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$\mathcal{R}(p,q)-$ ANALOGS OF DISCRETE DISTRIBUTIONS: GENERAL FORMALISM AND APPLICATIONS

MAHOUTON NORBERT HOUNKONNOU* AND FRIDOLIN MELONG

Dedicated to Professor Leonard Gross on the occasion of his 88th birthday

Abstract. In this paper, we define and discuss $\mathcal{R}(p,q)$- deformations of orthogonal polynomials, basic univariate discrete distributions of the probability theory. We focus mainly on binomial, Euler, Pólya and inverse Pólya distributions. We discuss relevant $\mathcal{R}(p,q)-$ deformed factorial moments of a random variable, and establish associated expressions of mean and variance. Furthermore, we derive recurrence relations for the probability distributions. Other known results in the literature are also recovered as particular cases.

1. Introduction

Deformed quantum algebras, namely $q-$ deformed algebras [16, 21, 22] and their extensions to $(p,q)-$analogs [4, 5], continue to attract much attention in mathematics and physics. Inspired by the $q-$ deformed quantum algebras, Chung and Kang developed a notion of $q-$ permutations and $q-$ combinations [9]. The $q-$ combinatorics and $q-$ hypergeometric series were examined in [1, 3, 8, 18]. A theory of $(p,q)-$ analogs of binomial coefficients and $(p,q)-$ Stirling numbers was elaborated in [10, 24]. In the same vein, from the $q-$ combinatorics, several authors investigated the $q-$ discrete probability distributions such as $q-$ binomial, $q-$ Euler, $q-$ hypergeometric, $q-$ Pólya, $q-$ contagious, and $q-$ uniform distributions [7, 17, 20]. Recently, our research group was interested in the study of quantum algebras, and, more specifically, in the generalization of the well-known $(p,q)-$ Heisenberg algebras. From Odzijewicz's work [21], we introduced the $\mathcal{R}(p,q)-$ deformed quantum algebras as generalizations of known deformed quantum algebras [13]. In [12], we performed the $\mathcal{R}(p,q)-$ differentiation and integration, and deduced all relevant particular cases of $q-$ and $(p,q)-$ deformations. This opens a novel route for developing the theory of $\mathcal{R}(p,q)-$ analogs of special numbers, combinatorics, and probability distributions. Then, the following question naturally arises: How to construct discrete probability distributions and deduce their properties from $\mathcal{R}(p,q)-$ deformed quantum algebras?

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This paper is organized as follows. In section 2, we introduce $\mathcal{R}(p,q)-$ deformed numbers, orthogonal polynomials, deformed factorials, binomial coefficients, derivative, and some basic notations and definitions. Section 3 is focused on $\mathcal{R}(p,q)-$ deformed combinatorics and analogs of discrete distributions. The mean value, or expectation, and variance of random variables of distributions are computed. Other known results are also recovered as particular cases. We end with some concluding remarks in Section 4.

2. $\mathcal{R}(p,q)-$ Deformed Numbers, Orthogonal Polynomials and Derivatives

2.1. $\mathcal{R}(p,q)-$ deformed numbers. Consider two real numbers p and q, such that $0<q<p\leq 1$, and a given rational function

$$\mathcal{R}(u,v)=\frac{\mathcal{P}(u,v)}{\mathcal{Q}(u,v)},$$

where $\mathcal{P}$ and $\mathcal{Q}$ are polynomial functions. We assume that $\mathcal{Q}(u,v)\neq 0$, for all $(u,v)\in D^\star\times D^\star$, where $D^\star:=\{z\in\mathbb{C}:|z|\leq 1, z\neq 0\}$. Furthermore, we assume that $\mathcal{R}(p^n,q^n)>0,\forall n\in\mathbb{N}$, and $\mathcal{R}(1,1)=0$.

Define the $\mathcal{R}(p,q)-$ deformed numbers by [13]:

$$[n]_{\mathcal{R}(p,q)}:=\mathcal{R}(p^n,q^n),\quad n\in\mathbb{N}. \tag{2.1}$$

These numbers are the principal Szegö-Jacobi parameters, $\{w_n\}_{n\geq 1}$ [23], of a random variable X, having finite moments of all orders, whose orthogonal polynomials with leading coefficient equal to 1, $\{f_n\}_{n\geq 0}$, satisfy the following recursive relation:

$$\big(X-\alpha_n\big)f_n(X)=f_{n+1}(X)+[n]_{\mathcal{R}(p,q)}f_{n-1}(X), \tag{2.2}$$

for all $n\geq 0$; with $f_0(X)=1,\quad f_{-1}(X)=0$ by convention. It follows that

$$f_1(X)=X-\mu_1, \tag{2.3}$$

where μ_1 is the first order moment of the random variable X. The Szegö-Jacobi parameters $\{\alpha_n\}_{n\geq 0}$ and the polynomials $\{f_n\}_{n\geq 1}$ can be determined recursively by the relation (2.2). As a matter of illustration, for $n=2$, we find

$$f_2(X)=X^2-\big(\mu_1+\alpha_1\big)X+\alpha_1\mu_1-\mathcal{R}(p,q), \tag{2.4}$$

and

$$\alpha_1=\frac{\mu_3-2\mu_1\mu_2+\mu_1^3}{\mu_2-\mu_1^2}. \tag{2.5}$$

The first and second order moments obey the relation: $\mu_2-\mu_1^2=\mathcal{R}(p,q)$.

For the three main $\mathcal{R}(p,q)-$ deformed distributions investigated in this paper (see Definitions 3.5, 3.6 and 3.7 in the sequel), we get:

(i) $\mathcal{R}(p,q)-$ deformed binomial distribution

$$f_0(X)=1,\quad f_1(X)=X-p_0\,\mathcal{R}(p^n,q^n),$$

and

$$f_2(X)=X^2-\Big(p_0\,\mathcal{R}(p^n,q^n)+\alpha_1\Big)X+\alpha_1p_0\,\mathcal{R}(p^n,q^n)-\mathcal{R}(p,q).$$

(ii) $\mathcal{R}(p,q)-$ deformed Euler distribution

$$f_0(X)=1,\quad f_1(X)=X-\theta\, E_{\mathcal{R}(p,q)}(-\theta)\, e_{\mathcal{R}(p,q)}(\epsilon_1\theta)$$

and

$$\begin{aligned} f_2(X) &= X^2-\Big(\theta\, E_{\mathcal{R}(p,q)}(-\theta)\, e_{\mathcal{R}(p,q)}(\epsilon_1\theta)+\alpha_1\Big)X+\\ &+ \alpha_1\,\theta\, E_{\mathcal{R}(p,q)}(-\theta)\, e_{\mathcal{R}(p,q)}(\epsilon_1\theta)-\mathcal{R}(p,q). \end{aligned}$$

(iii) $\mathcal{R}(p,q)-$ deformed Pólya distribution

$$f_0(X)=1,\quad f_1(X)=X-\frac{[n]_{\mathcal{R}(p^{-x},q^{-x})}\,[m]_{\mathcal{R}(p^{-x},q^{-x})}}{[m+u]_{\mathcal{R}(p^{-x},q^{-x})}}$$

and

$$\begin{aligned} f_2(X) &= X^2-\Big(\frac{[n]_{\mathcal{R}(p^{-x},q^{-x})}\,[m]_{\mathcal{R}(p^{-x},q^{-x})}}{[m+u]_{\mathcal{R}(p^{-x},q^{-x})}}+\alpha_1\Big)X\\ &+ \alpha_1\,\frac{[n]_{\mathcal{R}(p^{-x},q^{-x})}\,[m]_{\mathcal{R}(p^{-x},q^{-x})}}{[m+u]_{\mathcal{R}(p^{-x},q^{-x})}}-\mathcal{R}(p,q). \end{aligned}$$

The characterization of these distributions is given in the next section.

The $\mathcal{R}(p,q)-$ deformed numbers (2.1) are generalizing known particular numbers like the following:

- $q-$ Arick-Coon-Kuryskin deformation [2]

$$\mathcal{R}(u,v):=\mathcal{R}(1,v)=\frac{1-v}{1-q},\quad\text{and}\quad [n]_q=\frac{1-q^n}{1-q}.$$

- $(p,q)-$ Jagannathan-Srinivasa deformation [15]

$$\mathcal{R}(u,v)=\frac{u-v}{p-q},\quad\text{and}\quad [n]_{p,q}=\frac{p^n-q^n}{p-q}.$$

Define now the $\mathcal{R}(p,q)-$ deformed factorials (see [12] and [13])

$$[n]!_{\mathcal{R}(p,q)}:=\begin{cases}1 & \text{for}\quad n=0\\ \\ \mathcal{R}(p,q)\cdots\mathcal{R}(p^n,q^n) & \text{for}\quad n\geq 1,\end{cases}$$

and the $\mathcal{R}(p,q)-$ deformed binomial coefficients

$$\left[\begin{array}{c} m\\ n\end{array}\right]_{\mathcal{R}(p,q)}:=\frac{[m]!_{\mathcal{R}(p,q)}}{[n]!_{\mathcal{R}(p,q)}[m-n]!_{\mathcal{R}(p,q)}},\quad m,n\in\mathbb{N}\cup\{0\},\quad m\geq n$$

satisfying the relation

$$\left[\begin{array}{c} m\\ n\end{array}\right]_{\mathcal{R}(p,q)}=\left[\begin{array}{c} m\\ m-n\end{array}\right]_{\mathcal{R}(p,q)},\quad m,n\in\mathbb{N}\cup\{0\},\quad m\geq n.$$

We consider the following linear operators acting as follows (see [12] and [13]):

$$\begin{aligned} Q: f_n\longmapsto Qf_n(z): &= f_n(qz),\\ P: f_n\longmapsto Pf_n(z): &= f_n(pz), \end{aligned}$$

and the $\mathcal{R}(p,q)-$ derivative given by:

$$\partial_{\mathcal{R}(p,q)}:=\partial_{p,q}\frac{p-q}{P-Q}\mathcal{R}(P,Q)=\frac{p-q}{p^P-q^Q}\mathcal{R}(p^P,q^Q)\partial_{p,q}$$

extending known derivatives as follows:

(i) $q-$ Heine derivative [11]

$$\partial_q f_n(z) = \frac{f_n(z) - f_n(qz)}{z(1-q)}.$$

(ii) $q-$ Quesne derivative [22]

$$\partial_q f_n(z) = \frac{f_n(z) - f_n(q^{-1}z)}{z(q-1)}.$$

(iii) $(p,q)-$ Jagannathan-Srinivasa derivative [15]

$$\partial_{p,q} f_n(z) = \frac{f_n(pz) - f_n(qz)}{z(p-q)}.$$

(iv) $(p^{-1},q)-$ Chakrabarty - Jagannathan derivative [5]

$$\partial_{p^{-1},q} f_n(z) = \frac{f_n(p^{-1}z) - f_n(qz)}{z(p^{-1}-q)}.$$

(v) Hounkonnou-Ngompe generalization of $q-$ Quesne derivative [14]

$$\partial_{p,q} f_n(z) = \frac{f_n(pz) - f_n(q^{-1}z)}{z(q-p^{-1})}.$$

The algebra associated with the $\mathcal{R}(p,q)-$ deformation is a quantum algebra, denoted $\mathcal{A}_{\mathcal{R}(p,q)}$, generated by the set of operators $\{1, A, A^\dagger, N\}$ satisfying the following commutation relations:

$$AA^\dagger = [N+1]_{\mathcal{R}(p,q)}, \qquad A^\dagger A = [N]_{\mathcal{R}(p,q)}.$$
$$[N,\ A] = -A, \qquad \left[N,\ A^\dagger\right] = A^\dagger$$

with its realization on $\mathcal{O}(\mathbb{D}_R)$ given by:

$$A^\dagger := z, \qquad A := \partial_{\mathcal{R}(p,q)}, \qquad N := z\partial_z,$$

where $\partial_z := \frac{\partial}{\partial z}$ is the usual derivative on $\mathbb{C}$.

3. Characterization of $\mathcal{R}(p,q)-$ Deformed Combinatorics, Binomial Distribution, Euler Distribution and Pólya Distribution

3.1. $\mathcal{R}(p,q)-$ **deformed combinatorics.** Given the above results, we now introduce functions $\epsilon_i > 0$, with $i = 1, 2$, characterizing the previous particular deformations as follows:

(i) $q-$ Arick-Coon-Kuryskin deformation

$$\epsilon_1 = 1 \quad \text{and} \quad \epsilon_2 = q.$$

(ii) $(p,q)-$ Jagannathan-Srinivasa deformation

$$\epsilon_1 = p \quad \text{and} \quad \epsilon_2 = q.$$

We define the $j^{th}-$ order factorial of the $\mathcal{R}(p,q)-$ number by the factor:

$$[n]_{j,\mathcal{R}(p,q)} := \prod_{v=0}^{j-1}\mathcal{R}(p^{n-v},q^{n-v}), \quad j\in\mathbb{N}, \quad n\in\mathbb{R}$$

and the $\mathcal{R}(p,q)-$ deformed shifted factorial by:

$$\big(x\oplus y\big)^n_{\mathcal{R}(p,q)} := \prod_{i=1}^{n}\big(x\,\epsilon_1^{i-1} + y\,\epsilon_2^{i-1}\big), \quad \text{with} \quad \big(x\oplus y\big)^0_{\mathcal{R}(p,q)} := 1,$$

where $x,y\in\mathbb{N}$, while the $\mathcal{R}(p,q)-$ deformed Euler formula is developed as:

$$\big(x\oplus y\big)^n_{\mathcal{R}(p,q)} := \sum_{\kappa=0}^{n}\left[\begin{array}{c} n \\ \kappa \end{array}\right]_{\mathcal{R}(p,q)} \epsilon_1^{\binom{n-\kappa}{2}}\,\epsilon_2^{\binom{\kappa}{2}}\,x^{n-\kappa}\,y^{\kappa}, \quad x,y\in\mathbb{N}.$$

Similarly, we get

$$\big(x\ominus y\big)^n_{\mathcal{R}(p,q)} := \prod_{i=1}^{n}\big(x\,\epsilon_1^{i-1} - y\,\epsilon_2^{i-1}\big), \quad \text{with} \quad \big(x\ominus y\big)^0_{\mathcal{R}(p,q)} := 1,$$

where $x,y\in\mathbb{N}$, and

$$\big(x\ominus y\big)^n_{\mathcal{R}(p,q)} := \sum_{\kappa=0}^{n}\left[\begin{array}{c} n \\ \kappa \end{array}\right]_{\mathcal{R}(p,q)} (-1)^k\epsilon_1^{\binom{n-\kappa}{2}}\,\epsilon_2^{\binom{\kappa}{2}}\,x^{n-\kappa}\,y^{\kappa}, \quad x,y\in\mathbb{N}.$$

The case of $q-$factorial in [7, 20] corresponds to $\mathcal{R}(1,q)=1$, $\epsilon_1=1$ and $\epsilon_2=q$ leading to :

- The $j^{th}-$ order factorial of the $q-$ number:

$$[n]_{j,q} := \prod_{v=0}^{j-1}[n-v]_q, \quad j\in\mathbb{N}, \quad n\in\mathbb{R}.$$

- The $q-$ deformed shifted factorial:

$$\big(x\oplus y\big)^n_q := \prod_{i=1}^{n}\big(x + y\,q^{i-1}\big), \quad \text{with} \quad \big(x\oplus y\big)^0_q := 1, \quad x,y\in\mathbb{N}.$$

- The $q-$ deformed Euler formula:

$$\big(x\oplus y\big)^n_q := \sum_{\kappa=0}^{n}\left[\begin{array}{c} n \\ \kappa \end{array}\right]_q q^{\binom{\kappa}{2}}\,x^{n-\kappa}\,y^{\kappa}, \quad x,y\in\mathbb{N}.$$

Theorem 3.1. *The $\mathcal{R}(p,q)-$ deformed Vandermonde's formula is given by:*

$$[u+v]_{n,\mathcal{R}(p,q)} = \sum_{\kappa=0}^{n}\left[\begin{array}{c} n \\ \kappa \end{array}\right]_{\mathcal{R}(p,q)} \epsilon_1^{\kappa(v-n+\kappa)}\epsilon_2^{(n-\kappa)(u-\kappa)}[u]_{\kappa,\mathcal{R}(p,q)}[v]_{n-\kappa,\mathcal{R}(p,q)}, \tag{3.1}$$

or, equivalently,

$$[u+v]_{n,\mathcal{R}(p,q)} = \sum_{\kappa=0}^{n}\left[\begin{array}{c} n \\ \kappa \end{array}\right]_{\mathcal{R}(p,q)} \epsilon_1^{(n-\kappa)(u-\kappa)}\epsilon_2^{\kappa(v-n+\kappa)}[u]_{\kappa,\mathcal{R}(p,q)}[v]_{n-\kappa,\mathcal{R}(p,q)}, \tag{3.2}$$

where $n\in\mathbb{N}$.

Proof. For $n \in \mathbb{N}$, let us set:

$$T_n(u;v)_{\mathcal{R}(p,q)} := \sum_{\kappa=0}^{n} \left[\begin{array}{c} n \\ \kappa \end{array} \right]_{\mathcal{R}(p,q)} \epsilon_1^{\kappa(v-n+\kappa)} \epsilon_2^{(n-\kappa)(u-\kappa)} [u]_{\kappa,\mathcal{R}(p,q)} [v]_{n-\kappa,\mathcal{R}(p,q)}.$$

For $n = 1$, we have $T_1(u;v)_{\mathcal{R}(p,q)} = [u+v]_{\mathcal{R}(p,q)}$. Using the recursion relation

$$\left[\begin{array}{c} n \\ \kappa \end{array} \right]_{\mathcal{R}(p,q)} = \epsilon_1^{\kappa} \left[\begin{array}{c} n-1 \\ \kappa \end{array} \right]_{\mathcal{R}(p,q)} + \epsilon_2^{n-\kappa} \left[\begin{array}{c} n-1 \\ \kappa-1 \end{array} \right]_{\mathcal{R}(p,q)}, \quad \kappa \in \mathbb{N},$$

and

$$\begin{aligned} [u+v-n+1]_{\mathcal{R}(p,q)} [u]_{\kappa,\mathcal{R}(p,q)} [v]_{n-\kappa-1,\mathcal{R}(p,q)} &= \epsilon_2^{u-\kappa} [u]_{\kappa,\mathcal{R}(p,q)} [v]_{n-\kappa,\mathcal{R}(p,q)} \\ &+ \epsilon_1^{v-n+\kappa+1} [u]_{\kappa+1,\mathcal{R}(p,q)} [v]_{n-\kappa-1,\mathcal{R}(p,q)}, \end{aligned}$$

we obtain the first-order recursion relation

$$\begin{aligned} T_n(u;v)_{\mathcal{R}(p,q)} &= \sum_{\kappa=0}^{n-1} \left[\begin{array}{c} n-1 \\ \kappa \end{array} \right]_{\mathcal{R}(p,q)} \epsilon_1^{\kappa(v-n+\kappa+1)} \epsilon_2^{(n-\kappa)(u-\kappa)} [u]_{\kappa,\mathcal{R}(p,q)} [v]_{n-\kappa,\mathcal{R}(p,q)} \\ &+ \sum_{\kappa=0}^{n-1} \left[\begin{array}{c} n-1 \\ \kappa \end{array} \right]_{\mathcal{R}(p,q)} \epsilon_1^{(\kappa+1)(v-n+\kappa+1)} \epsilon_2^{(n-\kappa-1)(u-\kappa)} \\ &\times [u]_{\kappa+1,\mathcal{R}(p,q)} [v]_{n-\kappa-1,\mathcal{R}(p,q)} \\ &= \big[u+v-n+1\big]_{\mathcal{R}(p,q)} T_{n-1}(u;v)_{\mathcal{R}(p,q)}, \end{aligned}$$

with $T_1(u;v)_{\mathcal{R}(p,q)} = [u+v]_{\mathcal{R}(p,q)}$. Recursively, it follows that $T_n(u;v)_{\mathcal{R}(p,q)} = [u+v]_{n,\mathcal{R}(p,q)}$ giving (3.1). Finally, interchanging u and v, and replacing κ by $n-\kappa$, the expression (3.1) can be rewritten in the form (3.2). □

Taking $\mathcal{R}(1,q) = 1$, $\epsilon_1 = 1$ and $\epsilon_2 = q$, the $q-$ deformed Vandermonde's formula obtained in [6, 7] is recovered as:

$$\begin{aligned} [u+v]_{n,q} &= \sum_{\kappa=0}^{n} \left[\begin{array}{c} n \\ \kappa \end{array} \right]_q q^{(n-\kappa)(u-\kappa)} [u]_{\kappa,q} [v]_{n-\kappa,q} \\ &= \sum_{\kappa=0}^{n} \left[\begin{array}{c} n \\ \kappa \end{array} \right]_q q^{\kappa(v-n+\kappa)} [u]_{\kappa,q} [v]_{n-\kappa,q}, \quad n \in \mathbb{N}. \end{aligned}$$

Theorem 3.2. *The negative $\mathcal{R}(p,q)-$deformed Vandermonde's formula is given by:*

$$[u+v]_{-n,\mathcal{R}(p,q)} = \sum_{\kappa=0}^{\infty} \left[\begin{array}{c} -n \\ \kappa \end{array} \right]_{\mathcal{R}(p,q)} \epsilon_1^{\kappa(v+n+\kappa)} \epsilon_2^{(-n-\kappa)(u-\kappa)} [u]_{\kappa,\mathcal{R}(p,q)} [v]_{-n-\kappa,\mathcal{R}(p,q)},$$

or

$$[u+v]_{-n,\mathcal{R}(p,q)} = \sum_{\kappa=0}^{\infty} \left[\begin{array}{c} -n \\ \kappa \end{array} \right]_{\mathcal{R}(p,q)} \epsilon_1^{(-n-\kappa)(u-\kappa)} \epsilon_2^{\kappa(v+n+\kappa)} [u]_{\kappa,\mathcal{R}(p,q)} [v]_{-n-\kappa,\mathcal{R}(p,q)},$$

where $n \in \mathbb{N}$.

Proof. It is proved in a similar way as Theorem 3.1. □

These theorems are used to formulate the Pólya and inverse Pólya distributions in the sequel.

We recover the negative $q-$ Vandermonde's formula obtained in [7] by taking $\mathcal{R}(1,q)=1$, $\epsilon_1=1$ and $\epsilon_2=q$:

$$\begin{aligned}[x+y]_{-n,q} &= \sum_{\kappa=0}^{n} q^{-(n-\kappa)(x-\kappa)} \begin{bmatrix} -n \\ \kappa \end{bmatrix}_q [x]_{\kappa,q}\,[y]_{-n-\kappa,q}, \quad |q^{(x+y+1)}|<1 \\ &= \sum_{\kappa=0}^{n} q^{\kappa)(y+n+\kappa)} \begin{bmatrix} -n \\ \kappa \end{bmatrix}_q [x]_{\kappa,q}\,[y]_{-n-\kappa,q}, \ |q^{-(x+y+1)}|<1,\end{aligned}$$

where $0<q<1$. The following relations hold:

$$\frac{1}{[v]_{n,\mathcal{R}(p,q)}} = \sum_{\kappa=0}^{\infty} \left[\begin{array}{c} n+\kappa-1 \\ \kappa \end{array}\right]_{\mathcal{R}(p,q)} \epsilon_1^{n(u-\kappa)}\epsilon_2^{\kappa(v-n+1)} \frac{[u]_{\kappa,\mathcal{R}(p,q)}}{[u+v]_{n+\kappa,\mathcal{R}(p,q)}} \tag{3.3}$$

and

$$\frac{1}{[v]_{n,\mathcal{R}(p,q)}} = \sum_{\kappa=0}^{\infty} \left[\begin{array}{c} n+\kappa-1 \\ \kappa \end{array}\right]_{\mathcal{R}(p,q)} \epsilon_1^{\kappa(v-n+1)}\epsilon_2^{n(u-\kappa)} \frac{[u]_{\kappa,\mathcal{R}(p,q)}}{[u+v]_{n+\kappa,\mathcal{R}(p,q)}}, \tag{3.4}$$

where $u,v\in\mathbb{R}$.

Setting $\epsilon_1=1$ and $\epsilon_2=q$ provides the $q-$analogs of the formulae (3.3) and (3.4) of [8] as :

$$\begin{aligned}\frac{1}{[v]_{n,q}} &= \sum_{\kappa=0}^{\infty} \left[\begin{array}{c} n+\kappa-1 \\ \kappa \end{array}\right]_q q^{\kappa(v-n+1)} \frac{[u]_{\kappa,q}}{[u+v]_{n+\kappa,q}}, \quad |q^v|<1 \\ &= \sum_{\kappa=0}^{\infty} \left[\begin{array}{c} n+\kappa-1 \\ \kappa \end{array}\right]_q q^{n(u-\kappa)} \frac{[u]_{\kappa,q}}{[u+v]_{n+\kappa,q}}, \quad |q^{-v}|<1.\end{aligned}$$

Definition 3.3. The noncentral $\mathcal{R}(p,q)-$ Stirling numbers of the first and second kinds, $s_{\mathcal{R}(p,q)}(n,\kappa,j)$ and $S_{\mathcal{R}(p,q)}(n,\kappa,j)$ are defined via the relations:

$$[x-j]_{n,\mathcal{R}(p,q)} := \epsilon_2^{-\binom{n}{2}-j\,n} \sum_{\kappa=0}^{n} s_{\mathcal{R}(p,q)}(n,\kappa;j)\,[x]^{\kappa}_{\mathcal{R}(p,q)},$$

and

$$[x]^n_{\mathcal{R}(p,q)} := \sum_{\kappa=0}^{n} \epsilon_2^{\binom{\kappa}{2}+j\,\kappa} S_{\mathcal{R}(p,q)}(n,\kappa;j)\,[x-j]_{\kappa,\mathcal{R}(p,q)},$$

where $n\in\mathbb{N}\cup\{0\}$ and $x\in\mathbb{R}$.

This definition serves to introduce the moment factorials.

For $j=0$, we obtain $s_{\mathcal{R}(p,q)}(n,\kappa;0)=s_{\mathcal{R}(p,q)}(n,\kappa)$ and $S_{\mathcal{R}(p,q)}(n,\kappa;0)=S_{\mathcal{R}(p,q)}(n,\kappa)$, which are the $\mathcal{R}(p,q)-$ deformed Stirling numbers of the first and second kinds, respectively.

It is worth noticing that by setting $\mathcal{R}(1,q)=1$, $\epsilon_1=1$ and $\epsilon_2=q$, we recover the noncentral $q-$ Stirling numbers of the first and second kinds, $s_q(n,\kappa,j)$ and $S_q(n,\kappa,j)$, derived in [6, 7]:

$$[x-j]_{n,q} = q^{-\binom{n}{2}-j\,n} \sum_{\kappa=0}^{n} s_q(n,\kappa;j)\,[x]^{\kappa}_q,$$

and

$$[x]_q^n = \sum_{\kappa=0}^{n} q^{\binom{\kappa}{2}+j\,\kappa}\, S_q(n,\kappa;j)\,[x-j]_{\kappa,q},$$

where $n \in \mathbb{N}\cup\{0\}$ and $x \in \mathbb{R}$.

Let us provide now some characterizations of $\mathcal{R}(p,q)-$ deformed binomial, Euler and Pólya distributions. Specifically, we will focus on the deformation of factorial moments, mean values and variances of the random variable X, Stirling numbers of the first and second kinds, and recurrence relations for the probability distributions.

3.2. $\mathcal{R}(p,q)-$ deformed binomial distribution.

3.2.1. $\mathcal{R}(p,q)-$ *deformed factorial and binomial moments.* We introduce now the $\mathcal{R}(p,q)-$ deformed factorial and binomial moments. For that, we suppose X is a discrete non-negative integer valued random variable, and for $x \in \mathbb{N}\cup\{0\}$, we consider the distribution function g of the variable X such that $g(x) = P(X = x)$.

Definition 3.4. The $r^{th}-$ order $\mathcal{R}(p,q)-$ factorial moment of the random variable X is given by:

$$\mathbf{E}\Big([X]_{r,\mathcal{R}(p,q)}\Big) := \sum_{x=r}^{\infty} [x]_{r,\mathcal{R}(p,q)}\, g(x),$$

while the $r^{th}-$ order $\mathcal{R}(p,q)-$ binomial moment of the random variable X is provided by:

$$\mathbf{E}\Bigg(\left[\begin{array}{c} X \\ r \end{array}\right]_{\mathcal{R}(p,q)}\Bigg) = \sum_{x=r}^{\infty} \left[\begin{array}{c} x \\ r \end{array}\right]_{\mathcal{R}(p,q)} g(x), \quad r \in \mathbb{N}. \tag{3.5}$$

For $r = 1$, we obtain:

- The $\mathcal{R}(p,q)-$ mean value of the random variable X:

$$\mu_{\mathcal{R}(p,q)} := \mathbf{E}\Big(\big[X\big]_{\mathcal{R}(p,q)}\Big) = \sum_{x=1}^{\infty} \big[x\big]_{\mathcal{R}(p,q)}\, g(x).$$

- The $\mathcal{R}(p,q)-$ variance of the random variable X:

$$\sigma^2_{\mathcal{R}(p,q)} := \mathbf{V}\Big(\big[X\big]_{\mathcal{R}(p,q)}\Big) = \mathbf{E}\Big(\big[X\big]^2_{\mathcal{R}(p,q)}\Big) - \Big[\mathbf{E}\Big(\big[X\big]_{\mathcal{R}(p,q)}\Big)\Big]^2.$$

In terms of Stirling number, the binomial moment can be re-expresed as follows:

$$\mathbf{E}\Bigg[\left(\begin{array}{c} X \\ j \end{array}\right)\Bigg] = \sum_{m=j}^{\infty} (-1)^{m-j}\, \frac{(\epsilon_1-\epsilon_2)^{m-j}}{\epsilon_1^{-\binom{m}{2}+\tau(m-j)}}\, s_{\mathcal{R}(p,q)}(m,j)\,\mathbf{E}\Bigg(\left[\begin{array}{c} X \\ m \end{array}\right]_{\mathcal{R}(p,q)}\Bigg), \tag{3.6}$$

and the relation between the factorial moment and its $\mathcal{R}(p,q)-$ deformed counterpart is given by:

$$\mathbf{E}[(X)_j] = j! \sum_{m=j}^{\infty} (-1)^{m-j}\, \frac{(\epsilon_1-\epsilon_2)^{m-j}}{\epsilon_1^{-\binom{m}{2}+\tau(m-j)}}\, s_{\mathcal{R}(p,q)}(m,j)\, \frac{\mathbf{E}\big([X]_{m,\mathcal{R}(p,q)}\big)}{[m]_{\mathcal{R}(p,q)}!}, \tag{3.7}$$

where $j \in \mathbb{N}\backslash\{1\}$, $\tau \in \mathbb{N}\cup\{0\}$, and $s_{\mathcal{R}(p,q)}$ is the $\mathcal{R}(p,q)-$ Stirling number of the first kind.

The particular case of the $q-$ deformed binomial moment in [8] is retrieved as follows :

$$\mathbf{E}\bigg[\bigg(\begin{array}{c} X \\ j \end{array}\bigg)\bigg] = \sum_{m=j}^{\infty}(-1)^{m-j}(1-q)^{m-j}s_q(m,j)\mathbf{E}\bigg(\bigg[\begin{array}{c} X \\ m \end{array}\bigg]_q\bigg)$$

and the related factorial moment is linked to its $q-$ counterpart by the relation:

$$\mathbf{E}[(X)_j] = j!\sum_{m=j}^{\infty}(-1)^{m-j}(1-q)^{m-j}s_q(m,j)\frac{\mathbf{E}\big([X]_{m,q}\big)}{[m]_q!},$$

where $j \in \mathbb{N}$, and s_q is the $q-$ Stirling number of the first kind.

3.2.2. $\mathcal{R}(p,q)-$ *deformed binomial distribution.* The following relation holds:

$$\sum_{\kappa=0}^{n}\begin{bmatrix} n \\ \kappa \end{bmatrix}_{\mathcal{R}(p,q)} x^{\kappa}(y\ominus v)^{n-\kappa}_{\mathcal{R}(p,q)} = \sum_{\kappa=0}^{n}\begin{bmatrix} n \\ \kappa \end{bmatrix}_{\mathcal{R}(p,q)} y^{\kappa}(x\ominus v)^{n-\kappa}_{\mathcal{R}(p,q)}.$$

In particular, for $x = p_0$ and $y = 1$, we obtain

$$\sum_{\kappa=0}^{n}\begin{bmatrix} n \\ \kappa \end{bmatrix}_{\mathcal{R}(p,q)} p_0^{\kappa}(1\ominus v)^{n-\kappa}_{\mathcal{R}(p,q)} = \sum_{\kappa=0}^{n}\begin{bmatrix} n \\ \kappa \end{bmatrix}_{\mathcal{R}(p,q)} (p_0\ominus v)^{n-\kappa}_{\mathcal{R}(p,q)} = 1, \quad \forall p_0 \qquad (3.8)$$

where $x, y \in \mathbb{N}$. The $q-$analog computed in [20] is deduced by setting $\mathcal{R}(1,q) = 1$, $\epsilon_1 = 1$, and $\epsilon_2 = q$, as follows :

$$\sum_{\kappa=0}^{n}\begin{bmatrix} n \\ \kappa \end{bmatrix}_q x^{\kappa}(y\ominus v)^{n-\kappa}_q = \sum_{\kappa=0}^{n}\begin{bmatrix} n \\ \kappa \end{bmatrix}_q y^{\kappa}(x\ominus v)^{n-\kappa}_q.$$

For $x = p_0$ and $y = 1$, we obtain

$$\sum_{\kappa=0}^{n}\begin{bmatrix} n \\ \kappa \end{bmatrix}_q p_0^{\kappa}(1\ominus v)^{n-\kappa}_q = \sum_{\kappa=0}^{n}\begin{bmatrix} n \\ \kappa \end{bmatrix}_q (p_0\ominus v)^{n-\kappa}_q = 1, \quad \forall p_0,$$

where $x, y \in \mathbb{N}$.

The binomial distribution comes with a random variable X taking two values, 0 and 1, the probabilities $Pr(X=1) = p_0$ and $Pr(X=0) = 1-p_0$, and by letting $S_n = X_1 + \cdots + X_n$ be the sum of n independent random variables $(X_i)_{i\in\mathbb{N}}$ obeying the binomial law.

Definition 3.5. The $\mathcal{R}(p,q)-$deformed binomial distribution, with parameters n, p_0, p, and q, is given, for $0 \leq \kappa \leq n$ and $0 < q < p \leq 1$, by:

$$P_\kappa := P_r\big([S_n]_{\mathcal{R}(p,q)} = [\kappa]_{\mathcal{R}(p,q)}\big) = \left[\begin{array}{c} n \\ \kappa \end{array}\right]_{\mathcal{R}(p,q)} p_0^{\kappa}\,(1\ominus p_0)^{n-\kappa}_{\mathcal{R}(p,q)}.$$

For $\mathcal{R}(1,q) = 1$, $\epsilon_1 = 1$ and $\epsilon_2 = q$, we obtain the $q-$ binomial distribution given in [20]:

$$P_r\big([S_n]_q = [\kappa]_q\big) = \left[\begin{array}{c} n \\ \kappa \end{array}\right]_q p_0^{\kappa}\,(1\ominus p_0)^{n-\kappa}_q, \;\; 0 \leq \kappa \leq n;\; 0 < q < 1.$$

The $j^{th}-$ order $\mathcal{R}(p,q)-$ deformed factorial moment is given by:

$$\mu_{\mathcal{R}(p,q)}\big([S_n]_{j,\mathcal{R}(p,q)}\big) = [n]_{j,\mathcal{R}(p,q)}\,p_0^j, \quad j \in \mathbb{N},$$

and the factorial moment is expressed by the formula:

$$\mu_{\mathcal{R}(p,q)}[\big(S_n\big)_i] = i!\,\sum_{j=i}^{n}(-1)^{j-i}\left[\begin{array}{c} n \\ j \end{array}\right]_{\mathcal{R}(p,q)} p_0^j\,\frac{(\epsilon_1-\epsilon_2)^{j-i}}{\epsilon_1^{-\binom{j}{2}+\tau(j-i)}}\,s_{\mathcal{R}(p,q)}(j,i),$$

where $\tau \in \mathbb{N}\cup\{0\}$, $i \in \mathbb{N}$, and $s_{\mathcal{R}(p,q)}$ is the $\mathcal{R}(p,q)-$ Stirling number of the first kind. The recurrence relation for the $\mathcal{R}(p,q)-$ deformed binomial distributions takes the form:

$$P_{\kappa+1} = \frac{[n-\kappa]_{\mathcal{R}(p,q)}}{[\kappa+1]_{\mathcal{R}(p,q)}}\,\frac{p_0}{\epsilon_1^{n-\kappa}-\epsilon_2^{n-\kappa}\,p_0}\,P_\kappa, \quad \text{with} \quad P_0 = \big(1\ominus p_0\big)^n_{\mathcal{R}(p,q)}.$$

The recurrence relation for the $q-$ deformed binomial distributions is given by:

$$P_{\kappa+1} = \frac{[n-\kappa]_q}{[\kappa+1]_q}\,\frac{p_0}{1-q^{n-\kappa}\,p_0}\,P_\kappa, \quad \text{with} \quad P_0 = \prod_{j=1}^{n}\big(1-p_0\,q^{j-1}\big).$$

Now, let us consider, for $v\in\mathbb{N}$, the $l-$order differential operator

$$(vD_v)^l = \sum_{j=1}^{l}\frac{1}{[j-1]!_{\mathcal{R}(p,q)}}\Big(\sum_{t=0}^{j-1}\left[\begin{array}{c} j-1 \\ t \end{array}\right]_{\mathcal{R}(p,q)}(-1)^t q^{\binom{t}{2}}[j-t]^{l-1}_{\mathcal{R}(p,q)}\Big)v^j(D_v)^j,$$

where $D_v := d/d_{\mathcal{R}(p,q)}$ acting on v, giving, for $l=2$, the following second order differential operator:

$$(v\,D_v)^2 = \mathcal{R}(p,q)\,(v\,D_v) + \frac{1}{\mathcal{R}!(p,q)}\bigg(\mathcal{R}(p^2,q^2)-\mathcal{R}(p,q)\bigg)v^2(\,D_v)^2.$$

Besides, for $\epsilon_i>0$, with $i\in\{1,2\}$, such that $\forall p,q,\ 0<q<p\le 1$,

$$\mathcal{R}(p^{x-y},q^{x-y}) = \epsilon_1^{-y}\,\mathcal{R}(p^x,q^x) + \epsilon_1^{-y}\epsilon_2^{x-y}\,\mathcal{R}(p^y,q^y). \tag{3.9}$$

Applying the $\mathcal{R}(p,q)-$derivative on p_0 to the left and right hand sides of the relation (3.8) leads to

$$p_0D_{p_0}\sum_{\kappa=0}^{n}\begin{bmatrix} n \\ \kappa \end{bmatrix}_{\mathcal{R}(p,q)} p_0^\kappa(1\ominus u)^{n-\kappa}_{\mathcal{R}(p,q)} = \mu_{\mathcal{R}(p,q)}([S_n]_{\mathcal{R}(p,q)}),$$

and

$$p_0D_{p_0}\sum_{\kappa=0}^{n}\begin{bmatrix} n \\ \kappa \end{bmatrix}_{\mathcal{R}(p,q)}(p_0\ominus u)^{n-\kappa}_{\mathcal{R}(p,q)} = p_0\sum_{\kappa=0}^{n}\begin{bmatrix} n \\ \kappa \end{bmatrix}_{\mathcal{R}(p,q)}[n-\kappa]_{\mathcal{R}(p,q)}(p_0\ominus u)^{n-\kappa-1}_{\mathcal{R}(p,q)},$$

respectively. According to (3.9) and

$$\left[\begin{array}{c} x \\ \kappa \end{array}\right]_{\mathcal{R}(p,q)} = \epsilon_1^\kappa\left[\begin{array}{c} x-1 \\ \kappa \end{array}\right]_{\mathcal{R}(p,q)} + \epsilon_2^{x-\kappa}\left[\begin{array}{c} x-1 \\ \kappa-1 \end{array}\right]_{\mathcal{R}(p,q)}, \quad \kappa\in\mathbb{N}, \tag{3.10}$$

we obtain the mean value of the random variable sum S_n as:

$$\mu_{\mathcal{R}(p,q)}\big([S_n]_{\mathcal{R}(p,q)}\big) = p_0\,\mathcal{R}(p^n,q^n).$$

Besides,

$$\mu_{\mathcal{R}(p,q)}\big([S_n]^2_{\mathcal{R}(p,q)}\big) = (p_0\,D_{p_0})^2\sum_{\kappa=0}^{n}\left[\begin{array}{c} n \\ \kappa \end{array}\right]_{\mathcal{R}(p,q)} p_0^\kappa\,(1\ominus u)^{n-\kappa}_{\mathcal{R}(p,q)}.$$

Setting $\mathbf{X}=\frac{1}{\mathcal{R}!(p,q)}\bigg(\mathcal{R}(p^2,q^2)-\mathcal{R}(p,q)\bigg)$ yields the second order differential equation

$$(p_0\,D_{p_0})^2\sum_{\kappa=0}^{n}\left[\begin{array}{c} n\\ \kappa\end{array}\right]_{\mathcal{R}(p,q)}(p_0\ominus u)^{n-\kappa}_{\mathcal{R}(p,q)}=\mathcal{R}(p,q)\mu_{\mathcal{R}(p,q)}\big([S_n]_{\mathcal{R}(p,q)}\big)+Y_4,$$

where

$$Y_4 \quad=\quad \mathbf{X}\,p_0^2\sum_{\kappa=0}^{n}\left[\begin{array}{c} n\\ \kappa\end{array}\right]_{\mathcal{R}(p,q)}\mathcal{R}(p^{n-\kappa},q^{n-\kappa})\mathcal{R}(p^{n-\kappa-1},q^{n-\kappa-1})(p_0\ominus u)^{n-\kappa}_{\mathcal{R}(p,q)}.$$

Using the relation (3.10) and the $\mathcal{R}(p,q)-$ deformed shifted factorial, we obtain

$$\mu_{\mathcal{R}(p,q)}\big([S_n]^2_{\mathcal{R}(p,q)}\big)=\mathcal{R}(p,q)\,\mu_{\mathcal{R}(p,q)}([S_n]_{\mathcal{R}(p,q)})+\mathbf{X}\,p_0^2\mathcal{R}(p^n,q^n)\mathcal{R}(p^{n-1},q^{n-1}),$$

and from the mean value of S_n, we deduce the variance under the form:

$$\mathbf{Var}\big([S_n]_{\mathcal{R}(p,q)}\big)=p_0[n]_{\mathcal{R}(p,q)}\bigg(\mathcal{R}(p,q)+\mathbf{X}p_0[n-1]_{\mathcal{R}(p,q)}-p_0[n]_{\mathcal{R}(p,q)}\bigg),$$

where $\mathbf{X}p_0[n-1]_{\mathcal{R}(p,q)}>p_0[n]_{\mathcal{R}(p,q)}-\mathcal{R}(p,q)$, and

$$\mathbf{X}=\mathcal{R}(p,q)^{-1}\big(\mathcal{R}(p^2,q^2)-\mathcal{R}(p,q)\big).$$

Furthermore, applying the operator $p_0^r\bigg(D_{p_0}\bigg)^r\bigg|_{u=p_0}$ to formula (3.8), we obtain

$$p_0^r\big(D_{p_0}\big)^r\sum_{\kappa=0}^{n}\begin{bmatrix} n\\ \kappa\end{bmatrix}p_0^{\kappa}(1\ominus u)^{n-\kappa}_{\mathcal{R}(p,q)}\quad=\quad\mu_{\mathcal{R}(p,q)}\big([S_n]^r_{\mathcal{R}(p,q)}\big).$$

Moreover,

$$p_0^r\bigg(D_{p_0}\bigg)^r(p_0\ominus u)^{n-\kappa}_{\mathcal{R}(p,q)}\quad=\quad p_0^r\prod_{i=0}^{r-1}[n-\kappa-i]_{\mathcal{R}(p,q)}\,(p_0\ominus u)^{n-\kappa-r}_{\mathcal{R}(p,q)},$$

or equivalently,

$$\begin{aligned} p_0^r\bigg(D_{p_0}\bigg)^r(p_0\ominus u)^{n-\kappa}_{\mathcal{R}(p,q)}\quad&=\quad p_0^r\prod_{i=0}^{r-1}\epsilon_1^{-\kappa}[n-i]_{\mathcal{R}(p,q)}\,(p_0\ominus u)^{n-\kappa-r}_{\mathcal{R}(p,q)}\\ &+\quad p_0^r\prod_{i=0}^{r-1}\epsilon_2^{n-i}[-\kappa]_{\mathcal{R}(p,q)}(p_0\ominus u)^{n-\kappa-r}_{\mathcal{R}(p,q)}.\end{aligned}$$

Then,

$$p_0^r\bigg(D_{p_0}\bigg)^r\sum_{\kappa=0}^{n}\begin{bmatrix} n\\ \kappa\end{bmatrix}(p_0\ominus u)^{n-\kappa}_{\mathcal{R}(p,q)}=p_0^r\prod_{i=0}^{r-1}[n-i]_{\mathcal{R}(p,q)}$$

gives

$$\mu_{\mathcal{R}(p,q)}\big([S_n]^r_{\mathcal{R}(p,q)}\big)=p_0^r\prod_{i=0}^{r-1}[n-i]_{\mathcal{R}(p,q)},$$

and after computation, there follows the mean of the product $S_n(S_n-1)\cdots(S_n-r+1)$ as:

$$\mu_{\mathcal{R}(p,q)}\Big(\prod_{i=0}^{r-1}\epsilon_2^{-i}([S_n]^r_{\mathcal{R}(p,q)}-\epsilon_1^{r-i}[i]_{\mathcal{R}(p,q)})\Big)=p_0^r\prod_{i=0}^{r-1}[n-i]_{\mathcal{R}(p,q)}.$$

The particular case of the $q-$ deformation found in [20] corresponds to $\mathcal{R}(1,q)=1$, $\epsilon_1=1$, and $\epsilon_2=q$, and yields the mean value of S_n :

$$\mu_q\big([S_n]_q\big)=p_0\,[n]_q,$$

the variance of S_n :

$$\mathbf{Var}\big([S_n]_q\big)=[n]_q\,p_0\,(1-p_0),$$

and the mean of the product $S_n(S_n-1)\cdots(S_n-r+1)$:

$$\mu_q\Big(\prod_{i=0}^{r-1}q^{-i}([S_n]^r_q-[i]_q)\Big)=p_0^r\prod_{i=0}^{r-1}[n-i]_q.$$

3.3. $\mathcal{R}(p,q)-$ **deformed Euler distribution.** The $\mathcal{R}(p,q)-$ deformed exponential functions, denoted $E_{\mathcal{R}(p,q)}$ and $e_{\mathcal{R}(p,q)}$, are defined as follows:

$$E_{\mathcal{R}(p,q)}(z):=\sum_{n=0}^{\infty}\frac{\epsilon_2^{\binom{n}{2}}\,z^n}{\mathcal{R}!(p^n,q^n)}\quad\text{and}\quad e_{\mathcal{R}(p,q)}(z):=\sum_{n=0}^{\infty}\frac{\epsilon_1^{\binom{n}{2}}\,z^n}{\mathcal{R}!(p^n,q^n)}$$

with $E_{\mathcal{R}(p,q)}(-z)\,e_{\mathcal{R}(p,q)}(z)=1$. In the particular case where $\mathcal{R}(p,q)=1$, $\epsilon_1=p$, and $\epsilon_2=q$, they provide the Jagannathan-Srinivasa $q-$exponential functions [13]:

$$E_{p,q}(z)=\sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}\,z^n}{[n]_{p,q}!}\quad\text{and}\quad e_{p,q}(z)=\sum_{n=0}^{\infty}\frac{p^{\binom{n}{2}}\,z^n}{[n]_{p,q}!}$$

with $E_{p,q}(-z)\,e_{p,q}(z)=1$.

Definition 3.6. The $\mathcal{R}(p,q)-$ deformed Euler distribution, with parameters n, θ, p, and q, is defined by:

$$P_r(X=x)=E_{\mathcal{R}(p,q)}(-\theta)\ \frac{\epsilon_1^{\binom{x}{2}}\theta^x}{\mathcal{R}!(p^x,q^x)},\quad x\in\mathbb{N}\cup\{0\},$$

where $0<q<p\leq 1$, $0<\theta<1$ and $\sum_{x=0}^{\infty}P_r(X=x)=1$.

For $\mathcal{R}(1,q)=1$, involving $\epsilon_1=1$, and $\epsilon_2=q$, we retrieve the $q-$ Euler distribution obtained in [7]:

$$P_r(X=x)=E_q(-\theta)\ \frac{\theta^x}{[x]_q!},$$

where $x\in\mathbb{N}\cup\{0\}$, $0<q<1$ and $0<\theta<\frac{1}{1-q}$. The $j^{th}-$ order $\mathcal{R}(p,q)-$ factorial moment of X is given by

$$\mu_{\mathcal{R}(p,q)}\big([X]_{j,\mathcal{R}(p,q)}\big)=\theta^j\,\epsilon_1^{\binom{j}{2}}\,E_{\mathcal{R}(p,q)}(-\theta)\,e_{\mathcal{R}(p,q)}(\epsilon_1^j\theta),\quad j\in\mathbb{N}.\tag{3.11}$$

From the $j^{th}-$ order $\mathcal{R}(p,q)-$ factorial moment, we get

$$\mu([X]_{j,\mathcal{R}(p,q)}) = \sum_{x=j}^{\infty}[x]_{j,\mathcal{R}(p,q)}\, E_{\mathcal{R}(p,q)}(-\theta)\, \frac{\epsilon_1^{\binom{x}{2}}\theta^x}{\mathcal{R}!(p^x,q^x)}.$$

Using the relation

$$\mathcal{R}(p^x,q^x)^j\, \mathcal{R}!(p^{x-j},q^{x-j}) = \mathcal{R}!(p^x,q^x),$$

we obtain

$$\begin{aligned}\mu([X]_{j,\mathcal{R}(p,q)}) &= \theta^j\, \epsilon_1^{\binom{j}{2}}\, E_{\mathcal{R}(p,q)}(-\theta) \sum_{h=0}^{\infty} \frac{\epsilon_1^{\binom{h}{2}}\big(\epsilon_1^j\theta\big)^h}{[h]_{\mathcal{R}(p,q)}!}\\ &= \theta^j\, \epsilon_1^{\binom{j}{2}}\, E_{\mathcal{R}(p,q)}(-\theta)\, e_{\mathcal{R}(p,q)}(\epsilon_1^j\theta).\end{aligned}$$

Furthermore, exploiting the relation (3.11) and (3.7), we obtain the factorial moment as:

$$\mu_{\mathcal{R}(p,q)}(X)_i = i!\sum_{j=i}^{\infty}(-1)^{j-i}\, EU(j,i)\, s_{\mathcal{R}(p,q)}(j,i), \tag{3.12}$$

where

$$EU(j,i) = \frac{\theta^j\epsilon_1^{\binom{j}{2}}\, E_{\mathcal{R}(p,q)}(-\theta)\, e_{\mathcal{R}(p,q)}(\epsilon_1^j\theta)}{[j]_{\mathcal{R}(p,q)}!}\, \frac{(\epsilon_1-\epsilon_2)^{j-i}}{\epsilon_1^{-\binom{j}{2}+\tau(j-i)}},$$

$\tau \in \mathbb{N}\cup\{0\}$, $i \in \mathbb{N}$, and $s_{\mathcal{R}(p,q)}$ is the $\mathcal{R}(p,q)-$ deformed Stirling number of the first kind. Using the $\mathcal{R}(p,q)-$ factorials, and after computation, we get the recurrence relation for the associated $\mathcal{R}(p,q)-$ deformed Euler distributions as follows:

$$P_{x+1} = \frac{\theta\,\epsilon_1^x}{\mathcal{R}(p^{x+1},q^{x+1})}\, P_x, \quad \text{with} \quad P_0 = E_{\mathcal{R}(p,q)}(-\theta). \tag{3.13}$$

The particular case of the recurrence relation for the $q-$ deformed Euler distributions is given by:

$$P_{x+1} = \frac{\theta}{[x+1]_q}\, P_x, \quad \text{with} \quad P_0 = E_q(-\theta).$$

The results for the $q-$ deformation performed in [7], recovered with $\mathcal{R}(1,q)=1$, $\epsilon_1 = 1$, and $\epsilon_2 = q$, yield the $j^{th}-$ order $q-$ factorial moment of X:

$$\mu_q\big([X]_{j,q}\big) = \theta^j, \quad j \in \mathbb{N},$$

and the $q-$ factorial moments:

$$\mu_q(X)_i = i!\sum_{j=i}^{\infty}(-1)^{j-i}\,\frac{\theta^j}{[j]_q!}\,(1-q)^{j-i}\, s_q(j,i),$$

$i \in \mathbb{N}$, and s_q is the $q-$ deformed Stirling number of the first kind.

3.4. $\mathcal{R}(p,q)-$ deformed Pólya distribution. We assume here that boxes are successively drawn one after the other from an urn, initially containing r white and s black boxes. After each drawing, the drawn box is placed back in the urn together with x boxes of the same color. We suppose that the probability of drawing a white box at the i^{th} drawing, given that $j-1$ white boxes are drawn in the previous $i-1$ drawings, is given as:

$$P_{i,j} = \frac{[r+x(j-1)]_{\mathcal{R}(p,q)}}{[r+s+x(i-1)]_{\mathcal{R}(p,q)}} = \frac{[m-j+1]_{\mathcal{R}(p^{-x},q^{-x})}}{[m+u-i+1]_{\mathcal{R}(p^{-x},q^{-x})}},$$

where $j\in\mathbb{N}$, $i\in\mathbb{N}$, $0<q<p\leq 1$, $m=-r/x$, $u=-s/x$, and $x\in\mathbb{N}$. We call this model the $\mathcal{R}(p,q)-$ Pólya urn model. Setting $\mathcal{R}(1,q)=1$, $\epsilon_1=1$, and $\epsilon_2=q$, we obtain the probability for $q-$ Pólya urn model described in [6]:

$$P_{i,j} = \frac{[r+x(j-1)]_q}{[r+s+x(i-1)]_q} = \frac{[m-j+1]_{q^{-x}}}{[m+u-i+1]_{q^{-x}}}.$$

Let T_n be the number of white boxes drawn in n drawings. Then, we have:

Definition 3.7. The $\mathcal{R}(p,q)-$ deformed Pólya distribution, with parameters m, u, n, p, and q, is defined by:

$$P_\kappa := P_r(T_n=\kappa) = \Psi(p,q)\begin{bmatrix} n \\ \kappa \end{bmatrix}_{\mathcal{R}(p^{-x},q^{-x})} \frac{[m]_{\kappa,\mathcal{R}(p^{-x},q^{-x})}[u]_{n-\kappa,\mathcal{R}(p^{-x},q^{-x})}}{[m+u]_{n,\mathcal{R}(p^{-x},q^{-x})}},$$

where $0<q<p\leq 1$, $\kappa\in\mathbb{N}\cup\{0\}$, $\Psi(p,q) = \epsilon_1^{-x\kappa(u-n+\kappa)}\,\epsilon_2^{-x(n-\kappa)(m-\kappa)}$, $x\in\mathbb{N}\cup\{0\}$ and $\displaystyle\sum_{\kappa=0}^{n} P_r(T_n=\kappa)=1$.

For $\mathcal{R}(1,q)=1$, $\epsilon_1=1$, and $\epsilon_2=q$, we deduce the $q-$ Pólya distribution [6]:

$$P_r(T_n=\kappa) = q^{-x(n-\kappa)(m-\kappa)}\begin{bmatrix} n \\ \kappa \end{bmatrix}_{q^{-x}} \frac{[m]_{\kappa,q^{-x}}[u]_{n-\kappa,q^{-x}}}{[m+u]_{n,q^{-x}}},$$

where $0<q<1$, $\kappa\in\mathbb{N}\cup\{0\}$, and $x\in\mathbb{N}\cup\{0\}$.

The $j^{th}-$ order $\mathcal{R}(p,q)-$ deformed factorial moment is given by:

$$\mu_{\mathcal{R}(p,q)}\big([T_n]_{j,\mathcal{R}(p^{-x},q^{-x})}\big) = \frac{[n]_{j,\mathcal{R}(p^{-x},q^{-x})}\,[m]_{j,\mathcal{R}(p^{-x},q^{-x})}}{[m+u]_{j,\mathcal{R}(p^{-x},q^{-x})}},\ j\in\mathbb{N}.$$

Performing a similar computation like in the case of the Euler distribution, the factorial moment is expressed by:

$$\mu_{\mathcal{R}(p,q)}[(T_n)_i] = i!\sum_{j=i}^{n}(-1)^{j-i}\left[\begin{array}{c} n \\ j \end{array}\right]_{\mathcal{R}(p,q)} \frac{s_{\mathcal{R}(p^{-x},q^{-x})}(j,i)}{(\epsilon_1^{-x}-\epsilon_2^{-x})^{i-j}}P(j,i),$$

where

$$P(j,i) = \frac{\epsilon_1^{\binom{j}{2}}\,[m]_{j,\mathcal{R}(p^{-x},q^{-x})}}{\epsilon_1^{\tau(j-i)}\,[m+u]_{j,\mathcal{R}(p^{-x},q^{-x})}},$$

$\tau\in\mathbb{N}\cup\{0\}$, $i\in\mathbb{N}$ and $s_{\mathcal{R}(p^{-x},q^{-x})}$ is the $\mathcal{R}(p,q)-$ deformed Stirling number of the first kind. Besides, the recurrence relation for the $\mathcal{R}(p,q)-$ Pólya distributions is given as follows:

$$P_{\kappa+1} = \frac{\epsilon_2^{x(n+m-2\kappa-1)}}{\epsilon_1^{x(u-n+2\kappa+1)}}\frac{[n-\kappa]_{\mathcal{R}(p^{-x},q^{-x})}}{[u-n+\kappa+1]_{\mathcal{R}(p^{-x},q^{-x})}}\frac{[m-\kappa]_{\mathcal{R}(p^{-x},q^{-x})}}{[\kappa+1]_{\mathcal{R}(p^{-x},q^{-x})}}P_\kappa,$$

with

$$P_0 = \epsilon_2^{-x\,m\,n}\frac{[u]_{n,\mathcal{R}(p^{-x},q^{-x})}}{[m+u]_{n,\mathcal{R}(p^{-x},q^{-x})}}.$$

The recurrence relation for the $q-$ Pólya distributions is derived as:

$$P_{\kappa+1} = q^{x(n+m-2\kappa-1)}\frac{[n-\kappa]_{q^{-x}}}{[u-n+\kappa+1]_{q^{-x}}}\frac{[m-\kappa]_{q^{-x}}}{[\kappa+1]_{q^{-x}}}P_\kappa,$$

with

$$P_0 = q^{-x\,m\,n}\frac{[u]_{n,q^{-x}}}{[m+u]_{n,q^{-x}}}.$$

For $x=-1$, we obtain the $\mathcal{R}(p,q)-$ hypergeometric distribution:

$$P_r\big(T_n=\kappa\big) \quad = \quad \Phi(p,q)\left[\begin{array}{c} n \\ \kappa \end{array}\right]_{\mathcal{R}(p,q)}\frac{[m]_{\kappa,\mathcal{R}(p,q)}\,[u]_{n-\kappa,\mathcal{R}(p,q)}}{[m+u]_{n,\mathcal{R}(p,q)}},$$

where $0<q<p\leq 1$, $\kappa\in\mathbb{N}\cup\{0\}$, $\Phi(p,q)=\epsilon_1^{\kappa(u-n+\kappa)}\,\epsilon_2^{(n-\kappa)(m-\kappa)}$ and $\sum_{\kappa=0}^{n}P_r(T_n=\kappa)=1$.

The particular case of $q-$ deformation obtained in [6] is characterized by the $j^{th}-$ order $q-$ deformed factorial moment:

$$\mu_q\big([T_n]_{j,q^{-x}}\big) = \frac{[n]_{j,q^{-x}}\,[m]_{j,q^{-x}}}{[m+u]_{j,q^{-x}}}, \quad j\in\mathbb{N},$$

and the factorial moment:

$$\mu_q[(T_n)_i] = i!\sum_{j=i}^{n}(-1)^{j-i}\left[\begin{array}{c} n \\ j \end{array}\right]_q\frac{s_{q^{-x}}(j,i)}{(1-q^{-x})^{i-j}}\frac{[m]_{j,q^{-x}}}{[m+u]_{j,q^{-x}}},$$

where $i\in\mathbb{N}$ and $s_{q^{-x}}$ is the $q-$ deformed Stirling number of the first kind.

3.5. Inverse Pólya distribution. Let Y_n be a number of black boxes drawn until the n^{th} white box is drawn. Then, we have:

Definition 3.8. The inverse $\mathcal{R}(p,q)-$ deformed Pólya distribution with parameters n, m, u, and κ is given by:

$$P_y := P_r(Y_n=y) = F(p,q)\left[\begin{array}{c} n+y-1 \\ y \end{array}\right]_{\mathcal{R}(p^{-x},q^{-x})}\frac{[m]_{n,\mathcal{R}(p^{-x},q^{-x})}\,[u]_{y,\mathcal{R}(p^{-x},q^{-x})}}{[m+u]_{n+y,\mathcal{R}(p^{-x},q^{-x})}},$$

where $0<q<p\leq 1$, $y\in\mathbb{N}\cup\{0\}$, $F(p,q)=\epsilon_1^{n(u-x)}\,\epsilon_2^{-yx(m-n+1)}$, $x\in\mathbb{N}\cup\{0\}$, and

$$\sum_{t=0}^{\infty}F(p,q)\left[\begin{array}{c} n+y-1 \\ y \end{array}\right]_{\mathcal{R}(p^{-x},q^{-x})}\frac{[m]_{n,\mathcal{R}(p^{-x},q^{-x})}\,[u]_{y,\mathcal{R}(p^{-x},q^{-x})}}{[m+u]_{n+y,\mathcal{R}(p^{-x},q^{-x})}} = 1.$$

Note that the inverse $q-$ deformed Pólya distribution obtained in [6] can be recovered by taking $\mathcal{R}(1,q)=1$, $\epsilon_1=1$, $\epsilon_2=q$ and $F(p,q)=q^{-yx(m-n+1)}$ as:

$$P_r(Y_n=y):=q^{-yx(m-n+1)}\begin{bmatrix} n+y-1\\ y\end{bmatrix}_{q^{-x}}\frac{[m]_{n,q^{-x}}\,[u]_{y,q^{-x}}}{[m+u]_{n+y,q^{-x}}}$$

where $0<q<1$ and $y\in\mathbb{N}\cup\{0\}$. The $j^{th}-$ order $\mathcal{R}(p,q)-$factorial moment is given by:

$$\mu_{\mathcal{R}(p,q)}\big([Y]_{j,\mathcal{R}(p^{-x},q^{-x})}\big)=\frac{[n-j+1]_{j,\mathcal{R}(p^{-x},q^{-x})}[u]_{\mathcal{R}(j,p^{-x},q^{-x})}}{\epsilon_2^{jx(m-n+1)}[m+j]_{j,\mathcal{R}(p^{-x},q^{-x})}},$$

where $j\in\mathbb{N}$ and $m+j\neq 0$. Moreover, for $i\in\mathbb{N}$, the factorial moment yields:

$$\mu_{\mathcal{R}(p,q)}[(Y_n)_i]=i!\sum_{j=i}^{n}(-1)^{j-i}\begin{bmatrix} n+j-1\\ j\end{bmatrix}_{\mathcal{R}(p^{-x},q^{-x})}IP(j,i),$$

where

$$IP(j,i)=s_{\mathcal{R}(p^{-x},q^{-x})}(j,i)\,\frac{\epsilon_1^{\binom{j}{2}}}{\epsilon_1^{\tau(j-i)}}\frac{(\epsilon_1^{-x}-\epsilon_2^{-x})^{j-i}[u]_{j,\mathcal{R}(p^{-x},q^{-x})}}{[m+j]_{j,\mathcal{R}(p^{-x},q^{-x})}\,\epsilon_2^{jx(m-n+1)}},$$

$\tau\in\mathbb{N}\cup\{0\}$ and $s_{\mathcal{R}(p^{-x},q^{-x})}$ is the $\mathcal{R}(p,q)-$ deformed Stirling number of the first kind. The recurrence relation for the inverse $\mathcal{R}(p,q)-$ Pólya distibutions is provided by:

$$P_{y+1}=\frac{\epsilon_2^{-x(m-n+1)}[n+y]_{\mathcal{R}(p^{-x},q^{-x})}[u-y]_{\mathcal{R}(p^{-x},q^{-x})}}{[y+1]_{\mathcal{R}(p^{-x},q^{-x})}[m+u-n-y]_{\mathcal{R}(p^{-x},q^{-x})}}\,P_y,$$

with initial condition

$$P_0=\epsilon_1^{n(u-x)}\,\frac{[m]_{n,\mathcal{R}(p^{-x},q^{-x})}}{[m+u]_{n,\mathcal{R}(p^{-x},q^{-x})}}.$$

The recurrence relation for the inverse $q-$ Pólya distributions is given by the formula:

$$P_{y+1}=\frac{q^{-x(m-n+1)}[n+y]_{q^{-x}}[u-y]_{q^{-x}}}{[y+1]_{q^{-x}}[m+u-n-y]_{q^{-x}}}P_y,\quad\text{with}\quad P_0=\frac{[m]_{n,q^{-x}}}{[m+u]_{n,q^{-x}}}.$$

4. Concluding Remarks

$\mathcal{R}(p,q)$-deformed deformed numbers, orthogonal polynomials, and univariate dicrete binomial, Euler, Pólya and inverse Pólya distributions, induced by the $\mathcal{R}(p,q)$-deformed quantum algebras, have been investigated and discussed in a general framework. Results for known $q-$ deformed distributions have also been recovered as particular cases.

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