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COMPLEX HERMITE POLYNOMIALS: FROM THE SEMI-CIRCULAR LAW TO THE CIRCULAR LAW

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ABSTRACT. We study asymptotics of orthogonal polynomial measures of the form $|\mathcal{H}_N|^2 d\gamma$ where $\mathcal{H}_N$ are real or complex Hermite polynomials with respect to the Gaussian measure γ. By means of differential equations on Laplace transforms, interpolation between the (real) arcsine law and the (complex) uniform distribution on the circle is emphasized. Suitable averages by an independent uniform law give rise to the limiting semi-circular and circular laws of Hermitian and non-Hermitian Gaussian random matrix models. The intermediate regime between strong and weak non-Hermiticity is clearly identified on the limiting differential equation by means of an additional normal variable in the vertical direction.

1. Introduction

Let A^N be a $N \times N$ random Hermitian matrix from the Gaussian Unitary Ensemble (GUE)

$$\mathbb{P}(dX) = \frac{1}{Z} \exp\big(- \mathrm{Tr}(X^2)/2\big) dX \tag{1.1}$$

where dX is Lebesgue measure on the space of $N \times N$ Hermitian matrices X and Z the normalization constant. Equivalently, the entries A^N_{kl}, $1 \leq k \leq l \leq N$, of A^N are independent complex Gaussian variables with mean zero and variance one. It is a classical result going back to E. Wigner [10] that the empirical measure $\frac{1}{N}\sum_{k=1}^N \delta_{\lambda_k^N}$ on the (real) eigenvalues $\lambda_1^N, \ldots, \lambda_N^N$ of $A^N/2\sqrt{N}$ converges as $N \to \infty$ to the semi-circular law $\frac{2}{\pi}\,(1-x^2)^{1/2}\mathbf{1}_{\{|x|<1\}}dx$.

Consider now an independent copy B^N of A^N, and form the random matrix $A^N + iB^N$ whose entries are independent complex Gaussian variables with mean zero and variance two. This is a canonical non-Hermitian ensemble of random matrix theory widely referred to as the (complex) Ginibre Ensemble. Girko's classical theorem [3] indicates that the empirical measure on the (complex) eigenvalues of $(A^N+iB^N)/\sqrt{2N}$ converges as $N \to \infty$ towards the circular law $\frac{1}{\pi}\,\mathbf{1}_{\{x^2+y^2\leq 1\}}dxdy$ on the plane.

Interpolation from the Ginibre Ensemble to the GUE is provided by the family $A^N + i\rho B^N$ for some real parameter ρ, $|\rho| \leq 1$, yielding as limiting spectral measure Girko's elliptic law $\frac{1}{\pi ab}\,\mathbf{1}_{\{(x^2/a^2)+(y^2/b^2)\leq 1\}}dxdy$ where $a^2 = \frac{2}{1+\rho^2}$, $b^2 = \frac{2\rho^2}{1+\rho^2}$.

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These results extend to families of random matrices with non-Gaussian entries (cf. [10, 1, 3]...). In the Gaussian case, the determinantal structure of the joint law of the eigenvalues (cf. [8]) allows, by the so-called orthogonal polynomial method, for the analysis of the mean spectral measure through the Hermite polynomials. Following [9], let H_ℓ, $\ell \in \mathbb{N}$, be the Hermite polynomials defined by the generating series

$$\mathrm{e}^{\lambda x - \lambda^2/2} = \sum_{\ell=0}^{\infty} \frac{\lambda^\ell}{\sqrt{\ell!}}\, H_\ell(x), \quad \lambda \in \mathbb{R},\ x \in \mathbb{R}.$$

The family $(H_\ell)_{\ell\in\mathbb{N}}$ forms an orthonormal basis of the Hilbert space of real-valued square integrable functions with respect to the standard normal distribution $d\gamma(x) = \mathrm{e}^{-x^2/2}\frac{dx}{\sqrt{2\pi}}$ on $\mathbb{R}$. Extend now the polynomials H_ℓ to the complex plane. Then, for each $\tau > 1$, the family

$$\mathcal{H}^\tau_\ell(z) = (2\tau^2 - 1)^{-\ell/2} H_\ell\big(\tau x + i\sqrt{\tau^2-1}\,y\big), \quad z = x + iy \in \mathbb{C},\ \ell \in \mathbb{N},$$

defines an orthonormal sequence with respect to the standard Gaussian measure $d\gamma(z) = d\gamma(x)d\gamma(y)$ on $\mathbb{C}$. Here and below we identify $z = x + iy \in \mathbb{C}$ and $(x, y) \in \mathbb{R}^2$. The value $\tau = 1$ corresponds to the real case, while when $\tau \to \infty$, $\mathcal{H}^\tau_\ell(z) \to \frac{z^\ell}{\sqrt{2^\ell \ell!}}$ which form an orthonormal basis of the space of square integrable analytic functions on $\mathbb{C}$.

As presented in [8], the correlation functions of the random matrix ensemble $A^N + i\rho B^N$ are completely described by the Hermite kernel $\frac{1}{N}\sum_{\ell=0}^{N-1}\mathcal{H}^\tau_\ell(z)\mathcal{H}^\tau_\ell(\bar{z'})$ with $\tau = (1-\rho^2)^{-1/2}$. In particular, the mean eigenvalue density μ^N on the eigenvalues $\lambda_1^N, \ldots, \lambda_N^N$ of $A^N + i\rho B^N$ is given by

$$\langle f, \mu^N\rangle = \mathbb{E}\bigg(\frac{1}{N}\sum_{k=1}^{N} f(\lambda_k^N)\bigg) = \int_{\mathbb{C}} f(z)\,\frac{1}{N}\sum_{\ell=0}^{N-1}\big|\mathcal{H}^\tau_\ell(z)\big|^2 d\gamma(z) \tag{1.2}$$

for every bounded measurable function f on $\mathbb{C}$. Asymptotics of μ^N may thus be read off from the behavior of the probability density $\frac{1}{N}\sum_{\ell=0}^{N-1}|\mathcal{H}^\tau_\ell(z)|^2 d\gamma(z)$ on $\mathbb{C}$ suitably rescaled.

In the contribution [5], inspired by the work [4] by U. Haagerup and S. Thorbjørnsen, simple Markov integration by parts and differential equations on Laplace transforms have been emphasized in the context of real orthogonal polynomial ensembles, demonstrating in particular the underlying universal character of the arcsine law. It was shown namely that, properly rescaled, measures $H_N^2 d\gamma$ on the line converge weakly to the arcsine law $\frac{1}{\pi}(1-x^2)^{-1/2}\mathbf{1}_{\{|x|<1\}}dx$ (ξ will denote below a random variable with this distribution). By a simple averaging procedure, rescaled measures $\frac{1}{N}\sum_{\ell=0}^{N-1} H_\ell^2 d\gamma$ converge to the product $\sqrt{U}\,\xi$ where U is uniform on $[0,1]$ and independent from ξ, giving thus rise to the semi-circular law for the limiting spectral measure of the GUE by the representation (1.2) (cf. [5]).

The purpose of this note is to show that a similar structure arises in the context of non-Hermitian random matrices with complex eigenvalues, where the central role is now played by the uniform distribution on the unit circle. For any fixed ρ such that $0 < |\rho| \leq 1$, equivalently $\tau > 1$, we show namely that rescaled measures $|\mathcal{H}^\tau_N|^2 d\gamma$ converge to $(a\cos\Theta, b\sin\Theta)$ where Θ is uniform on the unit circle. Note that projections of Θ on diameters have the arcsine distribution. The analysis again relies on second order differential equations for Laplace transforms from which the limiting distribution may easily be identified.

These results and methods extend to non-compactly supported models aspects of the classical theory developed in [7] in which recurrence equations for orthogonal polynomials P_ℓ, $\ell \in \mathbb{N}$, of measures μ on the torus or with compact support on the line are used to show that measures $P_\ell^2 d\mu$ converge to the uniform distribution on the circle or the arcsine law. Following [5, 6], the same line of investigation should cover the complex extension of the other classical orthogonal polynomial ensembles.

After averaging, $\frac{1}{N}\sum_{\ell=0}^{N-1} |\mathcal{H}_\ell^\tau|^2 d\gamma$ converges weakly, for every fixed $\tau > 1$, to $\sqrt{U}(a\cos\Theta, b\sin\Theta)$ where U is uniform on $[0,1]$ and independent from Θ, giving thus rise to the elliptic law for the non-Hermitian random matrix models. Interestingly enough, the approach allows us to investigate as easily the intermediate regime studied by Y. Fyodorov, B. Khoruzhenko and H.-J. Sommers [2] concerned with weak non-Hermiticity as $\rho \to 0$ ($b \to 0$) with N. The differential equation approach indeed clearly identifies a Gaussian perturbation in the, properly rescaled, vertical direction, and it provides in particular a simple description of the limiting distribution put forward in [2].

In the first section of this note, we present the classical results in the strong non-Hermitian regime $A^N + i\rho B^N$ with $0 < |\rho| \le 1$ fixed. In the second part, we analyze the weak non-Hermitian regime in which $\rho \to 0$ with N.

2. Strong Non-Hermitian Random Matrices

In this section, we deal with strong non-Hermitian matrices given by the Ginibre Ensemble $A^N + i\rho B^N$ with $0 < |\rho| \le 1$ fixed. In the orthogonal polynomial description, $\tau > 1$ (possibly infinite) is therefore fixed independently of N. Following the strategy of [5], the first proposition describes the limiting distribution of the measures $|\mathcal{H}_N^\tau|^2 d\gamma$ properly renormalized by means of differential equations on Laplace transforms.

Proposition 2.1. *Let $\tau > 0$ be fixed, and let $Z_N = (X_N, Y_N)$ be a random variable with distribution $|\mathcal{H}_N^\tau|^2 d\gamma$ on $\mathbb{C}$. Then, as $N \to \infty$,*

$$\frac{Z_N}{\sqrt{2N}} \to (a\cos\Theta, b\sin\Theta)$$

in distribution, where Θ is uniform on the unit circle and where $a, b > 0$ are such that $a^2 = \frac{2\tau^2}{2\tau^2-1}$, $b^2 = \frac{2(\tau^2-1)}{2\tau^2-1}$.

Proof. As announced, we follow the general strategy of [5]. Along these lines, it is easy (although somewhat tedious) to show similarly that if we let

$$\varphi(t) = \int_{\mathbb{C}} \mathrm{e}^{t(\alpha x/a+\beta y/b)} \left|\mathcal{H}_N(z)\right|^2 d\gamma(z), \quad t \in \mathbb{R}, \tag{2.1}$$

where $\alpha, \beta \in \mathbb{R}$, $\alpha^2 + \beta^2 = 1$, then φ solves the second order differential equation

$$t\varphi'' + \left[1 + t^2(1-2c^2)\right]\varphi' - t\left[c^2t^2(1-c^2) + 2c^2 + 2N\right]\varphi = 0 \tag{2.2}$$

where $c^2 = \frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2}$. Considering $\varphi(t/\sqrt{2N})$ and letting $N \to \infty$, the limiting differential equation is given, whatsoever the choice of α and β, by $t\Phi'' + \Phi' - t\Phi = 0$ which is the characterizing differential equation of the Laplace transform of the arcsine law on $(-1, +1)$. Arguing as in [5] thus shows that, for every $\alpha, \beta \in \mathbb{R}$ with $\alpha^2 + \beta^2 = 1$,

$$\frac{1}{\sqrt{2N}} \Big(\alpha \, \frac{X_N}{a} + \beta \, \frac{Y_N}{b} \Big) \to \xi$$

in distribution where ξ is distributed according to the arcsine law on $(-1, +1)$. Since $\alpha \cos \Theta + \beta \sin \Theta$, where Θ is uniform on the unit circle, is distributed as ξ, the conclusion follows. □

Proposition 2.1 interpolates between the limiting cases $\tau \to 1$ for which $a = \sqrt{2}$, $b = 0$, which gives rise to the arcsine law (the distribution of $\cos \Theta$), and $\tau \to \infty$ for which $a = b = 1$, which gives rise to the uniform distribution on the unit circle.

To apply the preceding conclusions to the spectral measure of non-Hermitian random matrix models, we have to deal, according to the representation (1.2), with averages $\frac{1}{N} \sum_{\ell=0}^{N-1} |\mathcal{H}_\ell^\tau|^2 d\gamma$. This is accomplished by a suitable mixture with an independent uniform random variable. Namely, let $f : \mathbb{C} \to \mathbb{R}$ be bounded and continuous. Then

$$\int_{\mathbb{C}} f\Big(\frac{z}{\sqrt{2N}}\Big) \, \frac{1}{N} \sum_{\ell=0}^{N-1} \big|\mathcal{H}_\ell^\tau(z)\big|^2 d\gamma(z) = \frac{1}{N} \sum_{\ell=0}^{N-1} \int_{\mathbb{C}} f\bigg(\sqrt{\frac{\ell}{N}} \cdot \frac{z}{\sqrt{2\ell}}\bigg) \big|\mathcal{H}_\ell^\tau(z)\big|^2 d\gamma(z).$$

Together with Proposition 2.1 and Lebesgue's theorem, the limiting distribution as $N \to \infty$ is thus given by $\sqrt{U}\,(a \cos \Theta, b \sin \Theta)$ where U is uniform on $[0, 1]$ and independent from Θ. Now, the law of $\sqrt{U}\,(a \cos \Theta, b \sin \Theta)$ is uniform on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$. By (1.2), we thus easily recover Girko's elliptic law interpolating between the Ginibre Ensemble and the GUE.

Corollary 2.2. *Let A_N and B_N be independent copies from the GUE. For every fixed ρ, $0 < |\rho| \leq 1$, the mean spectral distribution μ^N of $(A_N + i\rho B_N)/\sqrt{2N}$ converges as $N \to \infty$ towards the uniform law on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$ where $a^2 = \frac{2}{1+\rho^2}$, $b^2 = \frac{2\rho^2}{1+\rho^2}$.*

As announced, for $\rho = \pm 1$, we recover the circular law, and when (formally) $\rho = 0$, the semi-circular law (the distribution of $\sqrt{U}\, \cos \Theta$).

3. Weak Non-Hermitian Random Matrices

In the weak non-Hermitian regime, the parameter ρ interpolating between the Ginibre Ensemble and the GUE tends to 0 with N. As investigated in the work [2] by Y. Fyodorov, B. Khoruzhenko and H.-J. Sommers, a new limiting distribution of the spectral measure develops in this regime, provided an appropriate zoom is put on the imaginary part of the eigenvalues. This deformation may be easily identified on the limiting differential equation on Laplace transforms on which it is actually seen that the elliptic law is deformed by a Gaussian variable in the vertical coordinate.

We start again with equation (2.2). Set, for a parameter $\kappa > 0$, $\varphi_\kappa(t) = \varphi(\kappa t)$, $t \in \mathbb{R}$, where φ is defined in (2.1), and

$$\psi(t) = \mathrm{e}^{-\beta^2\sigma^2 t^2/2}\varphi_\kappa(t), \quad t \in \mathbb{R},$$

where $\sigma^2 \geq 0$. From (2.2), it is easily seen that ψ solves the second order differential equation

$$\begin{aligned} t\psi'' + &\Big[1 + t^2\big(\kappa^2(1-2c^2) + 2\beta^2\sigma^2\big]\psi' \\ &- \bigg(\kappa^2 t^3\big[\kappa^2 c^2 t^2(1-c^2) - \beta^4\sigma^4 - \beta^2\sigma^2(1-2c^2)\big] \\ &\quad + t\big[\kappa^2(2c^2+2N) - 2\beta^2\sigma^2\big]\bigg)\psi = 0. \end{aligned} \tag{3.1}$$

Therefore, provided that $\kappa \sim \frac{1}{\sqrt{2N}}$ and $\kappa^2c^2 \sim \beta^2\sigma^2$, the limiting differential equation is given as above by $t\Psi'' + \Psi' - t\Psi = 0$. Thus, with $\sigma \sim \sigma_N$, $2Na_N^2 \to 0$ and $2N\sigma_N^2 b_N^2 \sim 1$,

$$\frac{1}{\sqrt{2N}}\Big(\alpha\,\frac{X_N}{a_N} + \beta\frac{Y_N}{b_N}\Big) \to \xi + \beta\sigma G$$

in distribution, where ξ is distributed according to the arcsine law on $(-1,+1)$ and G is an independent standard normal variable. As in the preceding section, we then conclude in particular to the following result.

Proposition 3.1. *Let $Z_N = (X_N, Y_N)$ be a random variable with distribution $|\mathcal{H}_N^\tau|^2 d\gamma$ on $\mathbb{C}$. Then, as $\sigma_N \to \sigma \geq 0$ and $\tau = \tau_N \to 1$ ($b_N \to 0$) such that $2N\sigma_N^2 b_N^2 \to \eta^2 > 0$, $0 < \eta < \infty$, $N \to \infty$,*

$$\Big(\frac{X_N}{2\sqrt{N}}, \sigma_N Y_N\Big) \to (\cos\Theta, \eta\sin\Theta + \sigma G)$$

in distribution, where Θ is uniform on the unit circle and G is an independent standard normal variable.

As in the preceding section, Proposition 3.1 may be translated at the level of the averages $\frac{1}{N}\sum_{\ell=0}^{N-1}|\mathcal{H}_\ell^\tau|^2 d\gamma$, and thus, by (1.2), for the mean spectral measure of the random matrix ensemble. Let $Z_N = (X_N, Y_N)$ be a random variable with law $|\mathcal{H}_N^{\tau_N}|^2 d\gamma$. For every bounded continuous function $f : \mathbb{C} \to \mathbb{R}$,

$$\begin{aligned} \int_{\mathbb{C}} f\Big(\frac{x}{2\sqrt{N}}, \tau_N y\Big)\frac{1}{N}\sum_{\ell=0}^{N-1}|\mathcal{H}_\ell^{\tau_N}|^2 d\gamma \\ = \int_{1/N}^1 \int_{\mathbb{C}} f\bigg(\sqrt{U_N(r)}\,\frac{x}{NU_N(r)}, \sigma_N y\bigg)\big|\mathcal{H}_\ell^{\tau_N}\big|^2 d\gamma\, dr \end{aligned} \tag{3.2}$$

where $U_N(r) = \ell/N$ for $\ell/N < r \leq (\ell+1)/N$, $\ell = 0, 1, \ldots, N-1$ ($U_N(0) = 0$). Since $U_N(r) \to r$, $r \in (0,1)$, and since $2NU_N\sigma_N^2 b_N^2 \to \eta^2 r$, it follows from Proposition 3.1 that

$$\int_{\mathbb{C}} f\Big(\frac{x}{2\sqrt{N}}, \sigma_N y\Big)\frac{1}{N}\sum_{\ell=0}^{N-1}|\mathcal{H}_\ell^{\tau_N}|^2 d\gamma \to \mathbb{E}\Big(f\big(\sqrt{U}\cos\Theta, \eta\sqrt{U}\sin\Theta + \sigma G\big)\Big)$$

where U is uniform on $[0,1]$ and independent from Θ and G. In the language of the spectral measure, we recover in this way the main result of [2].

Corollary 3.2. *Let A_N and B_N be independent copies from the GUE, and denote by $\lambda_1^N, \dots, \lambda_N^N$ the eigenvalues of $(A^N + i\rho_N B^N)/2\sqrt{N}$. Set, for each $N \geq 1$ and $k = 1, \dots, N$, $\widehat{\lambda}_k^N = \mathrm{Re}(\lambda_k^N) + 2i\sigma_N\sqrt{N}\,\mathrm{Im}(\lambda_k^N)$, and denote by $\widehat{\mu}^N$ the mean empirical measure on $\widehat{\lambda}_1^N, \dots, \widehat{\lambda}_N^N$. Then, as $\sigma_N \to \sigma \geq 0$ and $4N\sigma_N^2\rho_N^2/(1+\rho_N^2) \to \eta^2 > 0$, $0 < \eta < \infty$, $N \to \infty$, $\widehat{\mu}^N$ converges towards the distribution of*

$$\big(\sqrt{U}\cos\Theta, \eta\sqrt{U}\sin\Theta + \sigma G\big)$$

where U is uniform on $[0,1]$, Θ is uniform on $[0,2\pi]$, G is a standard normal variable, U, Θ and G being independent.

Typically, $\sigma_N = \sigma$ and $\rho_N \sim \frac{\eta}{2\sigma\sqrt{N}}$ as investigated in [2]. It is not difficult to represent the density of the limiting distribution in Corollary 3.2 as

$$\frac{1}{2\pi\eta}\int_{y-\eta\sqrt{1-x^2}}^{y+\eta\sqrt{1-x^2}} \mathrm{e}^{-t^2/2\sigma^2}\,\frac{dt}{\sqrt{2\pi\sigma^2}}\,, \quad (x,y) \in (-1,+1)\times\mathbb{R}$$

(cf. [F-K-S]).

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