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SCS 29: On the Closedness of the Set of Primes in Continuous Lattices

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TOPIC On the closedness of the set of primes in continuous lattices

REFERENCE [1] K.Keimel and M.Mislove SCS-Memo 9-30-76,
notably part 2

[2] K.H.Hofmann ,SCS-Memo 11-23-76

[3] O.Wyler, Algebraic theories of continuous lattices,
preprint (old title Compact complete lattices).

[4] K.H.Hofmann and J.D. Lawson, Irreducibility...,preprint.

The following question is fairly pressing:

(P) Let S be a CL-object. When is $\overline{\text{PRIME } S} = \text{PRIME } S$?

In [1] ,Keimel and Mislove have given a conclusive answer for distributive S , and in [4] the question of distributivity in CL was discussed at some length (Chapter 3). The condition Keimel and Mislove found to be necessary and sufficient in the distributive case was the following:

((0)) For all $a, x, y \in S$, the relations $a \ll x$ and $a \ll y$ imply $a \ll xy$.

This condition is evidently equivalent to any of the following

((0')) For any $s \in S$ the set $\text{int } \uparrow s$ is a filter. [2]

((0'')) For all $a, b, x, y \in S$, the relations $a \ll x$ and $b \ll y$ imply $ab \ll xy$.

((0''')) $\text{Graph } \ll$ is a subsemilattice of $S \times S$.

Keimel and Mislove give an example of a sublattice of the square violating this condition.

O.Wyler proves the following fact in [4] 12.5, old version:

If L is a lattice with 0 and 1 , then $\text{PRIME } \hat{L}$ is closed , where \hat{L} is the Z -dual of L (HMS -DUALITY).

West Germany: TH Darmstadt (Gierz, Keimel)
U. Tübingen (Mislove, Visit.)

England: U. Oxford (Scott)

USA: U. California, Riverside (Stralka)
LSU Baton Rouge (Lawson)
Tulane U., New Orleans (Hofmann, Mislove)
U. Tennessee, Knoxville (Carruth, Crawley)

The purpose of the Memo is to observe that this fact together with the observations by Keimel and Mislove in [1] suffice to show that ((0)) is sufficient for the closedness of $\text{PRIME } S$ regardless of distributivity. In order to make this a bit more self-contained, we present a proof of Wyler's proposition.

Recall that the \mathbb{Z} CL -topology of a CL-object S has a basis of the open sets

$$U(u; v_1, \dots, v_n) = \text{int } \uparrow u \setminus (\uparrow v_1 \cup \dots \cup \uparrow v_n),$$

$$\text{int } \uparrow u = \{x \in S : u \ll x\}.$$

LEMMA 1 . If $v_1 \dots v_n \leq u$, then $U(u; v_1, \dots, v_n) \cap \text{PRIME } S = \emptyset$.

Clear from the definitions. \square

PROPOSITION 2 . In a CL -object S , condition ((0)) implies $\overline{\text{PRIME } S} = \text{PRIME } S$.

Proof. Let $s \notin \text{PRIME } S$. Then there exist elements $x, y \notin \downarrow s$ with $xy \leq s$. Since S is a continuous lattice, we find elements $a \ll x$ and $b \ll y$ with $a, b \notin \downarrow s$. By condition ((0")) we know $ab \ll xy$. Thus $U(ab; a, b)$ is an open neighborhood of s which according to Lemma 1 ~~is~~ does not contain primes. \square

RECALL 3 . An algebraic lattice $S \in \underline{Z}$ is arithmetic if $K(S)$ is a sublattice (i.e. is closed under finite infs).

REMARK 4 . Any arithmetic lattice $S \in \underline{Z} \subseteq \underline{CL}$ satisfies condition ((0)).

Proof. Let $a \ll x, y$. Since $s = \sup(\downarrow s \cap K(S))$ for all $s \in S$, there are compact elements $c, k \in K(S)$ such that $a \ll c, k$ and $c \leq x$ and $k \leq y$. Since S is arithmetic, $ck \in K(S)$. Thus $a \leq ck \ll ck$, whence $a \ll ck$. \square

Start arithmetic [7]
COROLLARY 5 (Wyler) The set of primes is closed in any arithmetic lattice. This follows now from Proposition 2 and Remark 4.

COROLLARY 6 . If $S \in \underline{CL}$ and PS is as in ATLAS (the \underline{Z} -object of all lattice ideals), then $\text{PRIME } \text{PS}$ is closed. \square

For distributive S this occurs in the proof of 2.1 in Keimel-Mislove [1]. From ATLAS we recall the morphism $r_S: \text{PS} \rightarrow S$, $r_L(J) = \sup J$. By the preceding Corollary, $r(\text{PRIME } \text{PS})$ is a closed subspace of S which contains $\text{PRIME } S$ (since $p \in \text{PRIME } S$ implies $\downarrow p \in \text{PRIME } \text{PS}$). Keimel and

Mislove demonstrate that condition ((0)) implies that $J \in \text{PRIME } \mathcal{P}S$ always gives $\text{sup } J \in \text{PRIME } S$. We have

((n)) \Rightarrow ((n+1)):

((0))

((1)) If J is a prime ideal of S , then $\text{sup } J$ is a prime.

((2)) $\overline{\text{PRIME } S} = \text{PRIME } S$.

If S is distributive, then these conditions are equivalent. Example 4.2 in ATLAS satisfies ((2)), but not ((0)).

Question. Does ((2)) imply ((1))?

If S satisfies ((2)), then the distributive CL-subobject

$S' = \{x \in S : x = \text{inf}(\uparrow x \cap \text{PRIME } S)\}$ appears to play a role. It is not clear which. The closure operator $f: S \rightarrow S'$, $f(s) = \text{inf}(\uparrow s \cap \text{PRIME } S)$ is a lattice homomorphism. So what?