Communications on Stochastic Analysis

Volume 1 | Number 3 Article 7

12-1-2007

Almost any state of any amplitude Markov chain is recurrent

Luigi Accardi

Hiromichi Ohno

Recommended Citation

Accardi, Luigi and Ohno, Hiromichi (2007) "Almost any state of any amplitude Markov chain is recurrent," *Communications on Stochastic Analysis*: Vol. 1 : No. 3 , Article 7.
DOI: 10.31390/cosa.1.3.07
Available at: https://digitalcommons.lsu.edu/cosa/vol1/iss3/7

Communications on Stochastic Analysis
Vol. 1, No. 3 (2007) 441-453

ALMOST ANY STATE OF ANY AMPLITUDE MARKOV CHAIN IS RECURRENT

LUIGI ACCARDI AND HIROMICHI OHNO

Abstract. We introduce "amplitude Markov chains" associated to the matrix elements, in a fixed basis, of a unitary operator (discrete quantum dynamics). We prove the amplitude analogue of the relation between the recurrence probability of a state of a classical Markov chain and its first return probability.

This formula is then used to prove the universal property, mentioned in the title, which emphasizes a striking difference between the amplitude Markov chains and their classical analogues. This property is probably the statistical reflex of the reversibility of the quantum evolution. Finally note that, in the finite dimensional case, which is the most important one for the applications to quantum information, the word "almost" in the title can be omitted.

1. Introduction

The main difference between classical and quantum probability is that the former deals directly with probabilities while the latter deals with amplitudes from which the probabilities are obtained through the identity:

$$\text{probability } = |\text{amplitude}|^2 \in [0,1]. \tag{1.1}$$

One might therefore be tempted to develop a quantum probability calculus where "amplitudes, rather than probabilities, are added on disjoint events" (cf. [3]).

In some important cases (e.g. the two slit experiment [3]) this prescription leads to predictions in remarkable agreement with experiments. However, if one tries to apply this prescription literally, as a general principle, one might incur in contradictions. For example, if (A_n) is a sequence of disjoint events with amplitudes (ψ_n) one must have

$$\sum_n |\psi_n|^2 = 1,$$

but (by choosing for example $\psi_n = (\sqrt{6}/\pi)(1/n)$) one sees that it may happen that

$$\lim_{n\to\infty} |\psi_1 + \cdots + \psi_n| = +\infty.$$

In other words: the sum of probability amplitudes, corresponding to disjoint events, may not be a probability amplitude in the sense of (1.1). One can easily modify the above example so to obtain a finite set of disjoint events such that the square modulus of the sum of the corresponding amplitudes is > 1. In physics such

2000 *Mathematics Subject Classification.* Primary 60H40.

Key words and phrases. quantum probability, amplitude random walks, recurrent.

contradictions do not arise because the following empirical rule works remarkably well: “given a sequence of disjoint events, one should add their probabilities if these events are empirically distinguishable; one should add their amplitudes if they are empirically indistinguishable”. For example, in the two slit experiment, one adds probabilities if behind the screen there is an instrument allowing to distinguish through which slit the particle passed. One adds amplitudes otherwise.

From the mathematical point of view the problem amounts to the following: it is given a set A of complex numbers such that $\sum_{z\in F} |z|^2 \leq 1$ and one wants to know under which conditions, for any finite subset $F \subseteq A$ one has

$$\left|\sum_{z\in F} z\right| \leq 1. \tag{1.2}$$

Since a simple and easily applicable criterium, characterizing these sets, is not known, the best thing one can do at the moment, is to produce interesting examples of sets A and of families $\mathcal{F}$, of subsets of A with the following properties:

- **(i):** For each $F \in \mathcal{F}$, (1.2) is satisfied.
- **(ii):** On the sets of $\mathcal{F}$, the naive extension of the classical probabilistic techniques to amplitudes leads to coherent results.

Coherence has to be checked case by case because there is no general result saying that, even when restricted to the above mentioned sets, this generalization will not lead to inconsistencies, such as “probabilities” greater than 1. In the present note we discuss the possibility to extend to amplitudes and to quantum states the known relationship between the survival probability of a state of a classical discrete Markov chain and its first return probability.

We prove that the classical probabilistic analysis of this problem can be extended to amplitudes and leads to coherent results (i.e. to meaningful probabilities). We deduce an explicit form for the recurrence amplitude and we prove that, in the generic cas, the associated recurrence probability is equal to 1 independently of the unitary dynamics and for all states.

This is a striking difference with the classical case where, depending on the transition probability matrix of the chain, the recurrence probability of any state can vary continuously from 0 to 1 (cf. [2], Section 4).

From the point of view of physical interpretation, we believe that this surprising difference is a manifestation of the reversibility of the unitary quantum evolution. It may also be considered as a manifestation of the Poincarè recurrence theorem in classical mechanics.

An experiment to verify this property should keep in mind that all our considerations involve probability amplitudes, i.e. pure states, therefore no intermediate measurement should be performed before the recurrence takes place, otherwise this would introduce a decoherence and the amplitude should be replaced by a density matrix.

One possible candidate for such an experiment is to check the recurrence of the highest energy state of a three-level atom in Lambda–configuration. The atom should be initially brought to the upper level and then protected enough to guarantee that it can be considedered as an isolated system. If, after some time, a radiation is emitted, then one can be sure that the atom is returned to the upper state because, by definition of Lambda–configuration, no other jump up is allowed.

The results of the present note are also related to [1].

2. Amplitude Markov chains

Definition 2.1. Let Ω, S be countable sets. An *amplitude measure* on Ω conditioned on S is a family of map,

$$\psi_{ij} : \omega \in \Omega \to \psi_{ij}(\omega) \in \mathbb{C} \ ; \quad i, j \in S$$

such that

$$p_{ij} := \left| \sum_{\omega \in \Omega} \psi_{ij}(\omega) \right|^2$$

satisfy

$$\sum_{i \in S} p_{ij} = \sum_{j \in S} p_{ij} = 1.$$

In this definition, if S is a finite set, then the matrix (p_{ij}) is bi-stochastic.

Example. Let $S = \{1, \ldots, d\}$ and, for any $n \in \mathbb{N}(n \geq 1)$ let

$$\Omega = \Omega_n = S^n$$

the space of paths (or configurations) over the state space S. Let $U = (\psi_{ij})$ be a unitary $d \times d$ matrix. For $i, j \in S$, for each $n \in \mathbb{N}$ and

$$\omega := (j_1, \ldots, j_n) \in \Omega_n = S^n,$$

define

$$\psi^{(n)}_{ij}(\omega) = \psi_{ij_1} \psi_{j_1 j_2} \cdots \psi_{j_n j}. \tag{2.1}$$

Then,

$$\sum_{\omega \in \Omega_n} \psi^{(n)}_{ij}(\omega) = (U^n)_{ij}$$

so that the family of maps $\omega \mapsto \psi^{(n)}_{ij}(\omega)$ is an amplitude measure on Ω conditioned on S.

Definition 2.2. Given a unitary U in $\mathcal{B}(l^2(S))$, the amplitude measure $\{\psi^{(n)}_{ij}\}$ defined by (2.1) will be called an *amplitude Markov chain.*

In this definition, we allow that S is a infinite countable set. In the following we shall only consider amplitude Markov chains.

If $A \subseteq \Omega_n$ is any subset such that, defining

$$\psi^{(n)}_{ij}(A) := \sum_{\omega \in A} \psi^{(n)}_{ij}(\omega) \ ; \quad (i, j) \in S^2$$

one has

$$|\psi^{(n)}_{ij}(A)|^2 \leq 1, \tag{2.2}$$

then $\psi^{(n)}_{ij}(A)$ will be called the conditional amplitude of A given the boundary conditions (i, j). All the subsets $A \subseteq \Omega$ of the form

$$A = I_n \times \cdots \times I_1 \ ; \quad I_\alpha \subseteq S$$

are such that $\psi_{ij}^{(n)}(A)$ is a conditional amplitude of A because, in this case, this amplitude is:

$$\langle e_j, U \cdot P_{I_n} U \cdots P_{I_2} U \cdot P_{I_1} U \cdot e_i \rangle \tag{2.3}$$

where (e_i) is the canonical basis of $B(\ell^2(S))$ and $P_I (I \subseteq S)$ is the projection onto the space

$$\mathcal{H}_I = \overline{\text{span}}\{e_i : i \in I\}.$$

It is clear that these $\psi_{ij}(I_n \times \cdots \times I_1)$ satisfy (2.2).

These amplitudes can be experimentally realized by selective filters, without replacement of particles, followed by the U–dynamics between any two of them.

Notice that, since we do not normalize the wave function, the particles filtered out at the n–steps must be registered and kept into account in the statistics. More precisely, the experimental frequency, to be compared with the square modules of (2.3) will be N_j/N where:

N_j is the number of particles which have given positive response to the e_j–measurement at time $(n+1)$ and N is the total number of particles including those rejected by the intermediate filters.

In particular $\forall\, i, j \in S$ the amplitude to go from i to j in n steps is well defined because it corresponds to the set

$$A_n(i,j) := S \times S \times \cdots \times S = S^n$$

when $i = j$ this is called in physics the n step survival amplitude, (under the dynamics U) and the corresponding probability is called the n step survival probability.

The same is true for the amplitude of first arrival from i to j in n steps, which corresponds to the set

$$A_n(i,j) = (S \backslash \{j\}) \times (S \backslash \{j\}) \times \cdots \times (S \backslash \{j\}) = (S \backslash \{i\})^n.$$

In the theory of classical Markov chains, there is a famous relation between the probabilities of these two events (cf. [2]), chap. XII, sections 2, 3, 4).

Our goal in the next section is to prove that a similar relation is true for amplitudes.

3. Recurrence for Q–amplitudes

Let $S := \{1, \ldots, d\}$ $(d \leq +\infty)$ be an at most countable set and let $\mathcal{H} := \ell^2(S) \cong \mathbb{C}^d$ be the Hilbert space of square integrable sequences of complex numbers. For any unitary operator U on $\mathcal{H}$ and for any orthonormal basis (e_n) of $\mathcal{H}$, define the n–stemp transition amplitude from k to j:

$$\psi_{jk}(n) := (U^n)_{jk} = \langle e_j, U^n e_k \rangle$$

which, for $j = k$ gives the n-step survival amplitude of the quantum state j and notice that $\psi_{jk}(0) := \delta_{jk}$. The amplitude that, starting from j at time 0, the system arrives in k after n instants but not before is denoted by $\psi(n; j, k)$ and, in analogy with [2] (chap. XIII.2), we define $\psi(0; j, k) := \delta_{j,k}$.

Notice that, if one defines $\psi(n;j,k)$ by applying the analogue of the usual Markovian formula for the joint probabilities, then one has

$$\psi(n;j,k) := \sum_{j_1,\dots,j_{n-1}\neq k} U_{j_{n-1}k}U_{j_{n-2}j_{n-1}}\dots U_{j_1j_2}U_{jj_1}.$$

In analogy with the recurrence theory of classical Markov chains (cf. Fellr–I) we would like to introduce the amplitude that, starting from j the system before or later comes back to j (the return amplitude to j):

$$\rho(j) := \sum_{n=1}^{\infty} \psi(n;j,j). \tag{3.1}$$

However we must take into account a difference between amplitudes and probabilities. Namely that, while the return probability to e_j is always well defined as the sum of the probabilities of a disjoint family of events, the corresponding return amplitude is not always defined because, as we have explained in Section 1, the series (3.1) might not converge to something which is not a probability amplitude. For this reason, we postpone alter Proposition 1 a formal definition of the return amplitude.

Clearly one could interpret the definition in (6) in the sense that when the series on the right hand side converges then the equality defines the left hand side. However we will see that the analogy with the classical Markov chains suggests a more subtle point of view.

We will need the following.

Lemma 3.1. *For an arbitrary complex number $\psi = a + ib$ the condition*

$$Re\ (\psi) := a \geq -1/2 \tag{3.2}$$

is equivalent to

$$\left|\frac{\psi}{1+\psi}\right|^2 \leq 1 \tag{3.3}$$

and the equality holds iff

$$Re\ (\psi) = -1/2$$

Proof. In the above notations

$$\left|\frac{\psi}{1+\psi}\right|^2 = \frac{a^2+b^2}{(1+a)^2+b^2} = \frac{a^2+b^2}{a^2+b^2+1+2a}.$$

Therefore

$$\left|\frac{\psi}{1+\psi}\right| \leq 1 \quad \Leftrightarrow a^2+b^2 \leq a^2+b^2+1+2a \Leftrightarrow a \geq -1/2$$

and the equality holds iff $a = -1/2$. $\square$

Proposition 3.2. *Let U be a unitary operator on a Hilbert space $\mathcal{H}$ and let (e_j) be an orthonormal basis of $\mathcal{H}$. For $z \in \mathbb{C}$, $|z| < 1$, both series:*

$$\begin{aligned} \psi_j(z) &:= \sum_{n=1}^{\infty} \psi_{jj}(n) z^n = \sum_{n=1}^{\infty} \langle e_j, U^n e_j \rangle z^n = \langle e_j, \frac{zU}{1-zU} e_j \rangle \\ &= -1 + \langle e_j, \frac{1}{1-zU} e_j \rangle \end{aligned} \tag{3.4}$$

and

$$\rho_j(z) := \sum_{n=1}^{\infty} \psi(n; j, j) \cdot z^n \tag{3.5}$$

converge and the identity

$$\rho_j(z) = \frac{\psi_j(z)}{1 + \psi_j(z)} \tag{3.6}$$

holds.

Proof. The series (3.4) converges because $\|zU\| = |z| < 1$. The series (3.5) converges because

$$\begin{aligned} \psi(n; j, j) &= \sum_{\substack{j_1, \dots, j_{n-1} = 1 \\ j_1, \dots, j_{n-1} \neq j}}^{\infty} U_{jj_1} U_{j_1 j_2} \cdots U_{j_{n-1} j} \\ &= \langle e_j, U P_j^{\perp} \cdot U P_j^{\perp} \cdot \cdots \cdot U P_j^{\perp} \cdot U e_j \rangle \end{aligned} \tag{3.7}$$

so that

$$|\psi(n; j, j)| \leq \|e_j\| \cdot \|U P_j^{\perp}\|^{n-1} \|U e_j\| \leq 1.$$

By definition:

$$\psi_{jj}(n) = \langle e_j, U^n e_j \rangle$$

from which (3.4) follows. Now notice that

$$\psi_{jj}(n) = \langle j, U^n j \rangle = (U^n)_{jj} = \sum_{j_1, \dots, j_{n-1}} U_{jj_1} U_{j_1 j_2} \dots U_{j_{n-1} j}$$

where the sum is extended to the set $\mathcal{F}_n(S)$ of all functions

$$\pi : \{1, \dots, n-1\} \to S \; ; \quad n \geq 2$$

Now, denoting for $k = 1, \dots, n$,

$$\mathcal{F}_{n,k}(S) := \{\pi \in \mathcal{F}_n(S) : \pi(k) = j \; ; \; \pi(h) \neq j \; ; \; h < k\}$$

one has

$$\mathcal{F}_n(S) = \bigcup_{k=1}^{n} \mathcal{F}_{n,k}(S)$$

and the union is disjoint. Therefore

$$\begin{aligned}
\psi_{jj}(n) &= \sum_{k=1}^{n} \sum_{(j_1,\ldots,j_{n-1})\in\mathcal{F}_{n,k}(S)} U_{jj_1}\cdots U_{j_{n-1}j} \\
&= \sum_{k=1}^{n} \left(\sum_{\substack{(j_1,\ldots,j_{k-1}) \\ j_h\neq j}} U_{jj_1}\cdots U_{j_{k-1}j} \right) \sum_{(j_{k+1},\ldots,j_{n-1})} U_{jj_{k+1}}\cdots U_{j_{n-1}j} \\
&= \sum_{k=1}^{n} \psi(k;j,j)\psi_{jj}(n-k). \qquad (3.8)
\end{aligned}$$

The recurrence formula (3.8) implies that

$$\begin{aligned}
\psi_j(z) &= \sum_{n=1}^{\infty} \psi_{jj}(n)\cdot z^n = \sum_{n=1}^{\infty}\left[\sum_{k=1}^{n}\psi(k;j,j)\psi_{jj}(n-k)\right] z^n = \\
&= \psi(1;j,j)z \\
&+ \psi(1;j,j)\psi_{jj}(1)\cdot z^2 + \psi(2;j,j)\cdot z^2 + \\
&+ \psi(1;j,j)\psi_{jj}(2)\cdot z^3 + \psi(2;j,j)\cdot\psi_{jj}(1)\cdot z^3 + \psi(3;j,j)z^3 + \\
&\vdots \\
&+ \psi(1;j,j)\psi_{jj}(n-1)z^n + \psi(2;j,j)\psi_{jj}(n-2)\cdot z^n + \cdots + \psi(n;j,j)z^n \\
&+ \cdots \\
&= \psi(1,j,j)\cdot z\left[1+\sum_{n=1}^{\infty}\psi_{jj}(n)z^n\right] \\
&+ \psi(2;j,j)\cdot z^2\left[1+\sum_{n=1}^{\infty}\psi_{jj}(n)\cdot z^n\right] + \cdots \\
&= \sum_{k=1}^{\infty}\psi(k;j,j)\cdot z^k\cdot[1+\psi_j(z)] \\
&= \rho_j(z)\cdot[1+\psi_j(z)].
\end{aligned}$$

So that

$$\psi_j(z) = \rho_j(z)\cdot[1+\psi_j(z)]. \qquad (3.9)$$

In particular we see that either $\psi_j(z)$ or $\rho_j(z)$ can be zero if and only if $\psi_j(z)$ and $\rho_j(z)$ are both is identically zero. From now on we exclude this case.

Since we know that both $\rho_j(z)$ and $\psi_j(z)$ are analytic functions in $|z| < 1$, it follows that $\psi_j(z)$ can never be equal to -1. Therefore (3.9) is equivalent to (3.6) and this proves the statement. □

Definition 3.3. In the above notations the *return probability* to the state e_j for the dynamics U is defined by

$$P_U(j) := \lim_{s\uparrow 1}\left|\frac{\psi_j(s)}{1+\psi_j(s)}\right|^2. \qquad (3.10)$$

The corresponding return amplitude is defined by

$$\rho(j) := \lim_{s\uparrow 1} \frac{\psi_j(s)}{1+\psi_j(s)} = \lim_{s\uparrow 1} \rho_j(s). \tag{3.11}$$

Remark. It is not obvious from the definition that, if the limit (3.10) (resp. (3.11)) exists, then it defines a probability, i.e. $P_U(j) \in [0,1]$, (resp. an amplitude, i.e. $|\rho(j)| \leq 1$). In the following we will prove that this is indeed the case.

In both cases the definition is meant in the sense that, if the limit on the right hand side exists, then the objects on the left hand side are well defined.

Definition 3.4. A state j is called:

(1) *recurrent* if $p_U(j) = 1$.
(2) *non recurrent* if $p_U(j) < 1$.
(3) *strictly null* if $\psi_{jj}(n) = 0$. for each $n \in \mathbb{N}$, $n \geq 1$.
(4) *null* if $\psi_{jj}(n) \to 0$ as $n \to \infty$.
(5) *non null* if $\psi_{jj}(n) \not\to 0$ as $n \to \infty$.
(6) *periodic* if $1 \neq d_j < \infty$, where

$$d_j := MCD\{n : \psi(n; j, j) \neq 0\}.$$

If U has a fixed point except 0, that is, the point spectrum of U contains 1, then we can define the non-zero Hilbert subspace

$$\mathcal{H}_{\text{fix}} = \{\xi \in \mathcal{H} \,|\, U\xi = \xi\} = P_1\mathcal{H},$$

where P_1 is the spectral projection of U at 1. We put $\mathcal{H} = \mathcal{H}_{\text{fix}} \oplus \mathcal{H}_0$, where $\mathcal{H}_0$ is the orthogonal space of $\mathcal{H}_{\text{fix}}$. Then $1-U$ is injective on $\mathcal{H}_0$ and we can restrict our attention to the (possibly unbounded) operator $(1-U)^{-1}$ on $\mathcal{H}_0$ because all vectors in $\mathcal{H}_{\text{fix}}$ are clearly recurrent.

4. Classification of states

Lemma 4.1. *If U has discrete spectrum then it does not admit strictly null states.*

Proof. Let us represent U in the form

$$U = \sum_j e^{i\theta_j} |e_j\rangle\langle e_j|.$$

Then, for any unit vector $\xi \in \mathcal{H}$, we have

$$\langle \xi, U^n \xi\rangle = \sum_j \langle \xi, e_j\rangle\langle e_j, \xi\rangle e^{in\theta_j} = \sum_j |\langle \xi, e_j\rangle|^2 e^{in\theta_j}.$$

Let us denote

$$f_j := |\langle \xi, e_j\rangle|^2 \geq 0.$$

One has:

$$\sum_j f_j = 1. \tag{4.1}$$

Therefore, if ξ is strictly null, then for any $s \in (0,1)$:

$$\begin{aligned} 0 &= \sum_{n=1}^{\infty}\sum_j f_j e^{in\theta_j} s^n = \sum_j f_j \sum_{n=1}^{\infty} e^{in\theta_j} s^n = \sum_j f_j \frac{e^{i\theta_j}s}{1-se^{i\theta_j}} \\ &= \sum_j f_j s \frac{\cos\theta_j - s}{(1-s)^2 + 2s(1-\cos\theta_j)} \\ &+ i\sum_j f_j s \frac{\sin\theta_j}{(1-s)^2 + 2s(1-\cos\theta_j)} \end{aligned} \tag{4.2}$$

In particular the real part must be zero and, since $s \in (0,1)$, this implies that each $f_j = 0$, contradicting (4.1). Thus no strictly null state can exist. □

Remark. The assumption that U has discrete spectrum is essential. If U is the one–sided shift with respect. To the orthonormal basis $(e_j)_{j\in\mathbb{Z}}$ of $\mathcal{H}$, then

$$\langle e_j, U^n e_j\rangle = \langle e_j, e_{j+n}\rangle = 0 \ ; \qquad \forall\, n \geq 1$$

hence any vector e_j is strictly null.

Theorem 4.2. *If e_j is in the domain of $(1-U)^{-1}$, then the limit* (3.11) *exists, i.e. the return amplitude to j is well defined. Moreover the corresponding return probability is*

$$|\rho(j)|^2 = 1,$$

i.e. the state e_j is recurrent.

Proof. Introduce the spectral decomposition of U:

$$U = \int_0^{2\pi} e^{i\theta} E(d\theta)$$

and notice that, for $\theta \neq, 0, 2\pi$, one has:

$$\begin{aligned} \frac{1}{1-e^{i\theta}} &= \frac{1-\overline{e}^{-i\theta}}{|1-e^{-i\theta}|^2} = \frac{(1-\cos\theta) + i\sin\theta}{2(1-\cos\theta)} = \frac{1}{2} + \frac{i}{2}\frac{\sin\theta}{1-\cos\theta} \\ &= \frac{1}{2} + \frac{i}{2}\operatorname{cotg}\theta/2. \end{aligned}$$

Defining the operator

$$H := \int_0^{2\pi} \theta E(d\theta),$$

we write the operator

$$\begin{aligned} \frac{1}{1-U} &= \int \frac{1}{1-e^{i\theta}} E(d\theta) \\ &= \frac{1}{2}\int_0^{2\pi} E(d\theta) + \frac{i}{2}\int_0^{2\pi} \operatorname{cotg}\,(\theta/2) E(d\theta) \\ &= \frac{1}{2} + \frac{i}{2}\operatorname{cotg}\,(H/2) \end{aligned}$$

on the domain of $(1-U)^{-1}$.

This implies that, whenever e_j is in the domain of $(1-U)^{-1}$ the quantity

$$\psi_j(1) = -1 + \langle e_j, (1-U)^{-1} e_j\rangle = -\frac{1}{2} + \frac{i}{2}\langle j, \mathrm{cotg}(H/2)j\rangle \tag{4.3}$$

is well defined for all e_j in the domain of $(1-U)^{-1}$ and the above equality holds. Moreover if e_j is in this domain, then

$$\lim_{s\uparrow 1} \psi_j(s) =: \psi_j(1) \tag{4.4}$$

always exists and because of (4.3), satisfies

$$\mathrm{Re}\ \psi_j(1) = -\frac{1}{2}.$$

Therefore also the limit

$$\rho(j) = \lim_{s\uparrow 1} \rho_j(s) = \lim_{s\uparrow 1} \sum_{n\geq 1} \psi(n; j, j)\cdot s^n = \sum_n \psi(n, j, j)$$

exists and, by Lemma 1, satisfies $|\rho(j)| = |\rho_j(1)| = 1$. □

Corollary 4.3. *If $\mathcal{H}$ is finite dimensional, then any state ξ is recurrent.*

Proof. If $\xi \in \mathcal{H}_{\mathrm{fix}}$, then it is recurrent. If $\xi \in \mathcal{H}_0$, then it is in the domain of $(1-U)^{-1}$ and the thesis follows from theorem 4.2.

If $\xi = \alpha\xi_{\mathrm{fix}} + \beta\xi_0$ with $\alpha, \beta \neq 0$, $\xi_{\mathrm{fix}} \in \mathcal{H}_{\mathrm{fix}}\backslash\{0\}$ and $\xi_0 \in \mathcal{H}_0$, then we have

$$\begin{aligned} |\sum_{n=1}^{\infty}\langle \xi, s^n U^n \xi\rangle| &= |\sum_{n=1}^{\infty} |\alpha|^2 \langle \xi_{\mathrm{fix}}, s^n U^n \xi_{\mathrm{fix}}\rangle + |\beta|^2\langle \xi_0, s^n U^n \xi_0\rangle| \\ &= ||\alpha|^2 \frac{1}{1-s} + |\beta|^2 \langle \xi_0, (1-U)^{-1}\xi_0\rangle| \to \infty \end{aligned}$$

as $s \uparrow 1$. This implies that ξ is recurrent. □

Remark. We have already proved that, if e_j belongs to the domain of $(1-U)^{-1}$ then it is recurrent.

Now we are interested in the case when e_j is not in the domain of $(1-U)^{-1}$. This means that the integral

$$\int_0^{2\pi} (\mathrm{cotg}\theta/2)\mu_j(d\theta) \tag{4.5}$$

is either $\pm\infty$ or it does not exist, where $\mu_j(d\theta) = \langle e_j, E(d\theta)e_j\rangle$.

The following considerations show that a large class of states, which are not in the domain of $(1-U)^{-1}$, is recurrent. This is in some sense expected because a state not in the domain of $(1-U)^{-1}$ is "almost" a fixed point of U.

The main remark needed to prove this is that, if a state e_j is not in the domain of $(1-U)^{-1}$, but

$$\lim_{s\uparrow 1} |\psi_j(s)| = \lim_{s\uparrow 1} \left|\langle e_j, \frac{sU}{1-sU}\, e_j\rangle\right| = +\infty \tag{4.6}$$

in the sense that the limit exists and the equality holds, then

$$\lim_{s\uparrow 1}|\rho_j(s)| = \lim_{s\uparrow 1}\left|\frac{\psi_j(s)}{1+\psi_j(s)}\right| = \lim_{s\uparrow 1}\left|\frac{1}{1+1/\psi_j(s)}\right| = 1 \tag{4.7}$$

and this means the state e_j is recurrent.

Remark. The following arguments show that the only case when the state e_j may not be recurrent is when the spectral measure μ_j is such that the integrals

$$\int_{2\varepsilon}^{2(\pi-\varepsilon)} \operatorname{cotg}\theta/2\mu_j(d\theta)$$

are strongly oscillating as $\varepsilon \downarrow 0$.

Theorem 4.4. *If the integral* (4.5) *is either* $+\infty$ *or* $-\infty$ *then the state* e_j *is recurrent.*

Proof. First we get the identity:

$$\begin{aligned}\frac{1}{1-se^{i\theta}} &= \frac{1-se^{-i\theta}}{|1-se^{i\theta}|^2} = \frac{1-s\cos\theta+is\sin\theta}{|(1-s\cos\theta)-is\sin\theta|^2} \\ &= \frac{1-s\cos\theta+is\sin\theta}{1+s^2\cos^2\theta-2s\cos\theta+s^2\sin^2\theta} = \frac{1-s\cos\theta+is\sin\theta}{1+s^2-2s\cos\theta} \\ &= \frac{1-s\cos\theta}{1+s^2-2s\cos\theta}+\frac{is\sin\theta}{1+s^2-2s\cos\theta}. \end{aligned}\tag{4.8}$$

We only consider the imaginary part. As $s\uparrow 1$, the function

$$\frac{s\sin\theta}{1+s^2-2s\cos\theta} = \frac{\sin\theta}{\frac{1+s^2}{s}-2\cos\theta} =: I_j(s,\theta) \tag{4.9}$$

is monotone either decreasing or increasing for fixed θ (according to the sign of $\sin\theta$). Indeed, the function $\frac{1+s^2}{s}$ is decreasing on $s\in(0,1)$. Moreover, as $s\uparrow 1$, $I_j(s,\theta)$ converges to

$$\frac{\sin\theta}{2(1+\cos\theta)} = \frac{1}{2}\operatorname{cotg}(\theta/2).$$

If (4.5) is $+\infty$ then the negative part of the integral must be finite. Therefore, by the above remark, it will go to $+\infty$ monotonically increasing. Therefore we can apply Beppo Levi's monotone convergence theorem to (4.9) and conclude that

$$\lim_{s\uparrow 1}\ \mathrm{Im}\ (\psi_j(s)) = +\infty, \tag{4.10}$$

and this implies e_j is recurrent. A similar argument can be applied if (4.5) is $-\infty$. $\square$

Remark. It remains the case when the integral (4.5) does not exist. Even in this case the limit

$$\lim_{s\uparrow 1}\psi_j(s) =: I_j \tag{4.11}$$

may exist (cf. Lemma 4.5 below). If this happens and the real part of (4.8) convergences to 1/2, one has

$$\lim_{s\uparrow 1}\frac{\psi_j(s)}{1+\psi_j} = \frac{-1/2+iI_j}{1/2+iI_j}$$

and the state e_j is recurrent.

Lemma 4.5. *If the spectral measure $\mu_j(d\theta)$ has a C^1–density $p(\theta)$ with respect to the Lebesgue measure satisfying*

$$p(2\pi) = p(0) \ , \tag{4.12}$$

then the limit (4.11) *exists.*

Proof. In the notation (4.9) one has, for any $0 < \varepsilon < \pi/4$:

$$\begin{aligned}
\mathrm{Im}\ (\psi_j(s)) &= \int_0^{2\pi} I_j(s,\theta)p_j(\theta)d\theta \\
&= \int_0^{\varepsilon} I_j(s,\theta)p_j(\theta)d\theta + \int_{\varepsilon}^{2\pi-\varepsilon} I_j(s,\theta)p_j(\theta)d\theta \\
&+ \int_{2\pi-\varepsilon}^{2\pi} I_j(s,\theta)p_j(\theta)d\theta
\end{aligned}$$

for fixed $\varepsilon \in (0, \pi/4)$ the integrand of the middle term is bounded by a constant uniformly in s. Therefore, since μ_j is a bounded measure, the limit of the middle term as $s \uparrow 1$ exists and is equal to

$$\int_{\varepsilon}^{2\pi-\varepsilon} \cot\theta/2\mu_j(d\theta).$$

The density p can be extended to $\mathbb{R}$ by 2π–periodicity and condition (4.12) implies that this extension, still denoted p, is continuous.

Moreover, since p is C^1 in $[0, 2\pi]$, this extension is left and right differentiable at zero. With these notations, using the 2π–periodicity of $I_j(s,\theta)$ and of p and with the change of variables $\theta \to \theta - 2\pi$ becomes, and (4.12), we have

$$\int_{2\pi-\varepsilon}^{2\pi} I_j(s,\theta)p_j(\theta)d\theta = \int_{-\varepsilon}^{0} I_j(s,\theta)p(\theta)d\theta.$$

Therefore, since $I_j(s,-\theta) = -I_j(s,\theta)$:

$$\begin{aligned}
&\int_0^{\varepsilon} I_j(s,\theta)p_j(\theta)d\theta + \int_{2\pi-\varepsilon}^{2\pi} I_j(s,\theta)p_j(\theta)d\theta \\
= &\int_0^{\varepsilon} I_j(s,\theta)p(\theta)d\theta + \int_{-\varepsilon}^{0} I_j(s,\theta)p(\theta)d\theta \\
= &\int_0^{\varepsilon} I_j(s,\theta)[p(\theta) - p(-\theta)]d\theta.
\end{aligned} \tag{4.13}$$

As $s \uparrow 1$ the integrand converges to

$$\frac{\cos\theta/2}{\sin\theta/2}\,[p(\theta) - p(-\theta)]$$

which is integrable in $(0, \varepsilon)$ because the function $p(\theta) - p(-\theta)$ continuous and left and right differentiable at zero. Therefore the limit (4.13), as $s \uparrow 1$, exists by dominated convergence. □

Remark. If U is the one–sided shift with respect to the orthonormal basis $(e_j)_{j\in\mathbb{Z}}$ of $\mathcal{H}$, then

$$\langle e_j, U^n e_j\rangle = \langle e_j, e_{j+n}\rangle = 0 \ ; \qquad \forall\, n \geq 1$$

hence any state e_j is non recurrent.

Acknowledgment. The authors are grateful to Oleg Smolyanov for fruitful discussions and suggestions concerning the last section of the present note.

References

1. Accardi, L. and Fidaleo, F.: Entangled Markov chains, *Annali di Matematica Pura e Applicata* (2004) Preprint Volterra N. 556 (2003) http://www.springerlink.com/index/10.1007/s10231-004-0118-4.
2. Feller, W.: *An introduction to probability theory and its applications*, Vol. I, John Wiley, New York, London, Sydney, 1966.
3. Feynmann, R. P., Leighton, R.P., and Sands M.: *Lectures on Physics, III*, Addison–Wesley, 1966.

Centro Vito Volterra, Università di Roma "Tor Vergata", Roma I-00133, Italy

Graduate School of Information Sciences, Tohoku University, Sendai 980-8579, Japan