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DEFORMED GAUSSIAN OPERATORS ON WEIGHTED q-FOCK SPACES

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Dedicated to Professor Leonard Gross on the occasion of his 88th birthday

Abstract. We shall address the problem of reorganizing various approaches and examples concerning deformed Fock spaces and operators on them, scattered in papers [1][4][7][8][9][13], in a unified manner by the approach of Bożejko-Yoshida [10]. Moreover, we shall point out that discrete q-Hermite I polynomials (= symmetric Al-Salam-Carlitz I polynomials) can be treated within our approach. Due to this, one can provide various interesting examples of deformed Gaussian (field) operators, which were not referred in previous works [4][10].

1. Introduction

It is well-known that two theories of non-commutative probability theory were launched in the middle of 80's. One is quantum stochastic calculus on the Boson Fock space by Hudson-Parsatharathy and the other is free probability on the free Fock space by Voiculescu. The q-deformation of Brownian motions is examined by Bożejko-Speicher in [7] and the q-Gaussian processes are investigated by Bożejko-Kümmerer-Speicher in [6]. Their deformations interpolate Boson ($q = 1$), free ($q = 0$), and Fermion ($q = -1$) Fock spaces. A corresponding Gaussian process on a corresponding Fock space is defined as a sum of creation and annihilation operators on it.

On the other hand, deformed free Fock spaces by a single positive parameter are introduced in [8][9][13] to deform free independence, convolution and limit theorems from the point of statistics of pair partitions and continued fraction technique. In [10], Bożejko-Yoshida proposed the generalized q-Fock space and Gaussian field operator in such a way that [8][9][13] are contained as particular cases. In [4], Blitvić constructed the (q, t)-Fock space and realized the (q, t)-Gaussian operators on it. From now on, we shall denote a symbol "(q, t)" in the sense of [4] by "$\{q, t\}$" to avoid confusions with the (α, q)-deformation of [3][5].

Our main purpose of this paper is to reorganize various approaches and scattered examples in papers [1][4][7][8][9][13] from the point of Bożejko-Yoshida approach

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[10]. As a result, we shall present various and remarkable examples of deformed Gaussian (field) operators, which were not recognized in previous works [4][10].

2. Preliminaries

Let $\mathscr{H}$ be a complex Hilbert space equipped with the inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, where the inner product is linear on the right and conjugate linear on the left. Let $\mathcal{F}_{\rm fin}(\mathscr{H})$ denote the algebraic full Fock space over $\mathscr{H}$,

$$\mathcal{F}_{\rm fin}(\mathscr{H}) := \mathbb{C}\Omega \oplus \bigoplus_{n=1}^{\infty} \mathscr{H}^{\otimes n},$$

where Ω denotes the vacuum vector. We note that elements of $\mathcal{F}_{\rm fin}(\mathscr{H})$ are expressed as finite linear combinations of the elementary vectors $f_1\otimes\cdots\otimes f_n \in \mathscr{H}^{\otimes n}$. We equip $\mathcal{F}_{\rm fin}(\mathscr{H})$ with the inner product

$$\langle f_1\otimes\cdots\otimes f_m, g_1\otimes\cdots\otimes g_n\rangle_0 := \delta_{m,n}\prod_{k=1}^{n}\langle f_k, g_k\rangle, \quad f_k, g_k \in \mathscr{H}.$$

For $q \in (-1,1)$, define the q-symmetrization operator on $\mathscr{H}^{\otimes n}$ as

$$P_q^{(n)} = \sum_{\sigma\in\mathfrak{S}_n} q^{\ell(\sigma)}\sigma, \quad n \geq 1,$$
$$P_q^{(0)} = I_{\mathscr{H}^{\otimes 0}},\ P_0^{(n)} = I_{\mathscr{H}^{\otimes n}},$$

where we put $0^0 = 1$ and $\mathscr{H}^{\otimes 0} = \mathbb{C}\Omega$ by convention and

$$P_q = \bigoplus_{n=0}^{\infty} P_q^{(n)}$$

be the q-symmetrization operator on $\mathcal{F}_{\rm fin}(\mathscr{H})$. Since $P_q^{(n)}$ is known to be strictly positive,

$$\langle f_1\otimes\cdots\otimes f_m, g_1\otimes\cdots\otimes g_n\rangle_q := \langle f_1\otimes\cdots\otimes f_m, P_q(g_1\otimes\cdots\otimes g_n)\rangle_0$$

becomes an inner product and $\langle\cdot,\cdot\rangle_q$ is called the q-inner product with the convention $0^0 = 1$.

Definition 2.1. (1) For $q \in (-1,1)$, the (algebraic) full Fock space $\mathcal{F}_{\rm fin}(\mathscr{H})$ with respect to $\langle\cdot,\cdot\rangle_q$ is called the *q-Fock space* (the *Fock space of type* A) denoted by $\mathcal{F}_q(\mathscr{H})$ of Bożejko-Speicher [7]. In this paper, we do not take completion.
(2) Let $b_q^\dagger(f)$ be defined as the usual left creation operator,

$$b_q^\dagger(f)\Omega = f, \quad f \in \mathscr{H},$$
$$b_q^\dagger(f)(f_1\otimes\cdots\otimes f_n) = f\otimes f_1\otimes\cdots\otimes f_n, \quad n \geq 1,$$

and $b_q(f)$ be its adjoint with respect to $\langle\cdot,\cdot\rangle_q$, that is, $b_q = (b_q^\dagger)^*$. $b_q^\dagger$ and b_q are called the q-creation and q-annihilation operators, respectively.

The following proposition is a direct consequence of the definition.

Proposition 2.2. (1) *The q-annihilation operator b_q acts on the elementary vectors as follows:*

$$b_q(f)\Omega = 0, \quad b_q(f)f_1 = \langle f, f_1\rangle\,\Omega,$$
$$b_q(f)(f_1\otimes\cdots\otimes f_n) = \sum_{k=1}^{n} q^{k-1}\langle f, f_k\rangle\, f_1\otimes\cdots\otimes \overset{\vee}{f}_k\otimes\cdots\otimes f_n, \quad n\geq 2,$$

where $\overset{\vee}{f}_k$ means that f_k should be deleted from the tensor product.
(2) *The q-creation and q-annihilation operators satisfy the q-commutation relation (q-CCR)*

$$b_q(f)b_q^\dagger(g) - q\, b_q^\dagger(g)b_q(f) = \langle f, g\rangle\mathbf{1} \qquad f, g\in\mathscr{H}.$$

The readers can refer to [6][7] for details.

3. Weighted q-Deformation

Let $\{\tau_n\}_{n=1}^\infty$ be a sequence of strictly positive numbers and $[\tau_n]! := \prod_{i=1}^n \tau_i$.

Definition 3.1 ([10])**.** The *τ-weighted q-symmetrization operators* on $\mathscr{H}^{\otimes n}$ and $\mathcal{F}(\mathscr{H})$, respectively, are defined by

$$T_q^{(0)} = P_q^{(0)}, \quad T_q^{(n)} = [\tau_n]!P_q^{(n)},\ n\geq 1,$$
$$T_q = \bigoplus_{n=0}^\infty T_q^{(n)}.$$

Since $P_q^{(n)}$ and $\{\tau_n\}_{n=1}^\infty$ are a strictly positive operator and sequence, respectively, the τ-weighted q-inner product is defined by

$$\langle f_1\otimes\cdots\otimes f_m, g_1\otimes\cdots\otimes g_n\rangle_{q,\{\tau_n\}} :=\ \langle f_1\otimes\cdots\otimes f_m, T_q(g_1\otimes\cdots\otimes g_n)\rangle_0.$$

Let $\mathcal{F}_{q,\{\tau_n\}}(\mathscr{H})$ denote the τ-weighted (generalized) q-Fock space. The τ-weighted q-creation operator $b^\dagger_{q,\{\tau_n\}}(f)$ is defined as the usual left creation operator and $b_{q,\{\tau_n\}}(f)$ is its adjoint with respect to $\langle\cdot,\cdot\rangle_{q,\{\tau_n\}}$, that is, $b_{q,\{\tau_n\}} = (b^\dagger_{q,\{\tau_n\}})^*$.

Proposition 3.2. *(1) The τ-weighted q-annihilation operator $b_{q,\{\tau_n\}}$ acting on the elementary vectors is given as follows:*

$$b_{q,\{\tau_n\}}(f)\Omega = 0, \quad b_{q,\{\tau_n\}}(f)f_1 = \tau_1\langle f, f_1\rangle\,\Omega, \quad f\in\mathscr{H},$$
$$b_{q,\{\tau_n\}}(f)(f_1\otimes\cdots\otimes f_n) = \tau_n\sum_{k=1}^{n} q^{k-1}\langle f, f_k\rangle\, f_1\otimes\cdots\otimes \overset{\vee}{f}_k\otimes\cdots\otimes f_n, \quad n\geq 2,$$

where $\overset{\vee}{f}_k$ means that f_k should be deleted from the tensor product.
(2) The τ-weighted q-creation and annihilation operators satisfy

$$b_{q,\{\tau_n\}}(f)b^\dagger_{q,\{\tau_n\}}(g) - q\beta_N b^\dagger_{q,\{\tau_n\}}(g)b_{q,\{\tau_n\}}(f) = \langle f, g\rangle\tau_{N+1}, \quad f, g\in\mathscr{H},$$

where $\{\beta_n := \tau_{n+1}/\tau_n\}_{n=1}^\infty$ and operators β_N and τ_N are defined as

$$\begin{cases}\varphi_N\Omega = \Omega,\ \varphi_N(f_1\otimes\cdots\otimes f_n) = \varphi_n(f_1\otimes\cdots\otimes f_n), \quad n\geq 1,\\ \varphi\in\{\beta,\tau\}.\end{cases}$$

Proof. By direct computations, we have

$$
\begin{aligned}
&b_{q,\{\tau_n\}}(f)b^{\dagger}_{q,\{\tau_n\}}(g)f_1\otimes\cdots\otimes f_n\\
&=b_{q,\{\tau_n\}}(f)g\otimes f_1\otimes\cdots\otimes f_n\\
&=\tau_{n+1}\langle f,g\rangle f_1\otimes\cdots\otimes f_n+\tau_{n+1}\sum_{k=2}^{n+1}q^{k-1}\langle f,f_{k-1}\rangle\, g\otimes f_1\otimes\cdots\otimes \overset{\vee}{f}_{k-1}\otimes\cdots\otimes f_n\\
&=\tau_{n+1}\langle f,g\rangle f_1\otimes\cdots\otimes f_n+g\otimes\left(\beta_n\tau_n\sum_{k=1}^{n}qq^{k-1}\langle f,f_k\rangle\, f_1\otimes\cdots\otimes \overset{\vee}{f}_{k}\otimes\cdots\otimes f_n\right)\\
&=\tau_{n+1}\langle f,g\rangle f_1\otimes\cdots\otimes f_n+q\beta_n b^{\dagger}_{q,\{\tau_n\}}(g)b_{q,\{\tau_n\}}(f)f_1\otimes\cdots\otimes f_n.
\end{aligned}
$$

Hence our claim is obtained. □

Corollary 3.3. *Suppose $\tau_1 = 1$ and $\beta_n = Q > 0$ for $n \geq 1$. The following commutation relation holds:*

$$b_{q,\{\tau_n\}}(f)b^{\dagger}_{q,\{\tau_n\}}(g)-qQb^{\dagger}_{q,\{\tau_n\}}(g)b_{q,\{\tau_n\}}(f)=\langle f,g\rangle\tau_{N+1},\quad f,g\in\mathscr{H}.$$

Example 3.4. Suppose $\tau_1 = 1$ and $q \in (-1, 1)$.
(1) $Q = 1$ implies $\tau_n = \tau_2 > 0$, $n \geq 2$. If we set $\tau_2 = t$, then one can get $T_q^{(n)} = t^{n-1}P_q^{(n)}$ and the $(q,t)_W$-Fock space in the sense of Wojakowski [12]. Furthermore, if we take $q = 0$, one can derive the t-free deformation done by Bożejko-Wysoczańsky [8][9].
(2) If $Q = s^2, s \in (0, 1]$, then we have $\tau_n = s^{2(n-1)}$, $n \geq 1$. One can get $T_q^{(n)} = s^{n(n-1)}P_q^{(n)}$ and the $(q,s)_{BY}$-Fock space by Bożejko-Yoshida [10]. The s-free deformation of Yoshida [13] can be derived if $q = 0$. Moreover, one can see that a limiting case of $(q,s)_{BY}$ as $q \to 1$ coincides with the Q^N-deformation of the Boson Fock space [1].
(3) The Boolean Fock space can be derived as a limiting case of the $(0,t)_W$-Fock space as $t \to 0$ and also $(0,s)_{BY}$-Fock space as $s \to 0$.

One can derive a further deformation from (2) in Example 3.4. We shall show the relationship between Blitvić [4] construction and ours. In fact, if we replace s^2 by $t > 0$ in (2) of Example 3.4, then we have $\tau_n = t^{n-1}$ and $[\tau_n]! = t^{\binom{n}{2}}$ for $n \geq 1$. In addition, if one considers the (q/t)-symmetrization operator $P^{(n)}_{q/t}$, which is strictly positive for $|q| < t$, then one can consider the weighted (q/t)-symmetrization operator in forms of $T^{(0)}_{q/t} = P^{(0)}_{q/t}$ and $T^{(n)}_{q/t} = t^{\binom{n}{2}}P^{(n)}_{q/t}$, $n \geq 1$. From now on, we set

$$
\begin{aligned}
&Q^{(0)}_{q,t}=T^{(0)}_{q/t},\quad Q^{(n)}_{q,t}=T^{(n)}_{q/t},\ n\geq 1,\ |q|<t,\\
&Q_{q,t}=\bigoplus_{n=0}^{\infty}Q^{(n)}_{q,t},
\end{aligned}
$$

which are called the $\{q,t\}$-symmetrization operators on $\mathscr{H}^{\otimes n}$ and $\mathcal{F}(\mathscr{H})$, respectively. An inner product defined by

$$\langle f_1\otimes\cdots\otimes f_m, g_1\otimes\cdots\otimes g_n\rangle_{q,t} := \langle f_1\otimes\cdots\otimes f_m, Q_{q,t}(g_1\otimes\cdots\otimes g_n)\rangle_0$$

is called the $\{q,t\}$-inner product, which is the $(q/t,\sqrt{t})_{BY}$-inner product. The free Fock space equipped with this $\{q,t\}$-inner product is called the $\{q,t\}$-Fock space denoted by $\mathcal{F}_{q,t}(\mathscr{H})$. Therefore, we have seen the following propositions:

Proposition 3.5. *Suppose $q\in(-1,1),\ t\in(0,1]$ and $|q|<t$. The $(q/t,\sqrt{t})_{BY}$-Fock space is equivalent to the $\{q,t\}$-Fock space in the sense of* [4].

The $\{q,t\}$-creation operator $a^\dagger_{q,t}(f)$ is defined as the usual left creation operator and $\{q,t\}$-annihilation operator $a_{q,t}(f)$ as its adjoint with respect to $\langle\cdot,\cdot\rangle_{q,t}$.

Proposition 3.6. *(1) The $\{q,t\}$-annihilation operator $a_{q,t}$ acting on the elementary vectors is given as follows:*

$$a_{q,t}(f)\Omega=0,\quad a_{q,t}(f)f_1=\langle f,f_1\rangle\,\Omega,\quad f\in\mathscr{H},$$

$$a_{q,t}(f)(f_1\otimes\cdots\otimes f_n)=t^{n-1}\sum_{k=1}^{n}\left(\frac{q}{t}\right)^{k-1}\langle f,f_k\rangle\, f_1\otimes\cdots\otimes \overset{\vee}{f}_k\otimes\cdots\otimes f_n\quad n\geq 2, \tag{3.1}$$

where $\overset{\vee}{f}_k$ means that f_k should be deleted from the tensor product.
(2) The $\{q,t\}$-creation and annihilation operators satisfy

$$a_{q,t}(f)a^\dagger_{q,t}(g)-qa^\dagger_{q,t}(g)a_{q,t}(f)=\langle f,g\rangle t^N,\quad f,g\in\mathscr{H},$$

where the operator t^N is defined by

$$t^N\Omega=\Omega,\quad t^N(f_1\otimes\cdots\otimes f_n)=t^n f_1\otimes\cdots\otimes f_n,\quad n\geq 1.$$

Proof. It is easy to see our claim if we replace q by q/t and set $Q=t,\ \tau_n=t^{n-1}$ in Proposition 3.2 and Corollary 3.3. □

Remark 3.7. It is easy to see that the expression (3.1) can be written as

$$a_{q,t}(f)(f_1\otimes\cdots\otimes f_n)=\sum_{k=1}^{n}q^{k-1}t^{n-k}\langle f,f_k\rangle\, f_1\otimes\cdots\otimes \overset{\vee}{f}_k\otimes\cdots\otimes f_n,\quad n\geq 2.$$

We would like to consider the spectral measure (vacuum distribution) of the $\{q,t\}$-Gaussian (field) operator $g_{q,t}(f)$ on $\mathcal{F}_{q,t}(\mathscr{H})$ defined by

$$g_{q,t}(f):=a^\dagger_{q,t}(f)+a_{q,t}(f),\quad f\in\mathscr{H},$$

with respect to the vacuum state $\langle\Omega,\cdot\Omega\rangle_{q,t}$. Orthogonal polynomials play important roles to compute a distribution of such a field operator with respect to the vacuum state. In this paper we shall focus on

Definition 3.8. The $\{q,t\}$-*Hermite polynomials* are defined by the recurrence relation,

$$H_0(x;q,t)=1,\ H_1(x;q,t)=x,$$
$$xH_n(x;q,t)=H_{n+1}(x;q,t)+[n]_{q,t}H_{n-1}(x;q,t),\ n\geq 1,$$

where

$$[n]_{q,t} := \frac{t^n - q^n}{t-q}, \ (t \neq q)$$
$$= t^{n-1}[n]_{q/t}$$

and $[n]_q := [n]_{q,1} = \sum_{k=0}^{n-1} q^k$. Note that $[n]_{q,q} := \lim_{t\to q}[n]_{q,t} = q^{n-1}n$.

In [4], concrete examples of orthogonal polynomials and densities of orthogonalizing measures are not mentioned except for a very restricted case, $0 = q < t$. We have been seeking examples for $q \neq 0$, which can be treated within the $\{q,t\}$-deformation. In this paper, we shall present not only recognized examples, but also unrecognized ones in [4][10] as follows.

Example 3.9. Let us consider the $\{qs^2, s^2\}$-deformation for $q \in (-1,1), s \in (0,1]$. This deformation is of interest and quite fruitful.
(I) The $\{qs^2, s^2\}$-Gaussian (field) operator is equal to the $(q,s)_{BY}$-Gaussian (field) operator. The $\{q,s^2\}$-deformation is different from the $(q,s)_{BY}$ except for $q = 0$ or $s = 1$.
(II) In addition, the probability density for $(q,s)_{BY}$ case is known for the following three cases,

$$\begin{cases} (1)\ s = 1, & q \in (-1,1),\ \text{(in [6][7])} \\ (2)\ s \in (0,1], & q = 0,\ \text{(in [4][13])} \\ (3)\ s = \sqrt{|q|}, & |q| \in (0,1). \end{cases}$$

The case (1) is obvious at this time and provides the (Roger's continuous) q-Hermite polynomials. Therefore, one can obtain the q-Gaussian operator ([6][7]).

In case (2), it is known that the $\{0,t\}$-Hermite polynomials are the t-Chebyshev II polynomials ($q = 0 < t \leq 1$ and set $t = s^2$) . The $\{0,1\}$-Gaussian measure is the semicircular measure. If $t \neq 1$, the $\{0,t\}$-Gaussian measure is known to be a discrete probability measure with atoms at which are represented by the zeros of the t-Airy function (See [4] and references cited therein). The $\{0,t\}$-Gaussian (field) operator is the same as the $(0,\sqrt{t})_{BY}$-Gaussian (field) operator, which is nothing but the s-free Gaussian (field) operator [13]. Moreover, the limiting case $s \to 0$ implies the Boolean Gauss (field) operator, whose distribution is $\frac{1}{2}(\delta_1 + \delta_{-1})$. The case (3) is not referred as a particular example in [4][10]. Let us recall the discrete q-Hermite I polynomials given by the recurrence relation,

$$H_0(x;q) = 1,\ H_1(x;q) = x,$$
$$xH_n(x;q) = H_{n+1}(x;q) + q^{n-1}(1-q^n)H_{n-1}(x;q),\ n \geq 1,\ q \in (0,1),$$

which are a symmetric case of Al-Salam-Carlitz I polynomials (Al-Salam-Carlitz '65. See also [2] and [11]). The corresponding orthogonalizing measure is symmetric and uniquely given as

$$\mu_q = \sum_{k=0}^{\infty} \frac{(q^{k+1}, -q^{k+1}; q)_\infty q^k}{(q,-1,-q;q)_\infty} \left(\delta_{q^k} + \delta_{-q^k}\right), \quad q \in (0,1), \tag{3.2}$$

where δ_y denotes the Dirac measure at y and

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{k=1}^{n}(1-aq^{k-1}),$$

$$(a_1, a_2, \ldots, a_k; q)_n = \prod_{j=1}^{k}(a_j;q)_n, \quad n = 0,1,2,\ldots,\infty.$$

Therefore, one can show that the $\{q^2, |q|\}$-Hermite polynomials are identified as a rescaled version of discrete $|q|$-Hermite I polynomials $H_n(x;q)$. The orthogonalizing measure is given by

$$D_{1/\sqrt{1-|q|}}\mu_{|q|}, \quad |q| \in (0,1),$$

where D_λ denotes the dilation of a probability measure μ by $D_\lambda\mu(\cdot) = \mu(\cdot/\lambda)$, $\lambda \neq 0$. Moreover, the $\{q^2, |q|\}$-Gaussian (field) operator coincides with the $(q, \sqrt{|q|})_{BY}$-Gaussian (field) operator.
(III) Furthermore, since the $(q,s)_{BY}$-Fock space as $q \to 1$ coincides with the Q^N-deformation of the Boson Fock space mentioned in (2) of Example 3.4, a limiting case of the $\{qs^2, s^2\}$-Gaussian (field) operator as $q \to 1$ agrees with the Q^N-deformation of the classical Gaussian (field) operator [1]. It is this paper which first points out this nontrivial relationship of interest.

Remark 3.10. We note that the discrete q-Hermite I polynomials (= symmetric Al-Salam-Carlitz I) belong to the class IV of Brenke-Chihara polynomials, whose orthogonalizing measures are symmetric (Asai-Kubo-Kuo [2]). It is not possible in general to capture the class I, II, and III of the Brenke-Chihara polynomials, because orthogonalizing measures for these classes are not symmetric in general, but those of $\{q,t\}$-Hermite polynomials are symmetric.

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