* **26** (2019), 2, 1950009-1-1950009-16.
11. Ohmura, K., Watanabe, N.: On transmission efficiency of quantum modulations, *Quantum Studies: Mathematics and Foundations* (to appear).
12. Ohya, M.: Entropy transmission in C^*-dynamical systems, *J. Math. Anal. Appl.* **100** (1984), 222-235.
13. Ohya, M., Petz, D.: *Quantum Entropy and its Use*, Springer, Berlin, 1993.
14. Ohya, M., Umegaki, H.: *Quantum Theoretical Entropy*, Kyoritsu Pub., Tokyo, 1984.
15. Ohya, M., Watanabe, N.: *Foundation of Quantum Communication Theory*, Makino Pub. Co., Tokyo, 1998.
16. Petz, D.: Quasi-entropies for states of a von Neumann algebra, *Publ. RIMS KyotoUniv.* **23** (1985), 787-800.
17. Petz, D.: *Quantum Information Theory and Quantum Statistics*, Springer, Berlin, 2008.
18. Phelps, R. R.: *Lecture on Choquet's Theorem*, Van Nostrand, 1966.
19. Rényi, A.: On the foundations of information theory, *Rev. Int. Stat. Inst.* **33** (1965), 1-14.
20. Schatten, R.: *Norm Ideals of Completely Continuous Operators*. Springer, Berlin (1970).
21. Shannon, C. E.: Mathematical theory of communication, *Bell Systems Tech. J.* **27** (1948), 379-423 and 623-656.
22. Watanabe, N.: On quantum dynamical entropy for open systems, *Int. J. Quantum Inf.* **14** (2016), 4, 1640005-1-1640005-11.

Farrukh Mukhamedov: Department of Mathematical Sciences, College of Science, United Arab Emirates University, 15551 Al-Ain, United Arab Emirates

E-mail address: far75m@yandex.ru ; farrukh.m@uaeu.ac.ae

Kyouhei Ohmura: Department of Information Sciences, Tokyo University of Science, Noda, Chiba 278-8510, Japan

E-mail address: ohmura.kyouhei@gmail.com

Noboru Watanabe: Department of Information Sciences, Tokyo University of Science, Noda, Chiba 278-8510, Japan

E-mail address: watanabe@is.noda.tus.ac.jp