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High School 4th Mathematics: Precalculus for AP Calculus

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HIGH SCHOOL 4th MATHEMATICS: PRECALCULUS FOR AP CALCULUS

A Thesis

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ABSTRACT

The purpose of this thesis is to provide the needed instructional materials to those who are teaching a Precalculus course following Algebra I, Geometry, and Algebra II. The recent adoption of the Common Core State Standards in Mathematics (CCSSM) has left many teachers scrambling to find instructional materials that meet the graduation requirement as well as insuring that our students are college and career ready when they leave high school. Furthermore, the College Board's Advanced Placement (AP) Calculus curriculum is generally accepted as the model for a twenty-first century calculus course serving as prerequisite for STEM related fields of study at the college level. The path now needs to be set for a new precalculus course to align the AP goals and objectives with the CCSSM. For the 2014-2015 school year, high schools must offer AP courses in all four core content areas, math, ELA, science, and social studies (*www.louisianabelieves.com*). However, for students to be adequately prepared for AP Calculus there must be an effective precalculus course available to be taken first. This thesis, "High School 4th Mathematics: Precalculus for AP Calculus," is written specifically with the goal of meeting this requirement. In Appendix C of this thesis, high school mathematics teachers are provided with comprehensive lecture notes that contain lesson plans and student activities that are aligned with the CCSSM, and the Common Core State (+) Standards in Mathematics (CCS(+)SM). Each section of the lecture notes consists of a lesson plan that begins with a comprehensive overview of the major concepts, a list of the related CCSSM, a set of section learning objectives, lecture notes, and a variety of lesson activities that support the Common Core State Content Standards as well as Mathematical Practice Standards (MPS). Even though Appendix C can be used by any Precalculus teacher as a resource, it is designed specifically to go along with the textbook, *Precalculus* 8th Edition, written by Demana, Waits, Foley, and

Kennedy, the textbook which will be used in 2014-2015 by Southeastern Louisiana University for its Dual Enrollment Precalculus course, Math 165.

INTRODUCTION

Euclid once told King Ptolemy in the fourth century, B.C., there is no royal road to geometry. Still, to this date, neither is there a royal road to mathematical proficiency. In a world in which U.S. workers and industries are dealing with new forms of technology and facing the challenges of a global economy, it is increasingly urgent to ask: have we succeeded in fulfilling the goals of *A Nation at Risk*? This report, released in 1983, shocked the educational pride of the country and called for dramatic improvements of the quality of education. Yet, over three decades later, our students still fall off a mathematical cliff somewhere in the middle and upper grades. Employers report difficulties finding workers who have the skills and knowledge required to fill today's positions requiring technological sophistication. Internationally, America is the only industrialized country whose students fall further behind the longer they stay in school. The report *Trends in International Mathematics and Science Study* (TIMSS) has concluded that the quality of mathematics education in the United States is still declining and our students are not performing well at the international level.

In response to the growing national concerns about our students not having a competitive edge academically, the CCSS Initiative was launched in April 2009, based on the previous ground breaking work done by organizations like the National Council of Teachers of Mathematics (NCTM), the College Board, and many others (see Section 2.1). These nationally and internationally benchmarked K-12 academic standards for mathematics and English establish what students are expected to have learned when they graduate from high school and enter postsecondary education or the workplace. The final CCSS were released in June 2010 and have since been adopted by 43 states; the State of Louisiana is one of them (http://www.corestandards.org/standards-in-your-state/). The CCSS in mathematics are comprised of the

Standards of Mathematical Content and the Standards of Mathematical Practices in which procedural and conceptual knowledge as well as modeling skills and mathematical concepts of which need to be developed in order to be college and career ready by the end of grade twelve.

The CCSS are a fully state-led effort, and there is no overarching implementation process that can be applied in all states. Each state is individually responsible for implementing the standards in a manner best suited to their unique population of students. Unfortunately, the instructional shift in the recently adopted CCSS has left educators looking for appropriate instructional materials to prepare all students for the rigors of postsecondary education and be workforce ready by the time they graduate high school. There are some concerted state efforts in providing guidance for the first three high school mathematics classes (Algebra I, Geometry, and Algebra II) because these students will be assessed by one of two consortia who are designing new national CCSS aligned assessments, Smarter Balanced Assessment and Partnership for Assessment of Readiness for College and Careers (PARCC) courses. Louisiana is part of the PARCC consortium. However, there is very little guidance and quality instructional materials available for the fourth mathematics course following Algebra I, Geometry, and Algebra II. This problem is amplified by the fact that for the 2014-2015 school year, high schools in Louisiana must offer AP courses in all four core content subjects. In mathematics, for many high schools, this means offering AP Calculus.

As a classroom teacher who is teaching both Precalculus and AP Calculus classes, my thesis is a blue print of, or at least my idea of, what a CCSS aligned Precalculus course for AP Calculus should entail. With this in mind, the core of this thesis lies in Appendix C, Precalculus for AP Calculus: A Companion to Precalculus, 8th Edition, by Demana, Waits, Foley, and Kennedy: Comprehensive Resources that are AP Calculus Ready and CCSS and CCS(+)S

Aligned. In this 133- page appendix, Precalculus teachers are provided with Lesson Topics, Major Concepts, CCSSM and CCS(+)SM Alignments (see Appendix B), Objectives, Sequence of Lessons, Lecture Notes, and Student Activities. These lecture notes are a first draft and are intended for use as a companion to the Precalculus textbook by Demana et al. Unbeknownst to the book's authors, I am grateful because they have inspired me to write this thesis. After examining many textbooks, I selected this book for two reasons:

(1) Its balanced approach among the algebraic, numerical, graphical, and verbal methods of representing problems, which is very much like the AP approach.

(2) It is published by Pearson Education® which means MathXL (My Math Lab) is available. Even though the development of assignments using MathXL is not part of this thesis, web-based assignments will be available and used for the Dual Enrollment Precalculus course, Math 165, through Southeastern Louisiana University (see Section 1.3). I was first introduced to the MathXL platform in the summer of 2008 as I went through a Louisiana State University (LSU) workshop on using it in conjunction with a dual enrollment opportunity in College Algebra and Trigonometry with Southeastern Louisiana University. I received further exposure to MyMathLab over the course of the Master of Natural Science Program at LSU through the Louisiana Mathematics and Science Teachers Institute (LaMSTI). MathXL tutorial provides algorithmically generated practice exercises that are correlated at the chapter, section, and objective level to the exercise in the textbook. Every practice question is accompanied by an example and a guided solution designed to involve students in the solution process. Selected questions also include a video clip to help students visualize concepts. The software provides immediate feedback and can generate printed summary of students' progress.

To keep the integrity of these authors' work, to be truthful to their mission, and to best serve the students and teachers using this textbook, I have used their language, ideas, and examples as often as possible.

Many student activities cited in Appendix C are imported from Illustrative Mathematics, and easily recognizable by the icon:

Illustrative Mathematics.

Furthermore, included in the lecture notes are direct links to the uniform resource locator (url) of the specific problems in Illustrative Mathematics so that teachers can find helpful comments and solutions to the student activities at their fingertips. Such as:

Illustrative Mathematics **Comments and Solutions can be found at:**
http://www.illustrativemathematics.org/illustrations/599

In alignment with AP goals, writing assignments are incorporated throughout the lecture notes to help students understand the mathematics they study. These assignments are denoted by the icon:

Writing Assignment .

Lastly, conscientious efforts were made to point teachers to the connection between current Precalculus topics and future AP Calculus topics by adding alerts under the icon:

Looking Ahead to Calculus .

The body of this thesis is organized as follows. In Chapter 1, an overview is given of what high school 4th mathematics courses are, why our students need to take these courses, the role of high school 4th mathematics, and the instructional goal for a 4th year Precalculus course for AP Calculus. Chapter 2 discusses the history of mathematics standards and the reform

movements from the *New Math* of the 1960s to CCSSM in 2010. Chapter 3 is the literature review for this thesis. Finally, Chapter 4 explains why having a web-based assignment is an invaluable classroom tool in teaching and learning mathematics. This educational software platform is designed to generate multiple iterations, thus providing students with ample opportunities to improve their procedural fluency. And this allows teachers to maximize the instructional time in the classroom to help students in developing conceptual understanding of the topics to be studied.

CHAPTER 1. OVERVIEW OF HIGH SCHOOL 4th MATHEMATICS

1.1 Why 4th Mathematics?

Faced with tougher graduation requirements (Table 1) and scholarship offerings like the Taylor Opportunity Program for Students (Table 2), all college bound high school students are scheduled to take a 4th mathematics course past Algebra I, Geometry, and Algebra II .

Table 1: LA High School Graduation Requirement

College & Career Diploma	Career Diploma	Career/Technical Endorsement	Academic Endorsement

Basic Core Curriculum
(Incoming Freshmen 2008-2009 and beyond)

English - 4 Units

- English I, II, III, IV or Senior Applications in English

Math - 4 Units

- Algebra I (1 unit), Applied Algebra I (1 unit) or Algebra I-Pt. 1 and Algebra I-Pt. 2 (2 units)
- Geometry or Applied Geometry
- Remaining unit(s) from the following: Algebra II, Financial Mathematics, Math Essentials, Advanced Math—Pre-Calculus, Advanced Math—Functions and Statistics, Pre-Calculus, Calculus, Probability and Statistics, Discrete Mathematics, or a local math elective approved by BESE

LA Core 4 Curriculum
(Incoming Freshmen in 2008-2009 and beyond)

English - 4 Units

- English I, II, III, IV

Math - 4 Units

- Algebra I, Applied Algebra I, or Algebra I-Pt. 2
- Geometry or Applied Geometry
- Algebra II
- Remaining unit from the following: Financial Math, Math Essentials, Advanced Math/Precalculus, Advanced Math-Functions and Statistics, Pre-Calculus, Calculus, Probability and Statistics, Discrete Math, or a local math elective approved by BESE

Table 2: TOPS Core Curriculum

LOSFA
TOPS
Scholarship

TOPS Core Curriculum
For the Opportunity, Performance, and Honors Awards
For High School graduates of 2014 and thereafter

Units	Courses[1]
ENGLISH = 4 Units	
4 units	English I, II, III, & IV
MATH = 4 Units	
1 unit	Algebra I, or Integrated Mathematics I, or Applied Algebra I or Algebra I - Parts 1 & 2 (two units) or Applied Mathematics I & II (two units) or Applied Algebra 1A and 1B (two units)
1 unit	Algebra II or Integrated Mathematics II
2 units	Geometry, Applied Geometry, Pre-Calculus, Advanced Math-Pre-Calculus, Calculus, AP Calculus AB, Advanced Math-Functions and Statistics, Probability and Statistics, Discrete Mathematics, Applied Mathematics III, Integrated Mathematics III, or Algebra III

Not surprisingly, the study, *What Are ACT's College Readiness Benchmarks?*(2013), published by American College Testing (ACT) has found solid evidence that taking more mathematics courses correlates with greater success on students' college entrance examinations. In this report, of the students taking Algebra 1, Geometry, and Algebra 2, but no other mathematics courses, only 13% met the benchmark for readiness for college algebra. However, if students took at least one additional upper level mathematics course such as Precalculus or Trigonometry, 74% of them met the benchmark. ACT's College Readiness Benchmarks are the minimum ACT Test scores (Table 3) for students to have a high probability of success in credit-bearing college courses—English Composition, Social Sciences courses, College Algebra, or Biology. Students who meet a Benchmark on the ACT Test have approximately a 50% likelihood of earning a B or better and approximately a 75% chance of earning a C or better in the corresponding college course or courses (*What Are ACT's College Readiness Benchmarks?*, 2013).

Table 3: ACT's College Readiness Benchmarks

College Course or Course Area	ACT Subject-Area Test	The ACT Test Benchmark
English Composition	English	18
College Algebra	Mathematics	22
Social Sciences	Reading	21
Biology	Science	24

Figure 1 below contains the results for the ACT Mathematics test. Seven mathematics course sequences were examined in total; students taking six course sequences were compared to students who took less than Algebra 1, Geometry, and Algebra 2.

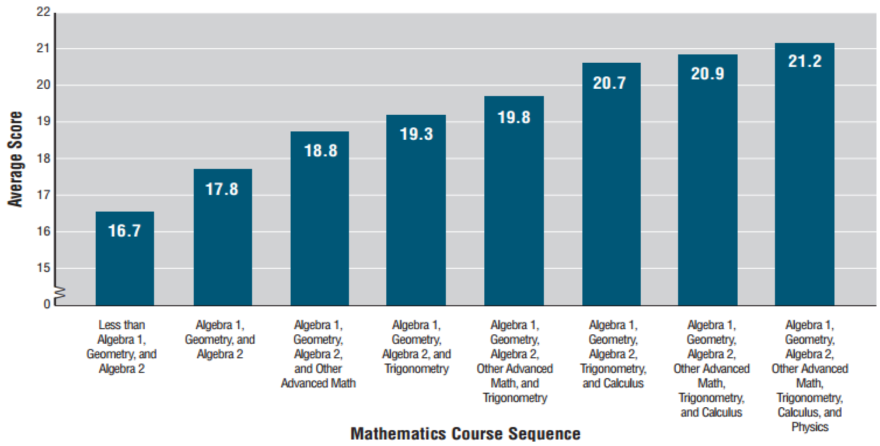


Figure 1: Average ACT Mathematics Score Associated with Mathematics Courses

In his groundbreaking report *Answers in the Toolbox*, Adelman (*Executive Summary*, 1999) found that the strongest predictor of graduating from college was that students who took mathematics higher than Algebra II (for example, Trigonometry, Precalculus, and Calculus) earned a college degree at twice the rate of those whose high school mathematics stopped at Algebra II. In this report, Adelman stated that 83.3% of 12th graders who had taken a Calculus course in 1992 graduated with a bachelor’s degree by 2000. For those whose most advanced course was Precalculus, 74.6% graduated, compared to 60% for Trigonometry, and 39.9% for Algebra II. *Answers in the Toolbox* is a study about what contributes most to long-term bachelor's degree completion of students who attend 4-year colleges. After all, degree completion is the true bottom line for college administrators, state legislators, parents, and most importantly, students. Needless to say, this data alone is a clear indication of how a well-developed high school Precalculus course would help students to be successful in colleges.

1.2 What are the 4th Mathematics Courses and Which Course Should One Choose?

Several reviews of state education policies indicate that it may be easy for many students to meet graduation requirements but bypass the mathematics and science courses and course sequences that contribute most to postsecondary readiness (Campbell et al., 2000; Mullis et al., 1998; National Research Council, 1999a; Potts, Blank, & Williams, 2002). For example, graduation requirements are often expressed in terms of credits. A credit is often defined as a Carnegie unit, an academic year course (e.g., the number of credits in various subject areas needed to graduate), rather than as specific academic courses (Potts et al., 2002). To the extent that high schools offer courses other than those in the college preparatory sequences, students may satisfy graduation requirements (i.e., number of credits) without taking the specific courses that would best prepare them for further education (and work). Consequently, many high school graduates do not possess the requisite knowledge for college-level work. College professors estimate that 42% of students are not adequately prepared for college, and 70% of college instructors report having to devote some of their first-year class time toward reviewing content that they feel should have been taught in high school. Only 28% of college instructors believe that public high schools adequately prepare students for the challenges of college (Achieve, 2004). Similarly, Conley (2007) argued that high schools often inadequately prepare students with the skills required for college-level courses, which are generally faster paced and require students to engage in more high-level tasks. The requisite skills include drawing inferences, interpreting results, analyzing conflicting sources of information, supporting arguments with evidence, and thinking deeply about material (Conley, 2007).

While some states offer students the option to pursue a truly rigorous course of study, a less rigorous set of course requirements remains the standard in almost every state. These

inadequacies may explain why so many students fulfill the requirements and graduate from high school but are ill-prepared for postsecondary level work. Unfortunately, without doubt, many high school students and parents believe that simply meeting the number of credits required for graduation will provide adequate preparation for college (Venezia et al., 2003). In any case, this assumption is incorrect. While students generally have multiple course options from which to choose to satisfy requirements, not all will adequately prepare students for postsecondary study (American Diploma Project, 2004). This is particularly true in mathematics and science (ACT, 2004; Schmidt et al., 1997).

In a high school setting, the sequence of the first three mathematics classes is widely accepted: Algebra I, Geometry, and Algebra II or Mathematics I, Mathematics II, and Mathematics III. In addition to the three courses, the CCSS suggests that "students should continue to take mathematics courses throughout their high school career to keep their mathematical understanding and skills fresh for use in training or course work after high school" (*CCSS, The Pathways*). However, there is much inconsistency about what a 4th high school mathematics course should entail.

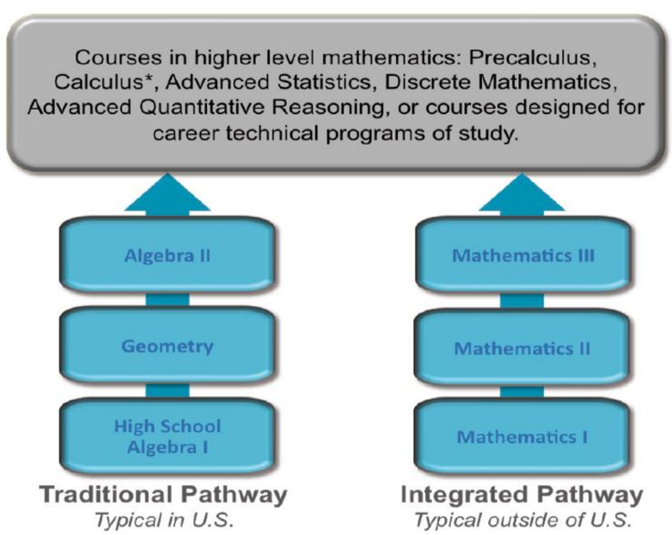


Figure 2: High School Mathematics Pathways

Again, there is compelling rationale for urging students to continue their mathematical education throughout high school, allowing students several rich options once they have demonstrated mastery of core content covered in Algebra I, Geometry, and Algebra II. The Pathways (Figure 2) describe possible courses for the first three years of high school. Quoting the statements from the CCSSM Appendix A:

> In addition, students going through the pathways should be encouraged to select from a range of high quality 4th mathematics options. For example, a student interested in psychology may benefit greatly from a course in discrete mathematics, followed by AP Statistics. A student interested in starting a business after high school could use knowledge and skills gleaned from a course on mathematical decision-making. A student interested in Science, technology, engineering, and mathematics (STEM) should be strongly encouraged to take Precalculus as the 4th mathematics, followed by AP Calculus.

In summary, appropriate and aligned standards, coupled with a core curriculum, can prepare students only if the courses are truly rigorous and challenging. It is more important for students to take the right kinds of courses rather than the right number of courses; the resources included in Appendix C in this thesis emphasize just that!

1.3 Role of High School 4th Mathematics: Precalculus for AP Calculus

This thesis targets the development of a CCSS aligned Precalculus high school mathematics course that should be taken after Algebra I, Geometry, and Algebra II in order to serve as a gateway and springboard to AP Calculus. While the debate goes on about what 4th mathematics courses should be, this thesis is focusing on the needs for a 4th mathematics course that serves as a prerequisite for AP Calculus. Since preparation for AP Calculus is the goal of this thesis: in Appendix C, conscientious efforts have gone into revising and reviewing both the topics and instructional approaches utilized in establishing a foundation for AP Calculus. Moreover, the lecture notes in Appendix C will be used to support a dual enrollment course

through Southeastern Louisiana University (SLU), Math 165. According to the SLU's course catalog, the official course description is:

> Math 165: Precalculus with Trigonometry (3 credit hours)
> Designed as a prerequisite for AP Calculus, this course covers the algebra and trigonometry necessary to prepare students for a standard university course in calculus and analytic geometry. The course will cover all of the Common Core (+) standards designed for the fourth year math course, except those in statistics.

So, what is Precalculus for AP Calculus? A simple, yet reasonable answer is the course to be taken before AP Calculus. For this reason, in order to understand what a good precalculus course should be, one needs to know what AP Calculus is.

The Advanced Placement (AP) program, at its inception in 1957, was designed to allow high school students to earn college credit, or at least advanced placement at college level course work, thereby avoiding unnecessary repetition once these students arrive in college. The program primarily served elite private high schools at the beginning. Even 50 years after its inception, the structure of the AP Program has not changed very much; however, its scope has changed dramatically. In 1960, 89 secondary schools participated in the AP Program. About a half century later, in 2013, that number has risen to 18,920 schools (College Board, 2013). In addition to the exam-taking that possibly earns students college credit, AP course-taking has become equally important in the role of identifying motivated, high achieving students in the college admission process. Furthermore, in an effort to put more academic rigors in high school curricula, many states' policy makers have begun mandating the inclusion of AP courses in their districts and high schools, and the State of Louisiana is no different. By the 2013-2014 school year, all Louisiana public high schools had to offer at least one AP course in three out of four core subject areas (math, ELA, science, and social studies) and by 2014-2015, schools must offer AP courses in all four core content areas (www.louisianabelieves.com). While taking more

courses is certainly better than taking fewer courses, the quantity of courses is not enough to guarantee that students will graduate ready for life after high school. According to Adelman (2004), the rigor of their high school curriculum is a key indicator for whether a student will graduate from high school and earn a college degree. Moreover, a study by the *U.S. Department of Education* found that the rigor of high school course work is more important than parent education level, family income, or race/ethnicity in predicting whether a student will earn a post-secondary degree.

According to the College Board, before studying AP Calculus, all students should complete four years of high school mathematics (e.g. Algebra I, Geometry, Algebra II, and 4th mathematics) designed for college-bound students. More importantly, students should have demonstrated mastery of material from these courses. The study of these courses should include algebra, geometry, coordinate geometry, and trigonometry, with the fourth year of study to include advanced topics in algebra, trigonometry, analytic geometry, and elementary functions. These functions include linear, polynomial, rational, exponential, logarithmic, trigonometric, inverse trigonometric, and piece-wise defined functions. In particular, students must be familiar with the properties of functions, the algebra of functions, and the graphs of functions. In addition, students must understand the language of functions (domain and range, odd and even, periodic, symmetry, zeros, intercepts, and so on) and know basic properties of trigonometric functions. The above mentioned learning topics are the focal points in Appendix C of this thesis.

1.4 Instructional Goals

A 2005-2006, the *ACT National Curriculum Survey* noted that college instructors want incoming college students to have a solid background in fundamental mathematical concepts and techniques in order to ensure success at the college level. In Calculus, students are expected to

solve problems that involve not only algebraic concepts that should have been learned in previous math classes, but also concepts that are introduced in the Calculus course. For example, in using the limit definition of the derivative to obtain the first derivative of the function, $f(x) = x^3$, many students first experience difficulties when expanding $(x + h)^3$ before taking the limit as h goes to 0. Hence, same algebraic mistakes occur before one can even start assessing an understanding of a calculus concept. Another example would be when a student is asked to find the critical points of the function $g(x) = sinx + cosx$. Students' limited knowledge of trigonometry may be an obstacle in finding the solution to $g'(x) = cosx - sinx = 0$. Smith (1981) noted some of the common mistakes students make as well as misconceptions that contributed to misapplication of the rules in algebra. Such mistakes include as expanded linearity property for the algebraic expression, $(x + y)^n = x^n + y^n$, $n \neq 1$, as well as applying a liberal canceling of terms for an expression such as $\frac{sinx}{siny} = \frac{x}{y}$ or $(sinx)(cosx) = sinx \Rightarrow cosx = 1$, by dividing both sides by *sinx*, instead of factoring, not realizing that in some cases *sinx* = 0. The correct approach to this question is $(sinx)(cosx) - sinx = 0$, then followed by factoring out $sinx$ such as $(sinx)(cosx - 1) = 0$. Using the zero product property, $sinx = 0 \ or \ cosx - 1 = 0$, would yield all solutions by solving for x.

Another problem area is factoring and solving questions like $x^{\frac{-2}{3}} + 2x^{\frac{-1}{3}} - 3 = 0$. Incidentally, the similar algebra deficiencies traced even as far back as more than eighty years ago (Murray, 1931). Let us look at the case to echo Murray's claim; students are faced with solving a trigonometric equation: $\sin(x) - \sin(2x) = 0, [0, 2\pi]$. Please see below (Figure 3):

Correct: Student A	Incorrect: Student B	Incorrect: Student C
$\sin(x) - \sin(2x) = 0$	$\sin(x) - \sin(2x) = 0$	$\sin(x) - \sin(2x) = 0$
$\sin(x) - 2\sin(x)\cos(x) = 0$	$\sin(x) - 2\sin(x)\cos(x) = 0$	$\sin(x) = \sin(2x)$
$\sin(x)\,(1 - 2\cos(x)) = 0$	$\sin(x) = 2\sin(x)\cos(x)$	$\frac{\cancel{\sin}(\cancel{x})}{\cancel{\sin}(\cancel{x})} = \frac{\sin(2\cancel{x})}{\cancel{\sin}(\cancel{x})}$
$\sin(x) = 0$ or $\cos(x) = \frac{1}{2}$	$\frac{\cancel{\sin(x)}}{\cancel{\sin(x)}} = \frac{2\,\cancel{\sin(x)}\cos(x)}{\cancel{\sin(x)}}$	$1 = 2$
$\therefore\ x = 0, \pi, 2\pi, \frac{\pi}{3}, \frac{5\pi}{3}$	$1 = 2\cos(x)$	$\therefore$ *no solutions.*
	$\therefore x = \frac{\pi}{3}, \frac{5\pi}{3}$	

Figure 3: Solving a Trigonometric Equation: $\sin(x) - \sin(2x) = 0, [0, 2\pi]$

For many people, the value of the AP Program is in its examinations which provide students with an opportunity to earn college credit while in high school. To AP teachers, AP is a comprehensive program that sets clear instructional goals and venues to guiding teachers in delivering a course that prepares students for success in introductory calculus-based STEM courses. For example, the philosophy statement of the AP Calculus Course Description (www.apcentral. collegeboard.com) emphasizes “broad concepts and widely applicable methods.” As stated in the AP Calculus Course Description, “although facility with manipulation and computational competence are important outcomes, they are not the core of the Calculus course.” To be successful in the AP Calculus examination, students are required to display solid reasoning and problem solving abilities. While students are still asked to “solve,” “simplify,” or “evaluate,” they are also expected to have knowledge beyond these skill-based tasks. Examining AP released exam questions, teachers will find that students are asked to “describe,” “interpret,” “explain,” and “justify.” These actions call for students to demonstrate a deeper understanding of the material that goes beyond simple procedural algorithms. AP Calculus ready students should be able to demonstrate analytical, graphical, numerical, and verbal understanding of mathematics.

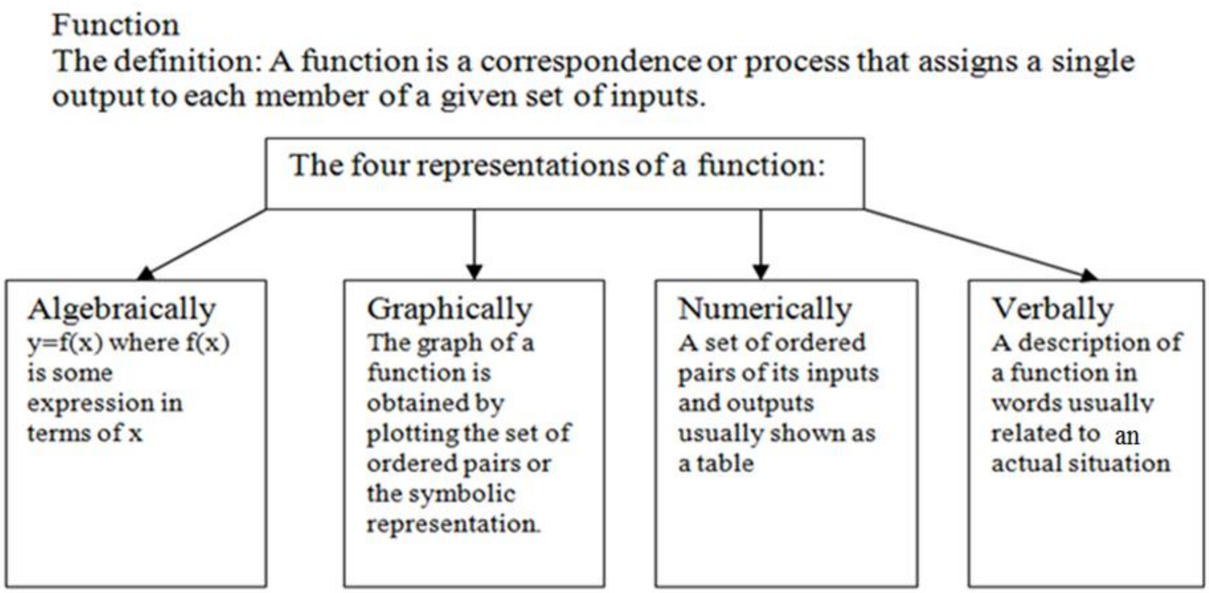


Figure 4: The Four Representations of a Function

While many mathematics teachers have already adopted the "rule of four," i.e., the use of words, tables, symbols, and graphs (Figure 4) as an instructional strategy in their mathematics classes, Precalculus for AP Calculus teachers can find added incentive for doing so from the goals for the AP Calculus program. The AP Course Description says that "Students should be able to work with functions represented in a variety of ways: graphical, numerical, analytical, or verbal. They should understand the connection among these representations" (AP Course Description, 2012). Since success in AP Calculus (*Topic Outline for AP Calculus AB at* (www.apcentral. collegeboard.com) is closely tied to the preparation students have had in courses leading up to their AP course, it should be emphasized that eliminating preparatory course work in order to take an AP course is not appropriate (College Board, Course Description). Furthermore, by default, the College Board's AP Calculus curriculum is accepted as the model for a twenty-first century calculus course. The path now needs to be set for a precalculus course to align the AP goals and objectives with the CCSS aligned course after taking Algebra I, Geometry, and Algebra II. For students to be adequately prepared for AP

Calculus, there must be an effective precalculus course available to be taken first. This thesis, "High School 4th Mathematics: Precalculus for AP Calculus," is written specifically with the goal of meeting this requirement. In Appendix C of the thesis, high schools teachers are provided with lesson plans and student activities that are aligned with CCSS and CCS(+)S to prepare students for AP Calculus. Each section lesson plan begins with a comprehensive overview of the major concepts, a list of the related Common Core State Standards for Mathematics, a set of section learning objectives, lecture notes, and a variety of lesson activities that support the CCSS as well as MPS.

Table 4: Appendix C: High School 4th Mathematics: Precalculus for AP Calculus Curriculum at a Glance

Unit 1	
Polynomial and Rational Functions	
Overview	• Review polynomial functions • Expand to higher order polynomial and rational functions and their properties • Modeling polynomial functions of higher degree • Solving polynomial and rational inequalities
AP Readiness	• Opportunity to look at these functions graphically, numerically, algebraically, and verbally both in and outside of contextual situations • Further developing the understanding of extrema. • Beginning to look at the continuity / discontinuity and concepts of a limit at infinity
Unit 2	
Exponential and Logarithmic Functions	
Overview	• Review exponential and logarithmic functions. • Review exponential and logarithmic properties • Study exponential and logarithmic modeling • Solving exponential and logarithmic equations
AP Readiness	• Opportunity to look at these functions graphically, numerically, algebraically, and verbally both in and outside of contextual situations.

(Table 4 Continued)

Unit 3	
Trigonometry	
Overview	• Review right triangle trigonometry. • Develop Unit Circle concepts with right triangle trigonometry and use this to connect the graphs of trigonometric functions as cyclic • Study identities • Study inverse trigonometric functions by restricting the domains • Apply inverse operation and identities to solve trigonometric equations and modeling real-world problems
AP Readiness	• Opportunity to study six trigonometric functions and their inverse functions, including concepts such as domain and range, odd and even, and period • Justifying mathematical relationship through written and verbal communication
Unit 4	
Applications of Trigonometry	
Overview	• Laws of Sine and Cosines • Vectors in the plane • Polar coordinates and Complex Numbers • DeMoivre's Theorem and nth root of polar coordinates
AP Readiness	• Communicate mathematics and explain solution both verbally and in writing • Technology to explore and interpret results, and support conclusions
Unit 5	
Optional (+) Standards	
Overview	• Matrix addition and multiplication • Multivariate linear systems and matrix • Ellipse and hyperbola • Binomial Theorem
AP Readiness	• Communicate mathematics and explain solution both verbally and in writing • Technology to explore and interpret results, and support conclusions

CHAPTER 2. MATHEMATICS STANDARDS

2.1 History of Standards-Based Reform and Common Core State Standards

For years, national reports have called for a greater focus in U.S. mathematics education. One of the defining moments in the history of mathematics education in America was the launching of Sputnik 1 by the Soviet Union in 1957. This marked the start of the space race between the United States and the Soviet Union. Concerned that the United States was falling behind in the areas of math and science triggered major national reforms in these areas. These reforms brought about the *New Math* of the 1960s and 1970s. The emphasis of the New Math was on set language and properties, proof, and abstraction. However, the New Math curriculum failed to meet the challenge of increasing the nation's mathematical prowess as a whole. Some would even say that the New Math created more math confusion than it eliminated, which brought about the trend of *Back to Basics* in the late 1970s and early 1980s. Back to Basics emphasized arithmetic computation and rote memorization of algorithms and basic arithmetic facts.

In the late 1980s, the focus shifted from "New Math" and "Back to Basics" to *Critical Thinking*. In 1989, the National Council of Teachers of Mathematics (NCTM) released a groundbreaking document, Curriculum and Evaluation Standards for School Mathematics. This publication, sometimes referred to as the *NCTM Standards*, stresses problem solving, communication, connections, and reasoning. In the 1990s, the major focus of reform in mathematics education was directed toward improving pedagogical skills in the teaching of mathematics. Numerous studies and articles promoted the use of manipulatives and technology in the classroom (Jeff et al., 2003; Kaput & Roschelle, 1997; NCTM, 2000). Key ideas of this era included the use of developmentally appropriate activities and the constructivist approach to

teaching. The NCTM Standards continued to gain support and popularity among mathematics educators, and many states developed grade-level scoping sequences and competency-based model programs that reflected these standards. Proficiency testing became more widespread, with some states requiring a certain level of competency in subject areas such as mathematics for grade promotion.

In spite of all efforts, the TIMSS reports (2004 & 2007) have concluded that mathematics education in the United States is still declining and our students are not performing well at the international level. The original purpose of the standards-based reform movement was to identify what students should know and be able to do at specific grade levels and to measure whether they were mastering that content. As the movement matured, it took on the additional purpose of applying consequences to schools whose students did not show mastery. In this case, the standards movement morphed into test-driven accountability. Standards-based reform originated in the late 1980s when the NCTM wrote a set of national standards for mathematics. The nation's governors and the administration of President George H.W. Bush subsequently adopted that approach for other subject areas and proposed the adoption of national academic education standards and national tests to measure how well students were learning. President Bush's successor, President Bill Clinton, continued to advocate for the basic approach of using standards and tests to reform education, but with a key variation. Rather than promoting national standards and tests, he urged states to develop their own standards and tests to measure student proficiency. President Bill Clinton's legislation was enacted, but, after great debate, that law did not include proposals to require states to provide the educational opportunities for students to reach those standards. Under President George H. W. Bush, all of the states were either in the process of implementing standards and aligned tests or had done so. The No Child Left Behind (NCLB) Act

proposed by President Bush, ramped up the intensity of President Clinton's laws by prescribing more extensive grade-level testing, setting a deadline of 2014 for all students to be proficient in English language arts and mathematics, and mandating specific actions that schools and school districts had to take if they did not reach the state-prescribed yearly goals for student proficiency.

The enactment of NCLB in 2002 was a turning point for the standards movement. Instead of academic standards serving as a focal point to raise the quality of instruction in schools, test driven accountability became the norm. Teachers understood that if their students did not pass the annual state accountability tests, their schools would be labeled as "failing" by the accountability system set by the states and the penalties prescribed by NCLB. In 2011, nearly half of U.S. schools did not meet their state targets for student proficiency (The Center on Education Policy, 2012). The standards and testing movement has resulted in an increased expectation for what should be taught in school. As a result, for the first time in American history, every state has made public its academic standards in the crucial areas of English Language Arts and Mathematics.

Moreover, the problems that emerged from having different standards in each of the 50 states spurred the nation's governors and chief state school officers to develop the Common Core State Standards (CCSS) in English language arts and mathematics, which have now been adopted by 43 states and the District of Columbia. It is a state-led effort coordinated by the National Governors Association Center for Best Practices (NGA Center) and the Council of Chief State School Officers (CCSSO). These rigorous, internationally aligned education standards establish a set of shared goals and expectations for what U.S. students should understand and be able to do in grades K-12 in order to be prepared for success in college and the workplace in the global economy. With clear academic expectations, teachers, parents, and

students can work toward shared goals. Furthermore, the standards draw from the best existing standards in the country and are benchmarked to top performing nations around the world, ensuring that our students are well prepared to compete not only with their peers, but also with students around the world, to reclaim America's academic competitive edge. Clearly, this increased rigor of the new standards and the development of new or revised curriculum materials that are aligned to the standards represent a major challenge for teachers. In January 2012, the Center on Education Policy reported that 30 of the 33 states who were adopting the CCSS will require new or substantially revised curriculum materials for the implementation of the CCSS. Again, this report itself indicates that there is a true need for developing a Precalculus course that is aligned with the current movement.

2.2 Common Core Content (+) Standards for Mathematics

The CCSSM consist of two interrelated sets of standards, the Standards for Mathematical Content and the Standards for Mathematical Practice. The Standards for Mathematical Contents are organized by grade level in Grades K–8; at the high school level, the standards are organized by conceptual categories – Number and Quantity, Algebra, Functions, Geometry, Statistics and Probability. Within each conceptual category, there are domains and clusters of which each consists of one or more standards. Most of the high school standards are meant to be mastered by the end of three years of mathematics courses. Additional mathematics that students should learn in 4th credit courses such as Precalculus is indicated by (+) standards. All standards without a (+) should be in the common curriculum for all college and career ready students. Standards with a (+) may also appear in courses intended for all students. All of the CCS(+)S designed for the 4th year mathematics course, except those in statistics, are addressed in Appendix C.

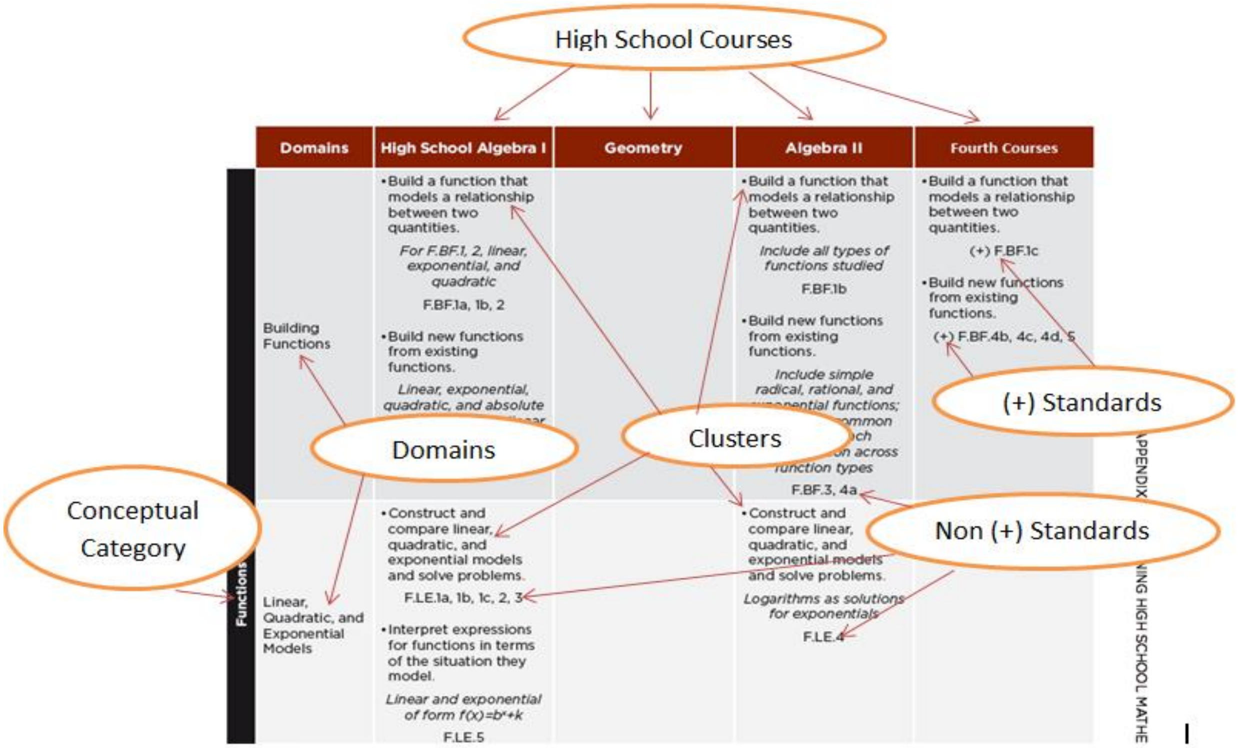


Figure 5: Overview of the Traditional Pathway for the Common Core State Standards and Common Core State (+) Standards for Mathematics

2.3 Common Core Standards for Mathematical Practices

The increased rigor is just another challenge that teachers encounter. The CCSSM require that students retain mathematical knowledge from the previous years and that they demonstrate sound mathematical practices as stated in the Mathematical Practice Standards (MPS). These eight MPS (for a complete list of the Standards for Mathematical Practices, see Appendix A) outline the need for students to be able to reason mathematically and demonstrate both a procedural and conceptual understanding of mathematics. The eight MPS are:

MP.1 Make sense of problems and persevere in solving them
MP.2 Reason abstractly and quantitatively
MP.3 Construct viable arguments and critique the reasoning of others
MP.4 Model with mathematics
MP.5 Use appropriate tools strategically
MP.6 Attend to precision
MP.7 Look for and make use of structure
MP.8 Look for and express regularity in repeated reasoning

These practices rest on important “processes and proficiencies” with longstanding importance in mathematics education. The first documents are the NCTM process standards of problem solving, reasoning and proof, communication, representation, and connection. The second documents are the standards of mathematical proficiency specified in the National Research Council’s report *Adding It Up* (2001). These eight practices should be at the core of good mathematics education in order for all students to arrive at mathematical proficiency. The MPS describe a classroom where students are actively engaged in challenging mathematical problems. These standards are not about skill-based content but are about establishing a classroom setting where students are given opportunities to solve problems collaboratively and discuss the solutions and prevailing mathematical ideas. Facilitation and utilization of these eight MPS are the anchors of Appendix C; the MPS are incorporated throughout the lecture notes, writing assignments, and student activities in Appendix C.

2.4 Connecting the Mathematical Practice to the Mathematical Content

The MPS describe ways in which students who are studying mathematics must engage with the subject matter as they grow in mathematical maturity throughout the elementary, middle, and high school years. Designers of curricula, assessments, and professional development should all attend to the need to connect the mathematical practices to mathematical content in mathematics instruction. The CCSSM are a balanced combination of procedure, understanding, and problem-solving. Expectations that begin with “understanding” are especially good opportunities to connect the practices to the content. Practices are those things that go beyond procedure and understanding. On one hand, students who lack understanding of a topic may rely on procedures too heavily. On the other hand, students who are fluent in procedural skills may falsely think they have a complete understanding of the topic area. Without a flexible

base from which to work, they may be less likely to consider analogous problems, represent problems coherently, justify conclusions, apply the mathematics to practical situations, use technology mindfully to work with the mathematics, explain the mathematics accurately to other students, step back for big ideas, or deviate from a known procedure to derive a shortcut. The high school content standards do not set explicit expectations for fluency, but fluency is important in the study of high school mathematics. In essence, it is difficult to teach conceptual understanding in mathematics without the supporting procedural skills, and procedural skills are weakened by a lack of understanding. And clearly, a lack of understanding prevents a student from meaningfully engaging in the mathematical practices.

In this respect, the content standards which set an expectation of conceptual understanding are potential “points of intersection” between the Standards for Mathematical Content and Standards for Mathematical Practice. These points of intersection should be weighted toward central and generative concepts in the school mathematics curriculum that most merit the time, resources, energies, and focus necessary to qualitatively improve the curriculum, instruction, assessment, professional development, and student achievement in mathematics. As you can see, having well developed classroom materials that allow students to facilitate Mathematical Practices will promote conceptual understanding. And student activities in Appendix C are written in a way to meet the expectation of improving students’ mathematical fluency as well as conceptual understanding.

2.5 How the Common Core Standards Prepare Students to Engage in AP Calculus

The goal of the CCSS is to establish a common set of rigorous expectations to prepare students for college and career readiness. The CCSS articulate the knowledge and skills students need to be ready to succeed in college and careers. They are designed to be (1) anchored in research and evidence; (2) aligned to college and workplace expectations; (3) rigorous, clear, and consistent; and (4) reflective of best practices in international frameworks. The report, Common Core State Standards Alignment, released by the College Board (2011), describes how the CCSS can prepare students to engage in AP English, AP Calculus AB, AP Calculus BC, AP Statistics, and AP Computer Science A. In the report, it is stated that

> The comparison is nuanced in that the Common Core State Standards are designed to articulate the knowledge and skills students need to be ready to succeed in college and careers, and AP courses and exams are designed to present the level of a first-year college course. In light of this distinction, alignment between the Common Core State Standards and AP Courses should not be interpreted as linkages of content or skills at the same level of rigor or challenge, but as areas where there is an identifiable bridge from one framework to another- a link from the Common Core State Standards to a specific AP Course. Similarly, “gaps” should not be interpreted as Common Core State Standards that do not prepare students for AP, but as valuable, general content that contributes to a student’s preparedness as a whole.

Then, it goes on to claim that in order for a CCSS to be considered aligned to an AP course, evidence of the extension of that CCSS must be cited from at least one document from the collection of AP materials. In Appendix C, *Looking Ahead to Calculus*, teachers can see how the study of the CCSS topics in Precalculus is directly connected to the future study of AP Calculus.

CHAPTER 3. LITERATURE REVIEW

3.1 What is College Ready and Why is College Ready Important?

According to Conley (Educational Policy Improvement Center, 2007), college readiness means that a student enters a college classroom, without remediation, and successfully completes entry-level college requirements. Being successful means a student can complete the first year college courses at a level of understanding and proficiency that makes it possible for the student to take the next course in the sequence or the next level of course in the subject area. Being college ready means being prepared for any postsecondary education or training experience, including study at two- and four-year institutions leading to a postsecondary credential (i.e. a certificate, license, Associates or Bachelor's degree). Being ready for college means that a high school graduate has knowledge and skills necessary to qualify for and succeed in entry-level, credit-bearing college courses without the need for remedial coursework.

In the last decade, research conducted by Achieve (2004) as well as others (Ali & Jenkins, 2002; Organization for Economic Cooperation and Development [OECD], 2009) show a convergence in the expectations of employers and colleges in terms of the knowledge and skills high school graduates need to be successful after high school. Economic reality reflects these converging expectations. Education is more valued and more necessary than ever before and all high school graduates need to be prepared for some postsecondary education and or training if they are to have options and opportunities in the job market. A high school diploma was once a ticket to the American Dream: a steady job that could launch a career that would support a family and raise a family's living standard. Times have changed.

The Organization for Economic Cooperation and Development (OECD, 2005) reported that 35 years ago, only 12% of U.S. jobs required some postsecondary training or an Associate's

degree and only 16% required a Bachelor's degree or higher; whereas, nearly eight in ten future job openings in the next decade in the U.S. will require postsecondary education or training. Forty-five percent will be in "middle skill" occupations, which require at least some postsecondary education and training, while 33% will be in highly skilled occupations for which a Bachelor's degree or more is required. By contrast, only 22% of future job openings will be "low skill" and accessible to those with a high school diploma or less. While the U.S. still ranks 3rd in the adult population (25-64 year olds) with an associate's degree or higher among 30 countries, we now rank 15th among 25-34 year olds with a two-year degree and above (OECD, 2010). Competing countries are catching up to – and even outpacing – the U.S. in the educational attainment of their new generation of adults. Higher levels of education lead to elevated wages, a more equitable distribution of income and substantial gains in productivity.

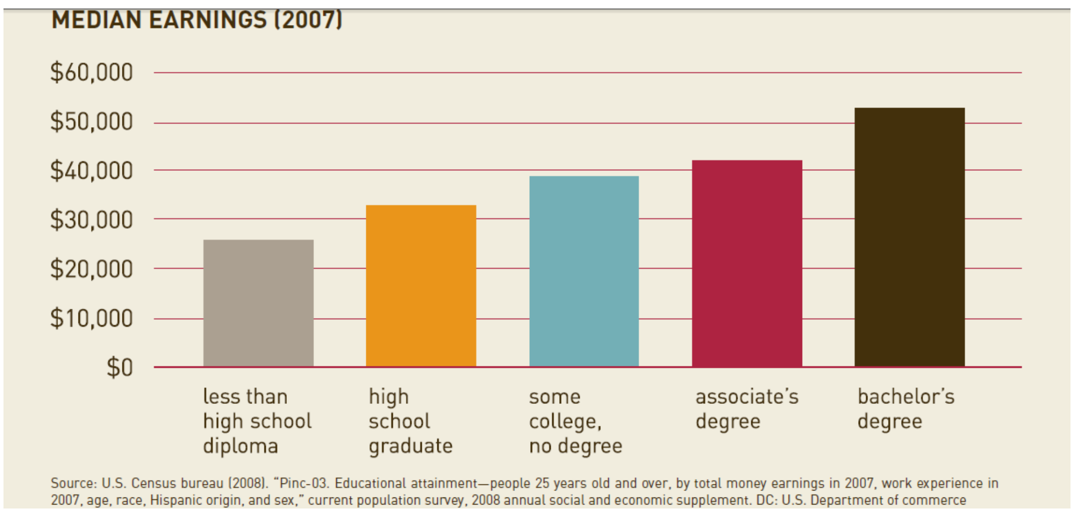


Figure 6: From Education to Work: A Difficult Transition for Young Adults with Low Levels of Education

Without a doubt, improving college readiness is crucial to the development of a diverse and talented labor force that is able to maintain and increase U.S. economic competitiveness throughout the world (ACT, 2004). Yet, many students graduate from high school without the essential skills needed to succeed in college. In particular, many students enter college mathematically underprepared or unprepared for collegiate level coursework. In fact, just 40% of ACT test-takers are ready for their first course in College Algebra (ACT, 2004). This data is provided by ACT research published in the 2004 article *Crisis at the Core*. As part of this study, the ACT organization established a score of 22 or higher on the mathematics portion of the ACT exam as the benchmark for college readiness in mathematics. This score predicts a higher probability of success for students in their first-year college mathematics course. Using this 22 benchmark, the *Crisis at the Core* document discusses the result of a national study comparing high school coursework and college readiness. The study had three primary conclusions:

1. Most students are not ready for college level mathematics when leaving high school.
2. The more math courses a student takes in high school, the more prepared they are for college.
3. In particular, every course taken beyond Advanced Algebra (Algebra II) in high school results in greater preparation for college mathematics.

Again, there is convincing statistics for encouraging students to continue their mathematical studies after Algebra II such as a Precalculus course in their high school years.

3.2 Educational Trend

To compete globally and keep up with expanding scientific and technical expertise, educators and policymakers have called for increasing emphasis on science, technology, engineering, and mathematics (STEM) course-taking in schools (President’s Council of Advisors on Science and Technology 2010). The percentage of high school graduates who earned credits

in advanced mathematics courses was greater in 2009 than in 2005, continuing the upward trend from 1990 (Table 5).

Table 5: Percentage of graduates earning credits in STEM courses, selected years: 1990–2009

	Year of graduation			
	1990	2000	2005	2009
STEM course	Percent			
Advanced mathematics	57*	74*	78*	84
Algebra II	53*	67*	70*	76
Other advanced mathematics	16*	20*	24*	30
Pre-calculus/analysis	14*	27*	30*	35
Calculus	7*	13*	14*	17

SOURCE: U.S. Department of Education, Institute of Education Sciences, National Center for Education Statistics, High School Transcript Study (HSTS), various years, 1990–2009

Seventy-six percent of graduates took Algebra II in 2009 compared to 53% in 1990. The percentage of graduates who took Precalculus in 2009 was 35% compared to 14% in 1990. Seventeen percent of graduates took Calculus in 2009 compared to 7% in 1990. Undoubtedly, having a solid precalculus course after Algebra II is an essential prerequisite for many, if not all, STEM programs at 2- or 4-year colleges.

3.3 Equity

Despite the much talked about changes in mathematics education, African American students continue to perform poorly in school mathematics (Secada, 1992). In 2009, and in all previous assessment years since 1992, the National Assessment for Educational Progress (NAEP) reported that the mathematics scores of African American 4^{th}, 8^{th}, and 12^{th} grade students were lower than the scores of their white counterparts (The Condition of Education,

2011). This disparity is known as an achievement gap – in the NAEP mathematics assessment; it is the difference between the average scores of two student subgroups on the standardized assessment. On one hand, Orr (1987) argues that African American students' poor mathematics performance is the result of a discontinuity that exists between students' home language and the perceived "precision" of mathematics and mathematical language. Lubienski (2001), on the other hand, suggests that African American students receive mathematics instruction that is not consistent with the recommendations suggested by the National Council of Teachers of Mathematics (NCTM). It is even further supported by the report that 58 percent of African American 8th grade students agreed that mathematics is mostly memorizing facts, which is significantly more than the 40 percent reporting nationally (Stretchens & Silver, 2000). However, few have situated the mathematics performance of African American students into the larger context of mathematics teaching and learning in U.S. schools. According to Lubienski (2001), most African American students are not experiencing instructional practices consistent with the recommendations suggested by the National Council of Teachers of Mathematics (NCTM), whereas more white students are experiencing NCTM standards-based instruction.

The data are clear: When we as a nation focus on something, we make progress. This is evident in our students' achievement after more than a decade of attention to improving elementary education. In 1996, nearly three of every four African American 4th graders could not perform at a basic level in mathematics. By 2007, that was down to 30 percent. But we cannot for a minute rest on this success. Far too many young people still enter high school underprepared and college unprepared. And the gaps separating the achievement of African American and Latino 12th graders from their white peers are bigger now than they were in the late 1980s. In 2009, White students at grade 12 scored 30 points higher on the National

Assessment for Educational Progress (NAEP) in mathematics than black students and 23 points higher than Hispanic students (The Condition of Education, 2011). NAEP mathematics scores range from 0 to 300 points and score gaps are calculated based on differences between unrounded scores. In 2009, achievement gaps between students in schools with high percentages of low-income students and students in schools with low percentages of such students exists at all three grade levels, 4th, 8th, and 12th. For this indicator, students are identified as attending schools with high percentages of low-income students if more than 75 percent of the students in the school are eligible for free or reduced-price lunch. In 2009, the low-income gap at grade 4 was -31 points, at grade 8, the gap was -38 points, and at grade 12, the gap was -36 points. To close the devastating gaps, we need to find the courage and the will to change a practice that continues to leave low socioecomomic students behind.

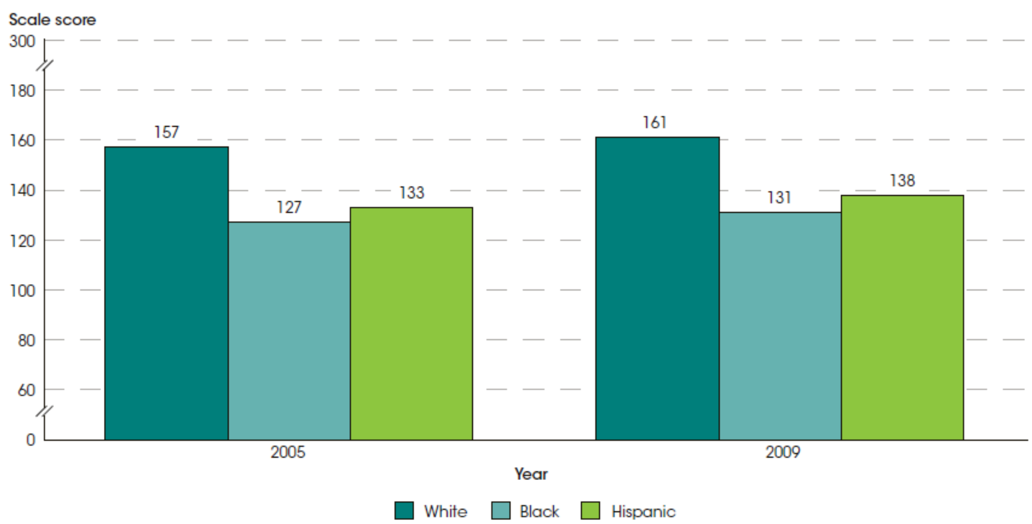


Figure 7: Average Mathematics Scale Scores of 12th Grade Students,
by Race / Ethnicity: 2005 and 2009

CHAPTER 4. INSTRUCTIONAL RESOURCES

4.1 Planning for the Use of MathXL as a Part of Teaching Precalculus

After examining many textbooks, I selected Precalculus 8th Edition, written by Demana, Waits, Foley, and Kennedy. This book is published by Pearson Education which means MathXL is available. As stated in the introduction of this thesis, development of assignments using MathXL is not part of this thesis, having an on-line access would be an invaluable asset in teaching and learning mathematics. MathXL is an innovative and user-friendly, web-based system from Pearson Education, which includes homework, tutorials, study plans, and assessments. MathXL helps students be self-reliant in learning – it is modular, self-paced, and adaptable to individual learning preferences. Some students need to be guided step by step, others work well by looking at an example and reading the text, and some work better when observing and hearing someone else work through a problem. All of these learning styles are available with MathXL.

After choosing the text, I first aligned each section from the text book with the CCSSM and CCS(+)SM (see Appendix B). Once done, I composed a preliminary scope and sequence for the High School 4th Mathematics: Precalculus for AP Calculus. Then, the selected sections were further developed into the lesson plans. These lesson plans contain my lecture notes and recommended student activities that are closely aligned with the CCSSM and the CCS(+)SM. Many of these activities are imported from the reputable site www.illustrativemathematics.org. Complete lesson plans are included in Appendix C.

This Precalculus course also will be used as a dual enrollment class with Southeastern Louisiana University (SLU), Math 165, Precalculus. For this reason, I consulted with Dr. Hudson and Becky Muller at Southeastern Louisiana University. As coordinator of the math dual

enrollment program at SLU, Becky Muller is composing the online component of this course's assignments using MathXL based on the scope and sequence for Appendix C: High School 4th Mathematics: Precalculus for AP Calculus.

This course is setup so that prior to working through problems and assessments online, students attend in-class lectures that introduce the lesson topics and discuss the process and procedures needed to successfully complete assignments as well as understand the materials. Students are expected to spend a certain amount of time in a monitored environment while each student is actively engaged in completing the work and is also required to spend additional time outside of class.

4.2 Why Have an Online Component in High School 4th Mathematics: Precalculus for AP Calculus?

The need to meet the challenges found in mathematics education has led to experimentation with many different approaches. Traditionally, teaching and learning mathematics has been built around the lecture model in which the teacher spends most of the time lecturing, answering homework questions, explaining rules and procedures, and working through numerous examples while students sit in rows watching the teacher's actions. More and more, other pedagogical methods are being explored, largely because of the perceived shortcomings of the traditional approach. Instead, more student-centered approaches are being advocated. However, regardless of which method or system is used, there is one constant component in every mathematics course – use of assignments to develop students' procedural fluency and content understanding. Students must do problems in order to learn. Furthermore, they need feedback on the correctness of their answers so that students are aware of their own level of understanding. Mathematics teachers are certainly aware of the importance of providing feedback to their students. However, for a variety of reasons, teachers often fail to do this.

The NCTM published *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989) which outlined the goals of mathematics reform. According to these goals, students in a standards based classroom are to learn to value mathematics, gain confidence in their mathematics ability, become problem solvers, and learn to communicate and reason mathematically. A major part of the mathematics reform movement is the increased use of technology in the mathematics classroom. NCTM strongly supports the use of technology in mathematics education in the Technology Principle. The Technology Principle states, "Technology is essential in teaching and learning mathematics; it influences the mathematics that is taught and enhances students' learning" (NCTM, 2000, p.1).

Encouraged by the NCTM, use of technology in the mathematics classroom has increased, and technology-enhanced classrooms are becoming more prevalent. A number of different technologies are being used in today's mathematics classrooms with varying degrees of success. Technologies such as graphing calculators allow students to explore more difficult problems than educators would have dared to assign years ago. Graphing calculators allow investigation of functions through tables, graphs and equations in ways that were not possible before their proliferation. Further, graphing calculators allow the focus to be on understanding and setting up and interpreting results (Dick, 1992; Hopkins, 1992).

Likewise, the positive effects of mathematics and technology instruction such as computer-integrated learning are becoming more prevalent in the mathematics classroom. Replacing "drill and kill" worksheets, software that is one-on-one, self-paced, and provides immediate feedback can help remediate and can enhance student understanding. Recent research indicates that the purposeful use of computers in classroom instruction can indeed enhance student outcomes (Archer, 1998; Milheim, 1995).

4.3 What is MathXL?

This educational software platform, MathXL provides students with a personalized, interactive learning environment. The program has a built-in tutorial system with menu driven tools such as: "help me solve this," "view an example," "video tutorial," "links to textbook pages," "interactive animation," and "ask my instructor." MathXL's practice exercises are correlated to the exercises in the textbook, and they regenerate algorithmically to give students unlimited opportunity for practice and mastery. With the random feature, not only are students allowed to practice as many times as they wish but also the computer records the highest score for each assignment. Most exercise problems are free-response and provide an intuitive math symbol palette for entering math notation. Exercises include guided solutions, sample problems, and learning aids for extra help at point-of-use, and they offer helpful feedback when students enter incorrect answers. More importantly, for teachers, this web-based assignment provides real-time data on where each student stands in his or her learning. Such data can be used to customize and personalize learning for individual students and the class as a whole. The digital tools allow teachers to spend more time with each student, rather than solely lecturing at the front of the classroom. In summary, the assignment is completed on the computer using MathXL, which provides the following characteristics:

- Not timed, since some students may need more or less time depending on their mathematical background and need to access learning aids.
- Questions are algorithmically generated and class assignments are designed in a way to give students unlimited opportunity for practice and mastery.
- Students receive immediate feedback for each answer submitted.
- May be entered and exited as often as necessary during the assigned time period as determined by the teacher.
- Work completed is saved each time. Students can re-enter to finish incomplete items.
- Gradebook allows students to track and view grades automatically calculated by the system.

- Study plan for self-paced learning is available in the Tutoring Center which generates a personalized plan for each student based on his or her test results linked to tutorial exercises for topics the student has not yet mastered.
- Students have access to multiple learning aids as selected by the instructor.

4.4 MathXL and Student Learning

Technology itself will not directly improve student achievement, but how technology is utilized into instruction will. Effective integration of technology will have impact on the student. Students are more motivated when using technologies that have a real purpose and provide a meaningful learning environment. Using MathXL had a great impact on my teaching and learning mathematics. The more I used MathXL, the more I was convinced that having an online component was a way of differentiating instruction and providing different avenues to meet the needs of my students. Teachers can provide only a finite set of static problems and take time to create, grade, and identify students' weaknesses. MathXL, however, is designed to overcome this issue with capability to generate multiple iterations, thus providing the students with ample opportunities. The computer gives a student undivided attention; it waits while the student works, and there is no pressure to complete a task in a prescribed time period as long as the assignments are done by the deadline. Unlike a teacher or tutor, the computer does not respond in an emotional fashion, nor does it mind repeating itself. The randomized items with contexts changing on each assignment provide the opportunity for enriching practices. This motivates students to practice more to enhance their content knowledge, to gain their mathematical confidence, and thus, to improve their achievement. Students spend more time doing math problems when using MathXL. There is less "teacher teaching" but more "student doing" with MathXL. Because grading of assignments, quizzes, and assessments is automated, teachers can invest their time in analyzing students' performance, using a detailed study plan for each student for individualized intervention. This also allows teachers to spend class time working one-on-one

with each student instead of standing in front of the class and spending the bulk of a class period delivering a lesson that may only resonate with a small group of students. Data-driven instruction means that teachers focus intervention in the area it will do the most good.

Above all, use of computer technology can motivate students to practice math problems while being provided with direct feedback (Kroesbergen & Van Luit, 2003). Provenzo et al. (1999) indicate that students are highly motivated when using technology that has a real purpose. Technology enables educators to introduce mathematical concepts to students at earlier stages of development (Provenzo et al., 1999). MacDonald and Caverly (1999) point out that computer programs and software incorporate a wide variety of skill levels where children can feel comfortable working at their own pace as they build up their confidence in mathematical exercises. For students to find learning motivating, they must first find learning enjoyable and rewarding (Schweinle, Meyer, & Turner, 2006). Motivation is important to students and teachers because of its effect on learning outcomes (Tavani & Losh, 2003). According to Tavani and Losh (2003), motivation is a significant predictor of academic performance. If motivation is beneficial to student learning outcomes, then it stands to reason that educators should strive to cultivate and enhance the motivation of students.

REFERENCES

Achieve. (2004). *Ready or not: Creating a high school diploma that counts.* Retrieved from http://www.achieve.org/files/ADPsummary_5.pdf.

ACT. (2004). *Crisis at the Core: Preparing all students for college and work*. Iowa City, IA: Authors.

ACT. (2005-2006). *National Curriculum Survey.* Iowa City, IA: Authors.

ACT. (2013). *What Are ACT's College Readiness Benchmarks?* Iowa City, IA: Authors.

ACT & The Education Trust. (2004). *On course for success: A close look at selected high school courses that prepare all students for college.* Iowa City, IA: Authors.

Adams, T. (1997). Technology makes a difference in community college mathematics teaching. *Community College Journal of Research & Practice*, *21*, 481-493.

Adelman, C. (1999). *Answers in the toolbox: Academic intensity, attendance patterns, and bachelor's degree attainment.* Washington, DC: U.S. Department of Education.

Adelman, C. (2004). *Principal indicators of student academic histories in postsecondary education, 1972-2000.* Washington, DC: U.S. Department of Education, Institute of Education Statistics.

Ali, R. & Jenkins, G. (2002). *The high school diploma: Making it more than an empty promise*. Oakland, CA: Education Trust West.

AP Course Description. (2012). Retrieved April 20, 2013, from www.apcentral.collegeboard.com.

Archer, J. (1998). The link to higher test scores, *Education Week 18*(5), 10-21.

Campbell, J. R., Hombo, C. M., & Mazzeo, J. (2000). *NAEP 1999: Trends in academic progress: Three decades of student performance.* Washington, DC: U.S. Department of Education.

Common Core State Standards for Mathematics. Retrieved July 25, 2012, from http://www.corestandards.org.

Conley, D. T. (2007). *Redefining college readiness.* Eugene, OR: Educational Policy Improvement Center.

Dick, T. (1992). Super calculators: Implications for calculus curriculum, instruction, and assessment. In J. T. Fey (Ed.), *Calculators in mathematics education, 1992 yearbook*, (pp. 145-157). Reston, VA: National Council of Teachers of Mathematics.

Hopkins, M. (1992). The use of calculators in assessment of mathematics achievement. In J. T. Fey (Ed.), *Calculators in mathematics education, 1992 yearbook*, (pp.158-166). Reston, VA: National Council of Teachers of Mathematics.

Jeff, T., Morrison, W., Messenheimer, T., Rizza, M., & Banister, S. (2003). A Retrospective Analysis of Technological Advancedment in Special Education. In D. L. Johnson, & C. D. Maddux (Eds.), *Technology in Education: A Twenty-Year Retrospective* (pp. 129-152). New York, NY: The Haworth Press.

Kaput, J., & Roschelle, J. (1997). Deepening the impact of technology beyond assistance with traditional formalism in order to democratize access to ideas underlying calculus. In E. Pehkonen (Ed.), Proceedings of the 21st Conference. *International Group for the Psychology of Mathematics Education*, (pp. 105-112). Helsinki, Finland: University of Helsinki.

Kroesbergen, E., & Van Luit, J. (2003). Mathematics Intervention for Children with Special Needs. *Remedial and Special Education*, *24*(2), 97-114.

Lubienski, S. (2006). Examining instruction, achievement, and equity with NAEP mathematics data. *Educational Evaluation and Policy Analysis*, *14*(14), 1-30.

MacDonald, L., & Caverly, D. (1999) Techtalk: Technology and Developmental Math. *Journal of Developmental Education, 22*(3), 1-5.

Milheim, W. (1995). Interactivity and computer-based instruction. *Journal of Educational Technology Systems,24*, 225-233.

Mullis, I. V. S., Martin, M. O., Beaton, A. E., Gonzalez, E. J., Kelly, D. L., & Smith, T. A. (1998). *Mathematics and science achievement in the final year of secondary school.* Chestnut Hill, MA: TIMSS International Study Center.

Murray, S. (1931). On certain deficiencies of college freshmen in algebra, *Mathematics News Letter*, pp. 14-15.

National Center for Education Statistics. (2011). *The Condition of Education: Mathematics Achievement Gaps*. National Center for Education Statistics.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics.* Reston, VA: The National Council of Teachers of Mathematics, Inc.

National Council of Teachers of Mathematics. (2000). *Principle and standards for school mathematics*. Reston, VA: The National of Teachers of Mathematics, Inc. Retrieved from http://www.nctm.org.

National Governor's Association, Council of Chief State School Officers. (2011). www.corestandards.org. Retrieved December 07, 2011, from Common Core State Standards Initiative: http://corestandards.org/.

National Research Council. (1999a). *Global perspectives for local action: Using TIMSS to improve U.S. mathematics and science education.* Washington, DC: National Academy Press.

National Research Council. (2001). *Adding it up*. Washington, DC: National Academy Press.

O'Callaghan, B. (1998). Computer-intensive algebra and students' conceptual knowledge of functions. *Journal for Research in Mathematics Education, 29*(2), 21-41.

OECD Programme for International Student Assessment (PISA). (2009). *PISA 2009 Result: What Students Know and Can Do: Student Performance in Reading, Mathematics, and Science.* PISA.

Potts, A., Blank, R. K., & Williams, A. (2002). *Key state education policies on K–12 education: 2002.* Washington, DC: Council of Chief State School Officers.

Provenzo, E., Brett, A., & McCloskey, G. (1999). *Computers, Curriculum, and Cultural Change.* Mahwah, NJ: Lawrence Erlbaum Associates.

Schmidt, W. H., McKnight, C. C., & Raizen, S. A. (1997). *A splintered vision: An investigation of U.S. science and mathematics education.* Boston: Kluwer Academic Publications.

Schweinle, A., Meyer, D., & Turner, J. (2006). Striking the Right Balance: Students' Motivation and Affect in Elementary Mathematics. *The Journal of Educational Research, 99*(5), 271-289.

Secada, W. (1992). Race, ethnicity, social class, language and achievement in mathematics. In *Handbook of research on mathematics teaching and learning*, edited by D. Grouws. New York: Macmillian.

Smith, B. (1981). Misguided mathematical maxim-makers, *The Two Year College Mathematics Journal*, pp. 309-316.

Tavani, C., & Losh, S. (2003). Motivation, self-confidence, and expectations as predictors of the academic performances among our high school students. *Child Study Journal, 33*(3). Available from Academic Search Premier.

Venezia, A., Kirst, M. W., & Antonio, A. L. (2003). *Betraying the college dream: How disconnected K–12 and postsecondary education systems undermine student aspirations.* Stanford, CA: The Bridge Project, The Stanford Institute for Higher Education Research.

APPENDIX A: STANDARDS FOR MATHEMATICAL PRACTICE

1. Make sense of problems and persevere in solving them.

Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution map rather than simply jumping into a solution attempt. They monitor and evaluate their progress and change course if needed. For instance, depending on the context of the problem, they transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, and search for regularity or trends. These students can also relate mathematical ideas from previously learned to current situation. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, "Does this make sense?" They can understand the approaches of others to solve complex problems and identify correspondences among different approaches.

2. Reason abstractly and quantitatively.

Mathematically proficient students make sense of quantities and their relationships in problem situations. They abstract a given situation and represent it symbolically, manipulate the representing symbols, and pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

3. Construct viable arguments and critique the reasoning of others.

Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that is flawed. If there is a flaw in an argument, then they can explain what it is.

4. **Model with mathematics.**

   Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in practical situation and map their relationships using such tools as diagrams, tables, graphs, flowcharts, and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

5. **Use appropriate tools strategically.**

   Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, or dynamic geometry software. Proficient students are sufficiently familiar with tools appropriate for their course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, mathematically proficient high school students analyze graphs of functions and solutions generated using graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. Mathematically proficient students are able to use technological tools to explore and deepen their understanding of concepts.

6. **Attend to precision.**

   Mathematically proficient students try to communicate precisely to other. They try to use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sigh consistently and appropriately. They are careful about specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context. By the time reaching high school, they have learned to examine claims and make explicit use of definitions.

7. Look for and make use of structure.

Mathematically proficient students look closely to discern a pattern or structure. In the expression, $x^2 + 9x + 14$, high school students can see the 14 as 2 x 7 and the 9 as 2 + 7. They recognize the significance of an existing line in a geometric figure and can use the strategy an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 – 3(x - y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real number x and y.

8. Look for and express regularity in repeated reasoning.

Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Noticing the regularity in the way terms cancel when expanding $(x - 1)(x + 1), (x - 1)(x^2 + x + 1), and\ (x - 1)(x^3 + x^2 + x + 1)$ might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

APPENDIX B: UNPACKING OF THE (+) STANDARDS AND SECTION MAPPING

Algebra		
Arithmetic with Polynomials and Rational Expressions	**A-APR**	
Use polynomial identities to solve problems		
CCStandards	Unpacking / Examples / Explanations	Sections
A-APR.5 (+) Know and apply the Binomial Theorem for the expansion of $(x + y)^n$ in powers of x and y for a positive integer n, where x and y are any numbers, with coefficients determined for example by Pascal's Triangle.	For small values of n, use Pascal's Triangle to determine the coefficients of the binomial expansion. Use the Binomial Theorem to find the nth term in the expansion of a binomial to a positive power. Examples: Use Pascal's Triangle to expand the expression $(2x-1)^4$. Find the middle term in the expansion of $(x^2+2)^{18}$. Pascal's Triangle 1 1 1 1 2 1 1 3 3 1 $(x+1)^3 = x^3+3x^2+3x+1$ 1 4 6 4 1 ↑ ↑ ↑ ↑ ↑ $_4C_0$ $_4C_1$ $_4C_2$ $_4C_3$ $_4C_4$	9.2
Rewrite rational expressions		
A-APR.7 (+) Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression; add, subtract, multiply, and divide rational expressions.	Simplify rational expressions by adding, subtracting, multiplying, or dividing. Understand that rational expressions are closed under addition, subtraction, multiplication, and division (by a nonzero expression).	2.6; Appendix A

Reasoning with Equations and Inequalities A-REI		
Solve systems of equations		
A-REI.8 (+) Represent a system of linear equations as a single matrix equation in a vector variable.	Write a system of linear equations as a single matrix equation. A matrix is a two dimensional array with rows and columns; a vector is a one dimensional array that is either one row or one column of the matrix. Students will use matrices of vectors to represent and transform geometric objects in the coordinate plane. They will explain the relationship between the ordered pair representation of a vector and its graphical representation.	7.3
A-REI.9 (+) Find the inverse of a matrix if it exists and use it to solve systems of linear equations (using technology for matrices of dimension 3 × 3 or greater).	Example: • Solve the system of equations by converting to a matrix equation and using the inverse of the coefficient matrix. $\begin{cases} 5x+2y=4 \\ 3x+2y=0 \end{cases}$ • Solution: Make matrix A with $A = \begin{bmatrix} 5 & 2 \\ 3 & 2 \end{bmatrix}$ Matrix $X = \begin{bmatrix} x \\ y \end{bmatrix}$ Matrix $B = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$ Matrix $A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{3}{4} & \frac{5}{4} \end{bmatrix}$ $X = A^{-1}B$ $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{3}{4} & \frac{5}{4} \end{bmatrix} \begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$	7.3

Functions		
Interpreting Functions F-IF		
Analyze functions using different representations		
F-IF.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★	F-IF.7.d (+) Graph rational functions, identifying zeros and asymptotes when suitable factorizations are available, and showing end behavior.	2.6
Building Functions F-BF		
Building a function that models a relationship between two quantities		
F-BF.1 Write a function that describes a relationship between two quantities.★	F-BF.1.c (+) Compose functions.	1.4
Build new functions from existing functions		
F-BF.4 Find inverse functions.	F-BF.4.b (+) Verify by composition that one function is the inverse of another.	1.5
	F-BF.4.c (+) Read values of an inverse function from a graph or a table, given that the function has an inverse.	1.5
	F-BF.4.d (+) Produce an invertible function from a non-invertible function by restricting the domain.	1.5
F-BF.5 (+) Understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents.		3.3; 3.4; 3.5

Trigonometric Functions F-TF		
Extend domain of trigonometric functions using the unit circle		
F-TF.3 (+) Use special triangles to determine geometrically the values of sine, cosine, tangent for π/3, π/4 and π/6, and use the unit circle to express the values of sine, cosine, and tangent for π– x, π+x, and 2π–x in terms of their values for x, where x is any real number.		4.3
CCStandards	Unpacking / Examples / Explanations	Sections
F-TF.4 (+) Use the unit circle to explain symmetry (odd and even) and periodicity of trigonometric functions.	Use the unit circle and periodicity to find values of sine, cosine, and tangent for any value of θ, such as π+θ, 2π+θ, where θ is a real number. Use the values of the trigonometric functions derived from the unit circle to explain how trigonometric functions repeat themselves. Use the unit circle to explain that f(x) is an even function if f(x) = f(-x) for all x, and an odd function if f(x) = -f(-x). Also know that an even function is symmetric about the y-axis.	4.3; 4.4; 4.5
Model periodic phenomena with trigonometric functions		
F-TF.6 (+) Understand that restricting a trigonometric function to a domain on which it is always increasing or always decreasing allows its inverse to be constructed.		4.7
F-TF.7 (+) Use inverse functions to solve trigonometric equations that arise in modeling contexts; evaluate the solutions using technology, and interpret them in terms of the context.★		4.7
Prove and apply trigonometric identities		
F-TF.9 (+) Prove the addition and subtraction formulas for sine, cosine, and tangent and use them to solve problems.	Prove the addition and subtraction formulas: sin (A $\pm$ B), cos (A $\pm$ B), tan (A $\pm$ B). Use the addition and subtraction formulas to determine exact trigonometric values such as sin(75°) or cos ($\frac{\pi}{12}$).	5.3

Geometry		
Similarity, Right Triangles, and Trigonometry G-SRT		
Apply trigonometry to general triangles		
G-SRT.9 (+) Derive the formula A = 1/2 ab sin(C) for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side.	For a triangle that is not a right triangle, draw an auxiliary line from a vertex, perpendicular to the opposite side and derive the formula, A = $\frac{1}{2}\, ab \sin(C)$, for the area of a triangle, using the fact that the height of the triangle is h = a $\sin(C)$.	5.5
G-SRT.10 (+) Prove the Laws of Sines and Cosines and use them to solve problems.	Using trigonometry and the relationship among sides and angles of any triangle, such as $\sin(C) = (\frac{h}{a})$, prove the Law of Sines. Using trigonometry and the relationship among sides and angles of any triangle and the Pythagorean Theorem to prove the Law of Cosines.	5.5; 5.6
G-SRT.11 (+) Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces).		5.5; 5.6
Expressing Geometric Properties with Equations G-GPE		
Translate between the geometric description and the equation for a conic section		
CCStandards	Unpacking / Examples / Explanations	Sections
G-GPE.3 (+) Derive the equations of ellipses and hyperbolas given the foci, using the fact that the sum or difference of distances from the foci is constant.	Given the foci, derive the equation of an ellipse, noting that the sum of the distances from the foci to any fixed point on the ellipse is constant, identifying the major and minor axis. Give the foci, derive the equation of a hyperbola, noting that the absolute value of the differences of the distances from the foci to a point on the hyperbola is constant, and identifying the vertices, center, transverse axis, conjugate axis, and asymptotes.	8.2

APPENDIX C: A COMPANION TO PRECALCULUS 8th EDITION, BY DEMANA, WAITS, FOLEY, AND KENNEDY: COMPREHENSIVE RESOURCES THAT ARE AP CALCULUS READY AND CCSSM AND CCS(+)SM ALIGNED

Precalculus for AP Calculus:

A COMPANION TO PRECALCULUS 8th EDITION, BY DEMANA, WAITS, FOLEY, AND KENNEDY: COMPREHENSIVE RESOURCES THAT ARE AP CALCULUS READY AND CCSSM AND CCS(+)SM ALIGNED

Sequence of Precalculus for AP Calculus

Unit 1: Polynomial and Rational Functions
Unit 2: Exponential and Logarithmic Functions
Unit 3: Trigonometry
Unit 4: Application of Trigonometry
Unit 5: Optional (+) Topics

Course Goal

Using algebraic function properties as a springboard to further in-depth study of transcendental functions and their properties is emphasized.
Throughout the course, students are engaged in applying their study of mathematics to the modeling of mathematical and real-world situations. The processes of representation, connection, and communication are important factors in molding students' reasoning and problem-solving strategies that are fundamental yet crucial to their success in the future study of the AP Calculus. Students extend their concept of function to operations with functions, including composition and transformations, and in-depth study of the behavior of functions from both a numerical and a graphical standpoint. Trigonometric functions are extended from the study of right triangles to encompass side and angle measurements in all triangles through the use of the Law of Sines and the Law of Cosines. Students make the transition in this course from trigonometric functions based on degree measure to working almost exclusively in radian measure, also a strong point in the future study of the AP Calculus. From this point forward, degrees are only used in situations involving specific degree measure application, such as solving triangles used in degree measure, as in the application of the Law of Sines (Cosines) or in problem involving angles of elevation or depression. The study of trigonometry is further extended to circular functions associated with the angles centered at the origin of the unit circle. Students see the connections between the right triangle, trigonometric function, and unit circle functions. Students develop in their study of Cartesian plane system to polar system. These relationships in two variables are then applied to modeling a number of mathematical and real-world relationships. Applications of these concepts are used to formulize transformations and visually and symbolically model problems involving motion. The Common Core Mathematical Practice Standards apply throughout each section and, together with the Common Core Content Standards, students experience the study of mathematics as a focused, coherent, and rigorous subject that helps them become problem solvers.

Table of Contents

Preface

Recent adoption of Common Core State Standards in Mathematics (CCSSM) has left many teachers scrambling to find instructional materials that would meet the graduation requirement as well as insuring that our students are college and career ready when they leave high school. Furthermore, the College Board's Advanced Placement (AP) Calculus curriculum is widely accepted as the model for a twenty-first century calculus course. The path now needs to be set for a new precalculus course designed to align with the AP goals and objectives. This paper, "High School 4^{th} Mathematics: Precalculus for AP Calculus," is written specifically with the goal of meeting this requirement. In this paper, Precalculus teachers are provided with lecture notes and student activities that are Common Core Standards and (+) Standards aligned and AP Calculus ready.

This work is a first draft and is intended for use as a companion to the textbook, Precalculus 8^{th} Edition, written by Demana, Waits, Foley, and Kennedy. Unbeknownst to the book's authors, I am grateful because they have inspired me to write this paper. After examining many textbooks, I selected this book for two reasons:

(3) Its balanced approach among the algebraic, numerical, graphical, and verbal methods of representing problems, which is very much like the AP approach.

(4) It is published by Pearson Education® which means MathXL (My Math Lab) is available.

To keep the integrity of these authors' work, to be truthful to their mission, and to best serve the students and teachers using this textbook, I have used their language, ideas, and examples as often as possible.

Even though development of assignments using MathXL is not part of this paper, web-based assignments will be available and used for the Dual Enrollment Precalculus course, Math 165, through Southeastern Louisiana University.

Many student activities are imported from Illustrative Mathematics, and easily recognizable by the icon: Illustrative Mathematics. Furthermore, included in the lecture notes are direct links to the url's of the specific problems in Illustrative Mathematics so that teachers can find helpful comments and solutions to the student activities at their fingertips.

In alignment with AP goals, writing assignments are incorporated throughout in this paper to help students to understand the mathematics they study. This is denoted by **Writing Assignment** . In addition, conscientious efforts were made to alert teachers to the connection between Precalculus and AP Calculus by using the title of **Looking Ahead to Calculus** . Advanced Placement Program (AP) is registered trademark of the College Board, which was not involved in the writing of, and does not endorse this paper.

Finally, it would have been too ambitious or even impossible to include all sections; I had to choose sections to meet the teaching time constraint. The decision of what topics should be included in this paper was made solely by me as a classroom teacher who is teaching Precalculus and AP Calculus. Consequently, it was necessary for these sections to be rearranged and renumbered. It was practical that new numbering appears before the topics to be studied and the original numbering from the textbook after. For example: U1S6: Fundamental Theorem of Algebra (2.5) means Fundamental Theorem of Algebra is listed in Unit 1 Section 6 in the paper and can be located in Chapter 2 Section 5 Precalculus written by Demana, et al. (2011).

Unit 1: Polynomial and Rational Functions

Overview: In this unit, we expand the study of basic functions from Algebra I and Algebra II. These functions can be seen using a graphing calculator, and their properties can be described using the notation and terminology will be introduced and studied in this unit. We will explore the theory and applications of specific families of functions. To start off, we will study three interrelated functions; polynomial, power, and rational functions. We will investigate the graphical behavior of polynomial and rational functions. This unit concludes with solving polynomial and rational inequalities.

Unit 1 Section 1: Functions and Their Properties (1.2)

Major Concept: Represent functions numerically, algebraically, and graphically. Determine the domain and range for functions, and analyze function characteristics such as extreme values, symmetry, asymptotes, and end behavior.

STANDARDS:

Algebra-Creating Equations A-CED

Create equations that describe numbers or relationships

A-CED.2 Create equations in two or more variables to represent relationships between quantities; graph equations on Coordinate axes with labels and scales.

Represent and Solve equations and inequalities graphically

A-REI.10 Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line).

Functions-Interpreting Functions F-IF

Understand the concept of a function and use function notation

F-IF.1 Understand that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range. If f is a function and x is an element of its domain, then f(x) denotes the output of f corresponding to the input x. The graph of *f* is the graph of the equation y = f(x).

F-IF.2 Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context.

Interpret functions that arise in applications in terms of the context

F-IF.4 For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.

F-IF.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes.

F-IF.6 Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from graph.

Analyze Functions using different representations

F-IF.9 Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions).

Building Functions F-BF

Build new functions from existing functions

F-BF.3 Recognize even and odd functions from their graphs and algebraic expression for them.

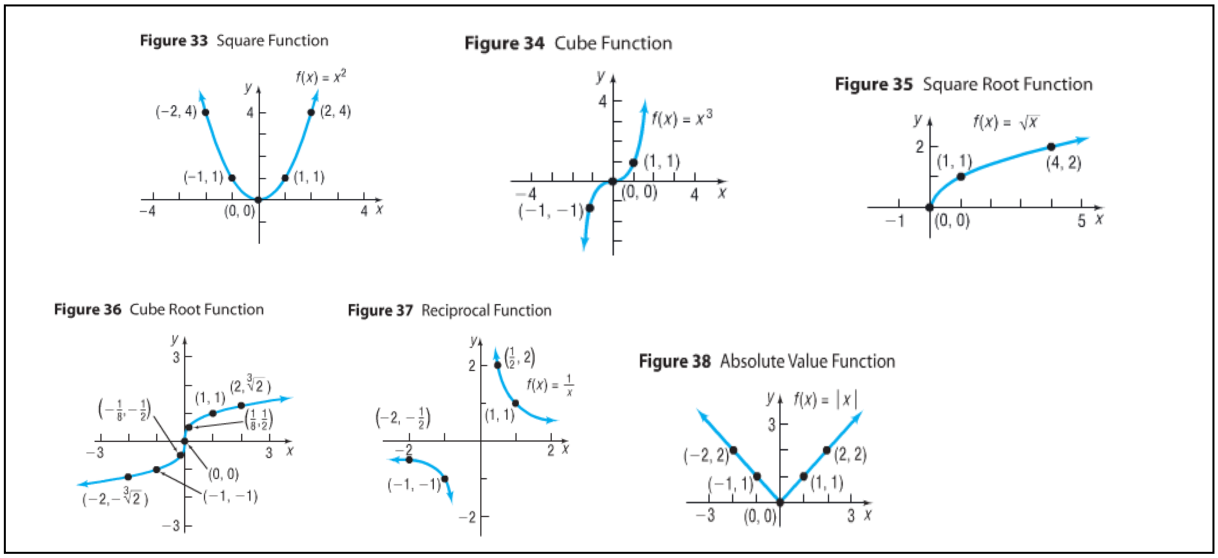
Unit 1 Section 1-A: Functions

Students will be able to:

- Determine the domain and range of a function algebraically and graphically.
- Compute the average rate of change.

Vocabulary:
domain, range, function notation, independent variable, dependent variable, mapping, graph of a function, implied domain, relevant domain, rate of change

1. **CONNECT** (10 minutes)
   Finding Domains: Graphical Approach



2. **INSTRUCT** (40 minutes)
   Conceptual Understanding: Functions
   Students can often give algebraic examples of functions, but do not understand the underlying definition.
   Give several examples of everyday items that may serve as functions.
   The concept they should have here is one of a machine. Next, the graphical understanding of a function, using the Vertical Line Test.

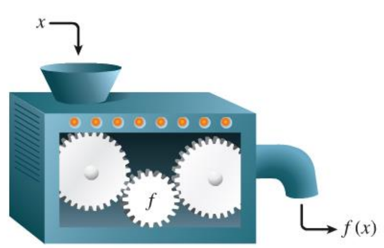


FIGURE 1.9 A "machine" diagram for a function.

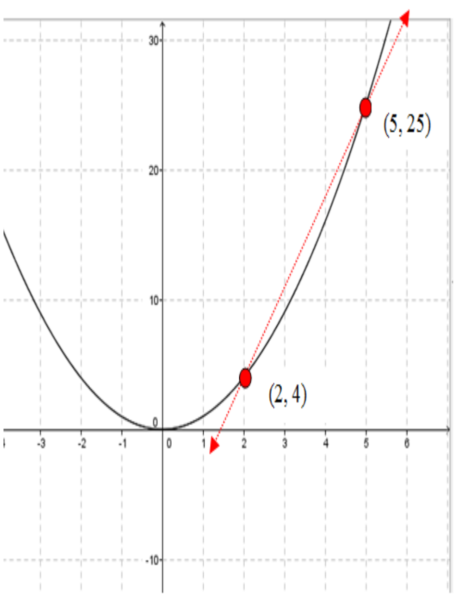
Conceptual Understanding: Intervals
Review the different forms that intervals may take, introducing the proper notation with each illustration.

Calculations: Domain
Have students discuss the domains of functions involving roots and fractions. Students should be able to calculate the domain of a function from an algebraic definition. Once this is properly understood, students should learn to use a graphing calculator to check their work. These techniques will lay important foundations for future calculations. Discuss the difference between implied and relevant domain.

Conceptual Understanding: Average Rate of Change
Students learned in Grade 8 that the “rate of change” of a linear function is equal to the “slope” of the line. And because the slope of a line is constant, that is, between any two points, it is the same. The rate of change has an unambiguous meaning for a linear function. For non-linear functions, however, rates of change are not constant, so we talk about average rates of change over as interval. For example, for the function, $f(x) = x^2$, the average rate of change from $x = 2 \; to \; x = 5$ is

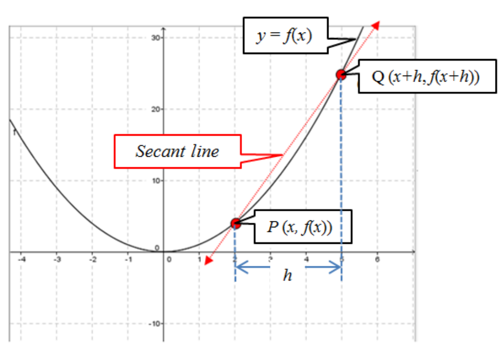
$$\frac{f(5) - f(2)}{5 - 2} = \frac{25 - 4}{3} = 7$$

This is the slope of the line from (2, 4) to (5, 25) on the graph of *f*. And if *f* is Interpreted as the area of a square of side x, then this calculation means that over this interval (2, 5), the area changes on average 7 squares units for each unit increase in the side length, *x*, of the square.



Average Rate of Change is also known as the **Difference Quotient**. The line joining P and Q (the line PQ is called a secant line) has slope:

$m = \frac{f(x+h)-f(x)}{h}$, $h \neq 0$.

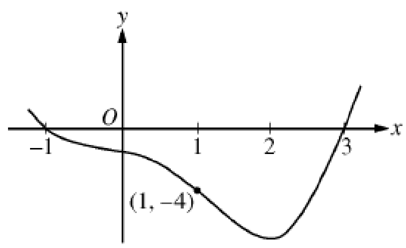


Looking Ahead to Calculus

AP® CALCULUS AB
2009 SCORING GUIDELINES (Form B)

Question 5

Let f be a twice-differentiable function defined on the interval $-1.2 < x < 3.2$ with $f(1) = 2$. The graph of f', the derivative of f, is shown above. The graph of f' crosses the x-axis at $x = -1$ and $x = 3$ and has a horizontal tangent at $x = 2$. Let g be the function given by $g(x) = e^{f(x)}$.



Graph of f'

(a) Write an equation for the line tangent to the graph of g at $x = 1$.

(b) For $-1.2 < x < 3.2$, find all values of x at which g has a local maximum. Justify your answer.

(c) The second derivative of g is $g''(x) = e^{f(x)}\left[(f'(x))^2 + f''(x)\right]$. Is $g''(-1)$ positive, negative, or zero? Justify your answer.

(d) Find the average rate of change of g', the derivative of g, over the interval $[1, 3]$.

Solution for (d)

(d) $\dfrac{g'(3) - g'(1)}{3 - 1} = \dfrac{e^{f(3)}f'(3) - e^{f(1)}f'(1)}{2} = 2e^2$

$2 : \begin{cases} 1 : \text{difference quotient} \\ 1 : \text{answer} \end{cases}$

Review:

Find and simplify: $\frac{f(x+h)-f(x)}{h}$

1. $f(x) = x^2$
2. $f(x) = \frac{1}{x}$
3. $f(x) = \frac{1}{x^2-3}$

Looking Ahead to Calculus

This slope concept is further developed into the "limit" concept in calculus. The limit used to define the slope of a tangent line is also used to define one of the two fundamental operations of calculus – **differentiation**.

Definition of the Derivative of a Function

The derivative of f at x is

$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ provided the limit exists.

Illustrative Mathematics **Comments and Solutions can be found at:**

http://www.illustrativemathematics.org/illustrations/599

Writing Assignment:

Using Function Notation: Given a function f, is the statement $f(x+h) = f(x) + f(h)$ true for any two numbers x and h? If so, prove it. If not, find a function for which the statement is true and a function for which the statement is false.

The purpose of the task is to explicitly identify a common error made by many students, when they make use of the "identity" $f(x+h) = f(x) + f(h)$. A function f cannot, in general, be distributed over a sum of inputs. This is an easy mistake to make because $f(x+h) = f(x) + f(h)$ is a true statement if f, x, h are real numbers and the operations implied by the parentheses are multiplication. The task has students find a single explicit example for which the identity is false, but it is worth emphasizing that in fact the identity fails for the vast majority of functions. Among continuous functions, the *only* functions satisfying the identity for all x and h are the functions $f(x) = ax$ for a constant a.

Comments and Solutions can be found at:
http://www.illustrativemathematics.org/illustrations/635

Finding Domain: Algebraic Approach

For the functions in (a)–(f), what restrictions does each operation place on the domain of the function?

a. $y = \dfrac{2}{x-3}$

b. $y = \sqrt{x-5} + 1$

c. $y = 4 - (x-3)^2$

d. $y = \dfrac{7}{4-(x-3)^2}$

e. $y = 4 - (x-3)^{1/2}$

f. $y = \dfrac{7}{4-(x-3)^{1/2}}$

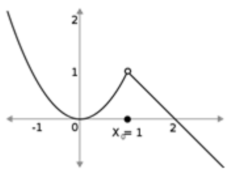
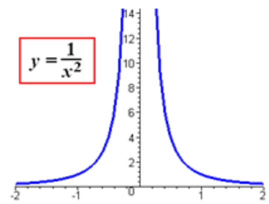
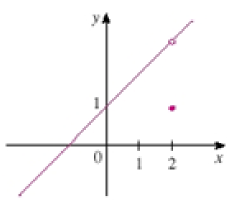
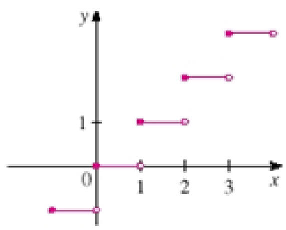
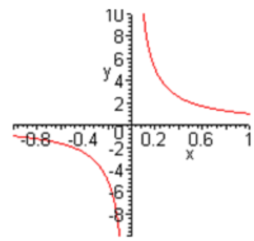
Unit 1 Section 1-B: Continuity and Increasing / Decreasing

Students will be able to:

- Determine graphically whether a function is continuous or discontinuous at a point.
- Classify the type of discontinuity graphically.
- Identify whether a function is increasing, decreasing, or constant based on a table of numerical data.
- Identify intervals on which a function is increasing or decreasing graphically.

Vocabulary:
continuous, removable discontinuity, jump discontinuity, infinite discontinuity, discontinuous, increasing, decreasing, constant

1. **INSTRUCT** (20 minutes)
   Continuity: Graphical Approach
   Introduce an informal concept of a function being continuous at a point if its graph does not come apart there. Another way to explain continuity is that the graph can be drawn through the point without picking up the pencil. Show the three different types of discontinuities (Removable discontinuity, Jump discontinuity, and Infinite discontinuity) that students will encounter with functions in this course, and have students look at several examples.

Removable Discontinuity:	**Jump Discontinuity:**	**Infinite Discontinuity:**
		
 (c) $f(x) = \begin{cases} \frac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$	 (d) $f(x) = [\![x]\!]$	

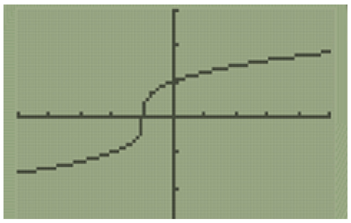
The simple geometric concept of an unbroken graph at a point is a visual notation that is extremely difficult to communicate accurately in the language of algebra. The key concept from the graphs seems to be that we want the point (x, $f(x)$) to slide onto the point (a, $f(a)$) without missing it from either direction. We say that $f(x)$ approaches $f(a)$ as a limit as x approaches a, and we write $\lim_{x \to a} f(x) = f(a)$. This "limit notation" reflects graphical behavior so naturally that we will use it throughout the course. In summary, a function is **continuous at *x* = *a*** if $\lim_{x \to a} f(x) = f(a)$. A function is **discontinuous at *x* = *a*** if it is not continuous at $x = a$.

Looking Ahead to Calculus

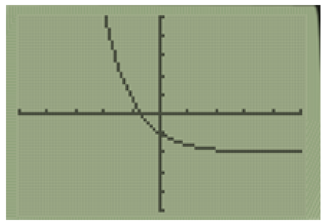
A limit exists on a removable discontinuity but not on a jump or infinite discontinuity. While the notation of limits is easy to understand, the *algebraic definition* of a limit can be a little intimidating and is best left to future courses.

Increasing and Decreasing Functions: Graphical Approach

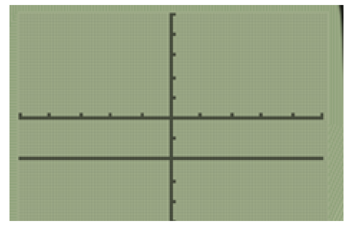
Here again an informal definition is given, with emphasis on a graphical understanding of a function increasing or decreasing on an interval. Students need to be able to identify intervals of increase or decrease from the graph of a function. It helps to point out what the function is doing as you move from right to left.



Increasing



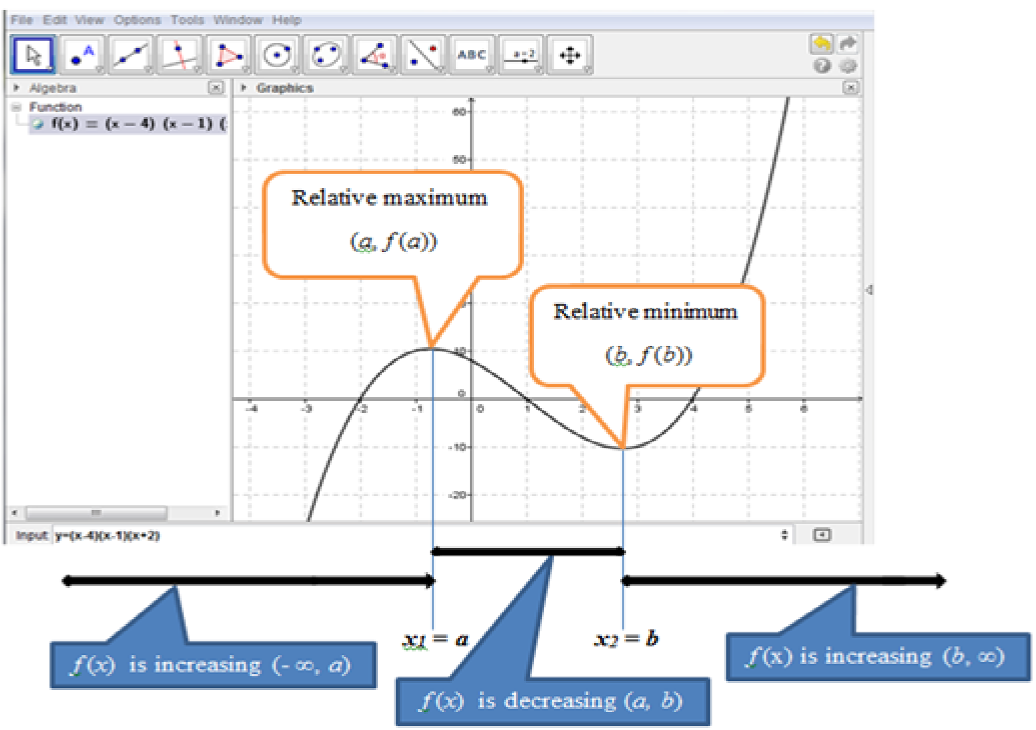
Decreasing



Constant

Increasing and Decreasing Functions

A function f is increasing on an interval if for any two numbers x_1 and x_2 in the interval, $x_1 < x_2$ implies $f(x_1) < f(x_2)$. A function f is decreasing on an interval if for any two numbers x_1 and x_2 in the interval $x_1 < x_2$ implies $f(x_1) > f(x_2)$.



Looking Ahead to Calculus

2013 AP Calculus AB Exam (MC without calculator)

If g is the function given by $g(x) = \frac{1}{3}x^3 + \frac{3}{2}x^2 - 70x + 5$, on which of the following intervals is g decreasing?

(A) $(-\infty, -10)$ *and* $(7, \infty)$ (B) $(-\infty, -7)$ *and* $(10, \infty)$ (C) $(-\infty, 10)$
(D) $(-10, 7)$ (E) $(-7, 10)$

2. **GROUP DISCUSSION** (10 minutes)
Discontinuities
Have students get into groups of three or four. Ask students to come up with an example of a function with a specific number of a particular type of discontinuity. For instance, you might ask students to find a function with exactly two removable discontinuities. They may use their graphing calculators to help. See which group can find an example first. Start with simple examples, then increase the difficulty level as students begin to see patterns.

3. **INDIVIDUAL EXPLORATION** (10 minutes)
Exploration: Increasing and Decreasing Functions
Have students investigate Exploration 1 on pages 86 and 87.

EXPLORATION 1 Increasing, Decreasing, and Constant Data

1. Of the three tables of numerical data below, which would be modeled by a function that is (a) increasing, (b) decreasing, (c) constant?

X	Y1
−2	12
−1	12
0	12
1	12
3	12
7	12

X	Y2
−2	3
−1	1
0	0
1	−2
3	−6
7	−12

X	Y3
−2	−5
−1	−3
0	−1
1	1
3	4
7	10

2. Make a list of $\Delta Y1$, the *change* in Y1 values as you move down the list. As you move from $Y1 = a$ to $Y1 = b$, the change is $\Delta Y1 = b - a$. Do the same for the values of Y2 and Y3.

X moves from	ΔX	$\Delta Y1$
-2 to -1	1	
-1 to 0	1	
0 to 1	1	
1 to 3	2	
3 to 7	4	

X moves from	ΔX	$\Delta Y2$
-2 to -1	1	
-1 to 0	1	
0 to 1	1	
1 to 3	2	
3 to 7	4	

X moves from	ΔX	$\Delta Y3$
-2 to -1	1	
-1 to 0	1	
0 to 1	1	
1 to 3	2	
3 to 7	4	

3. What is true about the quotients $\Delta Y/\Delta X$ for an increasing function? For a decreasing function? For a constant function?

4. Where else have you seen the quotient $\Delta Y/\Delta X$? Does this reinforce your answers in part 3?

Writing Assignment:

A continuous function f has domain all real numbers.
If $f(-1) = 5$ and $f(1) = -5$, explain why f must have at least one zero in the interval [-1, 1]. (This generalizes to a property of continuous functions known as the Intermediate Value Theorem.)

Unit 1 Section 1-C: Boundeness, Local/Absolute Extrema, and Symmetry

Students will be able to:

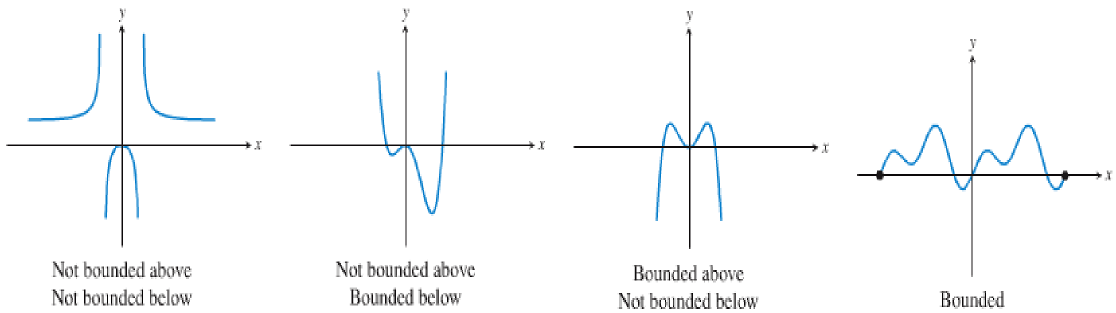
- **Determine** whether a function is bounded below or above graphically.
- **Identify** local and absolute extrema from the graph of a function.
- **Determine** numerically or graphically whether a function has odd or even symmetry.
- **Determine** algebraically whether a function has odd or even symmetry.

Vocabulary:
bounded below, lower bound, bounded above, upper bound, bounded, bounded on an interval, local minimum, absolute minimum, local maximum, absolute maximum, relative extrema, even, odd

1. **INSTRUCT** (25 minutes)

Conceptual Understanding: Bounded Functions

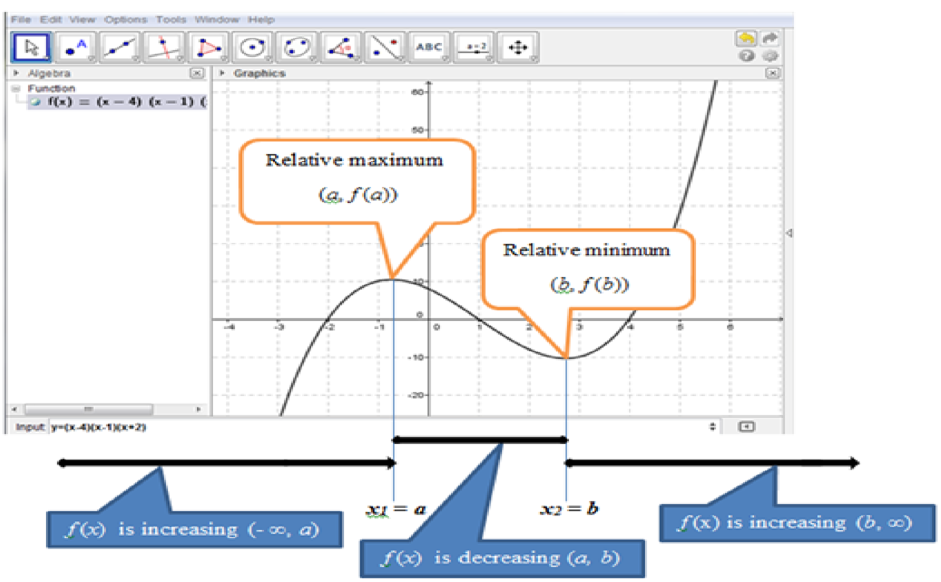
Explain the concepts of a function being bounded below, bounded above, and bounded. The graphical understanding to these definitions is key for students. Have students try to think of functions on their own that are unbounded, bounded below or above, and bounded.



Local(Relative) and Absolute Extrema:

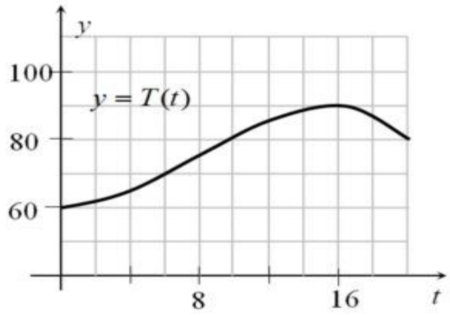
As students begin to understand functions being bounded, they will encounter graphical examples of extrema. Introduce the concepts of local (relative) and absolute maxima and minima, and have students identify these from the graph of a function.

Notice that a local maximum (or minimum) does not have to be *the maximum* (or *minimum*) value of a function; it only needs to be the maximum (or minimum) value of the function on *some* interval.



Comments and Solutions can be found at:
http://www.illustrativemathematics.org/illustrations/639

The figure shows the graph of T, the temperature (in degrees Fahrenheit) over one particular 20-hour period in Santa Elena as a function of time *t*.



1. Estimate T(14).
2. If $t = 0$ corresponds to midnight, interpret what we mean by T(14) in words.
3. Estimate the highest temperature during this period from the graph.
4. When was the temperature decreasing?
5. If Anya wants to go for a two-hour hike and return before the temperature gets over 80 degrees, when should she leave?

Looking Ahead to Calculus

A key theorem in calculus, the Extreme Value Theorem, states, if a function f is continuous on a closed interval [a, b], then f has both a maximum value and a minimum value on the interval.

Conceptual Understanding: Symmetry
Introduce the three kinds of symmetry: symmetry about the y-axis, about the x-axis, and about the origin. Have students try to see patterns among odd and even powers of x, and explain the meaning of the terms "odd function" and "even function." Have students try to come up with examples that are not powers of *x*. Finally, introduce the algebraic method for checking for symmetry.

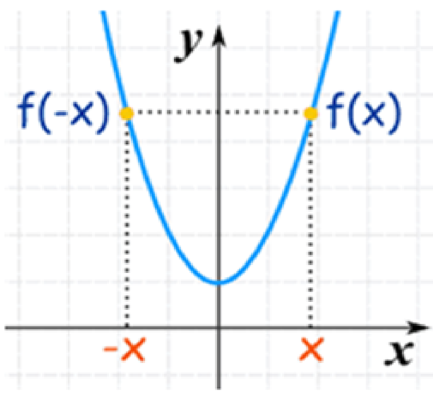
Definition: A function $f(t)$ is called **even** if $f(-t) = f(t)$ for all t.

The graph of an even function is symmetric about the *y*-axis. Here are some examples of even functions:

1. t^2, t^4, t^6, . . . , *any even power of t.*
2. cos(*at*).
3. A constant function is even.

Example: $f(x) = x^2 + 1$

Graphically



Numerically

$f(x) = x^2 + 1$

x	f(x)
-3	10
-2	5
-1	2
0	1
1	2
2	5
3	10

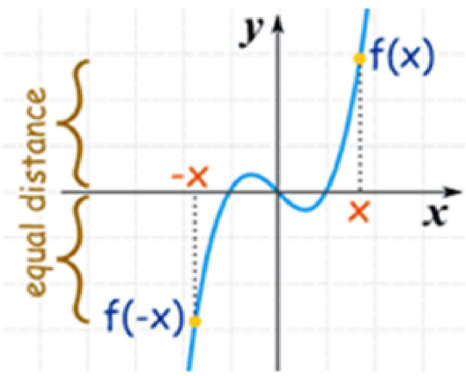
Algebraically:
Need to show that f(x) = f(-x)
Work: $f(x) = x^2 + 1 = (-x)^2 + 1 = f(-x)$

Definition. A function *f* (x) is called **odd** if *f* (-*t*) = - *f* (*t*) for all *t*.
The graph of an odd function is symmetric about the origin. Here are some examples of odd functions:

1. *t*, t^3, t^5, . . . , any *odd* power of *t*.
2. sin(*at*).
3. An identity function (*f* (*x*) = *x*) is odd.

Example: $f(x) = x^3 - x$

Graphically



Numerically

$f(x) = x^3 - x$

x	f(x)
-3	-24
-2	-6
-1	0
0	0
1	0
2	6
3	24

Algebraically:
Need to show that f(x) = -f(-x)
Work: $f(x) = x^3 - x = -[(-x)^3 - (-x)] = -[-x^3 + x] = -f(-x)$

2. **INDIVIDUAL EXPLORATION** (10 minutes)
   Exploration: Combining Odd and Even Functions
   Have students try to discover what happens when odd and even functions are added, subtracted, multiplied, or divided. Is the result even, odd, or neither? How is this related to even and odd numbers?

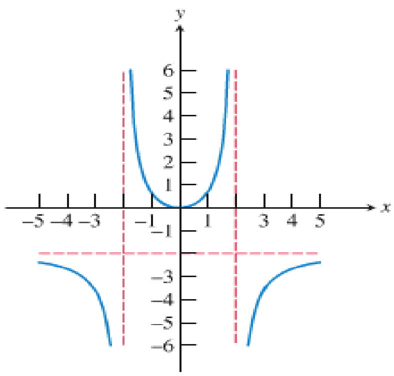
Unit 1 Section 1-D: Asymptotes and End Behavior

Students will be able to:
- **Identify** the horizontal and vertical asymptotes of a graph.
- **Identify** the end behavior of a function.

Vocabulary:
horizontal asymptote, vertical asymptote, end behavior

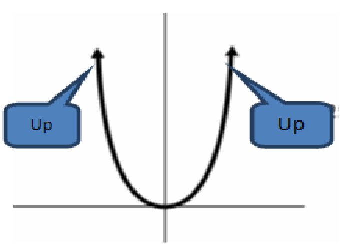
1. **INSTRUCT** (25 minutes)

Conceptual Understanding: Asymptotes (more on (2.6))
Asymptotes describe the behavior of the graph at its horizontal or vertical *extremities*. Using a graph, introduce the concept of vertical and horizontal asymptotes. Students should learn how to identify where likely vertical asymptotes are (by investigating the denominator of a rational function), and confirm this graphically.



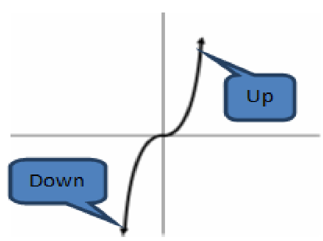
The graph of $f(x) = \frac{2x^2}{4-x^2}$ with the asymptotes shown as dashed lines.

Conceptual Understanding: End Behavior
Here, students need to discover how as *x* gets large, certain terms in a function become insignificant. Students should begin to develop an eye for which terms are important, and which can be neglected when considering end behavior. You can help them confirm this in two ways: numerically (have students plug in large numbers to see which terms are largest) and graphically (have students plot the graph of the function and the graph of its end behavior on one screen, then zoom out).



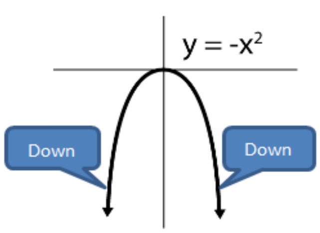
$y = x^2$

as $x \to -\infty$, $y \to \infty$; as $x \to \infty$, $y \to \infty$

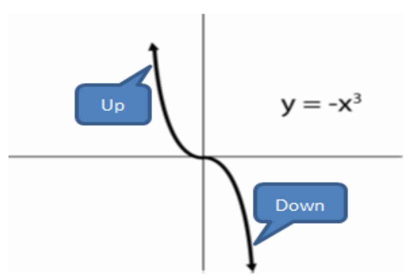


$y = x^3$

as $x \to -\infty$, $y \to -\infty$; as $x \to \infty$, $y \to \infty$



as $x \to -\infty$, $y \to -\infty$; as $x \to \infty$, $y \to -\infty$



as $x \to -\infty$, $y \to \infty$; as $x \to \infty$, $y \to -\infty$

Leading Term Test for Polynomials End Behavior Summary:

Based on our exploration and observation, there are four possible end behavior patterns for a polynomial function. In $f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_2 x^2 + a_1 x + a_0$, $a_n \neq 0$, the leading term, $a_n x^n$, determines the end behavior of a polynomial function. The power and coefficient of the leading term tell us which of the four patterns occurs:

$a_n > 0$				$a_n < 0$			
n is odd		**n is even**		**n is odd**		**n is even**	
Lt. End $x \to -\infty$	Rt. End $x \to \infty$	Lt. End $x \to -\infty$	Rt. End $x \to \infty$	Lt. End $x \to -\infty$	Rt. End $x \to \infty$	Lt. End $x \to -\infty$	Rt. End $x \to \infty$
Down $y \to -\infty$	Up $y \to \infty$	Up $y \to \infty$	Up $y \to \infty$	Up $y \to \infty$	Down $y \to -\infty$	Down $y \to -\infty$	Down $y \to -\infty$
$\lim_{x\to-\infty} f(x) = -\infty$	$\lim_{x\to\infty} f(x) = \infty$	$\lim_{x\to-\infty} f(x) = \infty$	$\lim_{x\to\infty} f(x) = \infty$	$\lim_{x\to-\infty} f(x) = \infty$	$\lim_{x\to\infty} f(x) = -\infty$	$\lim_{x\to-\infty} f(x) = -\infty$	$\lim_{x\to\infty} f(x) = -\infty$

2. **GROUP DISCUSSION** (10 minutes)
   Exploration: How Many Asymptotes?
   Have students form groups of three or four and discuss how many vertical and how many horizontal asymptotes a function can have. Have them give examples to back up their statements. A function can have at most two horizontal asymptotes, why?

Writing Assignment:
The Greek roots of the word "asymptote" mean "not meeting," since graphs tend to approach, but not meet, their asymptotes. Explain how and where a graph can cross its own asymptote.

Unit 1 Section 2: Composite Functions (1.4)

Major Concept: Build new functions from functions by adding, subtracting, multiplying, dividing, and composing functions.

Standards

Building Functions (F-BF)
Build a function that models a relationship between two quantities
F-BF.1.b Combine standard function types using arithmetic operations.
F-BF.1.c(+) Compose functions

Students will be able to:

- **Determine** the sum, difference, product, quotient, and composition of two given functions.
- **Find** the domains of the sum, difference, product, quotient, and composition of two given functions.

Vocabulary:
composition

1. **CONNECT** (10 minutes)
   Review: Prerequisites for Combining Functions
   Review finding domains in preparation for combining functions in various ways algebraically.

2. **INSTRUCT** (25 minutes)
   Calculations: Combining Functions Algebraically
   Most students will catch on very quickly to combining functions using addition, subtraction, multiplication and division. The more difficult aspect is finding the domain of these new functions. In the example given on page 118, students must understand why the domain of *gg* is not always all real numbers. Again, the concept of machine will help here.

   Composing Functions
   In the real world, functions quite frequently occur in which some quantity depends on a variable that, in turn, depends on another variable. Functions like these are called **composite functions**.

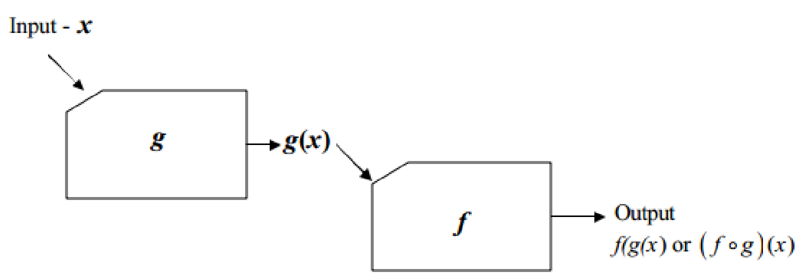
Suppose that an oil tanker is leaking oil and we want to be able to determine the area of the circular oil patch around the ship. It is determined that the oil is leaking from the tanker in such a way that the radius of the circular oil patch around the ship is increasing at a rate of 3 feet per minute. Find a function for the area of the oil spill as a function of time.

$$r(t) = 3t$$

$$A(r) = \pi r^2$$

Given two functions f and g, the composite function, denoted by *f* and *g* (read as " *f* composed with *g*"), is defined by $(f \circ g)(x) = f(g(x))$. The domain of $f \circ g$ is the set of all real numbers x in the domain of g such that $g(x)$ is in the domain of f.

The composition of *f* and *g* can be shown in a diagram.



The Formula for a Composition

The basic idea is to start with two functions *f* and *g* and produce a new function called their composition. There are two basic steps in this process. The first step is fairly mechanical, though perhaps somewhat unnatural. It involves combining the formulas for the functions *f* and *g* together to get a new formula. The next step is of varying complexity and involves analyzing how the domains and ranges of *f* and *g* affect those of the composition.

Suppose that $y = f(x)$ and that additionally the independent variable x is itself a function of a different independent variable t; i.e., $x = g(t)$. Then we can replace every occurrence of "x" in $f(x)$ by the expression "$g(t)$," thereby obtaining y as a function in the independent variable t. We denote this new function of t: $y = f(g(t))$.
Here, we refer $f(g(t))$ as the composition of *f* and *g* or the **composite function**.
The process of forming the composition of two functions is a mechanical procedure.

Procedure: *To obtain the formula for f(g(t)), replace every occurrence of "x" in f(x) by the expression "g(t)."*

It is natural to ask: What good is this whole business about compositions? One way to think of it is that we can use composite functions to break complicated functions into simpler parts. For example, $y = h(x) = \sqrt{x^2 + 1}$ can be written as the composition $f(g(x))$, where $y = f(z) = \sqrt{z}$ and $z = g(x) = x^2 + 1$. Each of the functions f and g is "simpler" than the original h, which can help when studying h.

Looking Ahead to Calculus

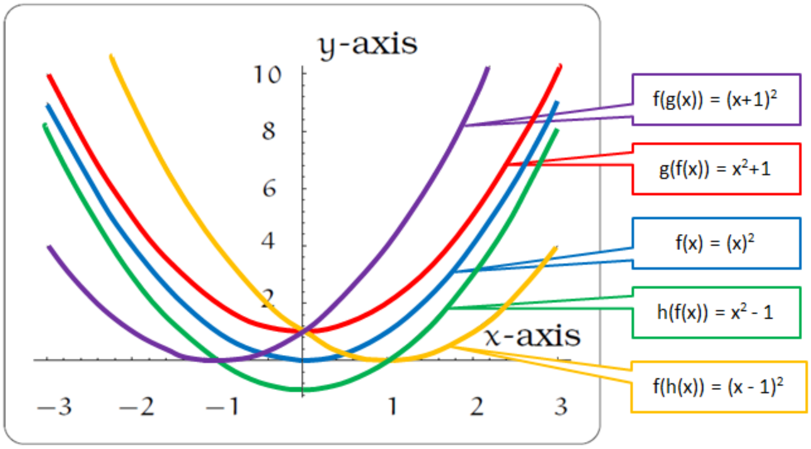
For integrating composite functions, the techniques call for two parts – *pattern recognition* and *change of variables*. Both techniques involve a u-substitution.

Theorem: Antidifferentiation of a Composite Function
$\int f\big(g(x)\big)g'(x)dx = F\big(g(x)\big) + C$. If $u = g(x)$, then $du = g'(x)dx$ and
$\int f(u)du = F(u) + C$.

Composite Functions: Algebraic Approach
Let $f(x) = x^2$, $g(x) = x + 1$ and $h(x) = x - 1$.
Find the formulas for $f(g(x))$, $g(f(x))$, $f(h(x))$ and $h(f(x))$. Discuss the relationship between the graphs of these four functions.

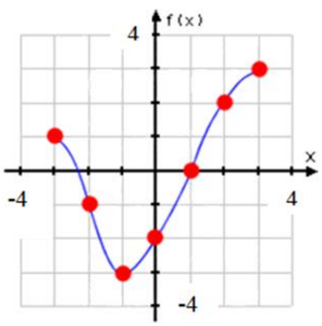
Solution. By applying procedure, we obtain the composition formulas. The four graphs are given on the domain $-3 \le x \le 3$.
$f(g(x)) = f(x + 1) = (x + 1)^2$
$g(f(x)) = g(x^2) = x^2 + 1$
$f(h(x)) = f(x - 1) = (x - 1)^2$
$h(f(x)) = h(x^2) = x^2 - 1$.
We can identify each graph by looking at its vertex:
- f(x) has vertex (0,0)
- f(g(x)) has vertex (-1,0)
- g(f(x)) has vertex (0,1)
- f(h(x)) has vertex (1,0)
- h(f(x)) has vertex (0,-1)

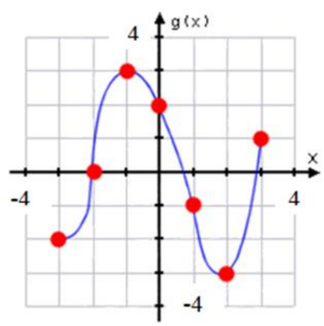
Horizontal or vertical shifting of the graph of f(x) = x^2 gives the other four graphs: See graph below. It is important that students see how composition is not commutative; the order in which you compose functions matters.



Composite Functions: Graphical Approach

Given *f*(*x*) and *g*(*x*) as shown below, find (*f* o *g*)(–1).





In this case, we read the points from the graph. We are asked to find (*f* o *g*)(–1) = *f*(*g*(–1)). This means that we first need to find *g*(–1). So look on the graph of *g*(*x*), and find *x* = –1. Tracing up from *x* = –1 to the graph of *g*(*x*), we arrive at *y* = 3. Then the point (–1, 3) is on the graph of *g*(*x*), and *g*(–1) = 3.
Now we plug this value, *x* = 3, into *f*(*x*). To do this, look at the graph of *f*(*x*) and find *x* = 3.
Tracing up from *x* = 3 to the graph of *f*(*x*), we arrive at *y* = 3. Then the point (3, 3) is on the graph of *f*(*x*), and *f*(3) = 3. Then **(*f* o *g*)(–1) = *f*(*g*(–1)) = *f*(3) = 3.**

Find: (*g* o *f*)(–3) = *g*(*f*(–3)) = *g*(1) = –1

Again, we get this answer by looking at *x* = –3 on the *f*(*x*) graph, finding the corresponding *y*-value of 1on the *f*(*x*) graph, and using this answer as our new *x*-value on the *g*(*x*) graph. That is, look at *x* = –3 on the *f*(*x*) graph, find that this leads to *y* = 1, uses *x* = 1 on the *g*(*x*) graph, and this leads to *y* = –1. Similarly:
(*g* o *f*)(–2) = *g*(*f*(–2)) = *g*(–1) = 3
(*g* o *f*)(–1) = *g*(*f*(–1)) = *g*(–3) = –2
(*g* o *f*)(0) = *g*(*f*(0)) = *g*(–2) = 0
(*g* o *f*)(1) = *g*(*f*(1)) = *g*(0) = 2
(*g* o *f*)(2) = *g*(*f*(2)) = *g*(2) = –3
(*g* o *f*)(3) = *g*(*f*(3)) = *g*(3) = 1

Evaluate each expression using the values given in the table.

x	**-3**	**-2**	**-1**	**0**	**1**	**2**	**3**
***f*(*x*)**	-7	-5	-3	-1	3	5	7
***g*(*x*)**	8	3	0	-1	0	3	8

(a) $(f \circ g)(1)$ (b) $(f \circ g)(-1)$ (c) $(g \circ f)(-1)$
(d) $(g \circ f)(0)$ (e) $(f \circ f)(-1)$ (f) $(g \circ g)(-2)$

Looking Ahead to Calculus

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	5	-1	1	2
2	4	-1	3	$\frac{3}{2}$
3	3	-1	4	1
4	2	-1	5	1
5	1	0	6	$-\frac{1}{2}$
6	2	1	4	-2

Part 1) Given $h_1(x) = f(x) + g(x)$, find $h_1'(2)$
Part 2) Given $h_2(x) = f(x) - g(x)$, find $h_2'(3)$
Part 3) Given $h_3(x) = f(x) \cdot g(x)$, find $h_3'(4)$
Part 4) Given $h_4(x) = \dfrac{f(x)}{g(x)}$, find $h_4'(2)$
Part 5) Given $h_5(x) = (f(x))^2$, find $h_5'(2)$
Part 6) Given $h_6(x) = f(g(x))$, find $h_6'(6)$

Comments and Solutions can be found at:
http://www.illustrativemathematics.org/illustrations/231

Example1: According to the U.S. Energy Information Administration, a barrel of crude oil produces approximately 20 gallons of gasoline. EPA mileage estimates indicate a 2011 Ford Focus averages 28 miles per gallon of gasoline.

1. Write an expression for G(x), the number of gallons of gasoline produced by x barrels of crude oil.
2. Write an expression for M(g), the number of miles on average that a 2011 Ford Focus can drive on g gallons of gasoline.
3. Write an expression for M(G(x)). What does M(G(x)) represent in terms of the context?
4. One estimate (from www.oilvoice.com) claimed that the 2010 Deepwater Horizon disaster in the Gulf of Mexico spilled 4.9 million barrels of crude oil. How many miles of Ford Focus driving would this spilled oil fuel?

In reference to the solution for part 4), in 2010, Ford sold just over 170,000 Focus models. The oil spilled from the Deepwater Horizon disaster would allow EACH Ford Focus sold in 2010 to drive over 15,000 miles.

Note that F.BF.1c does not require student facility with the notation $f \circ g$.

Illustrative Mathematics **Comments and Solutions can be found at:**
http://www.illustrativemathematics.org/illustrations/364

Example 2: Let f be the function that assigns to a temperature in degrees Celsius its equivalent in degrees Fahrenheit. Let g be the function that assigns to a temperature in degrees Kelvin its equivalent in degrees Celsius.

1. Explain what x and $f(g(x))$ represent in terms of temperatures, or explain why there is no reasonable representation.
2. Explain what x and $g(f(x))$ represent in terms of temperatures, or explain why there is no reasonable representation.
3. Given that $f(x) = \frac{9}{5}x + 32$ and $g(x) = x - 273$, find an expression for $f(g(x))$.
4. Find an expression for the function h which assigns to a temperature in degrees Fahrenheit its equivalent in degrees Kelvin.

Unit conversion problems provide a rich source of examples both for composition of functions (when several successive conversions are required) and inverses (units can always be converted in either of two directions). Note that the conversion function $g(x)$ is an approximation: The exact conversion formula is given by $g(x) = x - 273.15$.

Illustrative Mathematics **Comments and Solutions can be found at:**
http://www.illustrativemathematics.org/illustrations/616
Example 3: Let f be the function defined by $f(x) = x^2$. Let g be the function defined by $g(x) = \sqrt{x}$.

1. Sketch the graph of $y = f(g(x))$ and explain your reasoning.
2. Sketch the graph of $y = g(f(x))$ and explain your reasoning.

Note: This task addresses an important issue about inverse functions. In this case the function f is the inverse of the function g but g is not the inverse of f unless the domain of f is restricted.

Unit 1 Section 3: Inverse; One-to-One (1.5)

Major Concepts: Define One-to-One functions and construct inverse functions

Standards

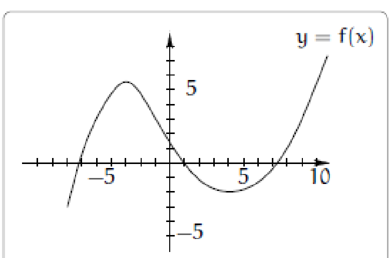
Building Functions (F-BF)
Build new functions from existing functions
F-BF.4.b (+) Verify by composition that one function if the inverse of another.
F-BF.4.c (+) Read values of an inverse function from a graph or a table, given that the function has an inverse.

Students will be able to:

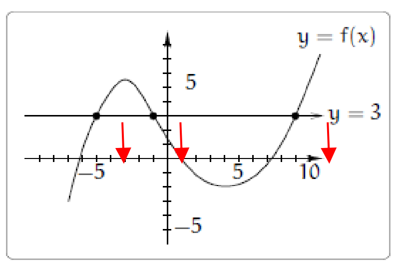
- **Determine** if a function is one-to-one using the Horizontal Line Test.
- **Determine** an inverse function algebraically.
- **Determine** an inverse function graphically.
- **Verify** that two functions are inverses of each other.

Vocabulary:
inverse relation, horizontal line test, one-to-one

1. **INSTRUCT** (35 minutes)
Inverse Relations: Numerically and Graphically
Once students understand relations as collections of ordered pairs, inverse relations are simply reversing the order of the coordinates in the ordered pairs. Demonstrate this numerically using a table. Ask students when this will be a function. Define what it means for a function to be one-to-one, and incorporate the Horizontal Line Test Vs. Vertical Line Test.



(a) Given the function $y = f(x)$.



(b) What values of x give $f(x) = 3$?

Inverse Functions: Algebraically and Graphically
Show students how to calculate an inverse function algebraically by switching *x* and *y*, then solving for *y*. Then present how to get the graph of an inverse function graphically. Discuss how to check that two functions are inverses of one another by showing $f(g(x)) = g(f(x)) = x$.

Conceptual Understanding: Inverse Functions
Ask students what they would do if they accidentally squared a number in their calculator, or if they accidentally cubed it. Use this concept of "undoing (or inverse operation)" a particular operation to introduce inverse functions. Give examples of the functions that the students are already familiar with and have them to determine if they are one-to-one. Then discuss how a function being one-to-one is important in defining the inverse function. Finally, have them find the inverse functions algebraically and graph them.

Comments and Solutions can be found at:
http://www.illustrativemathematics.org/illustrations/1374

The table below shows some input-output pairs of two functions f and g that agree for the values that are given but some of their output values are missing.

t	0	15	30	45	60	75	90	105	120
$f(t)$	0	0.5		1.3	2	2.7		4	
$g(t)$	0	0.5		1.3	2	2.7		4	

a. Complete the table in a way so that f could be invertible and so that g is definitely not invertible.

b. Graph both functions and explain from the graph why f is invertible and g is not.

c. Come up with two real life situations that f and g could be representing.

d. Find and interpret the value $f^{-1}(4)$ in terms of these contexts.

Note: This task illustrates several components of standard F-BF.B.4.c: Find Inverse Functions. Here, instead of presenting two functions and asking the students to decide which one is invertible, students are asked to complete a table of input-output pairs for the functions in such a way that one of the functions is invertible and the other one is not.

In part (c) students are asked to envision contexts plausibly represented in the table, and then in part (d), students are asked to find and interpret the "inverse context." This task also directly addresses part (c) of F-BF.4, which asks students to read values of inverse functions from a table. Finally, the task further gives students an opportunity to contrast an invertible and non-invertible function, which is mentioned in the Functions Progressions document as a reasonable extension of the standard. "...And although not required in the standard, it is reasonable to include, for comparison, a few examples where the input cannot be uniquely determined from the output." (pg. 13 Functions Progression). The task could be used for instruction or assessment.

Unit 1 Section 4: Graphical Transformations (1.6)

Major Concepts: Represent transformations (translations, reflections, stretches, and shrinks) algebraically and graphically.

Standards

Building Functions F-BF

Build new functions from existing functions

F-BF.3 Identify the effect on the graph of replacing f(*x*) by f(*x*) + k, kf(*x*), f(k*x*), and f(*x* + k) for specific values of k (both positive and negative); find the value of k given the graphs.

Unit 1 Section 4-A: Transformations

Students will be able to:

- **Determine** the equation of a function under a vertical /horizontal translation.
- **Determine** the equation of a function under a reflection.
- **Determine** the equation of a function under a vertical or horizontal stretching or shrinking.
- **Determine** the transformation induced by a given equation.

Vocabulary:
transformations, rigid transformations, non-rigid transformations, vertical translation, horizontal translation, reflections

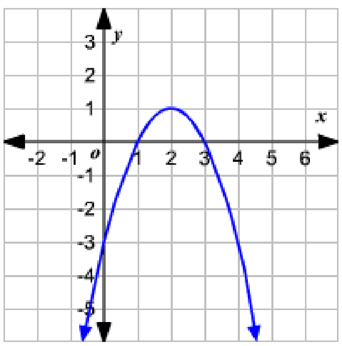
1. **CONNECT** (10 minutes)
   Review: Prerequisites for Transformations of Functions
   Have students review simplifying algebraic expressions in preparation for transformations of functions.

2. **INSTRUCT** (20 minutes)
   Conceptual Understanding: Translations
   Have students determine what they would do in order to shift the graphs of functions vertically and horizontally. Have them check their guesses using a graphing calculator. Then present the general rule for translations. Note how the horizontal translations work counter to the way vertical translations work algebraically.

   Conceptual Understanding: Reflections
   Present how to reflect functions about the *x*-axis and about the *y*-axis. Point out again how vertical changes take place on the outside of the function, while horizontal changes occur from the inside. Use this to have students determine how applying absolute value would affect the function.

3. INDIVIDUAL EXPLORATION (20 minutes)
Exploration: Compositions with Absolute Value
Given the graph of $y = f(x)$. The graph of $y = |f(x)|$ can be obtain by reflecting the portion of the graph below of the x-axis, leaving the portion above the x-axis unchanged. The graph of $y = f(|x|)$ can be obtained by replacing the portion of the graph to the left of the y-axis by a reflection of the portion to the right of the y-axis across the y-axis, leaving the portion to the right of the y-axis unchanged.

Have students investigate how applying absolute value affects the graph of a function. The graph of $y = f(x)$ is shown below. Graph the following equations. Please be ready to defend your answers.



1. $y = |f(x)|$
2. $y = f(|x|)$
3. $y = -|f(x+2)|$
4. $y = |f(|x|)|$

Exploration: Stretches and Shrinks
Have students investigate Exploration 3 on page 134 to determine how multiplication by a real number affects the graph of a function. Then discuss and summarize the general rule (p. 134).

Unit 1 Section 4-B: Combining Transformations

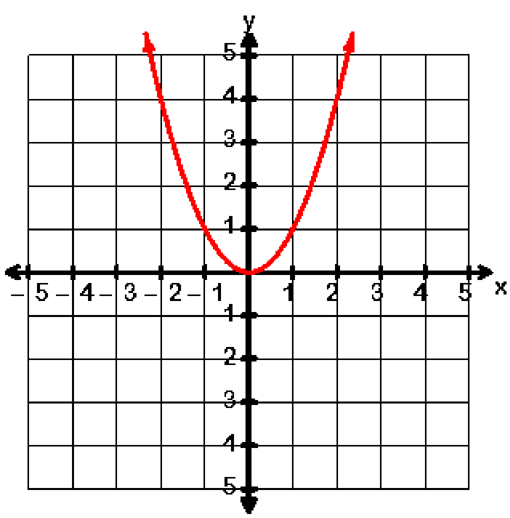
Students will be able to:
- **Determine** the equation of a function under multiple transformations.
- **Graph** the equation of a function under multiple transformations.

1. INSTRUCT (20 minutes)
Conceptual Understanding: Combining Transformations
Explain how to incorporate multiple transformations into the graph of a new function. Then explain how the analogous algebraic expression changes. Note how reordering the transformation yields a different function, so have students be careful of the order in which they transform.

Expression	Interpretation
f(a+2)	The output when the input is 2 more than a
f(a) + 2	2 more than the output when the input is a
2f(x) + 5	5 more than twice the output of f when the input is x
f(b) – f(a)	The change in output when the input changes from a to b

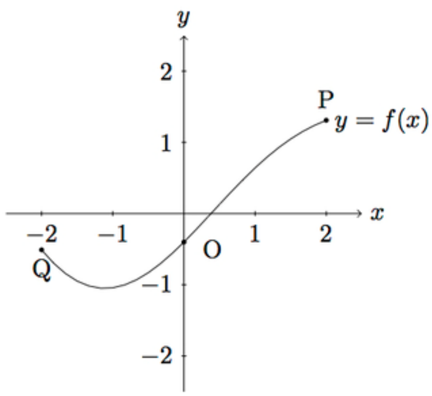
2. GROUP DISCUSSION (20 minutes)
Pass the Paper: Slope
In this particular activity, groups should begin by writing down one of the basic functions, then pass their paper to the next group. Groups should now write down instructions for three transformations; for instance, "shift left 2 units, shift down 3 units, then apply absolute value to the function." Then the papers should be passed (without actually doing the transformations!). Next, groups should attempt to graph the original function and then the new function after the three translations. Finally, they should try to find an algebraic expression for the new function. This exercise may be repeated multiple times.

Writing Assignment:
The function *f* defined by $f(x) = x^2$ is graphed in the *xy*-plane below.
The graph of the function *g* is obtained by reflecting the graph of f across the line $y = -1$.
Explain how you would find the value of *g*(-4).



Comments and Solutions can be found at
http://www.illustrativemathematics.org/illustrations/742

The figure below shows the graph of a function f whose domain is -2 ≤ *x* ≤ 2.



a. In (i)–(iii), sketch the graph of the given function and compare with the graph of f. Explain what you see.

   i. $g(x) = f(x) + 2$

   ii. $h(x) = -f(x)$

   iii. $p(x) = f(x+2)$

b. The points labelled Q, O, P on the graph of f have coordinates

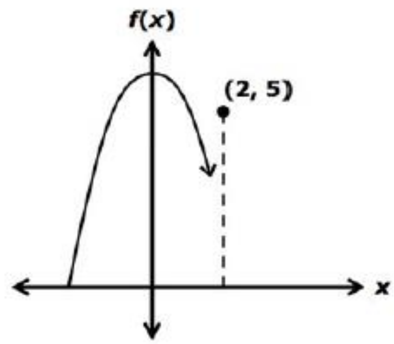
$$Q = (-2, -0.509), \quad O = (0, -0.4), \quad P = (2, 1.309).$$

What are the coordinates of the points corresponding to P, O, Q on the graphs of g, h, and p?

Comments and Solutions can be found at:
http://www.illustrativemathematics.org/illustrations/695

A computer game uses functions to simulate the paths of an archer's arrows. The x-axis represents the level ground on which the archer stands, and the coordinate pair $(2,5)$ represents the top of a castle wall over which he is trying to fire an arrow.

In response to user input, the first arrow followed a path defined by the function $f(x) = 6 - x^2$, failing to clear the castle wall.



The next arrow must be launched with the same force and trajectory, so the user must reposition the archer in order for his next arrow to have any chance of clearing the wall.

a. How much closer to the wall must the archer stand in order for the arrow to clear the wall by the greatest possible distance?

b. What function must the user enter in order to accomplish this?

c. If the user can only enter functions of the form $f(x+k)$, what are all the values of k that would result in the arrow clearing the castle wall?

Unit 1 Section 5: Polynomial Functions of Higher Degree with Modeling (2.3)

Major Concepts: Graph polynomial functions of higher degree, predict their end behavior, and find their zeros.

Standards

Arithmetic with Polynomials and Rational Expressions (A-APR)
Understand the relationship between zeros and factors of polynomials
A-APR.3 Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial.
Reasoning with Equations and Inequalities (A-REI)
Represent and solve equations and inequalities graphically.
A-REI.10 Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line).
Functions-Interpreting Functions (F-IF)
Analyze functions using different representations.
F-IF.7.c Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior.

Unit 1 Section 5-A: Polynomial Functions of Higher Degree

Students will be able to:

- **Identify** different components of a polynomial.
- **Bound** the number of roots of a polynomial given its degree.
- **Determine** the end behavior and dominant term of a polynomial.

Vocabulary:
cubic function, quartic function, term, standard form, coefficient, leading term

1. **CONNECT** (10 minutes)
   Review: Prerequisites for Higher-Degree Polynomials
   Review factoring and solving factored polynomial equations in preparation for graphing higher-degree polynomials.

2. **INSTRUCT** (25 minutes)
   Building Vocabulary: Polynomials
   Introduce any new vocabulary concerning polynomials to students. Give several examples to aid understanding. Recall that a polynomial function of degree n can be written in the form: $f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_2 x^2 + a_1 x + a_0,\ a_n \neq 0$. Also, every polynomial function is defined and continuous for all real numbers. Not only are graphs of polynomials unbroken without jumps or holes, but they are smooth, unbroken curves, with no sharp corners or cusps.

   Conceptual Understanding: Higher-Degree Polynomials
   First, explain how to apply principles of transformations to higher-degree polynomials, such as cubics, quartics, and quintics. Have students recall how quartic polynomials and quadratic polynomials look very similar; this will aid them in graphing.

   Graphing Calculators: Combining Monomial Terms
   Have students investigate how adding or subtracting a linear term from a higher-degree monomial changes the graph.

3. **INDIVIDUAL EXPLORATION** (10 minutes)
   Revisit: End Behavior of Power Functions
   Have students investigate to better understand the end behavior of power functions with positive integer powers.

Graph each function in the window [-5, 5] by [-15, 15].

1. (a) $f(x) = 2x^3$ (b) $f(x) = -x^3$
   (c) $f(x) = x^5$ (d) $f(x) = -0.5x^7$
2. (a) $f(x) = -3x^4$ (b) $f(x) = 0.6x^4$
   (c) $f(x) = 2x^6$ (d) $f(x) = -0.5x^2$
3. (a) $f(x) = -0.3x^5$ (b) $f(x) = -2x^2$
   (c) $f(x) = 3x^4$ (d) $f(x) = 2.5x^3$

Writing Assignment:
Describe the patterns you observed. More specifically, how do the values of the coefficient, a_n and the degree *n* affect the end behavior of $f(x) = a_n x^n$?

Unit 1 Section 5-B: Zeros of Polynomials and Modeling

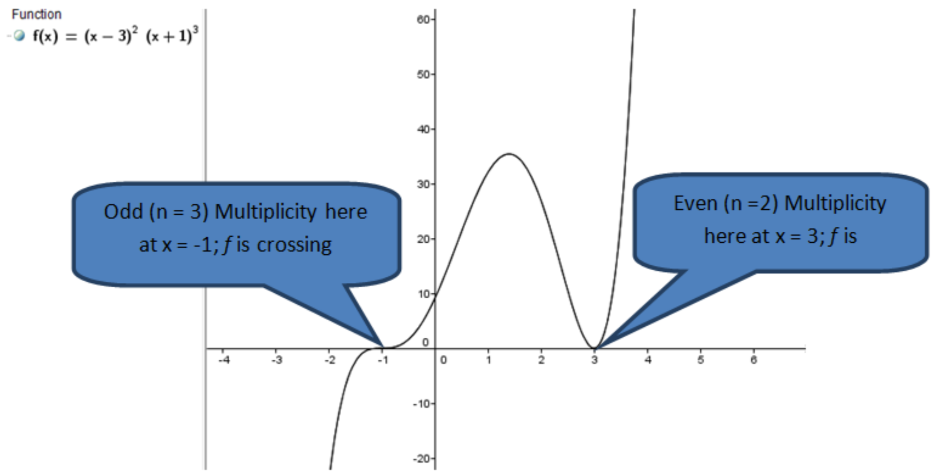
Students will be able to:
- **Determine** the zeros of a polynomial by factoring.
- **Approximate** the zeros of a polynomial using a graphing utility.
- **Sketch** the graph of a factored polynomial.

Vocabulary:
even and odd multiplicity, repeated zero, Intermediate Value Theorem

1. **INSTRUCT** (15 minutes)
   Conceptual Understanding: Zeros of a Polynomial
   Recall for students how zeros of a polynomial and the polynomial's factorization are related. Define the multiplicity of a zero (p. 189) for students. Explain (and show graphically) how the multiplicity of a zero has an effect on how the function looks near that zero (p. 190). Help students differentiate the behavior of even and odd multiplicities.

   Zeros of Polynomial Functions: Recall that finding the real number zeros of a function *f* is equivalent to finding the *x*-intercepts of the graph of $y = f(x)$ or the solutions to the equation $f(x) = 0$. In $f(x) = (x - 3)^2(x + 1)^3$, we say the polynomial function has a

repeated zero. This function has two repeated zeros; factor (x – 3) occurs twice or 3 is a zero of multiplicity 2 and factor (x + 1) occurs trice or -1 is a zero of multiplicity 3.



Zeros of Odd and Even Multiplicity: If a polynomial function f has a real zero c of odd multiplicity, then the graph of f crosses the x-*axis* at (c, 0) and the sign of f changes at $x = c$. If f has c of even multiplicity, then the graph of f touches the x-*axis* at (c, 0); consequently, the sign of f does not change at $x = c$.

2. GROUP DISCUSSION (10 minutes)

Pass the Paper: Graphing Factored Polynomials
Have groups write down a polynomial (in factored form) with three roots of varying multiplicities, then pass their paper. The groups must now graph the polynomial they received. Repeat this several times so students feel comfortable with behavior near zeros.

3. INSTRUCT (15 minutes)
Conceptual Understanding: Intermediate Value Theorem
Ask students whether all polynomials must have zeros. Is there a pattern to which ones must, and which ones might or might not? Once students see that odd powers of x must cross the x-axis somewhere, ask why? This will lead into a discussion involving the Intermediate Value Theorem. Introduce the theorem using a picture (not initially with the written form).

Looking Ahead to Calculus

The Intermediate Value Theorem tells us that a sign change implies a real zero.

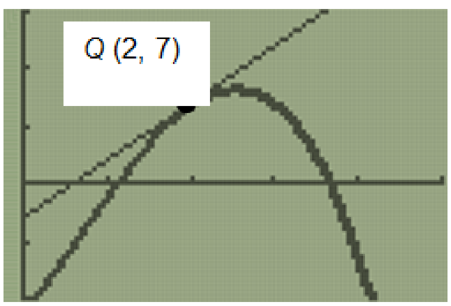
Writing Assignment:
Explain why $f(x) = x^9 - x + 50$ has at least one real zero.

Group Activity: Consider functions of the form $f(x) = x^3 + bx^2 + x + 1$ where b is a non-zero real number.

(a) Discuss as a group how the value of b affects the graph of the function.
(b) After completing (a), have each group member (individually) predict what the graphs of $f(x) = x^3 + 15x^2 + x + 1$ *and* $g(x) = x^3 - 15x^2 + x + 1$ will look like.
(c) Compare your predictions with each other. Confirm whether they are correct.

Looking Ahead to Calculus

The figure shows a graph of both $f(x) = -x^3 + 2x^2 + 9x - 11$ and the line L defined by $y = 5(x - 2) + 7$.



$[0, 5]$ by $[-10, 15]$

(a) Confirm that the point $Q(2,7)$ is a point of intersection of the two graphs.
(b) Zoom in at point Q to develop a visual understanding that $y = 5(x - 2) + 7$ is a *linear approximation* for $y = f(x)$ near $x = 2$.
(c) Recall that a line is tangent to a circle at a point P if it intersects the circle only at point P. View the two graphs in the window $[-5, 5]$ by $[-25, 25]$, and explain why that definition of tangent line is not valid for the graph of f.

Modeling: Designing a Box (p. 191)
Dixie Packaging Co. has contracted to make boxes with a volume of approximately 484 $(\text{in})^3$. Squares are to be cut from the corners of a 20-in by 25-in to make an open box. What are the dimensions of this open box?

Unit 1 Section 6: Fundamental Theorem of Algebra (2.5)

Major Concepts: Factor polynomials with real coefficients using factors with complex coefficients.

Standards:

The Complex Number System (N-CN)
Perform arithmetic operations with complex numbers.

N-CN.1 Know there is a complex number i such that $i^2 = -1$, and every complex number has the form $a + bi$ with a and b real.
N-CN.2 Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.
N-CN.3 (+) Find the conjugate of a complex number; use conjugates to find moduli and quotients of complex numbers.

Use complex numbers in polynomial identities and equations

N-CN.7 Solve quadratic equations with real coefficients that have complex solutions.
N-CN.8 (+) Extend polynomial identities to the complex numbers.
N-CN.9 (+) Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials.

Students will be able to:

- **Identify** the exact number of complex zeros of a given polynomial.
- **Determine** whether the complex conjugate of a zero is also a zero of a polynomial.
- **Determine** a polynomial given its zeros.
- **Factor** a polynomial with complex zeros.
- **Determine** whether a polynomial is irreducible over the reals.

Vocabulary:
complex zero, complex plane, real axis, imaginary axis, complex conjugate, irreducible over the reals

1. **CONNECT** (10 minutes)
   Review: Prerequisites for Complex Zeros
   Review calculations using complex numbers and factoring in preparation for studying complex zeros and the Fundamental Theorem of Algebra. **The Fundamental Theorem of Algebra:** A polynomial function of degree n has n complex zeros (real and nonreal) and some of these zeros may be repeated. This theorem is *existence theorem*. It tells of the existence of zeros but not how to find them.

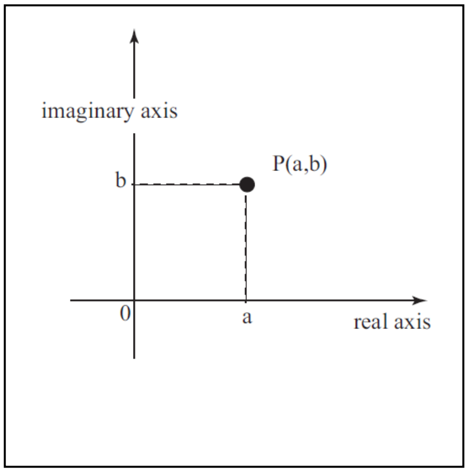
2. **INSTRUCT** (20 minutes)
   Conceptual Understanding: Fundamental Theorem of Algebra
   Recall the question about whether all polynomials have to have zeros. Introduce the Fundamental Theorem of Algebra and discuss how the answer to the previous question changes when complex numbers are being used. Have students note that the same relationship between linear factors and zeros of a polynomial exist with complex zeros.

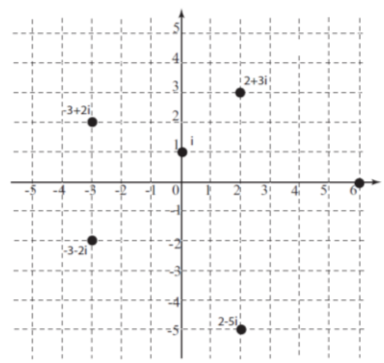
Conceptual Understanding: Complex Conjugates
Explain what a complex conjugate is, and show how for polynomials with real coefficients, complex zeros always come in conjugate pairs. Ask them why this should be in a quadratic polynomial.

Complex Numbers: A class of numbers including purely real numbers *a*, purely imaginary numbers *bi*, and numbers with both real and imaginary parts *a* + *bi*.

Complex Plane: A 2-dimensional representation of complex numbers established by a horizontal real axis and a vertical imaginary axis.

Complex Conjugate of $z = a + bi$: $\bar{z} = a - bi$





Complex plane showing several complex numbers.

Because the real part of z is plotted on the horizontal axis we often refer to this as the **real axis**. The imaginary part of z is plotted on the vertical axis and so we refer to this as the **imaginary axis**. Such a diagram is called a **Complex Plane**.

Note that purely real numbers lie on the real axis. Purely imaginary numbers lie on the imaginary axis. Another observation is that complex conjugate pairs (such as −3+2i and −3−2i) lie *symmetrically about the x axis*.

3. INDIVIDUAL EXPLORATION (10 minutes)
Exploration: Polynomials with Complex Coefficients
Have students investigate Exploration 1 below (p. 212) to determine what happens when a polynomial has complex coefficients. Do complex zeros come in conjugate pairs?

EXPLORATION 1 What Can Happen if the Coefficients Are Not Real?

1. Use substitution to verify that $x = 2i$ and $x = -i$ are zeros of $f(x) = x^2 - ix + 2$. Are the conjugates of $2i$ and $-i$ also zeros of $f(x)$?
2. Use substitution to verify that $x = i$ and $x = 1 - i$ are zeros of $g(x) = x^2 - x + (1 + i)$. Are the conjugates of i and $1 - i$ also zeros of $g(x)$?
3. What conclusions can you draw from parts 1 and 2? Do your results contradict the theorem about complex conjugate zeros?

4. INSTRUCT (10 minutes)

Conceptual Understanding: Irreducible Factors over the Reals

Discuss with students the meaning of a polynomial being irreducible over the real numbers (p. 214). A quadratic expression with real coefficients but no real zeros is irreducible over the reals, and this is not the same thing as having all complex zeros. In other words, the factors of a polynomial have real coefficients, the factorization can be accomplished with linear factors and irreducible quadratics factors.

Writing Assignment:

Answer yes or no. If yes, include an example. If no, give a reason. Is it possible to find a polynomial $f(x)$ of degree with real coefficient that has zeros $1+3i$ and $1-i$?

Comments and Solutions can be found at:
http://www.illustrativemathematics.org/illustrations/722

For this task, the letter i denotes the imaginary unit, that is, $i = \sqrt{-1}$.

a. For each integer k from 0 to 8, write i^k in the form $a + bi$.

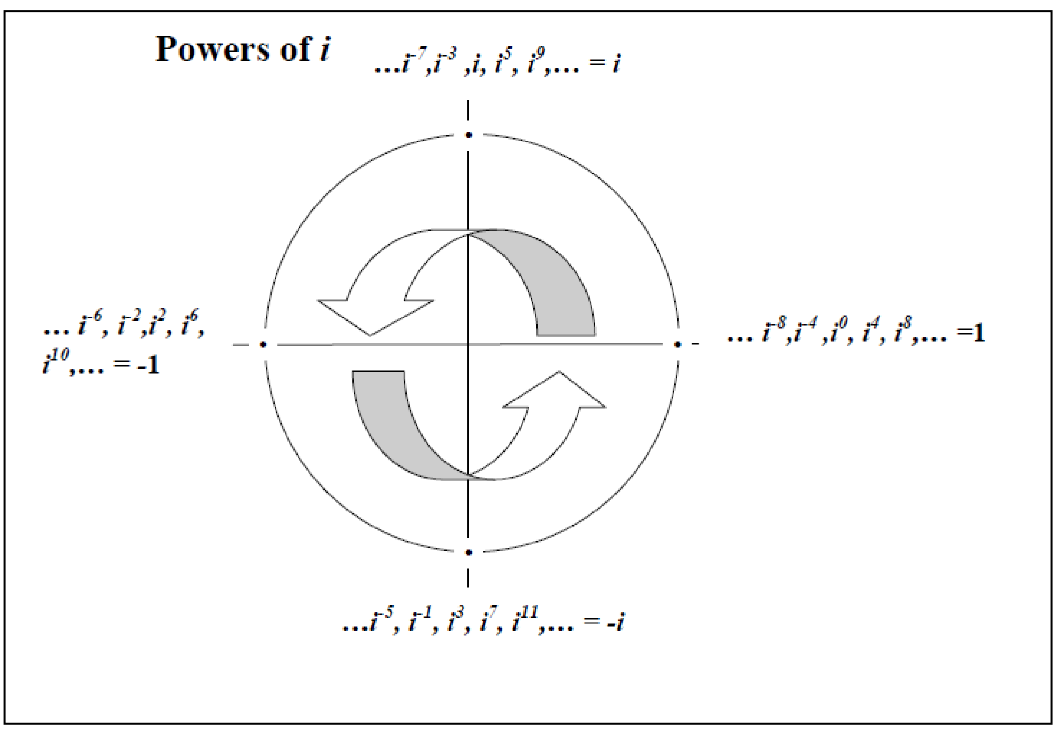
b. Describe the pattern you observe, and algebraically prove your observation. In particular, simplify i^{195}.

c. Write each of the following expression in the form $a + bi$:

- $i^2 + i + 1$
- $i^3 + i^2 + i + 1$
- $i^4 + i^3 + i^2 + i + 1$
- $i^5 + i^4 + i^3 + i^2 + i + 1$
- $i^6 + i^5 + i^4 + i^3 + i^2 + i + 1$
- $i^7 + i^6 + i^5 + i^4 + i^3 + i^2 + i + 1$
- $i^8 + i^7 + i^6 + i^5 + i^4 + i^3 + i^2 + i + 1$

d. Describe the pattern you observe, and algebraically prove your observation. In particular, compute

$$i^{195} + i^{194} + \cdots + i^2 + i + 1.$$



Unit 1 Section 7: Rational Functions (2.6)

Major Concepts: Graphs rational functions, identify asymptotes, and predict end behavior of rational functions.

Standards

Interpreting Functions (F-IF)
Analyze functions using different representations

F-IF.7.d (+)	Graph rational functions, identifying zeros and asymptotes when suitable factorizations are available, and showing end behavior.

Arithmetic with polynomials and Rational Expressions (A-APR)
Rewrite rational expressions

A-APR.6	Rewrite simple rational expressions in different forms; write a(x)/b(x) in the form q(x) + r(x)/b(x), where a(x), b(x), q(x), and r(x) are polynomials with the degree of r(x) less than the degree of b(x), using inspection, long division, or, for the more complicated examples, a computer algebra system.
A-APR.7 (+)	Understand that rational expressions from a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by nonzero rational expression; add, subtract, multiply, and divide

Unit 1 Section 7-A: Transformations and Limits of Rational Functions

Students will be able to:

- **Determine** the domain of a rational function.
- **Sketch** the graph of a rational function using transformations of the function 1/x.
- **Determine** the vertical and horizontal asymptotes of a rational function.

Vocabulary:
rational function, end-behavior asymptote, slant asymptote

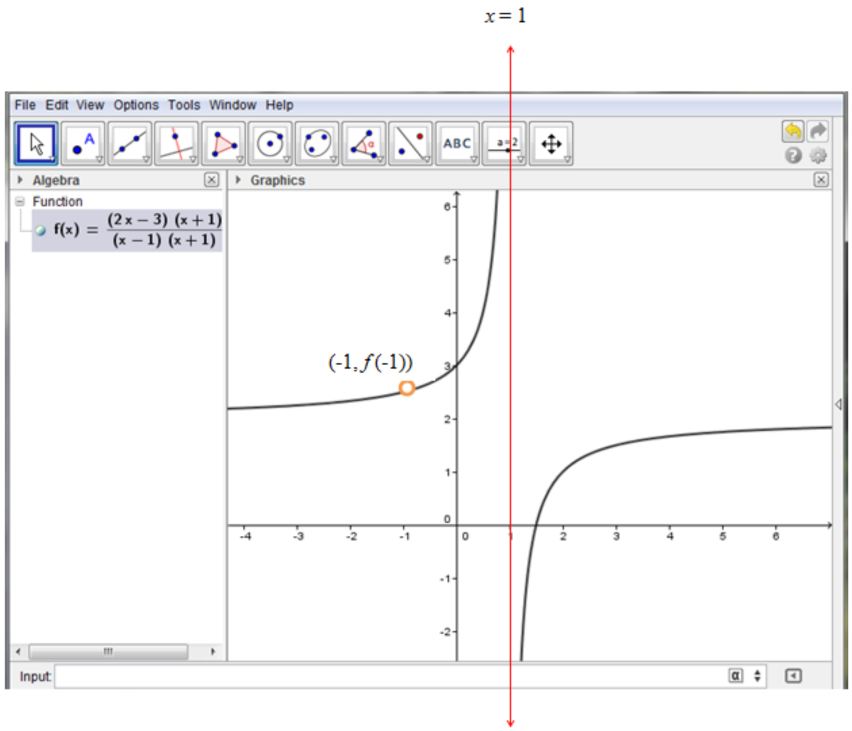
1. **CONNECT** (15 minutes)
   Review: Prerequisites for Rational Functions
   Review finding zeros of polynomials and polynomial division in preparation for studying rational functions.

2. **INSTRUCT** (20 minutes)
   Calculations: Finding Domains of Rational Functions
   Definition: Let P and Q be polynomial functions with $Q(x) \neq 0$. Then the function given by $R(x) = \frac{P(x)}{Q(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}$ is a rational function. The domain of a rational function is the set of all real numbers except the zeros of its denominator. Every rational function is continuous on its domain.

Students should probably be able to guess how to find the domain of a rational function, but there are several details that students should not forget. For instance, the domain of the function $f(x) = \frac{x}{x}$ is all real numbers except 0; students are likely, however, to cancel the x's and declare the domain to be all

Domain Restrictions: Vertical Asymptotes and Holes

In $f(x) = \frac{(2x-3)(x+1)}{(x-1)(x+1)}$, $(x + 1)$ is a common factor, thus when $x = -1$, $f(x)$ is undefined and 1, $f(-1)$) is the location where a hole occurs. See the graph below.



Steps to finding holes and vertical asymptotes:

1. Factor the rational function, cancel any common factors
2. Rewrite the simplified rational function.
3. Solve any common factor that was cancelled out, if any, for zero. These are the holes.
4. Solve the new denominator, if any, for zero. These are the vertical asymptotes.

Graphing: Transformations of 1/x

Show how the graph of any rational function of the form $(ax+b)/(cx+d)$ may be obtained from the function $1/x$ using transformations.

3. INDIVIDUAL EXPLORATION (10 minutes)
Exploration: Transformations of 1/x
Have students investigate Exploration 1 on page 219 to practice graphing transformations of 1/x and finding their equations.

EXPLORATION 1 Comparing Graphs of Rational Functions

1. Sketch the graph and find an equation for the function g whose graph is obtained from the reciprocal function $f(x) = 1/x$ by a translation of 2 units to the right.
2. Sketch the graph and find an equation for the function h whose graph is obtained from the reciprocal function $f(x) = 1/x$ by a translation of 5 units to the right, followed by a reflection across the x-axis.
3. Sketch the graph and find an equation for the function k whose graph is obtained from the reciprocal function $f(x) = 1/x$ by a translation of 4 units to the left, followed by a vertical stretch by a factor of 3, and finally a translation 2 units down.

Looking Ahead to Calculus

2013 AP Calculus AB Exam (MC without calculator)

Let f be the function given by $f(x) = \frac{(x+3)(x-2)^2}{(x-2)(x+1)}$. For which of the following values of x is f not continuous?

(A) -3 and -1 only (B) -3, -1, and 2 (C) -1 only (D) -1 and 2 only
(E) 2 only

Comments and Solutions can be found at:
http://www.illustrativemathematics.org/illustrations/825

The US Department of Energy keeps track of fuel efficiency for all vehicles sold in the United States. Each car has two fuel economy numbers, one measuring efficient for city driving and one for highway driving. For example, a 2012 Volkswagen Jetta gets 29.0 miles per gallon (mpg) in the city and 39.0 mpg on the highway.

Many banks have "green car loans" where the interest rate is lowered for loans on cars with high combined fuel economy. This number is not the average of the city and highway economy values. Rather, the combined fuel economy (as defined by the federal Corporate Average Fuel Economy standard) for x mpg in the city and y mpg on the highway, is computed as

$$\text{combined fuel economy} = \frac{1}{\frac{1}{2}\left(\frac{1}{x}+\frac{1}{y}\right)}.$$

a. What is the combined fuel economy for the 2012 Volkswagen Jetta? Give your answer to three significant digits.

b. For most conventional cars, the highway fuel economy is 10 mpg higher than the city fuel economy. If we set the city fuel economy to be x mpg for such a car, what is the combined fuel economy in terms of x? Write your answer as a single rational function, $a(x)/b(x)$.

c. Rewrite your answer from (b) in the form of $q(x)+\frac{r(x)}{b(x)}$ where $q(x), r(x)$ and $b(x)$ are polynomials and the degree of $r(x)$ is less than the degree of $b(x)$.

d. Use your answer in (c) to conclude that if the city fuel economy, x, is large, then the combined fuel economy is approximately $x+5$.

Unit 1 Section 7-B: Analyzing the Graph of a Rational Function

Students will be able to:

- **Determine** the end-behavior asymptote of a rational function.
- **Analyze** the graph of a rational function.

1. **INSTRUCT** (45 minutes)
   Conceptual Understanding: Asymptotes and End Behavior
   Explain (using examples) how to determine whether and where a rational function has a vertical or horizontal asymptote.

   Analysis: Rational Functions
   Practice analyzing the graphs of various rational functions, helping students particularly in finding horizontal asymptotes and end behavior asymptotes. Give students several examples to try on their own.

   Horizontal Asymptotes. A horizontal asymptote is a line $y = c$ such that the values of $f(x)$ get increasingly close to the number c as $x \to \infty$ or $x \to -\infty$. In $R(x) = \frac{f(x)}{g(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}$ and $R(x)$ is in a reduced form (no common factors),

   Case 1: If $n < m$, the end behavior of horizontal asymptote is y = 0.
   Case2: If $n = m$, then the end behavior of asymptote is y = $\frac{leading\ coefficient}{leading\ coefficient} = \frac{a_n}{b_m}$.
   Case 3: If $n > m$, then there is no horizontal asymptote. But the end behavior of asymptote is the quotient polynomial function y = $q(x)$, where $f(x) = g(x)q(x) + r(x)$. Special Case 3: if $n = m+1$, then there is an oblique asymptote. Use long division to locate the slant asymptote. **Oblique (Slant) Asymptote:** An oblique asymptote is an asymptote of the form y = ax + b, thus, also known as a slant asymptote. Rational functions have oblique asymptotes if the degree of the numerator is one more than the degree of the denominator.

Group Activity: Work in groups of two. Compare the functions $f(x) = \frac{x^2-9}{x-3}$ and $g(x) = x + 3$.

(a) Are the domains equal?
(b) Does f have a vertical asymptote? Explain
(c) Explain why the graphs appear to be identical.
(d) Are the functions identical?

Unit 1 Section 8: Solving Polynomial and Rational Inequalities (2.8)

Major Concepts: Solve inequalities involving polynomials and rational functions by using both algebraic and graphical techniques.

Standards

Seeing Structure in Expressions (A-SSE)
Interpret the structure of expressions
A-SSE.2 Use the structure of an expression to identify ways to rewrite it.
A-SSE.3a Factor a quadratic expression to reveal the zeros of the function it defines.
Reasoning with Equations and Inequalities (A-REI)
Understand solving equations as process of reasoning and explain the reasoning
A-REI.2 Solve simple rational and radical equation in one variable, and give examples showing how extraneous solutions may arise.
Solve equations and inequalities in one variable
A-REI.3 Solve linear equation and inequalities in one variable, including equations with coefficients represented by letters.

Unit 1 Section 8-A: Solving Polynomial Inequalities

Students will be able to:

- **Determine** where a factored polynomial is zero, positive, and negative.
- **Create** a sign chart for a factored polynomial.
- **Sketch** the graph of a polynomial from its sign chart.
- **Solve** a polynomial inequality using a sign chart.

Vocabulary:
sign chart

1. **CONNECT** (10 minutes)
   Review: Prerequisites for Solving Inequalities
   Review end behavior models, finding common denominators, and finding zeros of polynomials in preparation for solving inequalities in one variable.

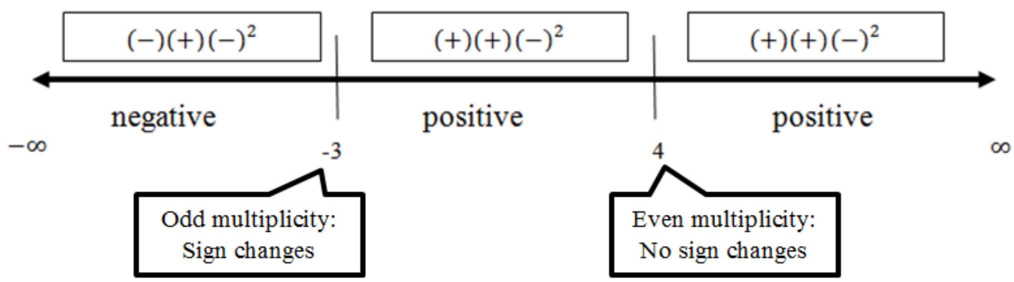
2. **INSTRUCT** (25 minutes)
   Calculations: Solving Polynomial Inequalities
   To aid students in solving polynomial inequalities, it is important to reassure them that the process is no more difficult than solving polynomial equalities. Students often have an aversion to inequalities. Demonstrate how the two procedures are related, and indicate the extra step of drawing a sign chart in order to solve an inequality. Have students check their work using their graphing calculator.

3. INSTRUCT AND CONNECT (20 minutes)
Sign Charts of Polynomials and graph (p. 236)

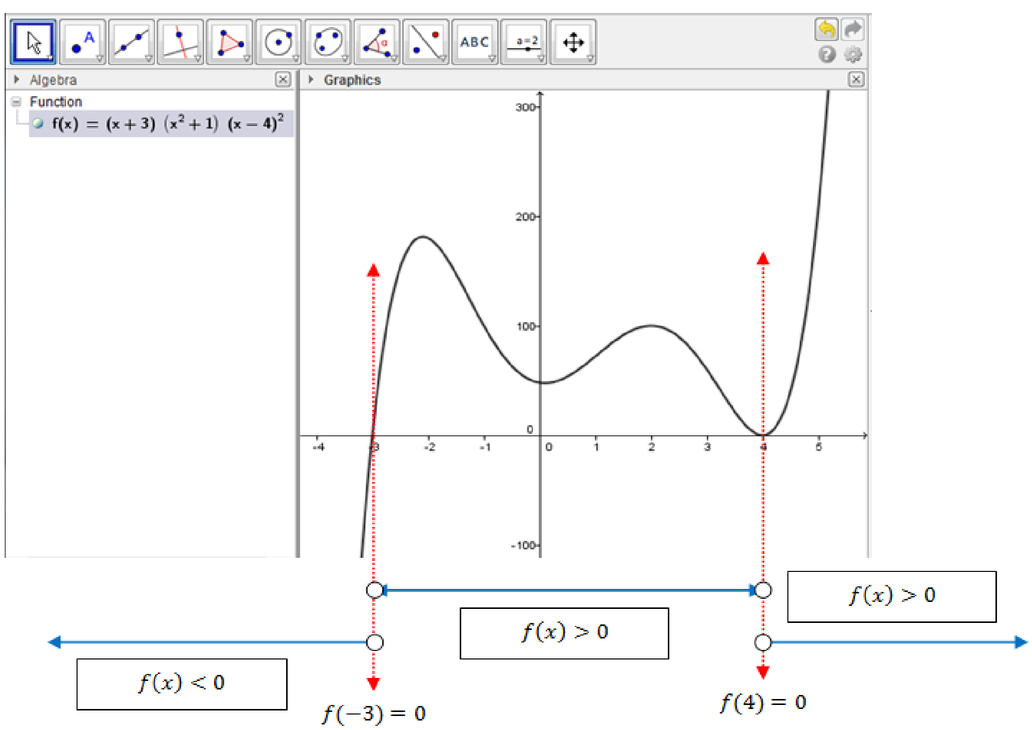
Example: Let $f(x) = (x+3)(x^2+1)(x-4)^2$. Find the (a) zeros and then use a sign chart to determine the real number values of x that result in (b) positive, and (c) negative. Graph $f(x)$.

Solution:

(a) The real zeros are -3, 4.
(b) $f(x)$ is positive when (-3, 4) ∪ (4, ∞)
(c) $f(x)$ is negative when (-∞, -3)



Graph below is done using GeoGebra:



Have students to practice either individually or as a group to practice Exploration 1 on page 237 to help them to gain the understanding of the connection between the sign chart of a polynomial and the graph of that polynomial.

EXPLORATION 1 Sketching a Graph of a Polynomial from Its Sign Chart

Use your knowledge of end behavior and multiplicity of real zeros to create a sign chart and sketch the graph of the function. Check your sign chart using the factor method of Example 1. Then check your sketch using a grapher.

1. $f(x) = 2(x - 2)^3(x + 3)^2$
2. $f(x) = -(x + 2)^4(x + 1)(2x^2 + x + 1)$
3. $f(x) = 3(x - 2)^2(x + 4)^3(-x^2 - 2)$

Unit 1 Section 8-B: Solving Rational Inequalities

Students will be able to:

- **Create** a sign chart for a rational function.
- **Solve** a rational inequality using a sign chart.
- **Solve** an inequality involving a radical using a sign chart.
- **Solve** an inequality involving absolute value using a sign chart.

1. INSTRUCT (35 minutes)

Calculations: Solving Rational Inequalities

The process here is quite similar to solving polynomial inequalities. The only additional steps are to get one common fraction, and use both the zeros and the places where the function is undefined when creating the sign chart.

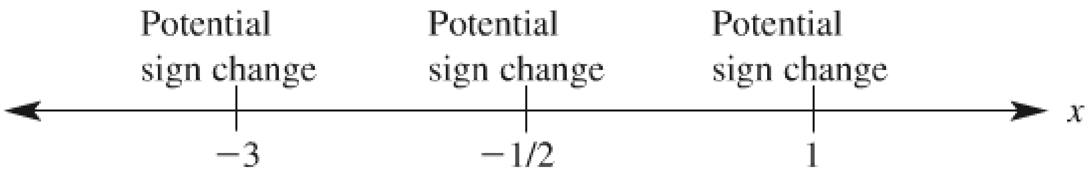
Example: Let $r(x) = \frac{2x+1}{(x+3)(x-1)}$. Determine the real number values of x that will result in $r(x)$ to be (a) zero, (b) undefined. Then make a sign chart to determine the real number values of x that will result in $r(x)$ to be (c) positive, (d) negative.

Solution:

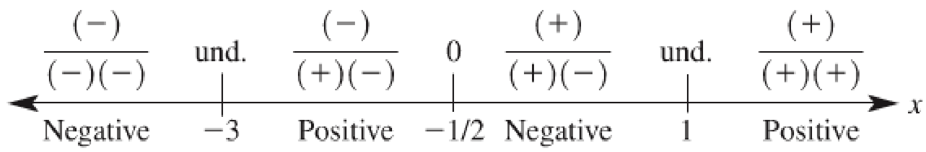
(a) The real zeros of $r(x)$ are the real zeros of the numerator. So $r(x) = 0$ if $= -\frac{1}{2}$.

(b) $r(x)$ is undefined when the denominator, $(x + 3)(x - 1) = 0$. So $r(x)$ is undefined if $x = -3$ *or* $x = 1$. Note: they are the vertical asymptotes.

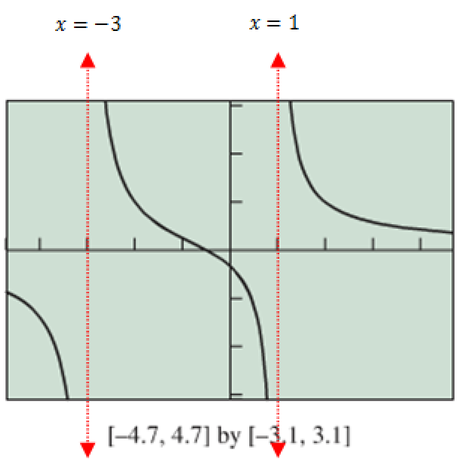
These findings lead to the following sign chart, with three points of potential sign change:



Additionally, by analyzing the factors of the numerator and denominator yields:



(c) So $r(x)$ is positive if $-3 < x < -\frac{1}{2}$ or $x > 1$: $(-3, -\frac{1}{2}) \cup (1, \infty)$.
(d) Similarly, $r(x)$ is negative if $x < -3$ *or* $-\frac{1}{2} < x < 1$: $(-\infty, -3) \cup (-\frac{1}{2}, 1)$.
The graph below supports the findings.



The graph of $r(x) = \frac{2x+1}{(x+3)(x-1)}$

Writing Assignment:

Let $f(x) = 3x - 5$. Assume x is in the interval defined by $|x - 3| < \frac{1}{3}$. Give a convincing argument that $|f(x) - 4| < 1$.

Illustrative Mathematics **Comments and Solutions can be found at:**
http://www.illustrativemathematics.org/illustrations/702

Chase and his brother like to play basketball. About a month ago they decided to keep track of how many games they have each won. As of today, Chase has won 18 out of the 30 games against his brother.

a. How many games would Chase have to win in a row in order to have a 75% wining record?

b. How many games would Chase have to win in a row in order to have a 90% winning record?

c. Is Chase able to reach a 100% winning record? Explain why or why not.

d. Suppose that after reaching a winning record of 90% in part (b), Chase had a losing streak. How many games in a row would Chase have to lose in order to drop down to a winning record below 55% again?

Unit 2: Exponential and Logarithmic Functions

Overview: In previous unit, we studied polynomial, power, and rational functions, also known as algebraic functions. They are obtained by adding, subtracting, multiplying, and dividing constants and an independent variable, and raising expressions to integer powers and extracting roots. In this unit, we study two interrelated functions: exponential and logarithmic functions, also known as transcendental functions, which go beyond, or transcend, these algebraic operations. Just like algebraic functions, these two functions have wide application. Exponential functions model growth and decay over time, such as population growth and the decay of radioactive materials. Logarithmic functions are the basis of the Richer scale of earthquake intensity and the pH acidity scale.

Unit 2 Section 1: Exponential Functions (3.1)

Major Concepts: Evaluate exponential expressions and identify and graph exponential functions

Standards

Creating Equations (A-CED)

Create equations that describe numbers or relationship

A-CED.1 Create equations and inequalities in one variable and use them to solve problems, including equations arising from linear and quadratic functions, and simple rational and exponential functions.

Interpreting Functions (F-IF)

Interpret functions that arise in applications in terms of the context

F-IF.4 For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.

Analyze functions using different representations

F-IF.7.e Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.

F-IF.8.b Use the properties of exponents to interpret expressions for exponential functions.

Linear, Quadratic, and Exponential Models (F-LE)

Construct and compare linear, quadratic, and exponential models and solve problems

F-LE.1.a Prove that linear functions grow by equal differences over equal intervals, and that exponential functions grow by equal factors over equal intervals.

F-LE.1.c Recognize situations in which a quantity grows or decays by a constant percent rate per unit interval relative to another.

F-LE.2 Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table).

F-LE.3 Observe using graphs and tables that a quantity increasing exponential eventually exceeds a quantity increasing linearly, quadratically, or (more generally) as a polynomial function

Students will be able to:

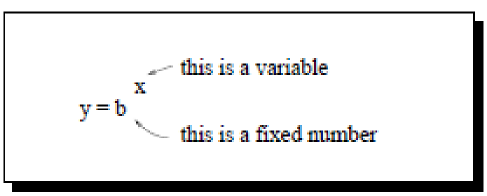
- **Identify** an exponential function.
- **Evaluate** an exponential function.
- **Graph** an exponential function by hand.
- **Determine** an exponential function from its table of values.
- **Graph** transformations of exponential functions.

Vocabulary:
exponential function, base, exponential growth function, growth factor, exponential decay function, decay factor, natural exponential function

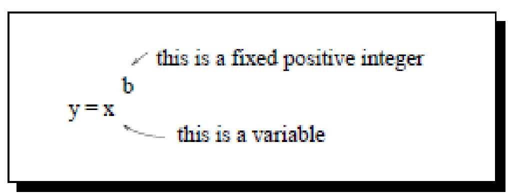
1. **CONNECT** (10 minutes)
   Review: Prerequisites for Exponential Functions
   Review evaluating expressions with rational exponents in preparation for studying exponential functions.

2. **INSTRUCT** (15 minutes)
   Exponential Functions
   Explain the definition of an exponential function, and help students begin to identify exponential functions from other types of functions. A word of warning: Students are always tempted to combine the constant in front with the exponential portion of the function, particularly when evaluating the function. It may help to recall order of operations with them. Help students understand why the domain of all exponential functions is a set of all real numbers. Find the corresponding *y*- values, the range. This is a good place to revisit one-to-one function again, in an anticipation of its inverse called a logarithmic function.

Note: Students are often confused between $y = x^2$ and $y = 2^x$ as to why the former is called a polynomial function with second degree; whereas, the latter is called an exponential function. One way to help your students to be able to distinguish is the location of the variable *x*. In $y = 2^x$, the changes occur at the exponential level; thus, it is called an exponential function.

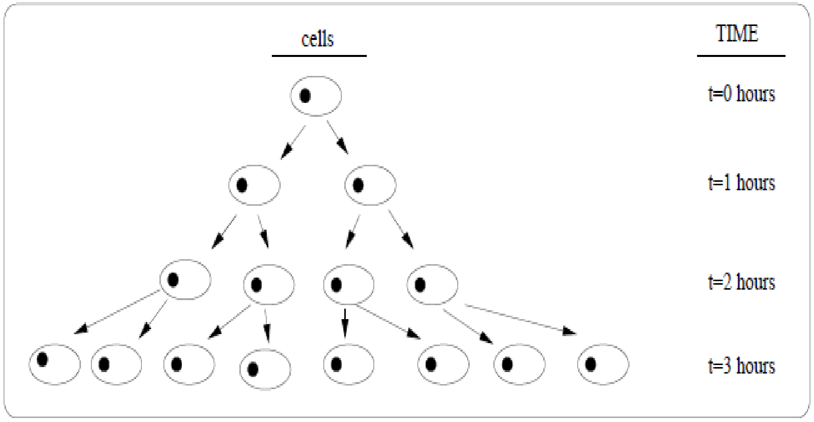


Exponential Picture



Monomial Picture

Conceptual Understanding: Doubling Effect
If we start with a single yeast cell under favorable growth conditions, then it will divide in one hour to form two identical "daughter cells." In turn, after another hour, each of these daughter cells will divide to produce two identical cells: we now have four identical "granddaughter cells" of the original parent cell. Under ideal conditions, we can imagine how this "doubling effect' will continue.



Total hours	Number of yeast cells
0	$1=2^0$
1	$2=2^1$
2	$4=2^2$
3	$8=2^3$
4	$16=2^4$
5	$32=2^5$
6	$64=2^6$

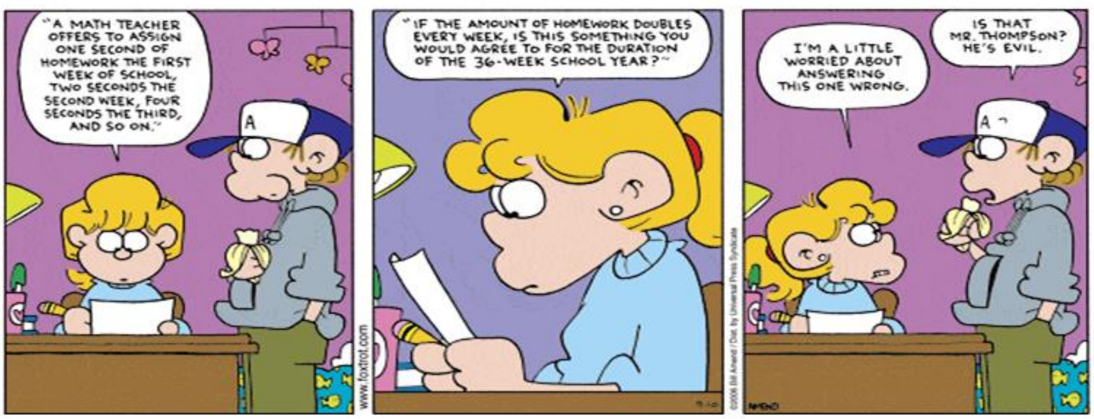


Table of Values for Exponential Functions

Demonstrate how to determine whether a function is exponential from its table of values. Consider a function of the form $f(x) = b^x$, where $b > 0$ and $b \neq 1$. Since $b > 0$ but $b \neq 1$, then $b > 1$ or $0 < b < 1$.

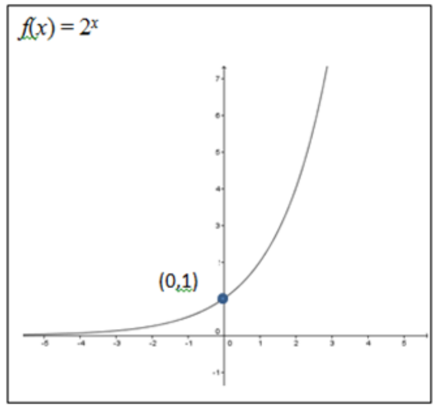
Case 1. b > 1: Exponential Growth Function

To examine this case, take a numerical example. Suppose that $b = 2$. $\Rightarrow f(x) = 2^x$

x	$f(x)$
-3	$\frac{1}{8}$
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4
3	8

Common Ratio = Constant Multiplier = Base of an Exponential Function.

Example, $\left(\frac{1}{4}\right) \div \left(\frac{1}{8}\right) = 2$



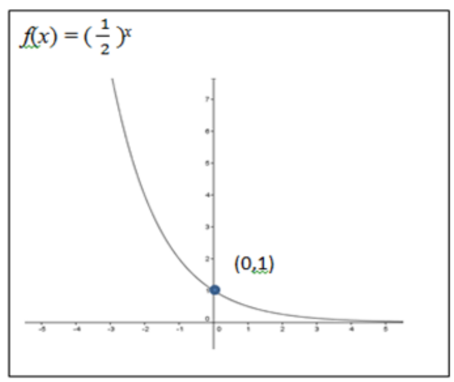
Case 2. 0< b < 1: Exponential Decay Function

To examine this case, take a numerical example. Suppose that $b = \frac{1}{2}$. $\Rightarrow f(x) = \left(\frac{1}{2}\right)^x$

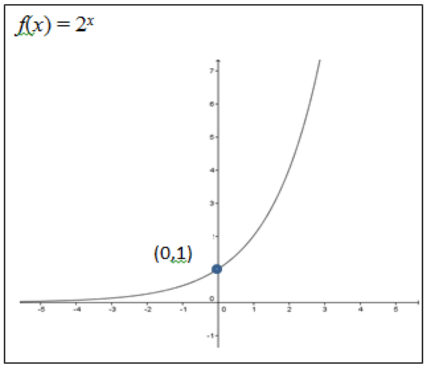
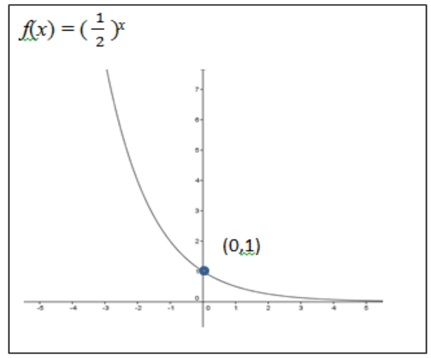
x	$f(x)$
-3	8
-2	4
-1	2
0	1
1	$\frac{1}{2}$
2	$\frac{1}{4}$
3	$\frac{1}{8}$

Common Ratio = Constant Multiplier = Base of an Exponential Function.

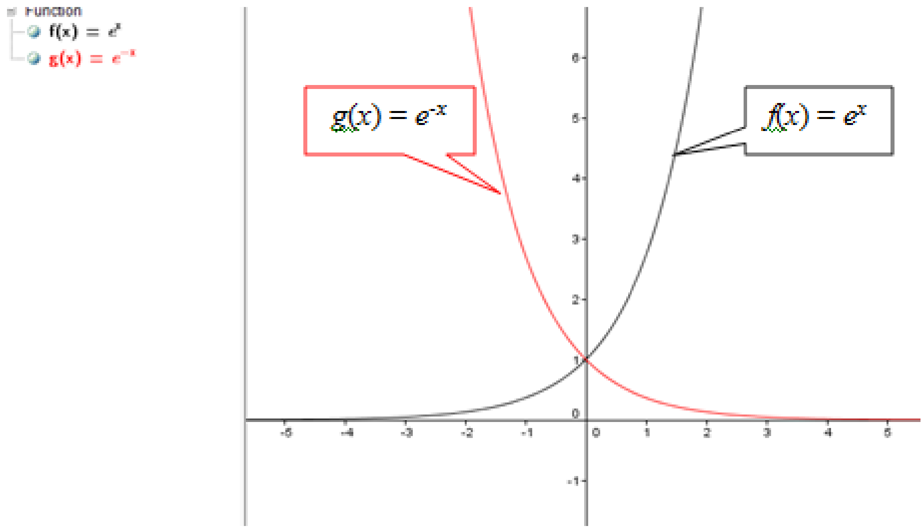
Example, $\left(\frac{1}{8}\right) \div \left(\frac{1}{4}\right) = \frac{1}{2}$



3. PROPERTIES OF EXPONENTIAL FUNCTIONS (20 minutes)

Graph		
Domain		
Range		
x-intercept		
y-intercept		
Asymptote		
Increasing Intervals		
Decreasing Intervals		

The Natural Base *e*: A particularly important example of an exponential function arises when b = e. You might recall that the number e is approximately equal to 2.718. Since $e > 1$ and $1/e < 1$, we can sketch the graphs of the exponential functions $f(x) = e^x$ and $g(x) = e^{-x} = (1/e)^x$.



4. INSTRUCT (10 minutes)

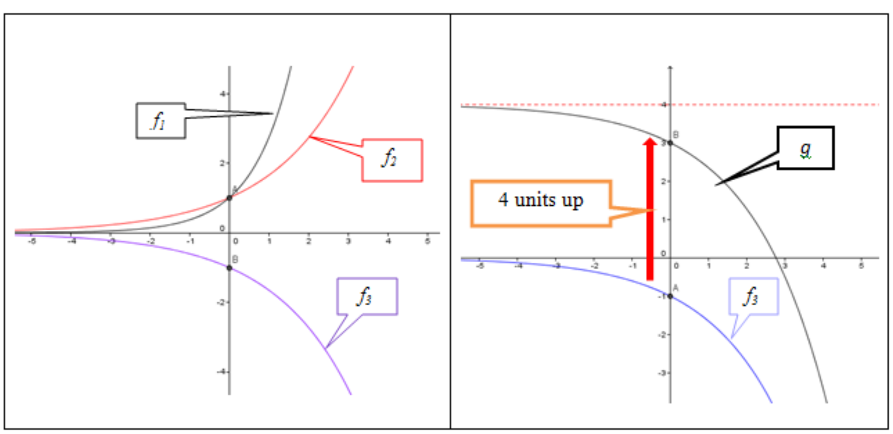
Graphing: Transformations of Exponential Functions

Demonstrate how to graph the transformation of an exponential function.
Be sure students are attentive to the order of the transformation steps.

Example: Let $g(x) = 4 - e^{x/2}$. Use transformations to explain how the graph of g is related to the graph of $f_1(x) = e^x$. Determine whether g is increasing or decreasing, find any asymptotes, and sketch the graph of g. The graph of g can be obtained from the graph of f_1 by a sequence of three transformations:

Solution:

$$f_1(x) = e^x \xrightarrow{\text{Horizontal stretch}} f_2(x) = e^{x/2} \xrightarrow{\text{Reflection in } x \text{ axis}} f_3(x) = -e^{x/2} \xrightarrow{\text{Vertical translation}} g(x) = 4 - e^{x/2}$$



Writing Assignment:

Describe how to transform the graph of $y = e^x$ into the graph of $y = 4 - e^{x/2}$.

Comments and Solutions can be found at:
http://www.illustrativemathematics.org/illustrations/1829

a. How do the values of the three functions $f(x) = 2x$, $g(x) = x^2$ and $h(x) = 2^x$ compare for large positive and negative values of x?

b. Explain your findings from part (a) for f and g by studying the quotients $\frac{g(x)}{f(x)}$.

c. Explain your findings from part (a) for g and h by studying $\frac{g(x+1)}{g(x)}$ and $\frac{h(x+1)}{h(x)}$ for different values of x.

Unit 2 Section 2: Exponential Modeling (3.2)

Major Concepts: Use exponential growth, decay, and regression to model real-life problems.

Standards

Linear, Quadratic, and Exponential Models (F-LE)

Construct and compare linear, quadratic, and exponential models and solve problems

F-LE.1.c Recognize situations in which a quantity grows or decays by a constant percent rate per unit interval relative to another.

Interpret expressions for functions in terms of the situation they model

F-LE.5 Interpret the parameters in a linear or exponential function in terms of a context.

Students will be able to:

- **Determine** growth and decay rates for exponential functions.
- **Calculate** the half-life of a radioactive substance.
- **Use** exponential functions to model various growth and decay problems, including bacteria growth, radioactive decay, and population growth.

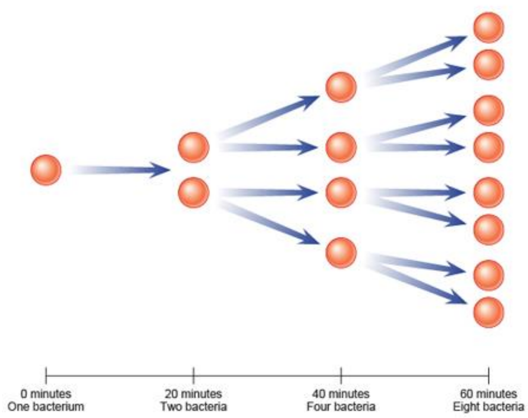
Vocabulary:
radioactive decay, half-life

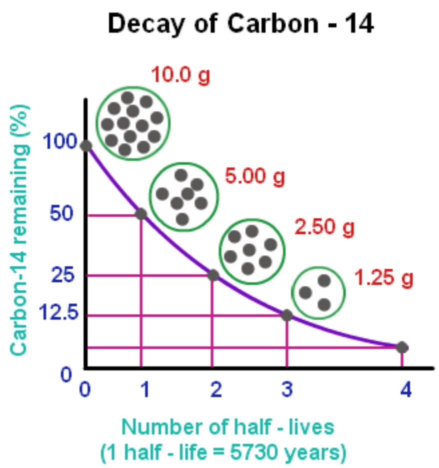
1. **CONNECT** (10 minutes)
Review: Prerequisites for Exponential Models
Review percent conversion and calculating interest in preparation for exponential models.

2. **INSTRUCT** (30 minutes)
Conceptual Understanding: Exponential Models
Explain how to use exponential functions to model situations in which a constant percentage rate is involved. Students should be able to determine the growth rate from a given exponential model, and the corresponding exponential model given the growth rate.

The number e is important as the base of an exponential function in many practical applications. In situations involving growth or decay of a quantity, the amount or number present at time t often is closely modeled by a function defined by
$A = Pe^{rt}$, where P is the amount or number present at time $t = 0$ and r is a growth rate ($r > 0$) or decay rate ($r < 0$).

Calculations: Bacterial Growth and Radioactive Decay
Demonstrate several applications of exponential growth, including bacterial growth and radioactive decay. Spend time in helping students understand and calculate the half-life of an element.





Example: The number B of bacteria in a petri dish culture after t hours is given by $B = 100e^{0.0693t}$. When will the number of bacteria be 200.

Solution: The initial bacteria present is 100 (*P* = 100) and the growth rate is 0.0693 or 6.93%. The time takes to reach the population 200 is doubling time.

Example: The amount *C* in grams of carbon-14 present in a certain substance after *t* years is given by $C = 20e^{-0.0001216t}$. Estimate the half-life of *C*.

Solution: In here, *P* is 20 grams, and the decay rate is 0.0001216 or 0.01216%. The half-life the substance *C* is the time takes to have *C* = 10.

Writing Assignment:

Explain why the half-life of a radioactive material is independent of the initial amount of the material that is present.

Comments and Solutions can be found at:
http://www.illustrativemathematics.org/illustrations/370

A hospital is conducting a study to see how different environmental conditions influence the growth of streptococcus pneumonia, one of the bacteria which causes pneumonia. Three different populations are studied giving rise to the following equations:

$$\begin{aligned} p_1(t) &= 1000e^{t/3}, \\ p_2(t) &= 1500e^{3t/8}, \\ p_3(t) &= 5000e^{t/4}. \end{aligned}$$

Here t represents the number of hours since the beginning of the experiment which lasts for 24 hours and $p_i(t)$ represents the size of the i^{th} bacteria population.

a. Explain, in terms of the structure of the expressions defining $p_1(t)$ and $p_2(t)$, why these two populations never share the same value at any time during the experiment.

b. Explain, in terms of the structure of the expressions defining $p_1(t)$ and $p_3(t)$, why these two populations will be equal at exactly one time during the experiment. Determine this time.

Comments and Solutions can be found at: http://www.illustrativemathematics.org/illustrations/760

In order to use Carbon 14 for dating, scientists measure the *ratio* of Carbon 14 to Carbon 12 in the artifact or remains to be dated. When an organism dies, it ceases to absorb Carbon 14 from the atmosphere and the Carbon 14 within the organism decays exponentially, becoming Nitrogen 14, with a half-life of approximately 5730 years. Carbon 12, however, is stable and so does not decay over time.

Scientists estimate that the ratio of Carbon 14 to Carbon 12 today is approximately 1 to $1,000,000,000,000$.

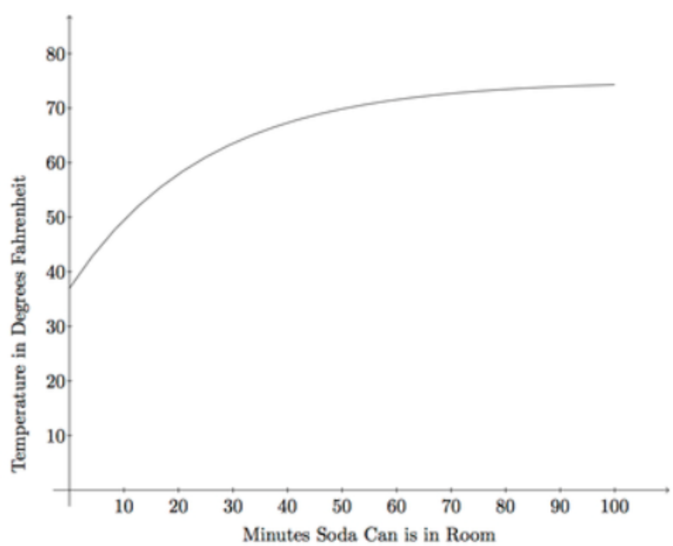
a. Assuming that this ratio has remained constant over time, write an equation for a function which models the ratio of Carbon 14 to Carbon 12 in a preserved plant t years after plant has died.

b. In a particular preserved plant, the ratio of Carbon 14 to Carbon 12 is estimated to be about 1 to $13,000,000,000$. What can you conclude about when plant lived? Explain.

c. Dinosaurs are estimated to have lived from about $230,000,000$ years ago until about $65,000,000$ years ago. Using this information and the given half-life of Carbon 14, explain why this method of dating is not used for dinosaur remains.

Comments and Solutions can be found at: http://www.illustrativemathematics.org/illustrations/569

Suppose a can of cold soda is left in a warm room on a summer day. The graph below shows the temperature of the soda as it gradually increased:



The function that describes the temperature, F, of the soda (in degrees Fahrenheit) after t minutes can be expressed by

$$F(t) = C - Re^{-kt},$$

for some positive values of C, R, and k.

a. Use the graph to estimate C.

b. Use the graph to estimate R.

c. What was the approximate room temperature? What was the initial temperature of the soda when placed in the room?

Unit 2 Section 3: Logarithmic Functions and Their Graphs (3.3)

Major Concepts: Convert equations between logarithmic form and exponential form, evaluate common and natural logarithms, and graph logarithmic functions.

Standards

Functions: Building Functions (F-BF)
Build new functions from existing functions

F-BF.5 (+)	Understand the inverse relationship between exponents and logarithmic and use this relationship to solve problems involving logarithms and exponents.

Unit 2 Section 3-A: Logarithmic Functions

Students will be able to:

- **Explain** the relationship between logarithmic and exponential functions.
- **Evaluate** a logarithmic expression.
- **Evaluate** simple logarithmic equations using exponential functions.
- **Analyze** the common logarithm function.

Vocabulary:
logarithmic function, base, common logarithm

1. **CONNECT** (10 minutes)
   Review: Prerequisites for Logarithmic Functions
   Review simplifying exponential expressions and using fractional exponents in preparation for studying logarithmic functions.

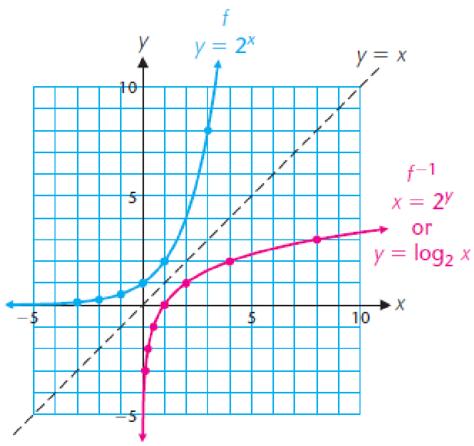
2. **INSTRUCT** (25 minutes)
Calculations: Evaluating Logarithmic Expressions
Students inevitably have trouble evaluating logarithmic expressions. Take time to describe the mental process they should use to evaluate logarithms. Explain the base and the number being substituted in are related. Give many examples using simple integer bases like 2 or 3. It is worth the time to get students to understand these functions.

Calculations: Common Logarithms
Now focus on logarithms with base 10. Give several examples and show how to evaluate these expressions with and without a calculator.

Graphing: Logarithmic Functions
Show how the graphs of the logarithmic and exponential functions are related. You will probably need to recall to students how inverse functions are related.

In solving $2^t = 10$, we have no algebraic way to solve this equation because we lacked a strategy to isolate the exponent t. However, the exponential function $f(x) = b^x$, where $b > 0$, $b \neq 1$, is a one-to-one function, and therefore has an inverse. Its inverse, denoted $f^{-1}(x) = \log_b x$ (read "log to the base b of x"), is called the logarithmic *function with base b.* Logarithms allow us to rewrite an exponential equation so that the exponent is isolated. Specifically, if $a = b^c$, then "c is the logarithm with base b of a" and is written as $\log_b a = c$ (We read as "log base b of a is c."). Using logarithms we can write $2^t = 10$ as $\log_2 10 = t$. These two expressions are equivalent, and in the expression $\log_2 10 = t$, t is isolated. Although this is a good thing, we still need a way to evaluate the expression $\log_2 10 = t$. We shall revisit this problem after learning of the properties of logarithms.

A point (x, y) lies on the graph of f^{-1} if and only if the point (y, x) lies on the graph of f; in other words, $y = \log_b x$ if and only if $x = b^y$. We can use this fact to deduce information about the logarithmic functions from our knowledge of exponential functions. For example, the graph of f^{-1} is the graph of f reflected in the line $y = x$; and the domain and range of f^{-1} are, respectively, the range and domain of f. Consider the exponential function $f(x) = 2^x$ and its inverse $f^{-1}(x) = \log_2 x$. Figure below shows the graphs of both functions and a table of selected points on those graphs. Because $y = \log_2 x$ if and only if $x = 2^y$. $\log_2 x$ is the exponent to which 2 must be raised to obtain x: $2^{\log_2 x} = 2^y = x$.



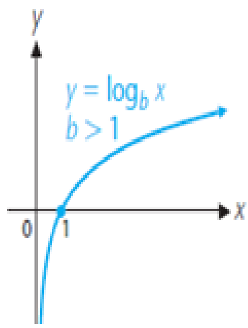
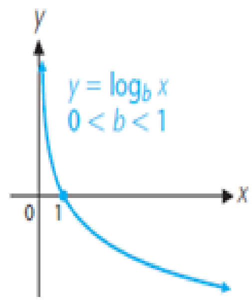
DOMAIN of $f = (-\infty, \infty)$ = RANGE of f^{-1}

RANGE of $f = (0, \infty)$ = DOMAIN of f^{-1}

f		f^{-1}	
x	$y = 2^x$	$x = 2^y$	y
-3	$\frac{1}{8}$	$\frac{1}{8}$	-3
-2	$\frac{1}{4}$	$\frac{1}{4}$	-2
-1	$\frac{1}{2}$	$\frac{1}{2}$	-1
0	1	1	0
1	2	2	1
2	4	4	2
3	8	8	3

Ordered pairs reversed

Definition: For $b > 0$, $b \neq 1$, the inverse of $f(x) = b^x$, denoted $f^{-1}(x) = \log_b x$, is the logarithmic function with base b. Hence, $y = \log_b x$ is equivalent to $x = b^y$. The log to the base b of x is the exponent to which b must be raised to obtain x. The domain of f^{-1} is $(0, \infty)$ and the range is $(-\infty, \infty)$.

	Case 1	Case 2
Graph		
Domain		
Range		
x-intercept		
y-intercept		
Asymptote		
Increasing		
Decreasing		

Comments and Solutions can be found at: **http://www.illustrativemathematics.org/illustrations/600**

Recall that $\log_b(x)$ is by definition the exponent which b must be raised to in order to yield x ($b > 0$).

Part I

a. Use this definition to compute $\log_2(2^5)$.

b. Use this definition to compute $\log_{10}(0.001)$.

c. Use this definition to compute $\ln(e^3)$.

d. Explain why $\log_b(b^y) = y$ where $b > 0$.

The above technique can be used to raise numbers to logarithmic powers by first simplifying the exponent.

Part II

a. Evaluate $10^{\log_{10}(100)}$.

b. Evaluate $2^{\log_2(\sqrt{2})}$.

c. Evaluate $e^{\ln(89)}$.

d. Explain why $b^{\log_b(x)} = x$ where $b > 0$.

Unit 2 Section 3-B: Natural Logarithmic Functions

Students will be able to:

- **Evaluate** a natural logarithm expression.
- **Analyze** the natural logarithm function.
- **Graph** transformations of logarithmic functions.

Vocabulary:
natural logarithm

1. **INSTRUCT** (35 minutes)

Calculations: Natural Logarithms

With the previous preparations, now introduce the natural logarithm, and show how to evaluate it at numbers that are powers of e. Point out that only powers of *e* will yield nice numbers from the natural logarithmic function (in much the same way that multiples of pi will yield nice numbers from trigonometric functions).

Graphing: Natural Logarithms

Analyze the graph of the natural logarithm function. Discuss how to transform this function graphically and algebraically.

Transformations of Logarithmic Functions

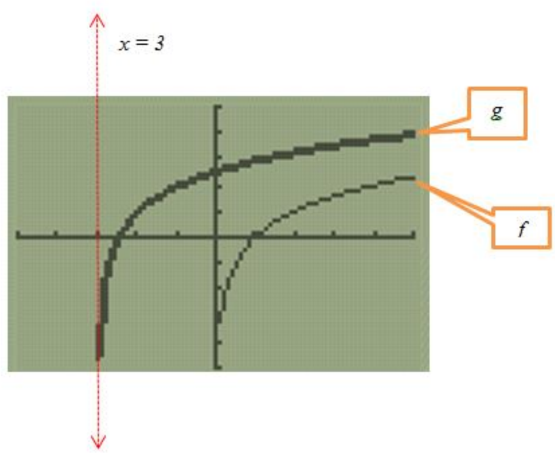
Let $g(x) = 1 + \log_2(x+3)$.

(A) Use transformations to explain how the graph of g is related to the graph of the logarithmic function $f(x) = \log_2 x$. Determine whether g is increasing or decreasing, find its domain and asymptote, and sketch the graph of g.

(B) Find the inverse of g.

Solutions

(A) The graph of g can be obtained from the graph of f by a horizontal translation (left 3 units) followed by a vertical translation (up 1 unit). The graph of g is increasing. The domain of g is the set of real numbers x such that $x + 3 > 0$, namely $(-3, \infty)$. The line $x = -3$ is a vertical asymptote (indicated by the dashed line in red)).



(B)	$y = 1 + \log_2(x+3)$	Subtract 1 from both sides
	$y - 1 = \log_2(x+3)$	Write in exponential form
	$x + 3 = 2^{y-1}$	Subtract 3 from both sides
	$x = 2^{y-1} - 3$	Interchange x and y
	$y = 2^{x-1} - 3$	

Therefore, the inverse of g is $g^{-1} = 2^{x-1} - 3$.

Unit 2 Section 4: Properties of Logarithmic Functions (3.4)

Major Concepts: Apply the properties of logarithms to evaluate expressions and graph functions.

Standards

Functions-Building Functions (F-BF)
Build new functions from existing functions

F-BF.5 (+)	Understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents.

Students will be able to:

- **Simplify** and expand logarithmic expressions using properties of logarithms.
- **Evaluate** a logarithmic expression by first changing the base.
- **Analyze** a general logarithmic function.

Vocabulary:
product rule, quotient rule, power rule

1. **CONNECT** (10 minutes)
   Review: Prerequisites for Logarithm Properties
   Review evaluating logarithmic expressions in preparation for studying logarithmic properties.

2. **INSTRUCT** (25 minutes)
   Calculations: Using Logarithmic Properties
   Introduce the properties of logarithms. Students should become proficient at both expanding and collapsing logarithmic expressions using the product and quotient rules. Next, introduce the formula for changing bases, and demonstrate how it is useful for evaluating logarithms with strange bases on a calculator.

Revisit from (Unit2 Section3) that was stated: Using logarithms, we can write $2^t = 10$ as $\log_2 10 = t$. These two expressions are equivalent, and in the expression $\log_2 10 = t$, t is isolated. Although this is a good thing, we still need a way to evaluate the expression $\log_2 10 = t$. We shall revisit this problem after learning of the properties of logarithms. Now, can you find t?

Unit 2 Section 5: Solving Exponential and Logarithmic Equations and Modeling (3.5)

Major Concepts: Apply the properties of logarithms to solve exponential and logarithmic equations algebraically and solve application problems using these equations.

Standards

Functions-Building Functions (F-BF)
Build new functions from existing functions
F-BF.5 (+) Understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents.
Functions-Linear, Quadratic, and Exponential Models (F-LE)
Construct and compare linear, quadratic, and exponential models and solve problems
F-LE.4 For exponential models, express as a logarithm solution to $ab^{ct} = d$ where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology.
Interpret expressions for functions in terms of the situation they model
F-LE. 5 Interpret the parameters in a linear or exponential functions in terms of a context.

Unit 2 Section 5-A: Solving Exponential and Logarithmic Equations

Students will be able to:
- **Solve** an exponential equation.
- **Solve** a logarithmic equation.

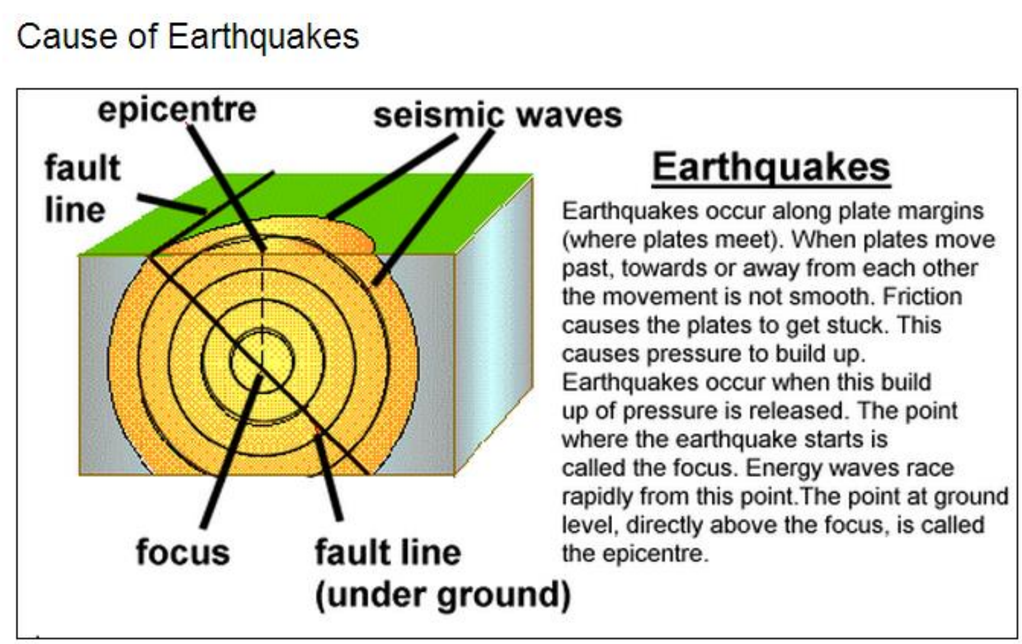
Vocabulary:
earthquake magnitude**,** pH, acidity, base

1. **CONNECT** (10 minutes)
   Review: Composing inverse functions and scientific notation in preparation for solving logarithmic and exponential equations.

2. **INSTRUCT** (25 minutes)
   Calculations: Solving Exponential and Logarithmic Equations
   Explain how to solve both exponential and logarithmic equations using inverse functions. It may help to form an analogy with other inverse functions used for solving (like the squaring and square root functions).

Conceptual Understanding:
Discuss the applications in various areas, including earthquake magnitudes and acidity of substances.

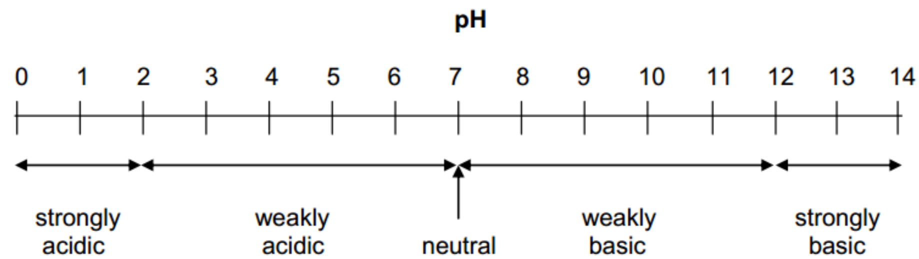
Earthquake: The Richter magnitude scale was developed in 1935 by Charles F. Richter of the California Institute of Technology as a mathematical device to compare the size of earthquakes. The magnitude of an earthquake is determined from the logarithm of the amplitude of waves recorded by seismographs. See the example on p. 295.

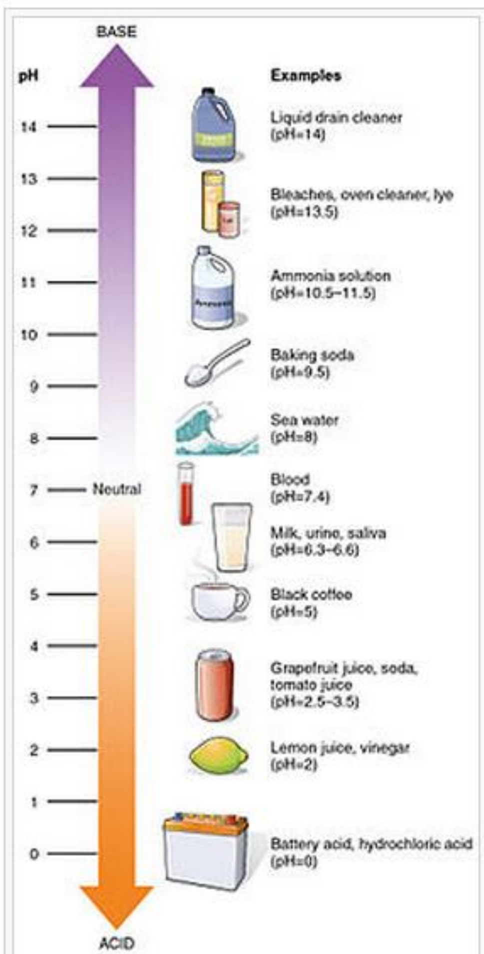


http://www.bennett.karoo.net/topics/earthquakes.html

Measuring Earthquakes: The magnitude (size) of an earthquake is measured using a seismometer. This is a machine that measures movements in the earth's surface.
The Richter Scale measures earthquakes on a logarithmic scale - this means that an earthquake of 6 is ten times more powerful than one with a score of 5.

Acidity: Chemists define the acidity or alkalinity of a substance according to the formula "pH = –*log*[H^+]" where [H^+] is the hydrogen ion concentration, measured in moles per liter. Solutions with a pH value of less than 7 are acidic; solutions with a pH value of greater than 7 are basic; solutions with a pH of 7 (such as pure water) are neutral. See the example on p. 296.





pH values of some common substances

pH values for liquids

Liquid	**pH**	**Concentration of positive ions (Moles per liter)**
Liquid Drainer	14	1.00E-14
Ammonia	11	1E-11
Detergent	9.5	3E-10
Blood	7.5	3.00E-08
Pure Water	7	1.00E-07
Milk	6.5	3.00E-07
Tomato Juice	3	1.00E-03
Vinegar	2.5	3.00E-02
Battery Acid	0	1.00E+00

http://en.wikipedia.org/wiki/PH

Unit 2 Section 5-B: Newton's Law of Cooling

Students will be able to:
- **Model** a cooling problem using Newton's Law of Cooling.
- **Fit** given data using regression models such as linear, logarithmic, exponential functions.

Vocabulary:
Newton's Law of Cooling, regression, linear regression, logarithmic regression,

1. **INSTRUCT** (35 minutes)
   Calculations: Newton's Law of Cooling
   An object that has been heated will cool to the temperature of the medium in which it is placed, such as the surrounding air or water. This model is called Newton's Law of Cooling. This will involve solving both exponential and logarithmic equations.

   Graphing Calculators: Regression
   Revisit regression, now armed with more possible functions than simple linear regressions. Show students how the layout of data points is an indication of the type of regression function that should be utilized. Try several examples in a graphing calculator.

Illustrative Mathematics **Comments and Solutions can be found at:**

http://www.illustrativemathematics.org/illustrations/382

A cup of hot coffee will, over time, cool down to room temperature. The principle of physics governing the process is Newton's Law of Cooling. Experiments with a covered cup of coffee show that the temperature (in degrees Fahrenheit) of the coffee can be modelled by the following equation

$$f(t) = 110e^{-0.08t} + 75.$$

Here the time t is measured in minutes after the coffee was poured into the cup.

a. Explain, using the structure of the expression $110e^{-0.08t} + 75$, why the coffee temperature decreases as time elapses.

b. What is the temperature of the coffee at the beginning of the experiment?

c. After how many minutes is the coffee 140 degrees? After how many minutes is the coffee 100 degrees?

Unit 3: Trigonometry

Overview: The trigonometric functions arise from the ratios within right triangles, the ultimate computational tool for engineers in the ancient world. As the civilization progressed from a flat Earth to a world of circles and sphere, trigonometry was soon to be used to understanding circular phenomena as well. The circular motion led to waves, and suddenly trigonometry was the indispensable tool for understanding everything from electrical current to modern telecommunications.

Unit 3 Section 1: Trigonometry Extended: The Circular Functions (4.3)

Major Concepts: Solve problems involving the trigonometric functions of real numbers and the properties of the sine and cosine as periodic

Standards

Functions-Trigonometric Functions (F-TF)

Extend the domain of trigonometric functions using the unit circle

F-TF.2 Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.

F-TF.3 (+) Use special triangles to determine geometrically the values of sine, cosine, and tangent for $\frac{\pi}{3}, \frac{\pi}{4}, and\ \frac{\pi}{6}$, and use the unit circle to express the values of sine, cosine, and tangent for π-x, π+x, and 2π-x in terms of their values fc x is any real number.

F-TF.4 (+) Use the unit circle to explain symmetry (odd and even) and periodicity of trigonometric functions.

Prove and apply trigonometric identities

F-TF.8 Prove the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$ and use it to find sin(θ), cos(θ), or tan(θ) given sin(θ), cos(θ), or tangent(θ) and the quadrant of the angle.

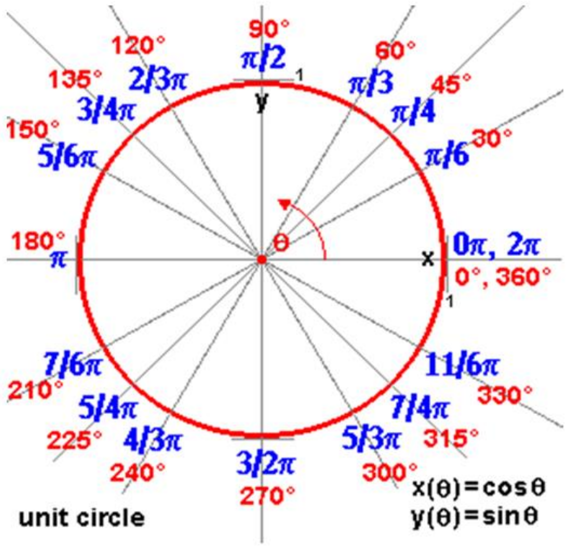
Unit 3 Section 1-A: Trigonometric Functions Using Coterminal Angles

Students will be able to:

- **Sketch** a non-acute angle in standard position on a coordinate axis.
- **Find** a coterminal angle for a non-acute angle.
- **Evaluate** a trigonometric function determined by a point in the plane.
- **Evaluate** a trigonometric function for a quadrantal angle.
- **Evaluate** other trigonometric functions using a given trigonometric ratio.

Vocabulary:
initial side, vertex, terminal side, angle measure, positive angle, negative angle, standard position, coterminal angle, reference triangle, quadrantal angle

1. **CONNECT** (10 minutes)
Review: Prerequisites for Extended Trigonometry
Review evaluating trigonometric functions of acute angles and converting angles from radians to degrees in preparation for evaluating trigonometric functions for larger angles. Students can easily find the conversion factor between central angle and circumference in observing a unit circle: $\frac{360^0}{2\pi}$ or $\frac{2\pi}{360^0}$.



They also learn that the radian measure of an angle can be defined as the quotient of arclength to radius. As a quotient of two lengths, radian measure is "dimensionless." That is why the "unit" is omitted when measuring angles in radians. In calculus, the benefits of radian measure are plentiful, leading to simple formulas for derivatives and integrals of trigonometric functions. In this course, there are two key benefits of using radians than degrees.

- arc length is simply $r\theta$, and
- $sin\theta \approx \theta$ for small θ.

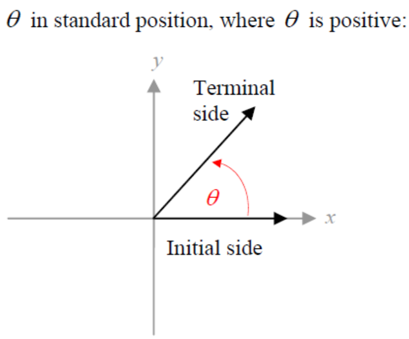
Looking Ahead to Calculus

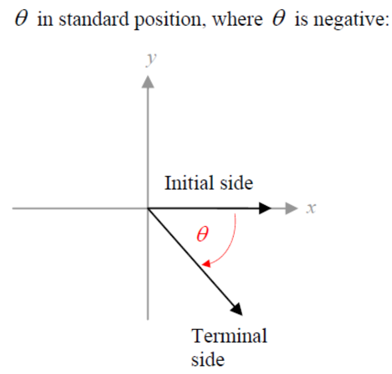
Those students who go on to study calculus later realize that the limit of $\frac{sin\theta}{\theta}$ as θ approaches zero is one $\Rightarrow \lim_{\theta \to 0} \frac{sin\theta}{\theta} = 1$

2. **INSTRUCT** (10 minutes)

Building Vocabulary: Angles on the Coordinate Plane

Describe the difference between the initial and terminal side of an angle, a positive and a negative angle, and the relationship between coterminal angles. Pictures are necessary with these explanations.





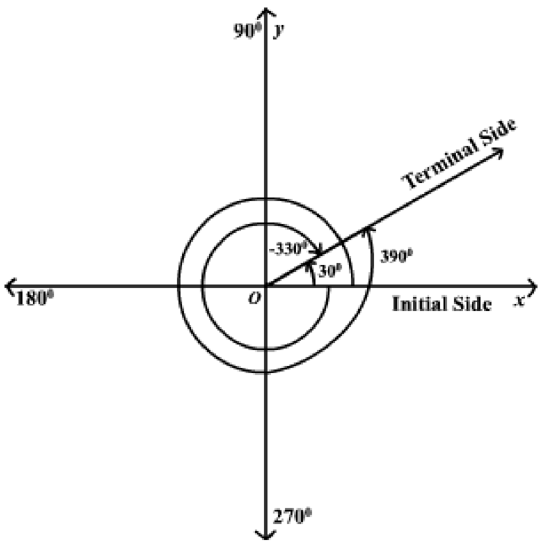
Calculations: Coterminal Angles

Demonstrate how to calculate coterminal angles in both degrees and radians. Be sure that students understand there are an infinite number of coterminal angles.

Coterminal angles are angles formed by different rotations, but with the same initial and terminal sides. To find measures of coterminal angles, add or subtract multiples of 360^{o}.

For example, 30°, –330° and 390° are all coterminal angles.

To find a positive and a negative angle coterminal with a given angle, you can add $360°k$, $k \in \mathbb{Z}$, if the angle is measured in degrees, or $2\pi k$, $k \in \mathbb{Z}$, if the angle is measured in radians.

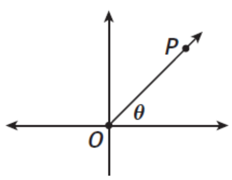


3. **INDIVIDUAL EXPLORATION** (5 minutes)
Exploration: First Quadrant Trigonometry
Have students investigate Exploration 1 on page 339 to better understand how to find trigonometric functions for angles in the first quadrant.

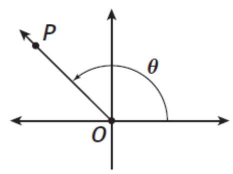
4. **INSTRUCT** (20 minutes)
Calculations: Evaluating Trig Functions for Other Angles
Show how to calculate each of the six trigonometric functions of an angle, given a particular point in the plane. Be sure students understand when to use appropriate negative signs. Then demonstrate the technique of using reference triangles.

Given an angle in standard position, a **reference triangle** is formed by drawing a perpendicular segment from a point on the terminal side of the angle to the *x*-axis. Each figure below shows an angle θ in standard position. Draw a perpendicular segment from point *P* to the *x*-axis to form a reference triangle.

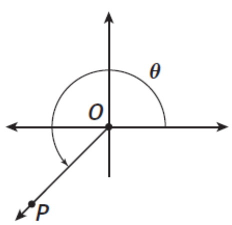
a.



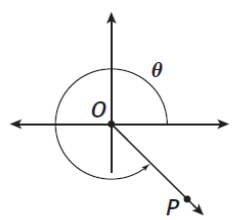
b.



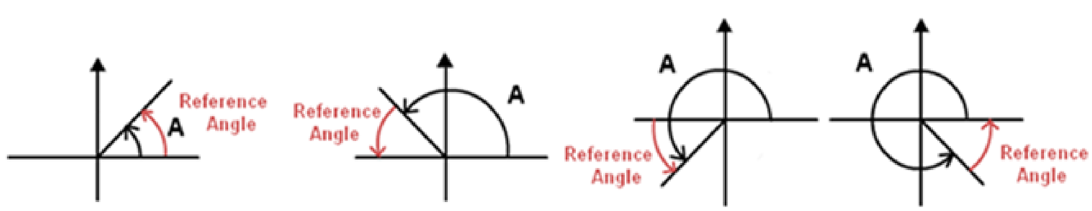
c.



d.



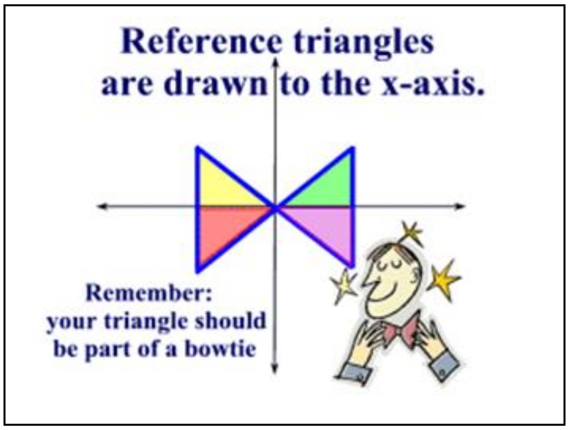
Once the reference triangle is studied, make a connection to **a reference angle** and where each reference angle is located in each quadrant. Have students to come up with a definition of a reference angle.



Q I **Q II** **Q III** **Q IV**

Let $0 \leq \theta < 2\pi$ a, then a reference angle, θ’, can be found by:

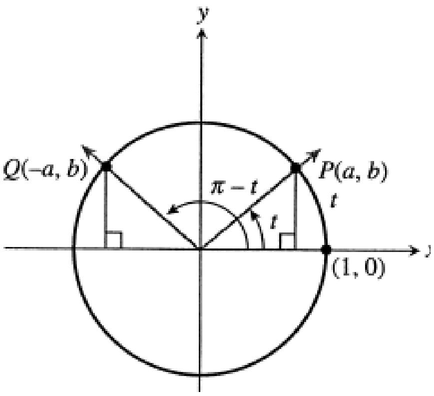
Q I	Q II	Q III	Q IV
$\theta' = \theta$	$\theta' = \pi - \theta$	$\theta' = \theta - \pi$	$\theta' = 2\pi - \theta$



http://www.regentsprep.org/Regents/math/algtrig/ATT3/referenceTriangles.htm

Once a reference triangle and a reference angle are studied, use the accompanying diagram below to explain each of the following.

(a) $\sin(\pi - t) = \sin t$
(b) $\cos(\pi - t) = -\cos t$



Unit 3 Section 1-B: The Unit Circle

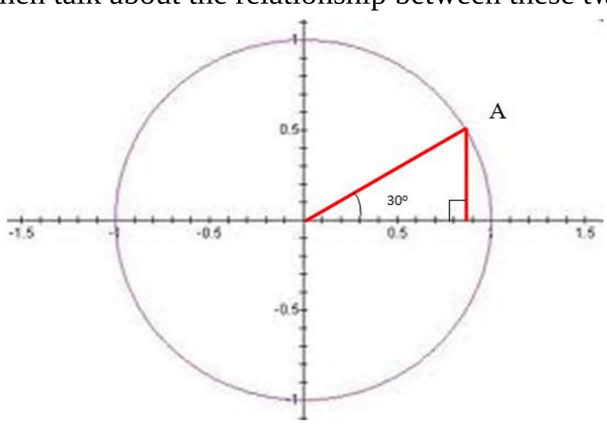
Students will be able to:

- **Determine** the sine and cosine of an angle determined on the unit circle.
- **Determine** the period of a given trigonometric function.

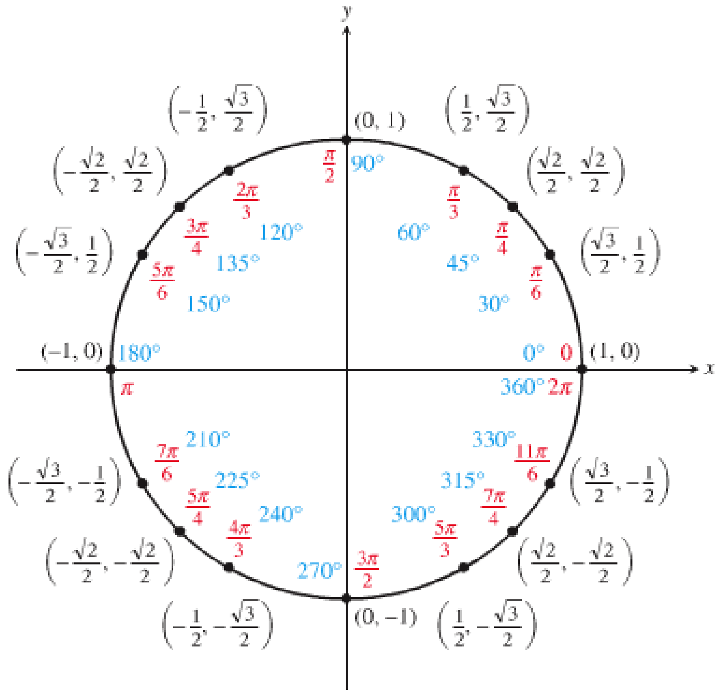
Vocabulary:
unit circle, circular function, periodic function, period

1. **INSTRUCT** (10 minutes)
   Calculations: Finding Other Trig Function Values from One
   Given the value of one trigonometric function and an additional piece of information (to specify the quadrant), students should be able to find the values of the other five trig functions. Be sure students know how to determine which quadrant is being discussed.

2. **CONNECT** (20 minutes)
   Making connection from students' prior geometry knowledge (a right triangle using trigonometric ratios) to the unit circle approach can be achieved by following steps. I do this by first drawing a right triangle with 30°-60°-90° on the board, and my students and I discuss the related ratio, 1: 2: $\sqrt{3}$. I label the ratio on the sides of the triangle and then draw a circle using the vertex where the hypotenuse and the base (longer leg) meet as the center and the length of the hypotenuse as the radius. I construct this circle with a string and chalk on the board. Of course, this circle has the radius of 2. I then fold the string in half, making the radius of 1, and construct a concentric circle, thus born is a *unit circle*. Lastly, I create right triangle by drawing a perpendicular line from the intersection of the unit circle and the hypotenuse, down to the base (*x*-axis).We then talk about the relationship between these two triangles.



3. INSTRUCT (20 minutes)
Calculations: Using the Unit Circle
Explain how to find the basic trig functions from the unit circle. With the unit circle and its values displayed, spend enough time asking students to identify various trig function values. It is very important for them to be able to use the unit circle easily.



Conceptual Understanding: Understanding the **Period** of trigonometric functions using the sign chart.

	sinθ	***cosθ***	***tanθ***
$0 < \theta < \frac{\pi}{2}$; QI	+	+	+
$\frac{\pi}{2} < \theta < \pi$; QII	+	−	−
$\pi < \theta < \frac{3\pi}{2}$; QIII	−	−	+
$\frac{3\pi}{2} < \theta < 2\pi$; QIV	−	+	−
$2\pi < \theta < \frac{5\pi}{2}$; QI	+	+	+
$\frac{5\pi}{2} < \theta < 3\pi$; QII	+	−	−
$3\pi < \theta < \frac{7\pi}{2}$; QIII	−	−	+
$\frac{7\pi}{2} < \theta < 4\pi$; QIV	−	+	−
Observation / Conclusion	**Period of 2π**	**Period of 2π**	**Period of π**

Unit 3 Section 2: Trigonometric Identities and Solving Trigonometric Equations (5.1)

Major Concepts: Use the fundamental identities to simplify trigonometric expressions and solve trigonometric equations.

Standards

Functions-Trigonometric Functions (F-TF)

Extend the domain of trigonometric functions using the unit circle

F-TF.4 (+) Use the unit circle to explain symmetry (odd and even) and periodicity of trigonometric functions.

Prove and apply trigonometric identities

F-TF.8 Prove the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$ and use it to find $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ given $\sin(\theta)$, $\cos(\theta)$, or tangent(θ) and the quadrant of the angle.

Unit 3 Section 2-A: Fundamental Identities

Students will be able to:

- **Evaluate** other trigonometric functions using a given trigonometric ratio and basic trigonometric identities.
- **Distinguish** which trigonometric functions are odd or even.
- **Simplify** trigonometric expressions using basic trigonometric identities.

Vocabulary:
identity

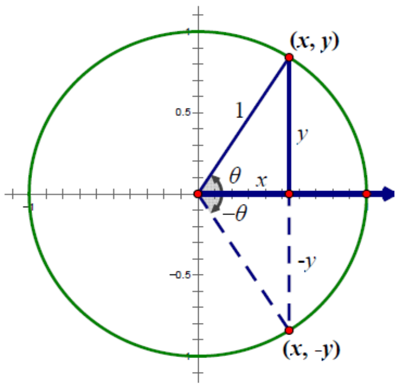
1. **CONNECT** (5 minutes)
   Review: Basic Trigonometric Identities: Reciprocal Identities and Quotient Identities

2. **INSTRUCT** (10 minutes)
   Introduce: Cofunction identities using a right triangle and **Even-Odd Identities** using a unit circle.

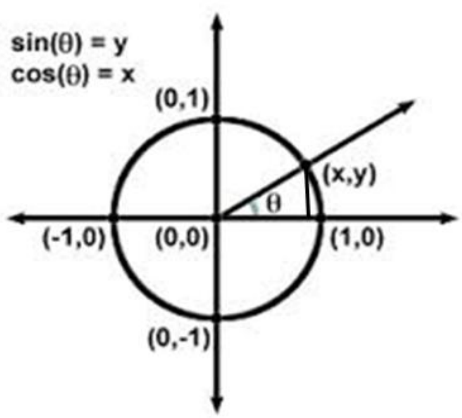
Sine and **cosine** are cofunctions.	**Tangent** and **cotangent** are cofunctions.	**Secant** and **cosecant** are cofunctions.
$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$	$\tan\theta = \cot\left(\frac{\pi}{2} - \theta\right)$	$\sec\theta = \csc\left(\frac{\pi}{2} - \theta\right)$
$\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$	$\cot\theta = \tan\left(\frac{\pi}{2} - \theta\right)$	$\csc\theta = \sec\left(\frac{\pi}{2} - \theta\right)$

Conceptual Understanding:

Even/Odd – Using the unit circle to explain symmetry (even and odd). Remember, in here, θ is input and x and y are the outputs. More specifically, $x = \cos(\theta)$ *and* $y = \sin(\theta)$. Since $\cos(\theta) = \cos(-\theta) = x$, thus cosine is even. For the same argument, $\sin(\theta) = -\sin(-\theta) = y$, thus sine is odd.



Pythagorean Identities: In a unit circle, it is observed that a point on the unit circle can be resented by the coordinates (cos θ, sin θ).

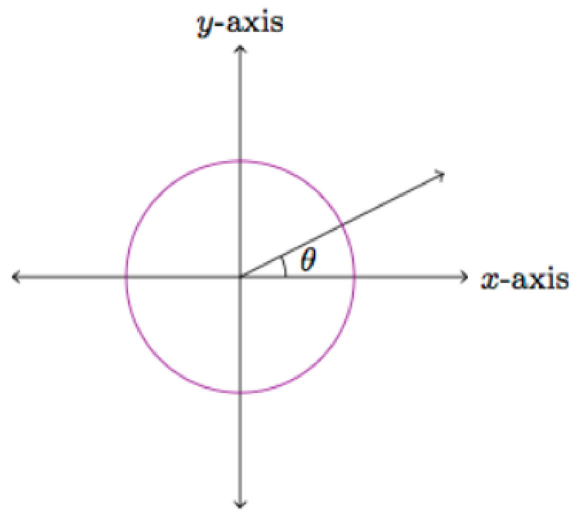


By substituting x and y with cos θ and sin θ, we now have $cos^2\theta + sin^2\theta = 1$. Also known as a Pythagorean Identity.

Have students to derive two additional Pythagorean Identities, $sec^2\theta$ and $csc^2\theta$ using the first Pythagorean Identity.

Comments and Solutions can be found at:
http://www.illustrativemathematics.org/illustrations/1704

Below is a picture of an angle θ in the x-y plane with the unit circle sketched in purple:



a. Explain why $\sin(-\theta) = -\sin\theta$ and $\cos(-\theta) = \cos\theta$. Do these equations hold for any angle θ? Explain.

b. Explain why $\sin(2\pi + \theta) = \sin\theta$ and $\cos(2\pi + \theta) = \cos\theta$. Do these equations hold for any angle θ? Explain.

Unit 3 Section 2-B: Solving Trigonometric Equations

Students will be able to:
- **Solve** a trigonometric equation by factoring.
- **Find** all solutions of a trigonometric equation.

1. **INSTRUCT** (45 minutes)
   Calculations: Identities
   Give several more examples of simplifying trigonometric expressions using the various trig identities. It may help students to consider their goal as they begin to use various expressions.

   Calculations: Solving Trigonometric Equations
   Using familiar techniques such as factoring, show students how to solve trigonometric equations. Students will tend to have the most difficulty finding when trigonometric functions have a specified value; reviewing the unit circle may be helpful here.

Writing Assignment:

Conceptual Understanding: Identities
Explain the difference between a general (conditional) equation and an identity.

Unit 3 Section 3: Graphs of Sine and Cosine (4.4)

Major Concepts: Generate the graphs of the sine and cosine functions and explore various transformations of these graphs.

Standards

Functions-Interpreting Functions (F-IF)
Analyzing functions using different representations
F-IF.7.e Graph trigonometric functions, showing period, midline, and amplitude.
F-IF.9 Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions).
Functions-Trigonometric Functions (F-TF)
Model periodic phenomena with trigonometric functions
F-TF.5 Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.

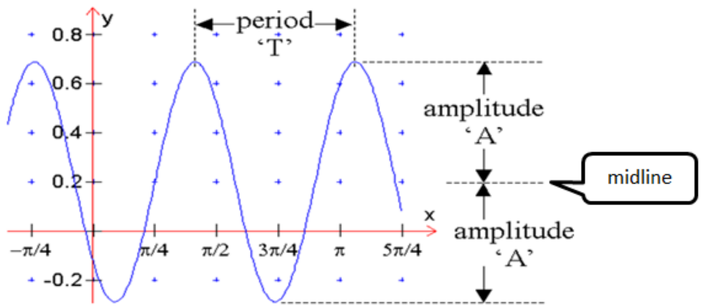
Unit 3 Section 3-A: Analyzing the Graphs of Trigonometric Functions

Students will be able to:
- **Analyze** the graphs of the sine and cosine functions.
- **Graph** sinusoids using transformations of sine and cosine functions.

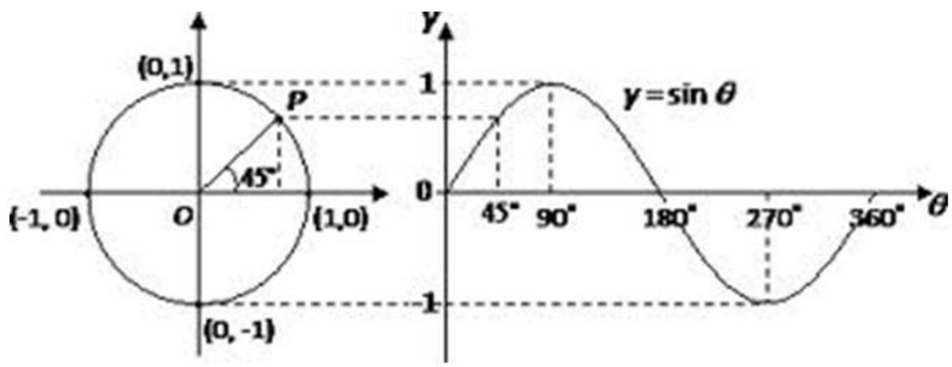
Vocabulary:
sinusoid, amplitude, frequency, phase shift, midline

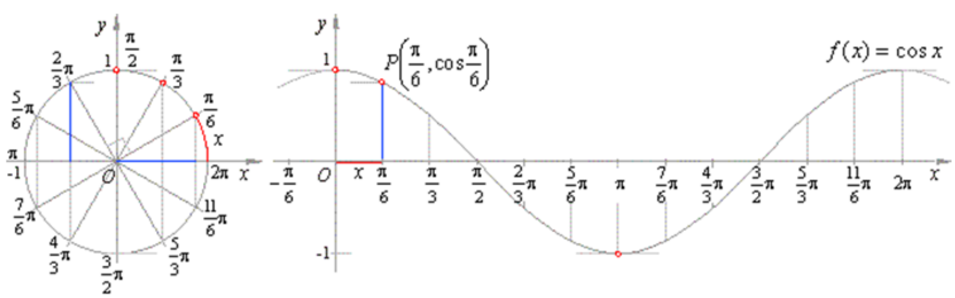
1. **CONNECT** (10 minutes)
   Review: Prerequisites for Sinusoids
   Review converting degrees to radians and finding the sign of various trigonometric functions in the four quadrants in preparation for finding sinusoids.

2. INSTRUCT (15 minutes)
Analysis: Sine and Cosine Functions
Graph sine and cosine functions, using the key terms: the domain, range, intercepts, zeroes, period, amplitude, and midline.



3. INDIVIDUAL EXPLORATION (10 minutes)
Exploration: Visualizing Wave Functions
Have students investigate Exploration 1 on page 350 to better understand the relationship between the sine function on a coordinate axis and the unit circle.

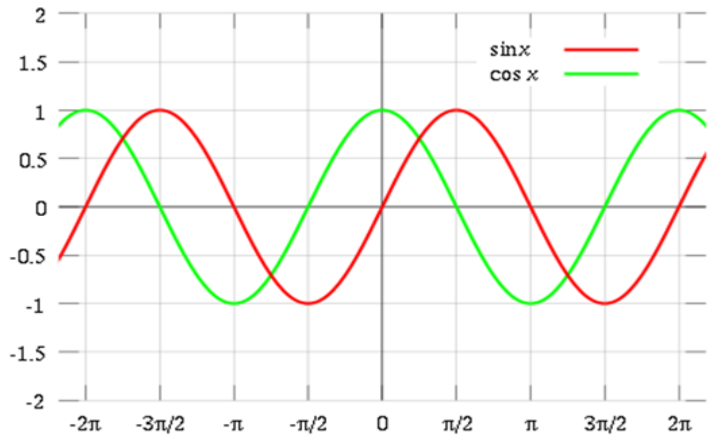




4. INSTRUCT (15 minutes)

Graphs: Sinusoids

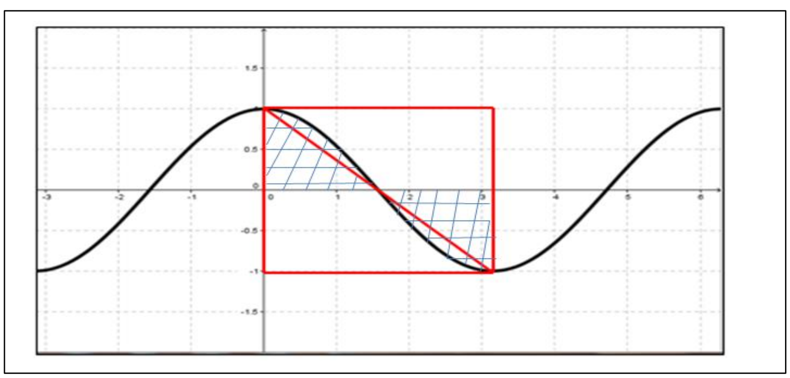
Have students note the relationship visually between the graphs of sine and cosine. (One is a transformation of the other.) Can you describe the graph of sin *x* in terms of cos *x* or vice versa?



Writing Assignment:

Graph $f(x) = cosx$, for $[0, \pi]$. Approximate the area bounded by the curve and the $x - axis$. Support your answer with a solid reasoning.

For teachers: This activity focuses on the idea of -1 ≤ $cosx$ ≤ 1. Also, with the restricted domain, $f(x) = cosx$ is now one-to-one function. Thus, it sets the anticipatory, yet very natural transitional setting for the upcoming inverse trigonometric functions section.

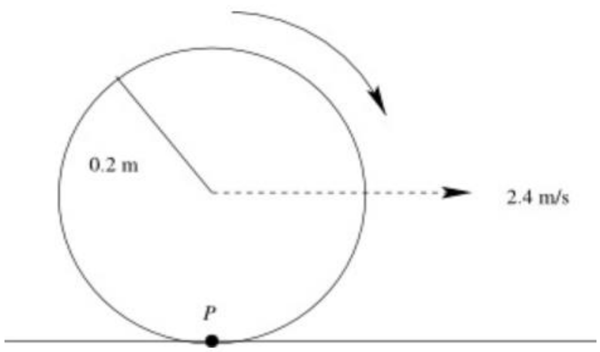


Looking Ahead to Calculus

Area = $2\int_0^{\frac{\pi}{2}} cosxdx = 2(sinx + c \,|_0^{\frac{\pi}{2}} = 2(sin\frac{\pi}{2} - \sin(0)) = 2(1-0) = 2$

Comments and Solutions can be found at:
https://www.illustrativemathematics.org/illustrations/595

A wheel of radius 0.2 meters begins to move along a flat surface so that the center of the wheel moves forward at a constant speed of 2.4 meters per second. At the moment the wheel begins to turn, a marked point P on the wheel is touching the flat surface.



a. Write an algebraic expression for the function y that gives the height (in meters) of the point P, measured from the flat surface, as a function of t, the number of seconds after the wheel begins moving.

b. Sketch a graph of the function y for $t > 0$. What do you notice about the graph? Explain your observations in terms of the real-world context given in this problem.

c. We define the horizontal position of the point P to be the number of meters the point has traveled forward from its starting position, disregarding any vertical movement the point has made. Write an algebraic expression for the function x that gives the horizontal position (in meters) of the point P as a function of t, the number of seconds after the wheel begins moving.

d. Sketch a graph of the function x for $t > 0$. Is there a time when the point P is moving backwards? Use your graph to justify your answer.

Unit 3 Section 3-B: Sinusoids as Transformations

Students will be able to:

- **Determine** the frequency, amplitude, and phase shift of a sinusoid.
- **Model** sinusoidal behavior using trigonometric functions.

1. EXPLORATION (50 minutes)

Sinusoidal Graphs

In this task students will explore the effect that changing the parameters in a sinusoidal function has on the graph of the function. Help them to find the period, frequency, and phase shift from given information.

Illustrative Mathematics **Comments and Solutions can be found at:**
https://www.illustrativemathematics.org/illustrations/1647

A general sinusoidal function is of the form
$y = A\,sin\big(B(x-h)\big)+k$ or $y = A\,cos\big(B(x-h)\big)+k$

a. Use graphing calculators to change the values of A, k, h, and B to create the functions in the table. Then describe the effect that changing each parameter has on the shape of the graph. Add more rows to the table, if necessary.

Function	Effect on $y = \sin x$
$y = 2\sin x$	
$y = -2\sin x$	
$y = \sin x + 2$	
$y = \sin x - 2$	
$y = \sin(x+2)$	
$y = \sin(x-2)$	
$y = \sin(2x)$	
$y = \sin(\frac{1}{2}x)$	
$y =$	
$y =$	

b. Describe how changing A, k, and h changes the graph of the function.

c. There seems to be a relationship between B and the period of the function but it is harder to describe than the other parameters. Experiment with different values of B and fill in the corresponding period in the table below. In the last row of the table, use the data you have collected to infer a general relationship between B and the period.

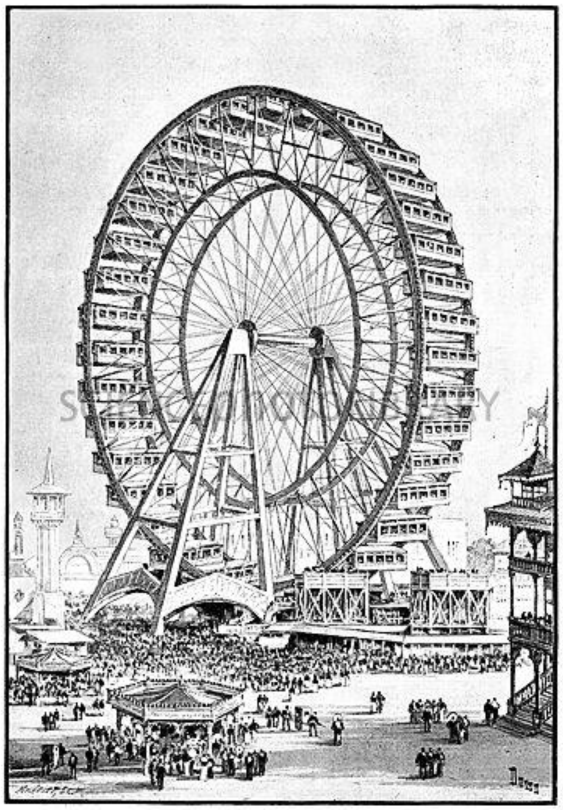
B	$y = \sin(Bx)$	Period
1	$y = \sin(x)$	
2	$y = \sin(2x)$	
4	$y = \sin(4x)$	
$1/2$	$y = \sin(\frac{1}{2}\,x)$	
-2	$y = \sin(-2x)$	
-4	$y = \sin(-4x)$	
$-1/2$	$y = \sin(-\,\frac{1}{2}\,x)$	
B	$y = \sin(Bx)$	

Note: This task serves as an introduction to the family of sinusoidal functions. It uses a graphing calculator to allow students explore the effect of changing the parameters in $y=A\sin(B(x-h))+k$ on the graph of the function. Sinusoidal functions are the perfect type of function to illustrate how new functions can be built from already known functions by shifting and scaling. Even though the task statement only considers transformation of $y=\sin(x)$ it would be easily modify the problem statement to explore transformations of $y=\cos(x)$.

The task can be used in a variety of ways. Students can work in class, possibly in groups, to experiment and then discuss their observations. Alternatively, students can do the exploration as homework and bring their observations to class for a whole class discussion and to pull together all the pieces.

Practice Standards: In this task students engage in SMP 7 - Look For and Make Use of Structure and SMP 8 - Look For and Express Regularity in Repeated Reasoning. Since students are asked to verbalize their observations, the instructor can also use the opportunity to work on SMP 6 - Attend to Precision, in this case precision of language and notation is important.

Applications: Ferris Wheel



The Ferris Wheel, a classic amusement park ride, was invented by George Ferris. Mr. Ferris was an American engineer who introduced his wheel at the 1893 World's Fair in Chicago.

Suppose that you are 4 feet off the ground in the bottom car of a Ferris Wheel and ready to ride.
If the radius of the wheel is 25 ft and it makes 2 revolutions per minute.

1. Sketch a graph that shows your height h (in ft) above the ground at time t (in sec) during the first 45 seconds of you ride.
2. Why is curve period? Explain in terms of the problem.
3. What is the period of your curve? Explain this number in terms of the problem.
4. Give a possible equation for the curve.
5. At what speed are you traveling on the Ferris Wheel (ft/sec)?
6. Suppose that the radius of the Wheel were decreased and that the Wheel still makes 2 revolutions per minute, would the (a) period change? Explain (If yes, indicate if it would increase or decrease). (b) amplitude change? Explain (If yes, indicate if it would increase or decrease). (c) speed change? Explain (If yes, indicate if it would increase or decrease).

Unit 3 Section 4: Graphs of Other Trigonometric Functions (4.5)

Major Concepts: Generate the graphs of the tangent, cotangent, secant, and cosecant functions and explore various transformations of these graphs.

Standards

Functions-Interpreting Functions (F-IF)
Interpreting functions that arise in applications in terms of the context
F-IF.4 For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.
Analyzing functions using different representations
F-IF.7.e Graph trigonometric functions, showing period, midline, and amplitude.
F-IF.9 Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions).

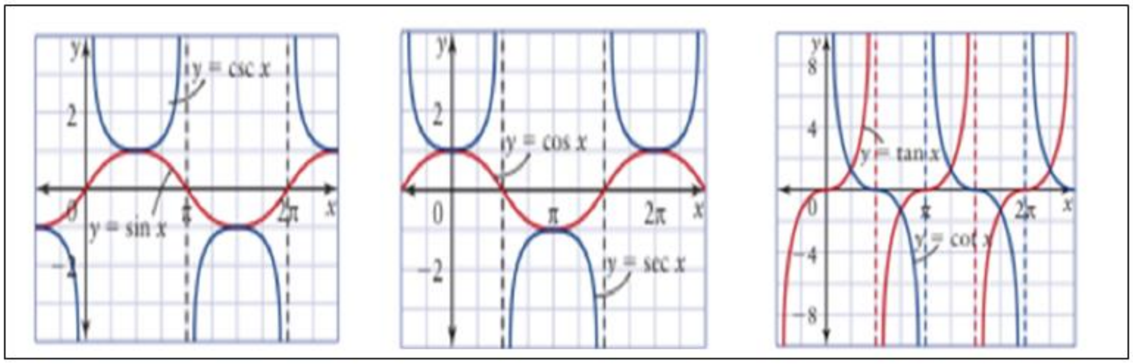
Students will be able to:

- **Analyze** the graphs of the tangent, cotangent, secant, and cosecant functions.
- **Graph** transformations of tangent, cotangent, secant, and cosecant functions.
- **Solve** trigonometric equations algebraically.

Vocabulary:
tangent, cotangent, secant, cosecant

1. **CONNECT** (10 minutes)
   Review: Prerequisites for the Other Trigonometric Functions
   Review finding vertical asymptotes and symmetry in preparation for studying tangent, cotangent, secant, and cosecant.

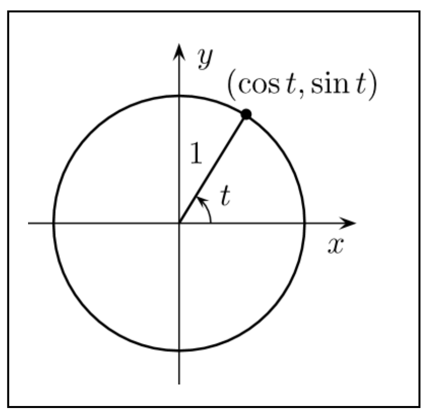
2. **INSTRUCT** (40 minutes)
   Analysis: The Other Trigonometric Functions
   Analyze each of the functions tangent, cotangent, secant, and cosecant, considering in particular the placement of asymptotes and zeros, and calculating symmetry.



Function	Amplitude	Period	x-intercepts	y-intercept	Asymptotes	Minimum	Maximum
$\boldsymbol{y = sinx}$	1	2π	kπ, kϵℤ	0	NA	-1	1
$\boldsymbol{y = cosx}$	1	2π	$\frac{(2k+1)\pi}{2}$, kϵℤ	1	NA	-1	1
$\boldsymbol{y = tanx}$	NA	π	kπ, kϵℤ	0	$\frac{(2k+1)\pi}{2}$, kϵℤ	NA	NA
$\boldsymbol{y = cscx}$	NA	2π	$\frac{(2k+1)\pi}{2}$, kϵℤ	NA	kπ, kϵℤ	NA	NA
$\boldsymbol{y = secx}$	NA	2π	kπ, kϵℤ	1	$\frac{(2k+1)\pi}{2}$, kϵℤ	NA	NA
$\boldsymbol{y = cotx}$	NA	π	$\frac{(2k+1)\pi}{2}$, kϵℤ	NA	kπ, kϵℤ	NA	NA

Writing Assignment:

Explain why it is correct to say $y = \tan(t)$ is the slope of the terminal side of angle *t* in standard position. *P* is on the unit circle.



Unit 3 Section 5: Graphs of Composite Trigonometric Functions (4.6)

Major Concepts: Graph sums, differences, and other combinations of trigonometric and algebraic functions.

Standards

Functions-Building Functions (F-BF)
Build functions that models a relationship between two quantities

F-BF.1.b	Combine standard function types using arithmetic operations
F-BF.1.c (+)	Compose functions

Students will be able to:

- **Determine** the period of a composite trigonometric function.
- **Identify and express** the sum of two sinusoids as a sinusoid.
- **Identify** the presence of damped oscillation in a function.

Vocabulary:
damping, damped oscillation, damping factor

1. **CONNECT** (10 minutes)
   Review: Prerequisites for Combining Trigonometric Functions
   Review transformations and compositions in preparation for combining trigonometric functions algebraically.

2. **INSTRUCT** (15 minutes)
   Conceptual Understanding: Combinations of Trigonometric and Algebraic Functions
   Give several examples of composing or otherwise combining trigonometric and algebraic functions together. Help students determine graphically whether the resulting functions are periodic or not. They should then confirm this algebraically.

3. **INDIVIDUAL EXPLORATION** (10 minutes)
   Exploration: Combinations of Sinusoids
   Have students investigate Exploration 1 on page 371 to understand when the sum or difference of two sinusoids is again a sinusoid.

4. **INSTRUCT** (10 minutes)
   Applications: Modeling damped oscillations
   Explain how multiplying a sinusoid by another function effectually changes the amplitude of the sinusoid. If the amplitude shrinks, the oscillation is said to be damped.

Unit 3 Section 6: Inverse Trigonometric Functions (4.7)

Major Concepts: Relate the concept of inverse function to trigonometric functions.

Standards

Functions-Building Functions (F-BF)
Build functions that models a relationship between two quantities
F-BF.1.c (+) Compose functions
Functions-Trigonometric Functions (F-TF)
Model periodic phenomena with trigonometric functions
F-TF.6 (+) Understand that restricting a trigonometric function to a domain on which it is always increasing or always decreasing allows its inverse to be constructed.
F-TF.7 (+) Use inverse functions to solve trigonometric equations that arise in modeling contexts; evaluate the solutions using technology, and interpret them in terms of the context.

Students will be able to:
- **Graph** the inverse sine, inverse cosine, and inverse tangent functions.
- **Evaluate** inverse trigonometric functions without a calculator.
- **Describe** the end-behavior of inverse trigonometric functions.

Vocabulary:
inverse sine function, arcsine, inverse cosine, arccosine, inverse tangent, arctangent

1. **CONNECT** (10 minutes)
   Review: Prerequisites for Inverse Trigonometric Functions
   Review signs of trig functions in each quadrant and evaluating trig functions exactly in preparation for finding inverse trig functions.

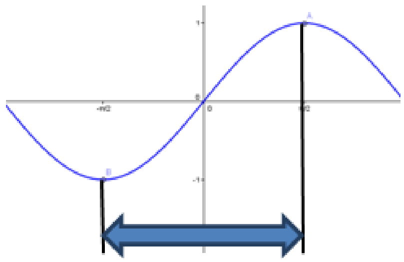
2. **INSTRUCT** (25 minutes)
   Conceptual Understanding: The Arcsine Function
   Introduce the inverse function for sine by first graphing the sine function, restricting domain on which it is increasing, $[\frac{-\pi}{2}, \frac{\pi}{2}]$, and then reflecting it about the line y = *x*.
   Be sure students do not confuse the notation between inverse sine and the reciprocal of sine.

$f(x) = \sin(x)$

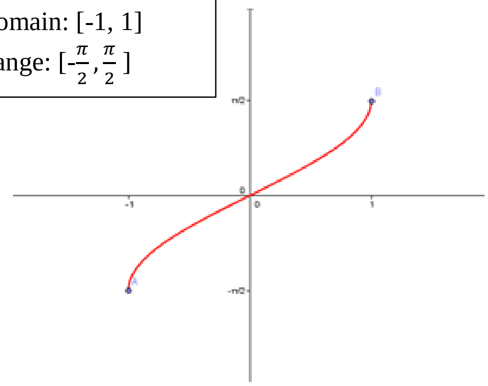
Domain: $(-\infty, \infty)$
Restricted Domain: $[-\frac{\pi}{2}, \frac{\pi}{2}]$
Range: $[-1, 1]$

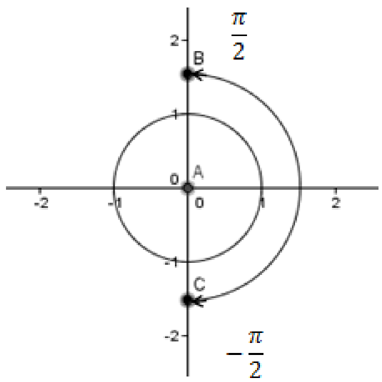
$g(x) = sin^{-1}(x)$

Domain: $[-1, 1]$
Range: $[-\frac{\pi}{2}, \frac{\pi}{2}]$



Restricted Domain
$[-\frac{\pi}{2}, \frac{\pi}{2}]$





The values of $y = \sin^{-1} x$ will always be found on the right hand side of the unit circle, between $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

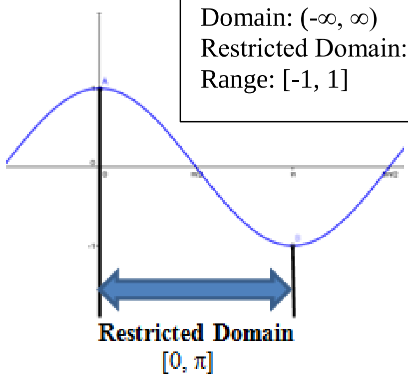
Calculations: Evaluating the Arcsine Function
Demonstrate how to evaluate the arcsine function for special values. Students need to understand how to reinterpret an inverse sine function as a question about the sine function.

Conceptual Understanding: The Arccosine Function

Repeat the previous process with the cosine function to obtain and evaluate the function arccosine.

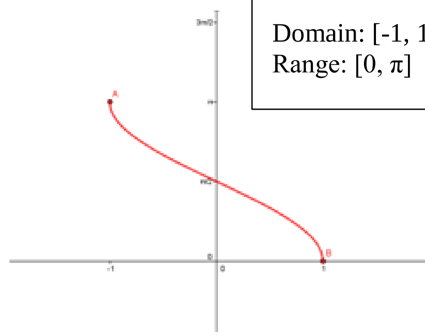
$f(x) = \cos(x)$

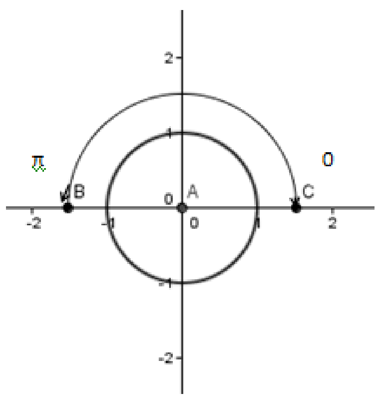
Domain: $(-\infty, \infty)$
Restricted Domain: $[0, \pi]$
Range: $[-1, 1]$



$g(x) = cos^{-1}(x)$

Domain: $[-1, 1]$
Range: $[0, \pi]$



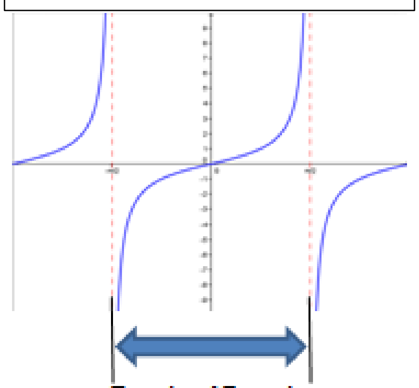


The values of $y = \cos^{-1} x$ will always be found on the top half of the unit circle, between $[0, \pi]$.

Conceptual Understanding: The Arctangent Function
Introduce the inverse function for tangent by first graphing the tangent function and then reflecting it about the line *y* = *x*. Note that this is not a function, but we can cut it so that the result is. Point out the end behavior of the arctangent function.

$f(x) = tan(x)$

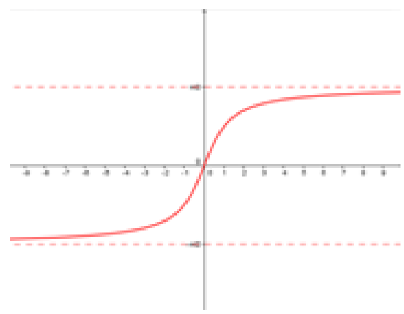
Domain: $x \in \mathbb{R}, x \neq \frac{\pi}{2} + k, k \in \mathbb{Z}$
Restricted Domain: $(-\frac{\pi}{2}, \frac{\pi}{2})$
Range: $(-\infty, \infty)$

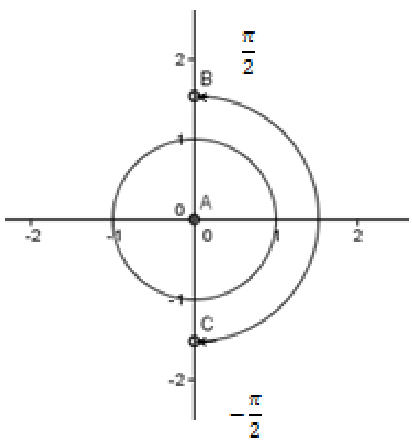


Restricted Domain
$(-\frac{\pi}{2}, \frac{\pi}{2})$

$g(x) = tan^{-1}(x)$

Domain: $(-\infty, \infty)$
Range: $(-\frac{\pi}{2}, \frac{\pi}{2})$





The values of $y = tan^{-1}(x)$ will always be found on the right hand side of the unit circle, between $(-\frac{\pi}{2}, \frac{\pi}{2})$

Writing Assignment:

Use an appropriately labeled triangle to explain why $sin^{-1}x + cos^{-1}x = \frac{\pi}{2}$.
For what values of *x* is the left-hand side of this equation defined?

Unit 3 Section 7: Sum and difference Identities (5.3)

Major Concepts: Prove and apply the identities for the sine, cosine, and tangent of a sum or difference.

Standards

Functions-Trigonometric Functions (F-TF)
Model periodic phenomena with trigonometric functions
F-TF.9 (+) Prove the addition and subtraction formulas for sine, cosine, and tangent and use them to solve problems.

Students will be able to:

- **State** the sum and difference formulas for sine and cosine.
- **Evaluate** a trigonometric expression using the sum and difference formulas.
- **Prove** a reduction formula using the sum and difference formulas.

Vocabulary:
law of additivity, reduction formula

1. **CONNECT** (10 minutes)
   Review: Prerequisites for Sum and Difference Identities
   Review angles in degrees and radians and function identities in preparation for deriving and using the trigonometric sum and difference identities.

2. **INDIVIDUAL EXPLORATION** (10 minutes)
   Exploration: Incorrect Sum and Difference Formulas
   Have students investigate Exploration 1 on page 421 to convince them that the sum and difference identities they are likely to try are incorrect.

3. **INSTRUCT** (25 minutes)
   Calculations: Sum and Difference Identities
   Demonstrate the origin of the sum and difference identities for sine and cosine, and their uses. Students do not need to memorize the identity for tangent, as long as they remember its relationship to sine and cosine.

Looking Ahead to Calculus

The following identities are used in calculus to prove important differentiation formulas.

$\frac{\sin(x+h)-sinx}{h} = sinx\left(\frac{cosh-1}{h}\right) + cosx\frac{sinh}{h}$ and
$\frac{\cos(x+h)-c0sx}{h} = cosx\left(\frac{cosh-1}{h}\right) - sinx\frac{sinh}{h}$

Illustrative Mathematics **Comments and Solutions can be found at:**
https://www.illustrativemathematics.org/illustrations/1116

In this task, you will show how all of the sum and difference angle formulas can be derived from a single formula when combined with relations you have already learned.

For the following task, assume that the sum angle formula for sine is true. Namely,

$$\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi.$$

a. To derive the difference angle formula for sine, write $\sin(\theta-\phi)$ as $\sin(\theta+(-\phi))$ and apply the sum angle formula for sine to the angles θ and $-\phi$. Use the fact that sine is an odd function while cosine is even function to simplify your answer. Conclude that

$$\sin(\theta-\phi) = \sin(\theta)\cos(\phi) - \cos(\theta)\sin(\phi).$$

b. To derive the sum angle formula for cosine, use what what you learned in (a) to show that

$$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi.$$

You may want to start with an exploration of $\sin\left(\frac{\pi}{2}-(\theta+\phi)\right)$.

c. Derive the difference angle formula for cosine,

$$\cos(\theta-\phi) = \cos\theta\cos\phi + \sin\theta\sin\phi.$$

d. Derive the sum angle formula for tangent,

$$\tan(\theta+\phi) = \frac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi}.$$

e. Derive the difference angle formula for tangent,

$$\tan(\theta-\phi) = \frac{\tan\theta-\tan\phi}{1+\tan\theta\tan\phi}.$$

Unit 3 Section 8: Multiple-Angle Identities (5.4)

Major Concepts: Apply the double angle identities, power-reducing identities, and half-angle identities.

Students will be able to:

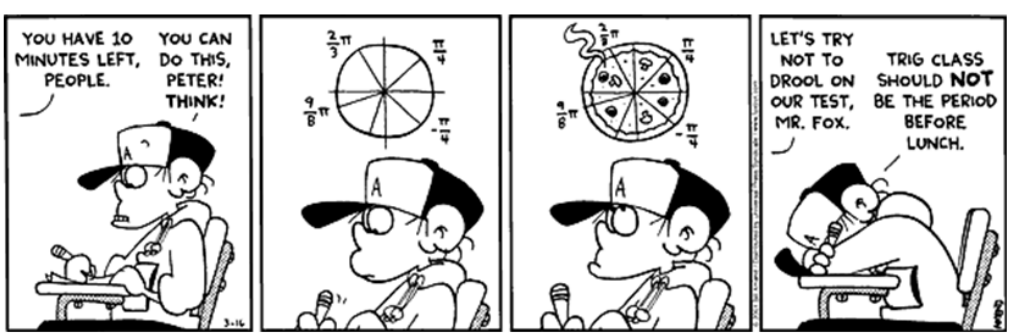
- **Prove** a double-angle formula using a sum formula.
- **Prove** a trigonometric identity using a double-angle or half-angle formula.
- **Solve** a trigonometric equation using a double-angle or half-angle formula.

Vocabulary:
double-angle identity, half-angle identity

1. **CONNECT** (10 minutes)
   Review: Prerequisites for Multiple-Angle Identities
   Review solving trigonometric equations in preparation for using multiple-angle identities.

2. **INSTRUCT** (10 minutes)
   Calculations: Double-Angle Formulas
   Demonstrate how the double-angle formulas follow from the sum and difference identities presented earlier. Work several examples.

3. **INDIVIDUAL EXPLORATION** (10 minutes)
   Exploration: Half-Angle Formulas
   Have students investigate Exploration 1 on page 429 to understand how half-angle formulas are derived and which root (positive or negative) to use in which case. The cartoon below, students can find the exact values of sine, cosine, and tangent of $\frac{9\pi}{8}$ using $(\frac{1}{2})(\frac{9\pi}{4})$.



4. INSTRUCT (15 minutes)

Calculations: Solving Trig Equations Using Multiple-Angle Identities

Demonstrate how to use multiple-angle identities to simplify equations so that one can factor them (and thus solve).

Looking Ahead to Calculus

One use for *cos 2u* is to derive the power-reducing identities. Some simple-looking functions like $y = sin^2u$ would be quite difficult to deal in certain calculus context were it not for the existence of these identities.

$$\cos 2u = cos^2 u - sin^2u = 2cos^2u - 1 = 1 - 2sin^2u$$

Power-Reducing Identities

$$\begin{cases} sin^2u = \frac{1-2cos2u}{2} \\ cos^2u = \frac{1+2cos2u}{2} \end{cases}$$

Unit 4: Applications of Trigonometry

Overview: In this unit, the study of trigonometry is further extended to the Law of Sines and the Law of Cosines. Students develop in their study of Cartesian plane system to polar system. Vectors are studied to understand in describing and representing the movement of objects.

Unit 4 Section 1: The Law of Sines (5.5)

Major Concepts: Prove the Law of Sines and apply the Law of Sines to solve a variety of problems.

Standards

Geometry-Similarity, Right Triangles, and Trigonometry (G-SRT)

Apply trigonometry to general triangles

G-SRT.9(+) Derive the formula A = ½ ab sin(C) for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side.

G-SRT.10(+) Prove the Laws of Sines and Cosines and use them to solve problems.

G-SRT.11(+) Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles.

Students will be able to:

- **Solve** a triangle given two angles and one side using the Law of Sines.
- **Solve** a triangle given two sides and one angle using the Law of Sines.
- **Find** areas of oblique triangles.

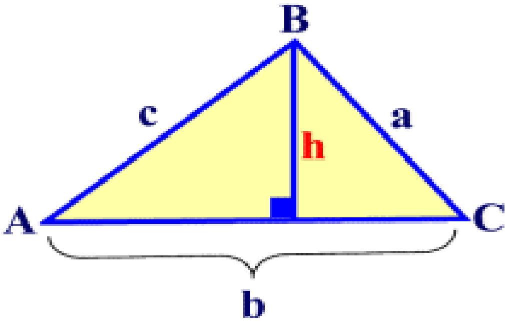
Vocabulary:

Law of Sines

1. **CONNECT** (10 minutes)
   Review: Prerequisites for the Law of Sines
   Review solving equations involving the sine function in preparation for applying the Law of Sines.

2. **INSTRUCT** (40 minutes)
   Calculations: Solving a Triangle Using the Law of Sines
   Derive the Law of Sines using right triangle trigonometry (p. 434).

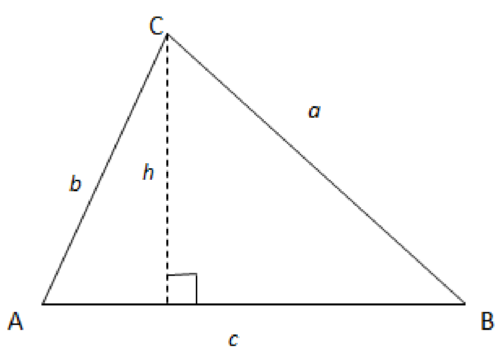
Triangle *ABC* below does not contain a right angle. A perpendicular is dropped from vertex *B*. It can now be observed that:



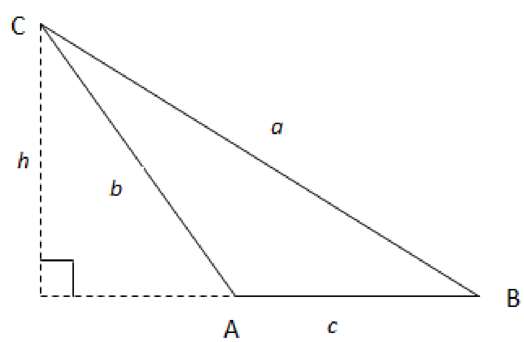
Sin A = $\frac{h}{c} \Rightarrow$ h = c Sin A and Sin C = $\frac{h}{a} \Rightarrow$ h = a Sin C, then h = c Sin A = a Sin C.
Since c Sin A = a Sin C, we have $\frac{Sin\ A}{a} = \frac{Sin\ C}{c}$.
Similarly, drop a perpendicular from vertex A. Now, we can observe that:
Sin B = $\frac{h}{c} \Rightarrow$ h = c Sin B and Sin C = $\frac{h}{b} \Rightarrow$ h = b Sin C, then h = c Sin B= b Sin C.
Since c Sin B = b Sin C, we have $\frac{Sin\ B}{b} = \frac{Sin\ C}{c}$.
Therefore, we have:
Law of Sines: $\frac{Sin\ A}{a} = \frac{Sin\ B}{b} = \frac{Sin\ C}{c}$

Practice using the Law of Sines for solving triangles in which AAS, ASA, or SSA are given. There is an ambiguous case when given SSA; discuss this ambiguity with students (p. 435). How to handle the ambiguous case is explained on p. 436, example 3.

Area Formula: The procedure used to prove the Law of Sines leads to a simple formula for the area of an oblique triangle. Note that each triangle has a height of $h = b \sin A$. To see this when A is obtuse, substitute the reference angle 180^{o} – A for A. Now the height of the triangle is given by $h = b \sin(180^{\circ} - A)$. Using the difference formula for sine, the height is given by $h = b(sin180^{\circ}cosA - cos180^{\circ}sinA) = b(0 \times cosA - (-1) \times sinA) = bsinA$. Consequently, the area of each triangle is given by Area = (1/2)(base)(height) = (1/2)(c)(b sinA) = (1/2) bc sinA. By similar arguments, we can derive the formulas: Area = (1/2) ab sin(C) = (1/2) ac sin(B).



A is acute.



A is obtuse

Unit 4 Section 2: The Law of Cosines (5.6)

Major Concepts: Prove the Law of Cosines and apply the Law of Cosines to solve a variety of problems.

Standards
Geometry-Similarity, Right Triangles, and Trigonometry (G-SRT)
Apply trigonometry to general triangles
G-SRT.10(+) Prove the Laws of Sines and Cosines and use them to solve problems.
G-SRT.11(+) Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles.

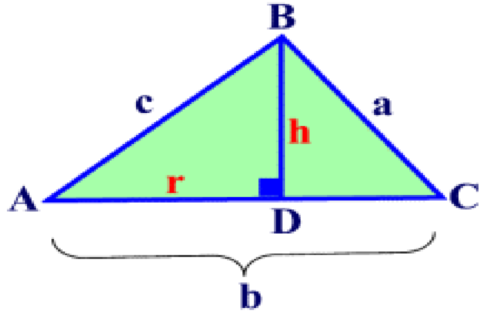
Students will be able to:
- **Solve** a triangle given three sides using the Law of Cosines.
- **Solve** a triangle given two sides and one angle using the Law of Cosines.
- **Calculate** the area of a triangle given SSS using Heron's Formula.

Vocabulary:
Law of Cosines

1. **CONNECT** (10 minutes)
   Review: Prerequisites for the Law of Cosines
   Review solving equations involving the cosine function in preparation for applying the Law of Cosines.

2. **DEMONSTRATE** (40 minutes)
   Derive: the Law of Cosines (SAS or SSS)

   Triangle *ABC* is not a right triangle.
   A perpendicular is dropped from vertex *B*. Thus, we have:
   Sin A = $\frac{h}{c} \Rightarrow$ h = c Sin A and Cos A = $\frac{r}{c} \Rightarrow$ r = c Cos A



Using the Pythagorean Theorem in triangle BCD, we have: $a^2 = h^2 + (b - r)^2$.
Substituting for h and r, we have:

$a^2 = (c \text{ Sin } A)^2 + (b - c \text{ Cos } A)^2$

$a^2 = c^2 \text{ Sin}^2 A + b^2 - 2bc \text{ Cos } A + c^2 \text{ Cos}^2 A$

$a^2 = c^2 (\text{Sin}^2 A + \text{Cos}^2 A) + b^2 - 2bc \text{ Cos } A$

$a^2 = c^2 (1) + b^2 - 2bc \text{ Cos } A$

$a^2 = b^2 + c^2 - 2bc \text{ Cos } A$

Similarly,

$b^2 = a^2 + c^2 - 2ac \text{ Cos } B$ and $c^2 = a^2 + b^2 - 2ab \text{ Cos } C$

Therefore, we have:

Law of Cosines: $a^2 = b^2 + c^2 - 2bc \text{ Cos } A$

$b^2 = a^2 + c^2 - 2ac \text{ Cos } B$

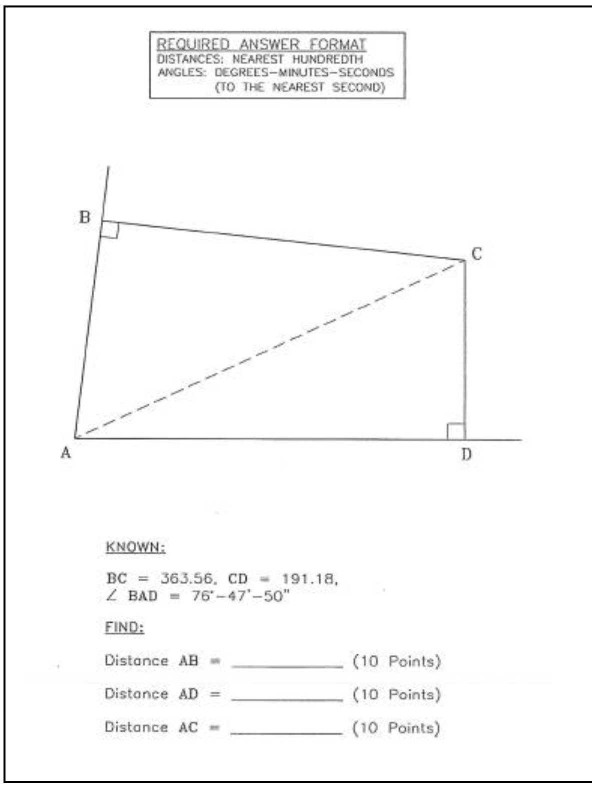
$c^2 = a^2 + b^2 - 2ab \text{ Cos } C$

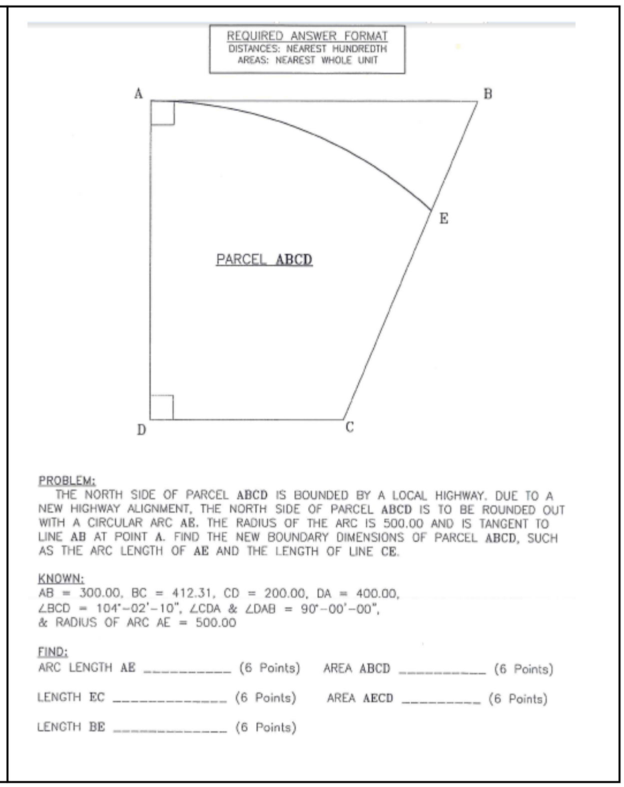
(Note: If angle C is a right angle, the cosine of angle C would be zero, and the Pythagorean Theorem would result.)

Calculations: Area of a Triangle

Derive Heron's Formula, p.445. Use this formula for find the area with given SSS.

Student Activity: Trig Star Questions 2013-2014





More Trig Star Survey questions and answers can be found at:
http://www.nsps.us.com/index.cfm?fuseaction=Page.viewPage&pageId=655&parentID=522&nodeID=2&stopRedirect=1

Unit 4 Section 3: Vectors in the Plane (6.1)

Major Concepts: Apply the arithmetic of vectors and use vectors to solve real-world problems.

Standards

Number and Quantity-Vector and Matrix Quantities (N-VM)

Represent and model with vector quantities.

N-VM.1(+) Recognize vector quantities as having both magnitude and direction. Represent vector quantities by directed line segments, and use appropriate symbols for vectors and their magnitudes (e.g., v, |v| (or ||v||), v).

N-VM.2(+) Find the components of a vector by subtracting the coordinates of an initial point from the coordinates of a terminal point.

N-VM.3(+) Solve problems involving velocity and their quantities that can be represented by vectors.

Perform operations on vectors

N-VM.4(+) Add and subtract vectors

a. Add vectors end-to-end, component-wise, and by the parallelogram rule. Understand that the magnitude of a sum of two vectors is typically not the sum of the magnitudes.

b. Given two vectors in magnitude and direction form, determine the magnitude and direction of their sum.

c. Understand vector subtraction **v** – **w** as v + (-w), where –w is the additive inverse of w, with the same magnitude as w and pointing in the opposite direction. Represent vector subtraction graphically by connecting the tips in the appropriate order, and perform vector subtraction component-wise.

N-VM.5(+) Multiply a vector by scalar

a. Represent scalar multiplication graphically by scaling vectors and possibly reversing their direction; perform scalar multiplication component-wise, e.g., as c(vx,vy) = (cvx, cvy).

b. Compute the magnitude of a scalar multiple c**v** using ||c**v**|| = |c|v. Compute the direction of c**v** knowing that when |c|v ≠ 0, the direction of c**v** is either along **v** (for c>0) or against **v** (for c<0).

Unit 4 Section 3-A: Two Dimensional Vectors

Students will be able to:

- **Determine** whether two vectors are equivalent.
- **Calculate** the magnitude of a vector.
- **Calculate** the sum of two vectors in component form.
- **Calculate** the sum of two vectors graphically by the tail-to-head representation.
- **Calculate** the sum of two vectors graphically by the parallelogram representation.
- **Calculate** a scalar product of a vector in component form.

1. **CONNECT** (10 minutes)
   Review: Prerequisites for Vectors
   Review solving for an angle in a triangle and in the coordinate plane in preparation for studying two-dimensional vectors.

2. **INSTRUCT** (10 minutes)
Building Vocabulary: Two-Dimensional Vectors
Introduce the concepts of vector, component, standard representation of a vector, magnitude, direction, initial point, terminal point, and equivalence of vectors.

- **Vector:** A mathematical object that has both magnitude and direction. Vectors can be expressed as **v**, or ⟨*a*, *b*⟩, or as a directed line segment (arrow) in the plane.

- **Scalar:** A real number. A scalar has magnitude but not direction.

- **Initial Point:** The point at the "tail" of the arrow representing a vector. *Often, the initial point is assumed to be (0, 0). This is the case in the notation ⟨a, b⟩***.**

- **Terminal Point:** The point at the "tip" of the arrow representing a vector.

- **Magnitude of a Vector:** The distance between a vector's initial and terminal points, denoted ||**v**|| or |**v**|. ||**v**|| = || ⟨*a*, *b*⟩ || = $\sqrt{a^2 + b^2}$ *Also called the length, or absolute value of the vector.*

- **Components of a Vector:** *a* and *b* in the vector ⟨*a*, *b*⟩.

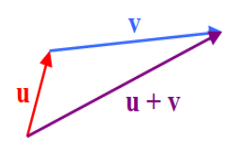
- **Parallel Vectors:** Two or more vectors whose directions are the same or opposite.

- **Equivalent Vectors:** Two or more vectors that have the same direction and magnitude—i.e., whose representations are the same in the form ⟨*a*, *b*⟩. *Note that equivalent vectors may not have the same initial and terminal points.*

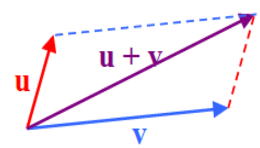
- **Zero Vector:** The vector ⟨0, 0⟩.

- **Resultant Vector:** The vector that results from adding two or more vectors.

- **Tail-to-Head Representation:** A geometric representation of vector addition **u + v** wherein the initial point of **v** is placed at the terminal point of **u**. The vector beginning at the initial point of **u** and ending at the (translated) terminal point of **v** represents **u + v**.



- **Parallelogram Representation / Parallelogram Rule:** A geometric representation of vector addition **u + v** wherein a parallelogram is formed by placing the initial points of **u** and **v** at the same place and letting each vector represent the sides of a parallelogram. The diagonal of the resulting parallelogram, starting at this shared initial point, represents **u + v**.



- **Velocity:** A vector whose magnitude is an object's speed (a scalar) and whose direction is the direction of the object's motion. *Note that speed is a scalar—magnitude, no direction—whereas velocity tells us how fast an object is moving and in what direction.*

- **Complex Numbers:** A class of numbers including purely real numbers a, purely imaginary numbers bi, and numbers with both real and imaginary parts $a + bi$.

- **Complex Plane:** A 2-dimensional representation of complex numbers established by a horizontal real axis and a vertical imaginary axis.

- **Rectangular Form of a Complex Number:** $a + bi$

- **cis θ:** Shorthand for $\cos\theta + i\sin\theta$

- **Polar Form of a Complex Number:** $r(\cos\theta + i\sin\theta) = r \operatorname{cis}\theta$

- **Complex Conjugate of $z = a + bi$:** $\bar{z} = a - bi$

- **Modulus of a Complex Number:** The distance between a number and 0 when plotted on the complex plane: $|z| = |a + bi| = \sqrt{a^2 + b^2}$ *Also called absolute value or magnitude.*

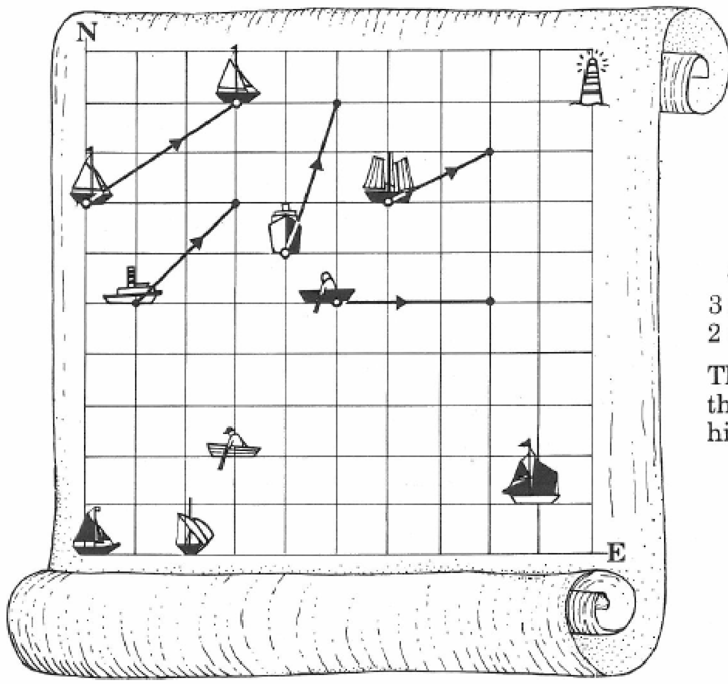
- **Argument of z, arg(z):** The angle—typically chosen in $(-\pi, \pi]$—formed by the positive-real axis and a segment connecting z to 0 in the complex plane.

- **Re(z):** a, the real part of the complex number $z = a + bi$

- **Im(z):** b, the coefficient of the imaginary part of the complex number $z = a + bi$

Student Activity: Vector Sea

Vector Sea



 has moved 3 squares East and 2 squares North.

The captain wrote the vector $\binom{3}{2}$ in his log.

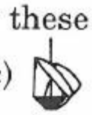
1)  has moved 1 square East and 3 North.

What vector did the captain write?

2) What vectors did the captains of these ships write:

a)    b)    c) 

3) Draw a map of Vector Sea and show how these ships moved:

a)  moved $\binom{2}{3}$ b)  moved $\binom{1}{4}$ c)  moved $\binom{6}{2}$

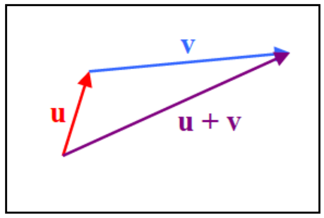
4)  moved $\binom{5}{1}$, then $\binom{3}{2}$

Draw the ship's path so far.

The ship then sailed straight to the light-house.
What vector did the captain write?

3. INDIVIDUAL EXPLORATION (10 minutes)
Exploration: Head Minus Tail Rule
If an arrow has initial point (x_1 , y_1) and terminal point (x_2 , y_2), it represents the vector $< x_2 – x_1 , y_2 – y_1 >$. Have students investigate Exploration 1 on page 457 to understand the Head Minus Tail Rule.

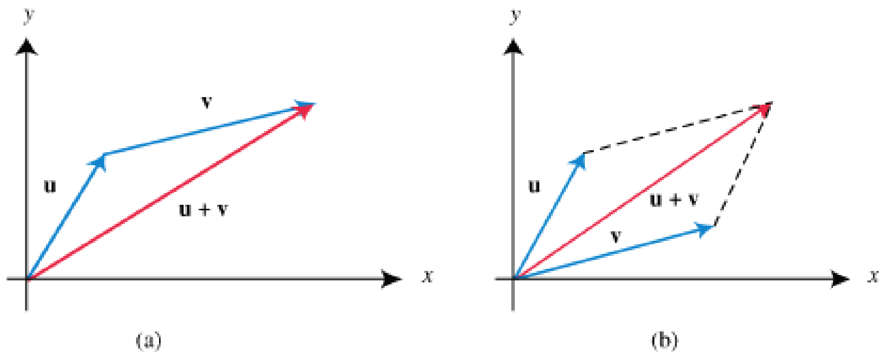
Tail-to-Head Representation: A geometric representation of vector addition **u + v** wherein the initial point of **v** is placed at the terminal point of **u**. The vector beginning at the initial point of **u** and ending at the (translated) terminal point of **v** represents **u + v**.



4. INSTRUCT (15 minutes)
Calculations: Magnitude of a Vector
Introduce the magnitude of a vector. Relate this concept with the Pythagorean theorem (or, equivalently, the distance formula).

Magnitude of a Vector: The distance between a vector's initial and terminal points, denoted ||**v**|| or |**v**|. $\|\mathbf{v}\| = \| \langle a, b\rangle \| = \sqrt{a^2 + b^2}$ *Also called the length, or absolute value of the vector.*

Calculations: Vector Operations
Explain the two operations of vector addition and scalar multiplication, both algebraically and graphically. Note both the tail-to-head and parallelogram methods for finding the sum of two vectors graphically.



Two ways to represent vector addition geometrically: (a) tail-to-head, and (b) parallelogram.

Unit 4 Section 3-B: Unit Vectors and Direction Angles

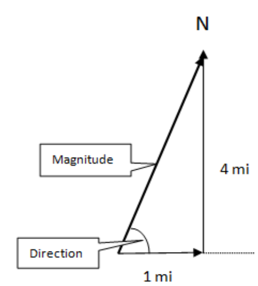
Students will be able to:

- **Determine** a unit vector in the direction of a given vector.
- **Express** a vector in component form as a linear combination of standard unit vectors.
- **Calculate** the components of a vector, given its magnitude and direction angle.

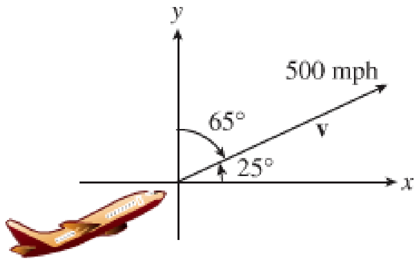
Vocabulary:
unit vector, standard unit vector, linear combination, horizontal component, vertical component, direction angle, resolving a vector, velocity, speed

1. **INSTRUCT** (35 minutes)
   Calculations: Unit Vectors, Direction Angles, and Components
   Explain the meaning of a unit vector, and demonstrate how to calculate it algebraically. Next, recall from trigonometry forming the right triangle from a line segment emanating from the origin in order to calculate the sides. Thus, students should readily see the method for computing both the direction angle and the components of a given vector.

   Revisit Vector Sea.

	Component form		Magnitude-direction form		
	horizontal	vertical	magnitude	direction	Drawing
1) 	1 mile East	4 miles North	$\sqrt{17}$ or about 4.12 miles	About 76 °	

Applications of Vectors: Velocity
The velocity of a moving object is a vector because velocity has both magnitude and direction. The magnitude of velocity is speed. Example 7 on p. 461.

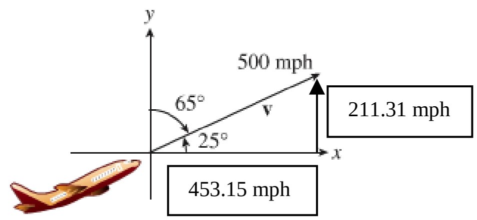


A DC-10 jet aircraft is flying on a bearing of 65° at 500 mph. Find the component form of the velocity of the airplane. The bearing is the angle that the line of travel makes with due north, measured clockwise.

Solution: Let **v** be the velocity of the plane. **A bearing of 65° is equivalent to a direction angle of 25°**. The plane's speed, 500 mph, is the magnitude of vector **v**; that is |**v**| = 500. The horizontal component of **v** is 500 cos 25° and vertical component is 500 sin 25°. Therefore, $v = (500 \cos 25^\circ)\mathbf{i} + (500 \sin 25^\circ)\mathbf{j}$

$$= \langle 500 \cos 25^\circ, 500 \sin 25^\circ \rangle \approx \langle 453.15, 211.31 \rangle$$

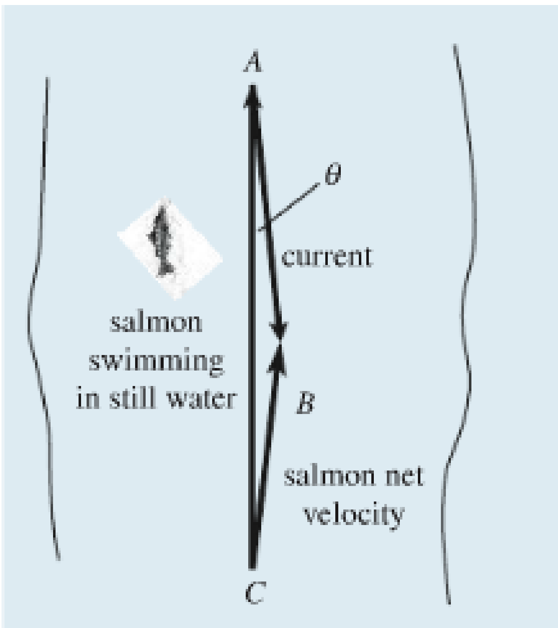
Conclusion: The components of the velocity give the eastward and northward speed. That is, the plane travels about 453.15 mph eastward and 211.31 mph northward as it travels at 500 mph on a bearing 65°.



Application: Student Activity (p. 463):

Young salmon migrate from the fresh water they are born in to salt water and live in the ocean for several years. When it is time to spawn, the salmon return from the ocean to the river's mouth, they follow the organic odors of their homestream to guide them upstream. Researchers believe the fish use currents, salinity, temperature, and magnetic field of the Earth to guide them. Some fish swim as far as 3500 miles upstream for spawning.

During one part of its migration, a salmon is swimming at 6mph, and the current is flowing at 3 mph at an angle of 7 degrees. How fast is the salmon moving upstream? Assume the salmon is swimming in a plane parallel to the surface of the water.



Solution (p. 463):

In the figure, vector $\overrightarrow{AB}$ represents the current of 3 mph, θ is the angle CAB, which is 7 degrees, the vector $\overrightarrow{CA}$ represents the velocity of the salmon of 6 mph, and the vector $\overrightarrow{CB}$ is the net velocity at which the fish is moving upstream.

So we have

$\overrightarrow{AB} = \langle 3\cos(-83°), 3\sin(-83°)\rangle \approx \langle 0.37, -2.98\rangle$

$\overrightarrow{CA} = \langle 0, 6\rangle$

Thus $\overrightarrow{CB} = \overrightarrow{CA} + \overrightarrow{AB} = \langle 3\cos(-83°), 3\sin(-83°) + 6\rangle$

$\approx \langle 0.37, 3.02\rangle$

The speed of the salmon is then $|\overrightarrow{CB}| \approx \sqrt{0.37^2 + 3.02^2} \approx 3.04$ mph upstream.

Unit 4 Section 4: Polar Coordinates (6.4) and Complex Numbers

Major Concepts: Convert points from polar to rectangular coordinates and vice versa.

Standards

Number and Quantity-The Complex Number System (N-CN)

Represent complex numbers and their operations on the complex plane.

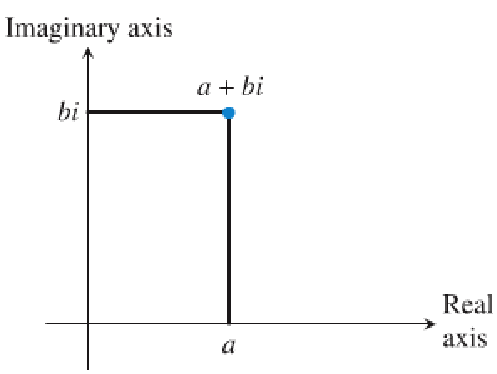
N-CN.4(+) Represent complex numbers on the complex plane in rectangular and polar form and explain why the rectangular and polar forms of a given complex number represent the same number.

Students will be able to:

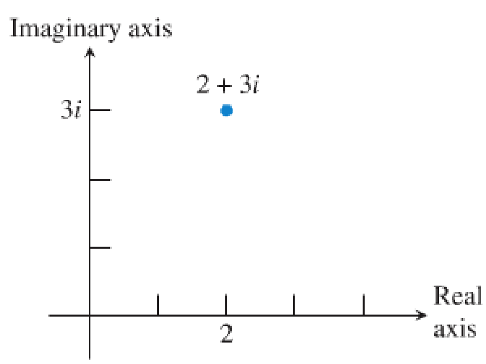
- **Plot** polar coordinates on the Cartesian plane.
- **Determine** all polar coordinates for a point in rectangular coordinates.
- **Convert** between rectangular and polar coordinates.

Vocabulary:
polar coordinates, pole, polar axis, directed distance, directed angle

1. **CONNECT** (10 minutes)
   Review: Prerequisites for Polar Coordinates
   Review Complex Plane. Every complex number can be associated with a point on the complex plane. This idea evolved through the work of Caspar Wessel (1745-1818), Jean-Robert Argrand (1768-1822), and Carl Friedrich (1777-1855). Real numbers are placed along the horizontal axis (the real axis) and imaginary numbers along the vertical axis (the imaginary axis), thus associating the complex number a + b*i* with the point (a, b). The diagram (b) below shows 2 + 3*i* as an example.



(a)



(b)

2. **INSTRUCT** (30 minutes)

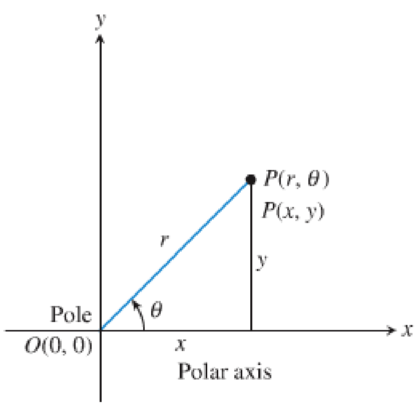
Conceptual Understanding: The Polar Coordinate System

From a complex plane, the modulus (magnitude) and the argument (direction) of a complex number, can be defined. These provide an alternative way of describing complex numbers, known as the polar form.

Useful Notation:

Name	***Meaning***	***Notation***	***Formula (assume z = a +bi)***
real part of *z*	*x*-coordinate of z	Re(z)	*a*
Imaginary part of *z*	y-coordinate of z	Im(z)	*b*
Complex conjugate of z	Reflection of *z* in real axis	$\bar{z}$	*a - bi*
Modulus of *z*	Length of *z*	|z|	$\sqrt{a^2 + b^2}$
Argument of *z*	Angle between positive *x*-axis and z	arg(*z*)	

Calculations: Converting Between Polar and Rectangular Coordinates



The distance of the point (*x*, *y*) from the origin is called the **modulus**, or **magnitude** of the complex number and has the symbol r. Alternatively, r is written as |z|. The modulus is always positive. The modulus can be found using Pythagorean Theorem, that is $|z| = r = \sqrt{a^2 + b^2}$

The angle between the positive *x* axis and a line joining (*x*, *y*) to the origin is called the **argument** of the complex number. Knowing values for *x* and *y*, trigonometry can be used to determine θ. Specifically, $\tan\theta = \frac{y}{x}$ $or\ \theta = tan^{-1}\frac{y}{x}$

Note: Care must be taken when using a calculator to find an inverse tangent that the solution obtained is in the correct quadrant. Drawing a complex diagram will always help to identify the correct quadrant. The position of a complex number is uniquely determined by giving its modulus and argument.

Coordinate Conversion Equations

Let the point P have polar coordinates (r, θ) and rectangular coordinates (x, y). Then

$$x = r\cos\theta, \qquad r^2 = x^2 + y^2,$$

$$y = r\sin\theta, \quad \tan\theta = \frac{y}{x}.$$

Unit 4 Section 5: DeMoivre's Theorem and nth Roots Polar Coordinates; Trigonometric Form of Complex Number and Complex Moduli (6.6)

Major Concepts: Represent complex numbers in the complex plane and write them in trigonometric form. Use the trigonometric form to simplify algebraic operations with complex numbers.

Standards

Number and Quantity-The Complex Number System (N-CN)

Perform arithmetic operations with complex numbers.

N-CN.1 Know there is a complex number i such that i^2 = -1, and every complex number has the form $a+bi$ with a and b real.

N-CN.2 Use the relation i^2 = -1and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.

N-CN.3(+) Find the conjugate of a complex number; use conjugates to find moduli and quotient of complex numbers.

Represent complex numbers and their operations on the complex plane.

N-CN.4(+) Represent complex numbers on the complex plane in rectangular and polar form and explain why the rectangular and polar forms of a given complex number represent the same number.

N-CN.5(+) Represent addition, subtraction, multiplication, and conjugation of complex numbers geometrically on the complex plane; use properties of this representation for computation. For example, $(-1+i\sqrt{3})^3 = 8$ because $(-1+i\sqrt{3})$ has modulus 2 and argument 120°.

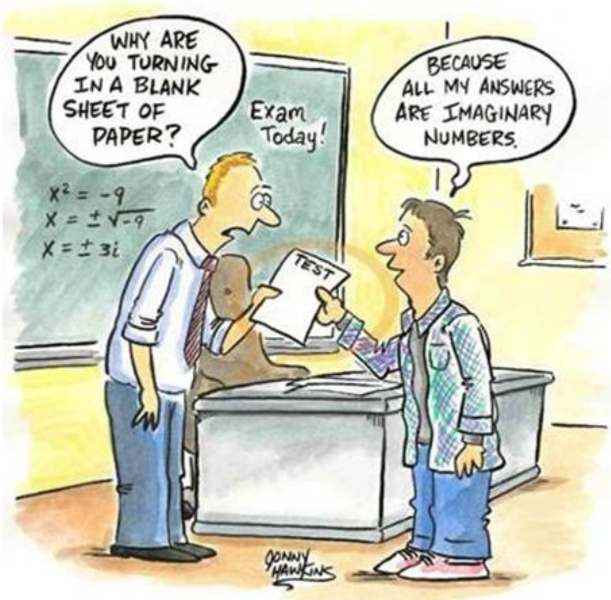
N-CN.6(+) Calculate the distance between numbers in the complex plane as the modulus of the difference, and the midpoint of a segment as the average of the numbers as its endpoints.

Unit 4 Section 5-A: Polar Form (or Trigonometric Form) of Complex Numbers

Students will be able to:

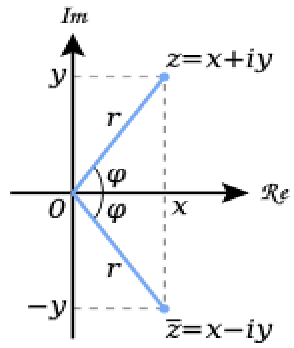
- **Plot** complex numbers in the complex plane.
- **Calculate** the modulus and argument of a complex number.
- **Convert** between the trigonometric and rectangular forms of a complex number.
- **Calculate** the product and quotient of two complex numbers using the

Vocabulary:
complex plane, real axis, imaginary axis, modulus, trigonometric form, polar form, argument



1. **CONNECT** (10 minutes)
   Review: Complex numbers
   Multiplying and dividing complex numbers

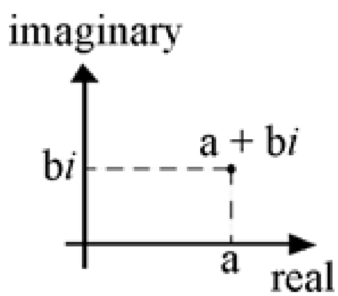
2. **INSTRUCT** (35 minutes)
   Conceptual Understanding: The Complex Plane
   Describe the complex plane, and how complex number can be plotted. Note the visual relationship between complex conjugates.

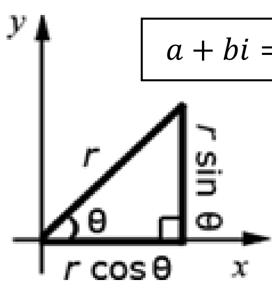


Geometric representation of z and its conjugate $\bar{z}$ in the complex plane

The complex conjugate is found by reflecting z across the real axis.

Then introduce the modulus and argument of a complex number, and note the similarity with polar coordinates.





$a + bi = r(cos\theta + isin\theta)$

If r is the distance of $z = a + bi$ from the origin and θ is the directional angle, then $z = r(cos\theta + isin\theta)$, which is the trigonometric form of z.

DEFINITION Trigonometric Form of a Complex Number

The **trigonometric form** of the complex number $z = a + bi$ is

$$z = r(\cos\theta + i\sin\theta)$$

where $a = r\cos\theta$, $b = r\sin\theta$, $r = \sqrt{a^2 + b^2}$, and $\tan\theta = b/a$. The number r is the *absolute value* or *modulus* of z, and θ is an **argument** of z.

When the modulus and argument of a complex number, z, are known, we write the complex number as z = r cis θ (pronounced "r siss theta"). For example, $r$cis θ is the abbreviation for r(cosθ + $i$sinθ).

Many calculators have keys that enable students to convert from rectangular form to polar form and vice versa. Using the TI-83 or TI-84, enter a complex number in rectangular form.
On the home screen enter ⇛ 4+3i
To convert to polar form:
Use MATH menu, curse to CPX.
Hit 7: Polar then enter ⇢ $5e^{36.87i}$
This means 5 cis 36.87°

However, Students are expected to do this without technology.
Convert 4+3i to polar form. To get the modulus:
r = $\sqrt{a^2+b^2} = \sqrt{4^2+3^2} = 5$ and to get the angle measure: tan θ = $\frac{3}{4}$ ⇛ $\tan^{-1}(\frac{3}{4}) \approx$ 36.87°.
∴ 4+3i = 5cis36.87°

Convert 2cis47° to rectangular form: 2(cos 47° + *i*sin 47) = 1.3639 + 1.4627*i*

Calculations: Trigonometric Form and Product and Quotient of Complex Numbers
The trigonometric form for complex numbers (involving the modulus and argument) is particularly convenient for multiplying and dividing complex numbers. **The product involves the product of the moduli and the sum of the arguments. The quotient involves the quotient of the moduli and difference of the arguments** (p. 505).

Product and Quotient of Complex Numbers

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$. Then

1. $z_1 \cdot z_2 = r_1 r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$.

2. $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)], \quad r_2 \neq 0.$

Illustrative Mathematics **Comments and Solutions can be found at:**
https://www.illustrativemathematics.org/illustrations/1094

In this problem, you will compute some values related to the complex numbers $2+i$ and $5-3i$.

a. Plot $2+i$ and $5-3i$ in the complex plane.

b. How far is $2+i$ from $5-3i$? That is, what is the length of the line segment between $2+i$ from $5-3i$?

c. What is the modulus of the difference of $2+i$ and $5-3i$?

d. What is the midpoint of the line segment between $2+i$ and $5-3i$?

e. What is the average of $2+i$ and $5-3i$?

f. Describe, using the complex plane, the relationship between your answers in (b) and (c).

g. Describe, using the complex plane, the relationship between your answers in (d) and (e).

Unit 4 Section 5-B: De Moivre's Theorem and Roots

Students will be able to:

- **Compute** powers of a complex number using De Moivre's Theorem.
- **Calculate** the nth roots of a complex number.

Vocabulary:
nth root of z

1. **INSTRUCT** (35 minutes)
   Conceptual Understanding: Powers of a Complex Number
   Have students use the multiplication rule to simplify the square of a complex variable. Have them guess the formula for the cube, and then provide **De Moivre's formula for powers of a complex number.**

De Moivre's Theorem

Let $z = r(\cos\theta + i\sin\theta)$ and let n be a positive integer. Then

$$z^n = [r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta).$$

Calculations: Roots of a Complex Number
Give the formula for finding roots of a complex number. Derivation of a general formula for finding the nth roots of a non-zero complex number is demonstrated on p. 508. Demonstrate the relationship between such roots visually in the complex plane.

Finding *n*th Roots of a Complex Number

If $z = r(\cos\theta + i\sin\theta)$, then the n distinct complex numbers

$$\sqrt[n]{r}\left(\cos\frac{\theta + 2\pi k}{n} + i\sin\frac{\theta + 2\pi k}{n}\right),$$

where $k = 0, 1, 2, \ldots, n - 1$, are the nth roots of the complex number z.

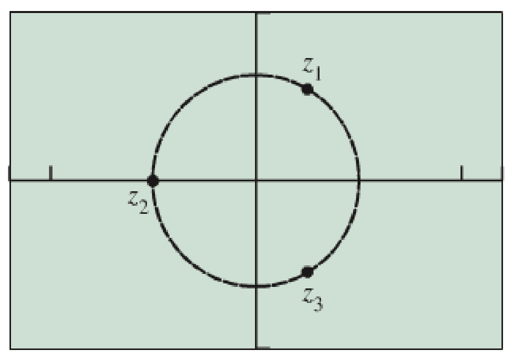
Example: Find the cube roots of -1 and plot them (p.509).
Solution: We first write the complex number $z = -1$ in trigonometric form $z = -1 + 0i = 1(cos\pi + isin\pi)$. The third roots of $z = -1 + 0i = 1(cos\pi + isin\pi)$ are the complex numbers

$cos\frac{\pi+2\pi k}{3} + isin\frac{\pi+2\pi k}{3}$, for $k = 0, 1, 2$. The three complex roots are:

$z_1 = cos\frac{\pi}{3} + isin\frac{\pi}{3} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$,

$z_2 = cos\frac{\pi+2\pi}{3} + isin\frac{\pi+2\pi}{3} = -1 + i0$,

$z_1 = cos\frac{\pi+4\pi}{3} + isin\frac{\pi+4\pi}{3} = \frac{1}{2} - i\frac{\sqrt{3}}{2}$.



[–2.4, 2.4] by [–1.6, 1.6]

The three cube roots of $z = -1$ are displayed on the unit circle.

Illustrative Mathematics **Comments and Solutions can be found at:**
http://www.illustrativemathematics.org/illustrations/1689

a. Let $z = 1 + i$ where $i^2 = -1$. Calculate z^2, z^3, and z^4.

b. Graph z, z^2, z^3, and z^4 in the complex plane. What do you notice about the positions of these numbers?

c. What is z^{100}? Explain.

Illustrative Mathematics **Comments and Solutions can be found at:**
https://www.illustrativemathematics.org/illustrations/1659

For each odd positive integer n, the only real number solution to $x^n = 1$ is $x = 1$ while for even positive integers n, $x = 1$ and $x = -1$ are solutions to $x^n = 1$. In this problem we look for all complex number solutions to $x^n = 1$ for some small values of n.

a. Find all complex numbers $a + bi$ whose cube is 1.

b. Find all complex numbers $a + bi$ whose fourth power is 1.

Unit 5: Optional (+) Topics: Additional Topics for (+) Standards

Unit 5 Section 1: Matrix Addition and Multiplication (7.2)

Major Concepts: Find sums, differences, products, and inverse of matrices.

Standards

Number and Quantity-Vector and Matrix Quantities (N-VM)

Perform operations on matrices and use matrices in applications.

N-VM.6(+)	Use matrices to represent and manipulate data.
N-VM.7(+)	Multiply matrices by scalars to produce new matrices, e.g., as when all of the payoffs in a game are doubled.
N-VM.8(+)	Add, subtract, and multiply matrices of appropriate dimensions.
N-VM.9(+)	Understand that, unlike multiplication of numbers, matrix multiplication for square matrices is not a commutative operation, but still satisfies the associative and distributive properties.
N-VM.10(+)	Understand that the zero and identity matrices play a role in matrix and multiplication similar to the role of 0 and 1 in the real numbers. The determinant of a square matrix is nonzero if and only if the matrix has a multiplicative inverse.
N-VM.11(+)	Multiply a vector (regarded as a matrix with one column) by a matrix of suitable dimensions to produce another vector. Work with matrices as transformations of vectors.
N-VM.12(+)	Work with 2x2 matrices as transformations of the plane, and interpret the absolute value of the determinant in terms of area.

Unit 5 Section 1-A: Matrix Addition and Multiplication

Students will be able to:

- **Determine** the order of a matrix.
- **Identify** the row and column subscripts of a matrix entry.
- **Compute** the sum and difference of two matrices.
- **Compute** a scalar multiple of two matrices.
- **Compute** the product of two matrices.
- **Compute** the sum, difference, and product of matrices using a graphing utility.

1. **INSTRUCT** (45 minutes)

   Building Vocabulary: Matrices

   Introduce the various vocabulary for matrices.

   - **Matrix:** a rectangular arrangement of numbers into rows and columns
   - **Determinant:** the product of the elements on the main diagonal minus the product of the elements off the main diagonal
   - **Dimensions or Order of a Matrix:** the number of rows by the number of columns
   - **Identity Matrix:** the matrix that has 1's on the main diagonal and 0's elsewhere
   - **Inverse Matrices:** matrices whose product (in both orders) is the Identity matrix
   - **Scalar:** in matrix algebra, a real number is called a scalar
   - **Zero Matrix:** a matrix whose entries are all zeros

https://www.georgiastandards.org/Common-Core/Pages/Math-9-12.aspx

The following is a **matrix**, a rectangular array of values, showing the wholesale cost of each item as well as the cost of decorations. "*wholesale*" and "*decorations*" are labels for the matrix **rows** and "*bears*", "*totes*", and "*shirts*" are labels for the matrix **columns**. The **dimensions of this matrix** called **A** are 2 rows and 3 columns and matrix **A** is referred to as a [**2 x 3] matrix**. Each number in the matrix is called an **entry**.

$$\begin{array}{ccc} & & \textit{Cost per Item} \\ & & \begin{array}{ccc} \textit{bears} & \textit{totes} & \textit{shirts} \end{array} \\ A = & \begin{array}{r} \textit{wholesale} \\ \textit{decorations} \end{array} & \begin{bmatrix} 4.00 & 3.50 & 3.25 \\ 1.25 & .90 & 1.05 \end{bmatrix} \end{array} \quad \boxed{\text{rows}}$$

$$\boxed{\text{columns}}$$

It is sometimes convenient to write matrices (plural of matrix) in a simplified format without labels for the rows and columns. Matrix A can be written as an array.

$$A = \begin{bmatrix} 4.00 & 3.50 & 3.25 \\ 1.25 & .90 & 1.05 \end{bmatrix} \text{ where the values can be identified as } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}.$$

In this system, the entry $\boldsymbol{a}_{22}$ = .90, which is the cost of decorations for tote bags.

Calculations: Matrix Addition and Scalar Multiplication

Define addition and scalar multiplication. These operations should not be difficult for students, as long as they remember to check the order of the matrices before adding. A **scalar** is a single number such as 3 and matrix **scalar multiplication** is done by multiplying each entry in a matrix by the same scalar. For example:

$$\text{Let } M = \begin{bmatrix} -2 & 0 & 5 \\ 1 & -3 & 4 \end{bmatrix}, \text{ then } 3M = \begin{bmatrix} -6 & 0 & 15 \\ 3 & -9 & 12 \end{bmatrix}.$$

Calculations: Matrix Multiplication

Matrix multiplication is usually conceptually difficult for students. First, talk about how the order of the matrices must match: an *m* x *r* matrix times an *r* x *n* matrix yields an *m* x *n* matrix (note how the r's match, and subsequently cancel out). Second, note how to get each entry in the product by proceeding across the row of the first and down the column of the second.

Constructing Matrix using Data:
At the beginning of November a stomach virus hits Central High School. Students in the Freshman and Sophomore classes are well, a little sick, or very sick. The following tables show Freshmen and Sophomores according to their levels of sickness and their gender.

Student Population		
Categories	Male	Females
Freshmen	250	300
Sophomores	200	275

% of Sick Students		
Categories	Freshmen	Sophomores
Well	20%	25%
Little Sick	50%	40%
Very Sick	30%	35%

Suppose school personnel needed to prepare a report and include the total numbers of well and sick male Freshmen and Sophomores in the school.

well Freshmen males + well Sophomore males = well males
$(.2)(250) + (.25)(200) = 100$
a little sick Freshmen males + a little sick Sophomore males = a little sick males
$(.5)(250) + (.4)(200) = 205$
really sick Freshmen males + really sick Sophomore males = really sick males
$(.3)(250) + (.35)(200) = 145$

Notice the positions of the values in these products. We are multiplying rows by columns to get the information we want. Translating the tables to matrices and using the **rows by columns pattern of multiplication** we get the following result.

$$\begin{array}{cc} & \begin{array}{cc} F & S \end{array} \\ \begin{array}{c} W \\ L \\ R \end{array} & \begin{bmatrix} .2 & .25 \\ .5 & .4 \\ .3 & .35 \end{bmatrix} \end{array} * \begin{array}{cc} & \begin{array}{cc} M & F \end{array} \\ \begin{array}{c} F \\ S \end{array} & \begin{bmatrix} 250 & 300 \\ 200 & 275 \end{bmatrix} \end{array} = \begin{bmatrix} (.2)(250)+(.25)(200) & (.2)(300)+(.25)(275) \\ (.5)(250)+(.4)(200) & (.5)(300)+(.4)(275) \\ (.3)(250)+(.35)(200) & (.3)(300)+(.35)(275) \end{bmatrix} = \begin{array}{cc} & \begin{array}{cc} M & F \end{array} \\ \begin{array}{c} W \\ L \\ R \end{array} & \begin{bmatrix} 100 & 128.75 \\ 205 & 260 \\ 145 & 186.25 \end{bmatrix} \end{array}$$

$$\begin{array}{cc} & \begin{array}{cc} F & S \end{array} \\ \begin{array}{c} W \\ L \\ R \end{array} & \begin{bmatrix} .2 & .25 \\ .5 & .4 \\ .3 & .35 \end{bmatrix} \end{array} * \begin{array}{cc} & \begin{array}{cc} M & F \end{array} \\ \begin{array}{c} F \\ S \end{array} & \begin{bmatrix} 250 & 300 \\ 200 & 275 \end{bmatrix} \end{array} = \begin{array}{cc} & \begin{array}{cc} M & F \end{array} \\ \begin{array}{c} W \\ L \\ R \end{array} & \begin{bmatrix} 100 & 128.75 \\ 205 & 260 \\ 145 & 186.25 \end{bmatrix} \end{array}$$

[level of sickness x class] * [class x gender] = [level of sickness x gender]

[3 x 2] * [2 x 2] = [3 x 2]

This procedure illustrates the multiplication of two matrices. In order to multiply two matrices, the number of columns of the matrix on the left must equal the number of rows of the matrix on the right. Also the labels of the columns of the left matrix must be the same as the labels of the rows of the right matrix. **If the dimensions of two matrices are not appropriately matched, it is not possible to multiply them.**

Conceptual Understanding: Identity Matrix

The matrix **I** = $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is called the 2 x 2 identity matrix because when you multiply any 2 x 2 matrix A by **I**, you get A as the answer. **I** acts like the number 1 in the multiplication of numbers.

Unit 5 Section 1-B: Inverse and Determinant of 2x2 matrices

Students will be able to:

- **Determine** whether two matrices are inverses of one another.
- **Determine** whether a 2x2 matrix is singular or nonsingular.
- **Compute** the determinant of a 2x2 matrix.
- **Compute** the minors and cofactors of a matrix.

1. INSTRUCT (45 minutes)

Graphing Calculators: Matrix Operations

Demonstrate to students how to input a matrix in their graphing calculator, and subsequently perform operations such as addition or multiplication.

Conceptual Understanding: Multiplicative Inverse (p. 535)

To help motivate the multiplicative inverse of a matrix, explain the idea behind the multiplicative inverse of a real number. When does a real number have one? What form does it have? Next, give an example of two matrices A and B such that AB does not equal BA. Point out how students must be careful in the order in which they multiply matrices. This will help to explain why we use the notation A^{-1} for inverse (and not $\frac{1}{A}$).

Calculations: Determinant of a 2 x 2 Matrix

A unique number associated with every square matrix is called the determinant. Only square matrices have determinants. Give the formula for the determinant of a 2 x 2 matrix, and talk about how this tells whether a matrix has an inverse.

The number $ad - bc$ is the determinant of the 2 x 2 matrix A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and is denoted det A = $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

Calculations: Inverse of a 2 x 2 Matrix
Once students understand the determinant, present the formula for the inverse of a 2x2 matrix. Students should immediately see why the determinant must be nonzero for the matrix to have an inverse. Give several examples and have them check through multiplication that the matrices really are inverses of one another.

Inverse of a 2 x 2 Matrix
If $ad - bc \neq 0$, then $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & c \end{bmatrix}$.

Unit 5 Section 1-C: Inverse and Determinant of n x n matrices

Students will be able to:

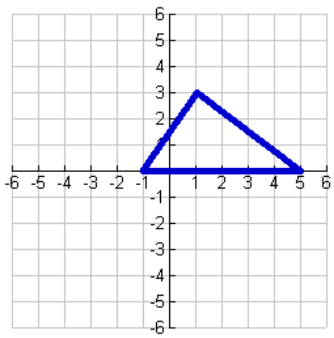
- **Compute** the determinant of an *n* x *n* matrix.
- **Determine** whether an *n* x *n* matrix is singular or nonsingular.
- **Compute** the inverse and determinant of a matrix using a graphing utility.

1. **INSTRUCT** (30 minutes)
Calculations: Graphing Calculator
Using graphing calculator, calculate the determinant of a 3x3 matrix. The determinant of a matrix can be used to find the area of a triangle. If (x_1, y_1), (x_2, y_2), and (x_3, y_3) are vertices of a triangle, the area of the triangle is

$$\text{Area} = \pm\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

where the symbol ($\pm$) indicates that the appropriate sign should be chosen to yield a positive area.

Example: Given a triangle with vertices (-1, 0), (1, 3), and (5, 0), find the area using the determinant formula. Verify that area you found is correct using geometric formulas.



2. **INDIVIDUAL EXPLORATION** (10 minutes)
Exploration: Definition of Determinant
Have students investigate Exploration 2 on page 536 to better understand how determinants are calculated.

3. **INSTRUCT** (5 minutes)
Calculations: Inverses of n x n Matrices
The process for calculating the inverse of an n x n matrix is too complicated for this setting, but students should be shown how to calculate it on a graphing calculator. They can use the multiplication function to check that the inverse is accurate.

Unit 5 Section 2: Solving Systems of Equation (7.3)

Major Concepts: Solve systems of linear equations using an inverse matrix.

Standards
Algebra-Reasoning with Equations and Inequalities (A-REI)
Solve systems of equations
A-REI.8(+) Represent a system of linear equations as a single matrix equation in a vector variable.
A-REI.9(+) Find the inverse of a matrix if it exists and use it to solve systems of linear equations (using technology for matrices of dimension 3 x 3 or greater).

Students will be able to:
- **Construct** a matrix equation for a system of linear equations.
- **Solve** a linear system of equations using inverse matrices.

1. **INSTRUCT** (45 minutes)
Conceptual Understanding: Matrix Equation

A system of equations such as $\begin{cases} ax + by = c \\ dx + ey = f \end{cases}$ can be written as a matrix equation where $\begin{bmatrix} a & b \\ d & e \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c \\ f \end{bmatrix}$ or [A][V] = [B] where [A] is the coefficient matrix, [V] is the variables matrix, and [B] is the matrix for the constant terms. The unknown variable matrix can be isolated with the following algebraic reasoning:

$[A][V] = [B]$ *write the system of linear equations as a matrix equation*
$[A]^{-1}[A][V] = [A]^{-1}[B]$ *multiply both sides of the equation by the **inverse** of matrix [A]*
$[I] [V] =[A]^{-1}[B]$ *the inverse of matrix [A] times matrix [A] equals the **identity** matrix*
$[V]=[A]^{-1}[B]$ *the unknown variables equal the inverse of [A] times [B]*

Calculations: Solve the system $\begin{cases} 3x - 2y = 0 \\ -x + y = 5 \end{cases}$

Solution: First, we write the system as a matrix equation.

Let [A] = $\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$, [V] = $\begin{bmatrix} x \\ y \end{bmatrix}$, and [B] = $\begin{bmatrix} 0 \\ 5 \end{bmatrix}$, then [V]=[A]$^{-1}$[B] = $\begin{bmatrix} 10 \\ 15 \end{bmatrix}$.

The solution of the system is $x = 10, y = 15$, or (10, 15)

Solving systems of equations of higher order can be accomplished using a similar format and a TI-84 graphing calculator to find the inverse of the coefficient matrix and find the necessary products.

Unit 5 Section 3: Ellipse (8.2)

Major Concepts: Derive the equations of ellipses given the foci.

Standards

Geometry-Expressing Geometric Properties with Equations (G-GPE)

Translate between the geometric description and the equation for a conic section

G-GPE.3(+) Derive the equations of ellipses and hyperbolas given the foci, using the fact the sum or difference of distances from

Unit 5 Section 3-A: Geometry of an Ellipse

Students will be able to:

- **Calculate** the focal axis, foci, vertices, major and minor axes, and semi-major and semi-minor axes of an ellipse.
- **Analyze** the graph of an ellipse.

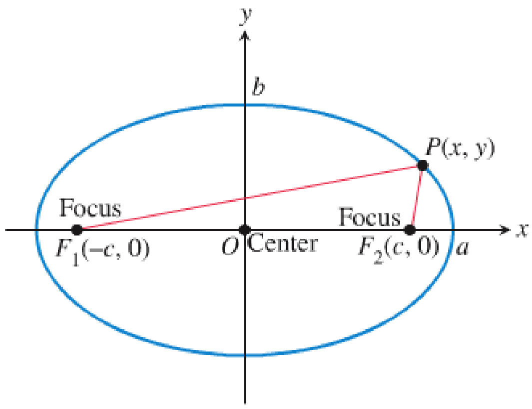
Vocabulary:

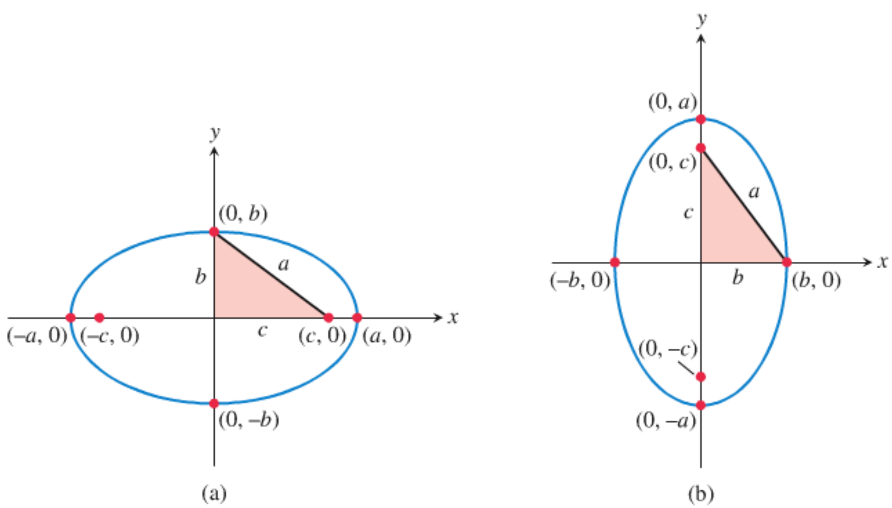
ellipse, focus, focal axis, center, vertex, chord, major axis, minor axis, semi-major axis,
semi-minor axis

1. **CONNECT** (10 minutes)
   Review: Prerequisites for Ellipses
   Completing the square.

2. **INSTRUCT** (30 minutes)
   Building Vocabulary: Ellipses

Introduce the various vocabulary associated with an ellipse, including the standard equation, focal axis, foci, vertices, semi-major axis, and semi-minor axis. Students should be able to calculate each of these, given an equation for the ellipse. One point to warning: students will often confuse the focal axis (a line), the major and minor axes (line segments), and the semi-major and semi-minor axes (numbers).





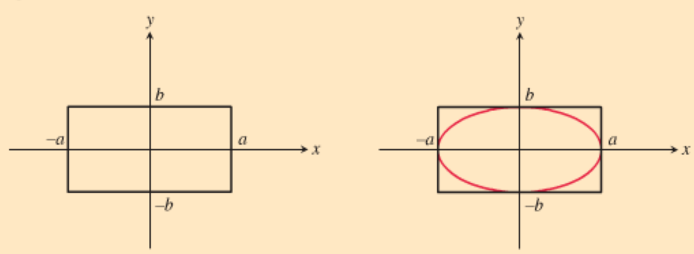
Ellipses centered at the origin with foci on (a) the *x*-axis and (b) the *y*-axis. In each case, a right triangle illustrating the Pythagorean relation is shown.

Review: How to graph Ellipse with students

Drawing Lesson

How to Sketch the Ellipse $x^2/a^2 + y^2/b^2 = 1$

1. Sketch line segments at $x = \pm a$ and $y = \pm b$ and complete the rectangle they determine.
2. Inscribe an ellipse that is tangent to the rectangle at $(\pm a, 0)$ and $(0, \pm b)$.



Unit 5 Section 3-B: Translations of Ellipses

Students will be able to:

- **Calculate** the focal axis, foci, vertices, major and minor axes, and semimajor and semiminor axes of a translated ellipse.
- **Determine** the equation of an ellipse given its foci and major/minor axis.
- **Calculate** the eccentricity of an ellipse.
- **Describe** the effect of eccentricity on the shape of an ellipse.

Vocabulary:
ellipse, focus, focal axis, center, vertex, chord, major axis, minor axis, semimajor axis, semiminor axis, eccentricity, ellipsoid of revolution

1. **INSTRUCT** (20 minutes)
   Conceptual Understanding: Translating Ellipses
   Demonstrate how the graph and the algebraic expression of a function change due to translation of the graph of an ellipse. Students should be able to explain how each of the focal axis, foci, vertices, semimajor axis, and semiminor axis subsequently change. Given information about the ellipse, students should be able to write its equation.

2. **GROUP DISCUSSION** (15 minutes)
   Calculations: Eccentricity
   Introduce eccentricity, and then have students investigate Exploration 2 on page 597 to see how eccentricity affects the graph of an ellipse.

   The noun eccentricity comes from the adjective eccentric, which means off-center. Mathematically, the eccentricity is the ratio of c to a. The larger c is, compare to a, the more off-center the foci are. In any ellipse, $a > c \geq 0$. Dividing this inequality by a shows that $0 \leq e < 1$. So the eccentricity of an ellipse is between 0 and 1.

DEFINITION Eccentricity of an Ellipse

The **eccentricity** of an ellipse is

$$e = \frac{c}{d} = \frac{\sqrt{a^2 - b^2}}{a},$$

where a is the semimajor axis, b is the semiminor axis, and c is the distance from the center of the ellipse to either focus.

Unit 5 Section 4: Hyperbola (8.3)

Major Concepts: Derive the equations of hyperbolas given the foci.

Standards
Geometry-Expressing Geometric Properties with Equations (G-GPE)
Translate between the geometric description and the equation for a conic section
G-GPE.3(+) Derive the equations of ellipses and hyperbolas given the foci, using the fact the sum or difference of distances from the foci is constant.

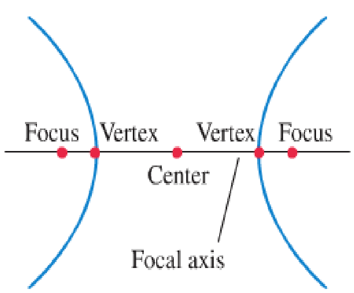
Unit 5 Section 4-A: Geometry of a Hyperbola

Students will be able to:

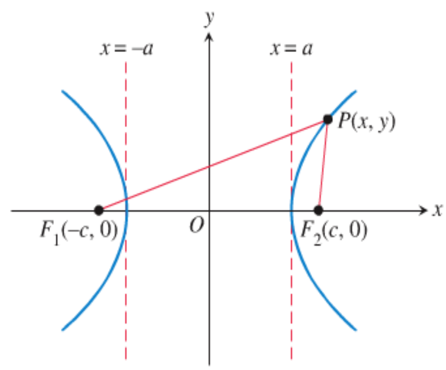
- **Calculate** the focal axis, foci, vertices, semitransverse axis, semiconjugate axis, and asymptotes of a hyperbola.
- **Analyze** the graph of a hyperbola.

Vocabulary:
hyperbola, focus, focal axis, center, vertex, chord, transverse axis, conjugate axis, semitransverse axis, semiconjugate axis

1. **INSTRUCT** (30 minutes)
   Building Vocabulary: Hyperbolas
   Introduce the various vocabulary associated with a hyperbola, including the standard equation, focus, focal axis, center, vertex, chord, transverse axis, conjugate axis, semitransverse axis, and semiconjugate axis. Students should be able to calculate each of these, given an equation for the hyperbola.



Key points on the focal axis of a hyperbola



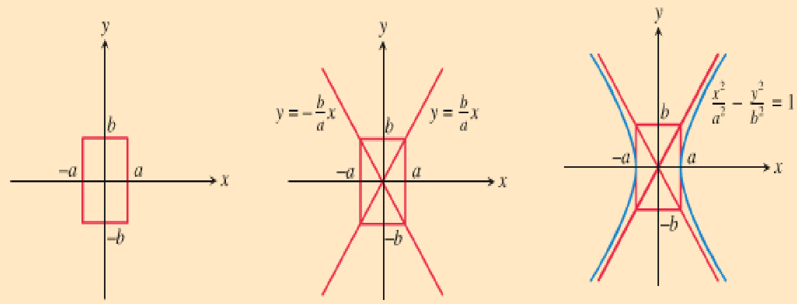
Structure of a Hyperbola: The difference of the distances from the foci to each point on the hyperbola is a constant

2. INSTRUCT (30 minutes)
Graph: Hyperbola

Drawing Lesson

How to Sketch the Hyperbola $x^2/a^2 - y^2/b^2 = 1$

1. Sketch line segments at $x = \pm a$ and $y = \pm b$, and complete the rectangle they determine.
2. Sketch the asymptotes by extending the rectangle's diagonals.
3. Use the rectangle and asymptotes to guide your drawing.



Hyperbolas with Center (0, 0)

• **Standard equation**	$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$	$\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1$
• **Focal axis**	x-axis	y-axis
• **Foci**	$(\pm c, 0)$	$(0, \pm c)$
• **Vertices**	$(\pm a, 0)$	$(0, \pm a)$
• **Semitransverse axis**	a	a
• **Semiconjugate axis**	b	b
• **Pythagorean relation**	$c^2 = a^2 + b^2$	$c^2 = a^2 + b^2$
• **Asymptotes**	$y = \pm \dfrac{b}{a} x$	$y = \pm \dfrac{a}{b} x$

Unit 5 Section 4-B: Translations of a Hyperbola

Students will be able to:

- **Calculate** the focal axis, foci, vertices, semitransverse axis, semiconjugate axis, and asymptotes of a translated hyperbola.
- **Determine** the equation of a hyperbola given its transverse and conjugate axes.

1. **INSTRUCT** (40 minutes)
Conceptual Understanding: Translating Hyperbola
When a hyperbola with center (0, 0) is translated horizontally by h units and vertically by k units, the new center of the hyperbola is (h, k). Such a translation does not have any change on the length of the transverse or conjugate axis or the Pythagorean relation.

Hyperbolas with Center (h, k)		
• **Standard equation**	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$	$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$
• **Focal axis**	$y = k$	$x = h$
• **Foci**	$(h \pm c, k)$	$(h, k \pm c)$
• **Vertices**	$(h \pm a, k)$	$(h, k \pm a)$
• **Semitransverse axis**	a	a
• **Semiconjugate axis**	b	b
• **Pythagorean relation**	$c^2 = a^2 + b^2$	$c^2 = a^2 + b^2$
• **Asymptotes**	$y = \pm\frac{b}{a}(x-h) + k$	$y = \pm\frac{a}{b}(x-h) + k$

Unit 5 Section 5: Binomial Theorem (9.2)

Major Concepts: Expand a power of a binomial using the Binomial Theorem or Pascal's Triangle. Find the coefficient of a given term of a binomial expansion.

Standards
Algebra-Arithmetic with Polynomials and Rational Expressions (A-APR)
Perform arithmetic operations on polynomials
A-APR.5(+) Know and apply the Binomial Theorem for the expansion of $(x+y)^n$ in powers of *x* and *y* for a positive integer *n*, where x and y are any numbers, with coefficients determined for example by Pascal's Triangle.

Students will be able to:
- **Write out** Pascal's triangle.
- **Expand** a power of a binomial expression using the binomial theorem.
- **Determine** the coefficient of a term in a binomial expansion.

Vocabulary:
binomial, Pascal's triangle, binomial theorem

1. **CONNECT** (10 minutes)
Review: Prerequisites for the Binomial Theorem
Review multiplying out binomial expressions in preparation for the binomial theorem.

2. **INDIVIDUAL EXPLORATION** (10 minutes)
Exploration: Binomial Coefficients
Have students investigate Exploration 1 on page 652, comparing their results with the expansions on the same page, to better understand the relationship between binomial coefficients and combinations.

Powers of Binomials

Many important mathematical discoveries have begun with the study of patterns. In this chapter, we want to introduce an important polynomial theorem called the Binomial Theorem, for which we will set the stage by observing some patterns.

If you expand $(a + b)^n$ for $n = 0, 1, 2, 3, 4,$ and 5, here is what you get:

$$\begin{aligned}
(a+b)^0 &= 1\\
(a+b)^1 &= 1a^1b^0 + 1a^0b^1\\
(a+b)^2 &= 1a^2b^0 + 2a^1b^1 + 1a^0b^2\\
(a+b)^3 &= 1a^3b^0 + 3a^2b^1 + 3a^1b^2 + 1a^0b^3\\
(a+b)^4 &= 1a^4b^0 + 4a^3b^1 + 6a^2b^2 + 4a^1b^3 + 1a^0b^4\\
(a+b)^5 &= 1a^5b^0 + 5a^4b^1 + 10a^3b^2 + 10a^2b^3 + 5a^1b^4 + 1a^0b^5
\end{aligned}$$

3. **INSTRUCT** (30 minutes)
Calculations: Pascal's Triangle
Introduce Pascal's triangle, noting how to proceed to the next row from the previous one, and relate this to the coefficient work done previously (p. 653). State the recursion formula that results from this relationship.

Pascal's Triangle

If we eliminate the plus signs and the powers of the variables a and b in the "triangular" array of binomial coefficients with which we began this section, we get:

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \\ \iddots & & & & & \vdots & & & & & \ddots \end{array}$$

This is called **Pascal's triangle** in honor of Blaise Pascal (1623–1662), who used it in his work but certainly did not discover it. It appeared in 1303 in a Chinese text, the *Precious Mirror*, by Chu Shih-chieh, who referred to it even then as a "diagram of the old method for finding eighth and lower powers."

Calculations: Binomial Theorem (p. 654)
Present the Binomial Theorem and explain how it can be used to expand a binomial expression quickly. Do several examples.

The Binomial Theorem

We now state formally the theorem about expanding powers of binomials, known as the Binomial Theorem. For tradition's sake, we will use the symbol $\binom{n}{r}$ instead of ${}_nC_r$.

The Binomial Theorem

For any positive integer n,

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \cdots + \binom{n}{r}a^{n-r}b^r + \cdots + \binom{n}{n}b^n,$$

where

$$\binom{n}{r} = {}_nC_r = \frac{n!}{r!(n-r)!}.$$

VITA

Yong Suk Lowery was born in Taegu, South Korea. She has been teaching for sixteen years. She is currently teaching Precalculus, Trigonometry, and AP Calculus. She received her Bachelor of Science degree in Secondary Mathematics Education, Bachelor of General Studies degree, and Master of Education in Curriculum and Instruction from the Southeastern Louisiana University.