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SCS 31: The Lattice of Ideals of a C*-Algebra

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TOPIC The lattice of ideals of a C*-algebra

REFERENCE SCS Memo Keimis 12-15-76
and other references given below (end of memo)

One recalls that the model of an algebraic lattice is the lattice of ideals of a ring (indeed more generally the lattice of congruences of a universal algebra); the compact elements (pardon me: finite elements) typically are the finitely generated ideals (respectively, congruences). Indeed there are representation theorems for xxxxxxxx algebraic lattices as congruence lattices of suitable algebras.The situation is typical.

This is of little consolation for the functional analyst who has to do with tological algebras, notably topological rings. The appropriate ideals have to be closed in order to yield decent quotient rings,and for the most part they emerge as kernels of continuous representations, and hence must be closed (where,as usual, we assume that everything in sight is Hausdorff unless it is a continuous lattice in the Scott topology or a spectrum in the Jacobson topology). We notice that the sup of a family $\mathcal{J}$ of closed two sided ideals is $(\sum \mathcal{J})^-$, and so a finitely generated ideal or even a prinipcal ideal $\langle x\rangle$ (= smallest closed ideal containing x) is no longer a compact element in the lattice of all closed ideals.

Everybody knows that C*-algebras are crucial objects in the study of unitary representations of topological groups and in the study of operator algebras in general.

DEFINITION 1. A C*-algebra is a complex Banach algebra with an involution $a \mapsto a^*$ (* is a * automorphism satisfying $(ca)^* = \bar{c}a^*$ and $(ab)^* = b^*a^*$) such that $\|a^*a\| = \|a\|^2$.

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Standard references are the books by Dixmier [1] and Sakai [2].

DEFINITION 2. Let A be a C*-algebra. We denote with Id A the lattice of closed two sided ideals of A.

Every $I \in$ Id A is self adjoint. Further, $I, J \in$ Id A implies $IJ = I \cap J$ and $I + J = I \vee J$ [1, Dixmier 16-20].

The purpose of this memo is to discuss the following observation:

PROPOSITION 3. *If A is a C*-algebra then Id A is a continuous Brouwerian lattice.*

Thus C*-algebras provide an example of a relevant class of topological rings for which the ideal theory is governed by continuous lattice theory rather than algebraic lattice theory. This seems fitting.

In order to illustrate several aspects of the proposition, I will give several proofs. The first uses the primitive spectrum of a C*-algebra and the connection between local compactness of a space X and O(X) being continuous [see Keimis loc.cit. Corollary 5]. Recall that it is not unambiguously clear when a non-Hausdorff space should be called locally compact. We agree:

DEFINITION 4. A space X is called locally quasicompact if for every point x in any open set $U \subseteq X$ there is an open set V and a quasicompact set K such that $x \in V \subseteq K \subseteq U$.

Notice that a quasicompact space need not at all be locally quasicompact. Do locally quasicompact spaces occur?

FACT 5. Let A be a C*-algebra and let Prim $A \subseteq$ Id A be the space of all primitive ideals in the Jacobson topology. Then Prim A is locally quasicompact.

[see Dixmier 1, p.65]. Recall that an ideal is primitive, iff it is the kernel of an irreducible representation. Primitive ideals of a C*-algebra are automatically closed. Every primitive ideal is prime; the converse does not generally hold.

LEMMA 6. If X is locally quasicompact, then O(X) is a continuous lattice. Moreover, for $U, V \in O(X)$ we have $U \ll V$ iff there is a quasicompact subset K of X with $U \subseteq K \subseteq V$.

Proof. If the last assertion is proved, then by Definition 4, O(X) is a continuous lattice. Clearly $U \subseteq K \subseteq V$ with a quasicompact K implies $U \ll V$. If, conversely, $U \ll V$, let $\mathcal{U}$ be the set of all open $W \subseteq V$ for which there is a quasicompact $K_W \subseteq V$ with $W \subseteq K_W \subseteq V$. By Definition 4 we have $V = \bigcup \mathcal{U}$. Hence there are $W_1,\dots,W_n \in \mathcal{U}$ with $U \subseteq W_1 \cup \dots \cup W_n$. Then $K = K_{W_1} \cup \dots \cup K_{W_n}$ is quasicompact and satisfies $U \subseteq K \subseteq V$. []

FIRST PROOF OF PROPOSITION 3.

For any set $X \subseteq A$ we set $h(X) = \{I \in \text{Prim } A : X \subseteq I\}$ (the hull of X) and define $U(X) = \text{Prim } A \setminus h(I)$.
Then $I \mapsto U(I) : \text{Id } A \longrightarrow O(\text{Prim } A)$ is an isomorphism of lattices. [Dixmier 1, p.62]. By FACT 5 and LEMMA 6 we are done. []

REMARK 7. Let A be a C*-algebra and $I,J \in \text{Id } A$. Then the following conditions are equivalent:

(1) $I \ll J$.
(2) There is a quasicompact set W such that $U(I) \subseteq W \subseteq U(J)$.

This is clear from Lemma 6 and the isomorphism $\text{Id } A \cong O(\text{Prim } A)$. []

SECOND PROOF OF PROPOSITION 3.

Let A be a C*-algebra. A C*-seminorm p on A is a seminorm $p: A \longrightarrow \mathbb{R}^+$ with $p(a^*a) = p(a)^2$ for all $a \in A$ and $p(ab) \le p(a)p(b)$ for all $a,b \in A$. If we let $SN(A)$ denote the set of all C*-seminorms, then $SN(A) \subseteq C(A, \mathbb{R}^+)$; note that $\ker p = \{x \in A : p(x)=0\}$ is a two sided closed ideal and that the norm on $A/\ker p$ induced by p is a C*-norm hence must agree with the unique C*-quotient norm [Dixmier 1, p.7,16]. Thus $p(a) \le \|a\|$; because of $|p(a) - p(b)| \le p(a - b) \le \|a - b\|$ this implies the continuity of p. Conversely, if $I \in \text{Id } A$, then $p_I : A \longrightarrow \mathbb{R}^+$ given by $p_I(a) = \|a + I\|_{A/I}$ is a C*-seminorm with $\ker p_I = I$. Also $p_{\ker p} = p$. Clearly $I \subseteq J \Leftrightarrow p_I \ge p_J$. The set $SN(A)$ is closed under arbitrary pointwise sups (check!). We have observed:

LEMMA 8. The function $I \mapsto p_I : \text{Id } A \longrightarrow SN(A)^{op}$ is a lattice isomorphism with inverse $p \mapsto \ker p$. []

Now we note that $SN(A) \subseteq \prod\{[0,\|a\|]: a \in A\}$ and that SN(A) is closed in the pointwise topology. Thus SN(A) is a closed sup -subsemilattice of the compact Lawson semilattice $\prod_{a \in A}[0,\|a\|]$ (relative to the sup- operation). Hence SN(A) is in CL ,hence Id A is a continuous lattice by Lemma 8. ▯

The distributivity followed in proof 1 from the distributivity of O(Prim A) and will not be discussed again. Recall that in meet continuous complete lattices distributivity means being Brouwerian.

The device with the C*-norms is due to Fell; Maurice Dupré noticed that this device equips Id A with the structure of a compact topological abelian and idempotent semigroup.

REMARK 9. Let A be a C*-algebra and $I,J \in$ Id A. The the following conditions are equivalent:

(1) $I \ll J$

(3) There exist elements $a_1,\dots,a_n \in A$ and real numbers $r_1,\dots,r_n$ with $p_J(a_k) < r_k \leq p_I(a_k)$,$k=1,\dots,n$ such that $q \in SN(A)$ and $q(a_k) < r_k$, $k=1,\dots,n$ implies $q \leq p_I$.▯

THIRD PROOF OF PROPOSITION 3.

Here we use the theory developed by Gert Kjærgård PEDERSEN in half a dozne articles in Math.Scand.between 1966-1970. References are to be found in Memoir AMS 169 [3].

A subvector space V in a C*-algebra is called hereditary,iff $0 \leq a \leq v \in V$ implies $a \in V$. All closed ideals $I \in$ Id A are automatically herediary; non-closed two sided ideals need not be hereditary.

FACT 10. Let A be a C*-algebra. Then A contains a unique two sided hereditary dense ideal A^o which is minimal relative to these properties.

Example: Let X be a locally compact but not compact space and $C_o(X)=A$ the C*algebra of a continuous complex valued functions vanishing at infinity. Then $A^o = C_{oo}(X)$, the ideal of continuous functions with compact support.

EXAMPLE 11. a) Let $X = [0,\Omega[$ be the space of ordinals up to the first uncountable one (or,alternativly, the long half line). If $A=C_o(X)$, then A does not have an identity, but still $A^0 = C_{oo}(X) = C_o(X) = A$.

b) Let A be the C*-algebra of all bounded operators on an inseparable Hilbert space whose range has countable dimension. Then $A^o = A$ and A does not have an identity.

REMARK 12. Let A be a C*-algebra and $I, J \in \text{Id } A$. Then the following statements are equivalent:

(1) $I \ll J$.

(4) There is an element $a \in J^o$, $0 \leq a$ such that $I \subseteq \langle a \rangle$ (the closed ideal generated by a in A).

Proof. (1) => (4): Let $\mathcal{J} = \{\langle x \rangle : 0 \leq x \in J^o\}$. If $0 \leq x, y \in J^o$, then $0 \leq x + y \in J^0$, hence $\langle x+y \rangle \in \mathcal{J}$; but $0 \leq x \leq x + y$ and $\langle x+y \rangle$ is hereditary, whence $x \in \langle x+y \rangle$, and thus $\langle x \rangle \subseteq \langle x+y \rangle$. Thus $\mathcal{J}$ is upwards directed. Evidently $J^0 \subseteq \cup \mathcal{J}$, whence $J \subseteq (\cup \mathcal{J})^-$. Hence by (1) there is an $a \in (J^o)^+$ with $I \subseteq \langle a \rangle$.

(4) => (1): Let $\mathcal{J}$ be any up-directed family in Id A with $J \subseteq (\cup \mathcal{J})^-$. Let $\mathcal{J}' = \{J \cap K : K \in \mathcal{J}\}$. Then $J = (\cup \mathcal{J}')^-$ [Let $x \in J^+$, then $x = \lim x_m$ with $x_m \in \cup \mathcal{J}$; thus $xx_m \in J \cap \cup \mathcal{J} = \cup \mathcal{J}'$, hence $x^2 = \lim xx_m \in (\cup \mathcal{J}')^-$, hence $x \in (\cup \mathcal{J}')^-$ by the functional calculus in C*-algebras.] Now $\cup \mathcal{J}'$ is a hereditary dense two sided ideal of J, hence contains J^0 by minimality of the Pedersen ideal. Thus by (2) there is some member $K \in \mathcal{J}$ with $a \in J \cap K \subseteq K$, whence $I \subseteq \langle a \rangle \subseteq K$. □

In order to finish the THIRD PROOF OF PROPOSITION 3 we note that for each $J \in \text{Id } A$ the ideal $\cup \{\langle a \rangle : 0 \leq a \in J^o\}$ contains J^o, hence is dense in J, whence $J = \sup\{\langle a \rangle : 0 \leq a \in J^0\}$.

Example: Let $A = LC(H)$ be the C*-algebra of compact operators on a Hilbert space H. Then A^0 is the set of the finite rank operators. If a is any non-zero finite rank operator, then $\langle a \rangle = A$, since $\text{Id } A = \{\{0\}, A\}$. And indeed $A \ll A$ since Id A is finite, hence $K(\text{Id } A) = \text{Id } A$.

~~LEMMA 13. Let A be a C*-algebra and $I \in \text{Id } A$. Suppose $I \subseteq \langle a \rangle \cap \langle b \rangle$. Then $I \subseteq \langle ab \rangle = \langle abb^*a^* \rangle$.~~

SUMMARY and PROBLEMS

Let A be a C*-algebra. Then Id A is a Brouwerian continuous lattice and for two ideals I,J ⊆ Id A the following are equivalent: (1) I <<J. (2) There is a quasicompact set K with U(I) ⊆ K ⊆ U(J). (4) There is a positive element a ⊆ J^{o} (the Pedersen ideal of J) with I ⊆ <a>.

We recall that we have PRIME Id A = IRR Id A and that $(\text{PRIME Id A})^{-}$ (closure in the CL-topology of Id A) is the smallest (order) generating closed subset. We also know that PRIME Id A is closed iff condition ((1)) holds (i.e. iff I << J_1,J_2 implies I << J_1J_2). We know Prim A ⊆ PRIME Id A and that Prim A is order generating.

It is known that for separable A we have Prim A = PRIME Id A; I do not believe that the general situation is known.

The CL-topology on Id A can be obtained from the topology of pointwise convergence of SN(A) via the isomorphisms given in Lemma 8.

Problem 1. Is there a direct relation between conditions (2) and (3) above?

Problem 2. Is condition ((1)) satisfied for Id A?

Problem 3. Is Prim A closed in Id A in the CL - topology?

(This would imply Prim A = PRIME A =$(\text{PRIME A})^{-}$ and settle the question on the primitivity of primes in Id A.)

Problem 4 . Are there any alternative proofs of Proposition 3 (and alternative characterisations of I << J)?

I thank Maurice Dupré for having talked with me on these matters.

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