Seminar on Continuity in Semilattices

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SCS 30: Continuous Semilattices and Duality

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SEMINAR ON CONTINUITY IN SEMILATTICES (SCS)

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TOPIC Continuous Semilattices and Duality

REFERENCE

HMS duality and ATLAS duality have played important roles in the development of the theory of continuous lattices. I would like to offer herewith a third duality. In the HMS duality homomorphisms are taken into 2 with the discrete topology to form the dual. ATLAS duality can be viewed as taking homomorphisms into 2 with the topology $\{2, \emptyset, \{0\}\}$. Since only complete lattices are considered, the inverse image of 1 is a "closed" filter, i.e., closed under infs and hence has a least element. Thus the dual of L may be identified with L^{op}. Then the left adjoint of $f: L \to M$ is, with this identification, just $\hat{f}: \hat{M} \to \hat{L}$.

Herein we consider the third possibility, namely having $\{1\}$ open, but not $\{0\}$. Here inverse images of 1 correspond to "open" filters, i.e. open in the sense of Scott. The appropriate objects here are "continuous semilattices" and the category turns out to be self-dual.

West Germany: TH Darmstadt (Gierz, Keimel)
U. Tübingen (Mislove, Visit.)

England: U. Oxford (Scott)

USA: U. California, Riverside (Stralka)
LSU Baton Rouge (Lawson)
Tulane U., New Orleans (Hofmann, Mislove)
U. Tennessee, Knoxville (Carruth, Crawley)

1. Continuous Semilattices

1.1. DEFINITION A semilattice $(S, \wedge)$ is a continuous semilattice if

(i) S has a 1,

(ii) Every up-directed subset has a supremum (Sometimes called upper Dedekind completeness),

(iii) For all $x \in S$, $x = \sup\{y : y \ll x\}$ and this set is up-directed. (Here $y \ll x$ if for any up-directed set D with $\sup D \geq x$, then $d \geq y$ for some $d \in D$).

Note that if S is a lattice then it is upper complete and conditionally complete. Hence if in addition it possesses a least element it is a complete lattice, and thus a continuous lattice.

Condition (iii) may be replaced by either of the following equivalent conditions:

(iii)* For all $x \in S$, there exists an up-directed set D with $x = \sup D$ where $y \ll x$ $\forall\, y \in D$.

or

(iii)′ $\forall\, x \in S$, $x = \sup\{z : \exists\, y$ with $z \ll y \ll x\}$ and this set is up-directed. (Requires a little work)

1.2 PROPOSITION. Let S be a continuous semilattice. Then $\ll$ has the interpolation property, i.e., if $z \ll x$, then $\exists\, y$ with $z \ll y \ll x$. Hence $x \ll y \Leftrightarrow x \lll y$ ($x \lll y$ if for any directed set D, $y \leq \sup D \Rightarrow x \ll d$, $\exists\, d \in D$).

1.3 DEFINITION. Let $(S, \wedge)$ be a semilattice with 1. A (non-empty) filter F is an open filter if $\forall$ up-directed sets D, $\sup D \in F \Rightarrow d \in F$, $\exists d \in D$.

The open filter F is locally bounded if $\forall y \in F$, $\exists$ an open filter $G \subset F$ and $\exists z \in F$ such that $y \in G$ and $G \subset \uparrow z$.

1.4 PROPOSITION. Let S be a continuous semilattice. $x \ll y \Rightarrow \exists$ an open filter G such that $x \notin G$, $y \in G$, and $G \subset \uparrow x$.

Proof. By induction and the interpolation property, construct $\{x_n\}_{n \in \omega}$ satisfying $x \ll x_{n+1} \ll x_n \ll y$ $\forall n$. Let $G = \bigcup_n (\uparrow x_n)$.

1.5 PROPOSITION. Suppose S is an upper Dedekind complete semilattice with 1. Then S is a continuous semilattice $\Leftrightarrow$ every open filter is locally bounded and the open filters separate the points of S.

Proof. $\boxed{\Rightarrow}$. Let F be an open filter, $y \in F$. $\exists x \ll y$ with $x \in F$. By 1.4 $\exists G$ with $y \in G$ and $G \subset \uparrow x$.

Suppose $y \neq z$. Wlog $y \nleq z$. Then $\exists x \ll y \ni x \nleq z$. By 1.4 $\exists G \ni y \in G \subset \uparrow x$. Hence $z \notin G$.

$\boxed{\Leftarrow}$ Let $x \in S$ and $w < x$. $\exists$ an open filter $F \ni x \in F$, $w \notin F$. F locally bounded $\Rightarrow \exists$ an open filter G and $z \in F \ni x \in G \subset \uparrow z$. Note $z \nleq w$.

We show $z \ll x$. Suppose D is up-directed, $\sup D \geq x$. Then $\sup D \in G \Rightarrow d \in G$, $\exists\, d \in D$. Hence $z \leq d$. Thus $z \ll x$.

Certainly

$$x \geq \sup\{z : \exists \text{ an open filter } G \text{ with } x \in G \subset \uparrow z\}.$$

We have just shown for any $w < x$, $\exists\, z$ in this set with $z \nleq w$. Hence equality must hold. Since as above for each z in this set, $z \ll x$, the proof is completed by showing this set is up-directed and appealing to (iii)*.

Let F_1, F_2 be open filters, $x \in F_1 \cap F_2$, and $F_1 \subset \uparrow z_1$, $F_2 \subset \uparrow z_2$. Since $F_1 \cap F_2$ is an open filter, local boundedness implies $\exists$ an open filter $G \subseteq F_1 \cap F_2$ and $z_3 \in F_1 \cap F_2 \ni x \in G \subset \uparrow z_3$. Then $z_1 \leq z_3$ and $z_2 \leq z_3$.

2. The Scott Topology and the Category CS.

2.1 PROPOSITION. Let S be a continuous semilattice. Then the topology generated by the basis of open filters is precisely the Scott topology.

Proof. Let $\mathcal{U}$ denote the Scott topology, i.e., all "open" increasing sets. Let $x \in U \in \mathcal{U}$. $x = \sup\{y : y \ll x\}$, an up-directed set $\Rightarrow \exists\, y \ll x \ni y \in U$. By 1.4 $\exists$ an open filter $G \ni x \in G \subset \uparrow y \subset U$. Hence the open filters form a base for the Scott topology. ∎

2.2 PROPOSITION (Scott). Let S and T be continuous semilattices and let $f : S \to T$ be order-preserving.

Then f is continuous in the Scott topologies $\Leftrightarrow x_\alpha \nearrow x \Rightarrow f(x_\alpha) \nearrow f(x)$ ($x_\alpha \nearrow x$ means x_α is an increasing net with supremum x).

2.3 DEFINITION. The category $\underline{CS}$ of continuous semilattices has as its objects all continuous semilattices and as morphisms all identity preserving homomorphisms which also preserve sups of up-directed sets (i.e. are continuous for the Scott topologies).

3. Duality of Continuous Semilattices

3.1 DEFINITION. For $S \in \underline{CS}$, we define the dual $\hat{S}$ of S to be $\mathrm{Hom}_{\underline{CS}}(S, 2)$ (where 2 is $\{0, 1\}$). The order in the dual $\hat{S}$ is defined by $f \leq g \Leftrightarrow f(x) \leq g(x)\ \forall x \in S$.

Note that $f: S \to 2$ is a morphism in $\underline{CS}$ $\Leftrightarrow f^{-1}(1)$ is an open filter. Hence $\hat{S}$ may be identified with the set of open filters on S ordered by inclusion. $\hat{S}$ has an identity, namely S itself, and the semilattice operation is intersection. Furthermore, if $\mathcal{F}$ is a collection of open filters on S which is up-directed, then $\cup \mathcal{F}$ is an open filter which is the sup of the collection. Hence $\hat{S}$ is upper Dedekind complete.

3.2 PROPOSITION. Let S be a continuous semilattice. Then for open filters $G, F \in \hat{S}$, $G \ll F \Leftrightarrow \exists\, z \in F \ni G \subset \uparrow z$. Furthermore $\forall F \in \hat{S}$, $\mathcal{D}_F = \{G \in \hat{S} : G \ll F\}$ is up-directed and $F = \bigcup \mathcal{D}_F$. Hence $\hat{S}$ is a continuous semilattice.

Proof. Let $F \in \hat{S}$. Set
$\mathcal{D} = \{G \in \hat{S} : \exists\, z \in F \ni G \subset \uparrow z\}$. By 1.5
$F = \bigcup \mathcal{D}$. We show $\mathcal{D}$ is ascending. Let $G_1, G_2 \in \mathcal{D}$; $\exists\, z_1, z_2 \in F \ni G_1 \subset \uparrow z_1$, $G_2 \subset \uparrow z_2$. Then $z_1 \wedge z_2 \in F$. By 1.5 $\exists\, G_3$ and $w \in F \ni z_1 \wedge z_2 \in G_3 \subset \uparrow w$. Hence $G_1 \cup G_2 \subset G_3 \in \mathcal{D}$.

We show $G \ll F$ $\forall\, G \in \mathcal{D}$. Let $\{F_\alpha\}$ be an up-directed set of open filters, and suppose $F \subset \bigcup_\alpha F_\alpha$. If $G \in \mathcal{D}$, $\exists\, z \in F \ni G \subset \uparrow z$. Now $z \in F_\alpha$ for some α; hence $G \subset F_\alpha$. Thus $G \ll F$.

Conversely suppose $G \ll F$ in $\hat{S}$. Since $F = \bigcup \mathcal{D}$ and $\mathcal{D}$ is ascending, $\exists\, G' \in \mathcal{D} \ni G \subset G'$. But since G' has a lower bound in F, so also does G. Thus $G \in \mathcal{D}$.

3.3 DEFINITION. If $f: S \to T$ in $\underline{CS}$, define $\hat{f}: \hat{T} \to \hat{S}$ by $\hat{f}(\alpha) = \alpha \circ f$ for $\alpha \in \hat{T}$, $\alpha: T \to 2$. If $\hat{T}$ is identified with the open filters of T, then $\hat{f}(F)$ is the open filter $f^{-1}(F)$ in S. It is straightforward to verify that $\hat{f}: \hat{T} \to \hat{S}$ is also a $\underline{CS}$-morphism. Hence "$\wedge$" is a contravariant functor from $\underline{CS}$ into itself.

Also we define $\varepsilon_S : S \to \hat{\hat{S}}$ to be the standard evaluation mapping. In terms of filters $\varepsilon_S(X) = \{F \in \hat{S} : x \in F\}$. It is straightforward to verify that this set is an open filter in $\hat{S}$ and hence in $\hat{\hat{S}}$.

3.4. THE DUALITY THEOREM. The category $\underline{CS}$ is self-dual with respect to the functor $\wedge : \underline{CS} \to \underline{CS}$ and the natural isomorphism $\varepsilon : 1_{\underline{CS}} \to \hat{\hat{}}$.

Proof. We omit the straightforward proof that ε is a natural transformation, and show only that ε_S is an isomorphism for each S.

ε_S is injective. An immediate consequence of the definition of ε_S and the fact the open filters of S separate points (1.5).

ε_S is surjective. Let $\Gamma \in \hat{\hat{S}}$, i.e. Γ is an open filter on $\hat{S}$. Let $D = \{s \in S : s \text{ is a lower bound for some } F \in \Gamma\}$. We show D is up-directed. Let s resp. t be a lower bound for F_1 resp. F_2 where $F_1, F_2 \in \Gamma$. Since Γ is a filter, $F_1 \wedge F_2 = F \in \Gamma$. By 3.2 F is the union of the up-directed set of open filters G which have a lower bound in F. Since Γ is open, there exists an open filter G and $z \in F \ni$ $G \in \Gamma$ and $G \subset \uparrow z$. Hence $z \in D$. Also $s \leq z$ since $z \in F \subset F_1$. Similarly $t \leq z$. Thus D is up-directed.

Let $x = \sup D$. We claim $\Gamma = \mathcal{E}_s(x)$. Let $F \in \Gamma$. As in the preceding paragraph there exists an open filter G and $z \in F \ni G \in \Gamma$, $G \subset \uparrow z$. Thus $z \in D$ and hence $z \le x$. Since F is a filter, $x \in F$. Hence $F \in \Gamma \Rightarrow x \in F$.

Conversely, suppose F is an open filter in S and $x \in F$. Since $x = \sup D$, D is up-directed, and F is open, there exists $s \in D \ni s \in F$. But $\exists\, G \in \Gamma \ni s$ is a lower bound for G. Hence $G \subset \uparrow s \subset F$. Since Γ is a filter, $F \in \Gamma$. Thus $\Gamma = \mathcal{E}_s(x)$. ∎

4. Some Computations

(0) For S finite, $\hat{S} = S^{op}$, S with the order reversed.

(1) For $S \cong [0,1]$, $\hat{S} \cong (0, \tfrac{1}{2}] \cup \{1\}$.

(2) S is a lattice $\Leftrightarrow$ S is upper complete
$\Leftrightarrow$ $\hat{S}$ has property ((0)), i.e.,
$a \ll x$ and $a \ll y \Rightarrow a \ll x \wedge y$.

(3) S has a 0 $\Leftrightarrow$ $\{1\}$ is an open filter in $\hat{S}$
$\Leftrightarrow$ $1 \ll 1$ in $\hat{S}$

Combining (2) and (3) gives

(4) $S \in \underline{CL} \Leftrightarrow \hat{S}$ has property ((0)) and $1 \ll 1$ in $\hat{S}$.

We know $f: S \to T$ is a morphism in $\underline{CL}$ if it is a morphism in $\underline{CS}$ and has a left-adjoint $g: T \to S$, i.e. $s \ge g(t) \Leftrightarrow f(s) \ge t$.

(5) Let S and T be continuous semilattices. Let $f: S \to T$ be a morphism in <u>CS</u>. Then $f: S \to T$ has a left adjoint $g: T \to S \Leftrightarrow \hat{f}: \hat{T} \to \hat{S}$ preserves $\ll$, i.e. for $x, y \in \hat{T}$, $x \ll y$, then $\hat{f}(x) \ll \hat{f}(y)$.

It is somewhat surprising that in (2) and (5) we encounter conditions that have already arisen in the study of <u>CL</u>.

5. Questions

(1) The subbase $\{S \setminus \uparrow x : x \in S\} \cup \{\Uparrow x : x \in S\}$ makes a continuous semilattice S into a (Hausdorff) topological semilattice. Which continuous semilattices are locally compact for this topology? (I don't have an example of one that is not yet.) What is the dual notion? Characterize the continuous homomorphisms algebraically for this topology.

(2) Do arbitrary products and coproducts exist in <u>CS</u>? If not, for what important subcategories do they exist?

(3) What are the dual notions of distributivity and modularity? Some undertaking along the lines of the first part of Chpt. 3 of HMS Duality might be in order here.