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SCS 25: Observations

Karl Heinrich Hofmann

Technische Universität Darmstadt, Germany, hofmann@mathematik.tu-darmstadt.de

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SEMINAR ON CONTINUITY IN SEMILATTICES (SCS)

NAME(S) HOFMANN

DATE M D Y

November 23 76

TOPIC OBSERVATIONS

REFERENCE KEIMEL - MISLOVE SCS 9 - 30-76 (rec'd Tulane 11-12-76)

1) Keimel and Mislove establish in SCS 9-30 the following

LEMMA. Let $L \in \underline{CL}$ and S a closed subsemilattice ($1 \in S!$), $s \in L \setminus S$. Then there is a closed subsemilattice T of L such that

(i) $s \notin T$. (ii) $S \subseteq \text{int } T$.

I feel the necessity to prove this again.

Proof. Since L has small subsemilattices and by Nachbin's theorem there is a closed convex subsemilattice neighborhood U of s with $U \cap S = \emptyset$. (Nachbin's theorem says in our context that the convex closed neighborhoods of s form a basis; we observe that the ^{closed} convex hull of a semilattice neighborhood is a semilattice neighborhood.)

Let $u = \min U$, and set $m = \min \uparrow u \cap S$. I claim there is an $a \ll m$ such that $\uparrow a \cap U = \emptyset$ (for if not, then for every $a \ll m$ we would have an $u_a \in \uparrow a \cap U$; if v is a cluster point of the net $(u_a)_{a \ll m}$ then $u \leq m \leq v$ with $u, v \in U$; by the convexity of U we would have $m \in U$, which would contradict $m \in S$ because of $U \cap S = \emptyset$). We observe

(1) If $t \in \uparrow u \cap S$; then $t \in \text{int } \uparrow a$; also $s \notin \uparrow a$; indeed $\uparrow u \cap S \subseteq \uparrow m \subseteq \text{int } \uparrow a$. The ~~last~~ ^{second} assertion is due to $\uparrow a \cap U = \emptyset$.
Let ~~$I(u) = S \cap \uparrow u$~~ $I(u) = S \cap \uparrow u$. Then $s \notin S \setminus \text{int } \uparrow u$,

West Germany: TH Darmstadt (Gierz, Keimel)
U. Tübingen (Mislove, Visit.)

England: U. Oxford (Scott)

USA: U. California, Riverside (Stralka)
LSU Baton Rouge (Lawson)
Tulane U., New Orleans (Hofmann, Mislove)
U. Tennessee, Knoxville (Carruth, Crawley)

since $s \in \text{int } U \subseteq \text{int } \uparrow u$. We record (in view of $I(u) \subseteq \text{int } I(u)^-$):

(2) If $t \in S \setminus \uparrow u$, then $t \in \text{int } I(u)^-$; also $s \notin I(u)^-$.

We now set $T = I(u)^- \cup \uparrow a$; since $I(u)^-$ is an ideal, this is a closed subsemigroup. Because of (1) and (2) ~~ix~~ it satisfies the requirements □

However, I wonder, whether this is the most economic approach to the

PROPOSITION (Keimel, Mislove). If $L \in \underline{CL}$, then $\dot{\Gamma}(L) = \{S \in \Gamma(L) : 1 \in S = SS\}$ is a continuous lattice.

Proof. Let $A, B \in \dot{\Gamma}(L)$. Then AB is the smallest closed subsemilattice containing $A \cup B$: Indeed, if S is this subsemilattice, then $(A \cup B)^2 \subseteq S$, but $(A \cup B)^2 = A \cup B \cup AB = AB$ (since $A = A1 \subseteq AB$ and $B = 1B \subseteq AB$). On the other hand, AB is a subsemigroup containing A, B hence $A \cup B$, whence $S \subseteq AB$. Thus, ~~the~~ while the (inf) operation on $\Gamma(L)$ is $(A, B) \mapsto A \cup B$, the (inf) operation on $\dot{\Gamma}(L)$ is $(A, B) \mapsto AB$. Now, when Schliemann excavated Troya he found a ceramic tile in Troya 4 (fourth from top) on which there was engraved: "If thou takest a compact abelian monoid, then thou findest that the space of closed submonoids is compact and a topological monoid under the multiplication of sets." There is also a reference of that in Hofmos "Elements of Compact Semigroups" A.7.1 and B.4.2. □

Let me notice that Keimel and Gierz utilized $\dot{\Gamma}(L)$ in "Topologische Darstellung von Verbänden" M.Z.150 (1976), p.90.

The Proposition incidentally illustrates a situation in which for Lemma 1.13 in SCS -Darmstadt 8-1-76 we have a kernel operator $k: S \rightarrow S$ (Scott continuous) such that the \underline{CL} -topology of $k(S)$ is the one induced from that of S (which is not normally the case), while the inf operations of S and $k(S)$ still differ. ~~For~~

Furthermore notice that L is a \underline{CL} -retract of $\dot{\Gamma}(L)$, the retractor being \inf , the coretraction $s \mapsto \uparrow s$.

2) Proposition 2.1 in Keimel-Mislove is very nice and I wish Jimmie and I could have had that in the long ramblings of "irreducibility and generation".

Condition (C) is something that is not entirely unfamiliar since ATLAS. Notice that (C) is equivalent to

(C') For each $x \in L$, $\text{int } \uparrow x$ is a filter.

Even in \underline{Z} , $\text{int } \uparrow x$ is often badly behaved (ATLAS, Example 4.2)

Notice that in a \underline{CL} -object L with (C') for each $x \in L$ the element $\bar{x} = \inf \text{int } \uparrow x$ is in $A(L)$ (ATLAS 4.1 and Hofmann-Lawson, Irred 2.13). Clearly $x \mapsto \bar{x}$ is monotone and idempotent and satisfies $x \leq \bar{x}$, hence is a closure operator, whose image is precisely $A(L)$. Thus condition (C') is sufficient that $A(L)$ is a complete lattice and indeed an inf subsemilattice. In the language of ATLAS Section 4 ~~the image of the map $x \mapsto \bar{x}$ is $A(L)$~~ , this does not secure $A(L) = B(L)$ but I wonder whether much of that Section does not apply here with $A(L)$ in place of $B(L)$. Does anyone have an example of an $L \in \underline{CL}$ satisfying (C) [\Leftrightarrow (C')] with $L \neq A(L)$? Well, cancel the question and look at the standard Cantor semilattice. In that case $A(L)$ is obtained by deleting all cofinite elements (formerly we said cocompact and therefore $A(L) \cong I$). Is it alright if I formulate a few ~~the~~ questions at this point?

Q1. Let $L \in \underline{CL}$, $c: L \rightarrow L$ a closure operator. When is $c(L) \in \underline{CL}$?
When is the corestriction $c': L \rightarrow c(L)$ in \underline{CL} ?

Note c' is right adjoint to the inclusion. It happens sometimes, but not too often, that a right adjoint is in \underline{CL} . Q1 cannot be answered

by just dualizing what we know about kernel operators, since the CL-theory is not inf-sup symmetric.

Q2. In the light of Q1, when ~~xxxx~~ is ~~xxx~~ $A(L) \in \underline{CL}$ under the hypothesis (C)?

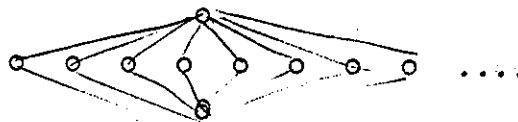
3) Re congruences on a CL-object

Perhaps there is a one page proof ⁱⁿ pp.6-7 for a 20 page paper.

In another vein:

Let $L \in \underline{Z}$. Then $(\text{Cong}^* L, \cap)$ by HMS-Duality is isomorphic to $\sum (K(L))$, the semilattice of sup-subsemilattices $A \subseteq K(L)$ containing \emptyset relative to the operation $(A, B) \mapsto A \vee B$. Since Keimel and Mislove show that L has to be stably zero dimensional, if $(\text{Cong}^* L, \cap) \in \underline{CL}$ we might just as well ask for a (discrete) semilattice S (with identity) such that the complete semilattice $\sum S$ of all subsemilattices under the set product is a continuous lattice, and we may assume that S is without order dense chains.

I do not fully understand which $S \in \underline{S}$ qualify. I notice that $T \in K(\sum S)$ if $S \setminus T$ is finite. Thus $\sum S$ is algebraic if every subsemilattice is the intersection of cofinite ones. Here is one that comes close but does not quite make it:



Here is one that does: S an arbitrary chain with maximum (and no order dense subchain if \hat{S} is to be dimensionally stable).

4) Literature: Oswald Wyler, Compact complete lattices, Preprint 48 pp. (address Carnegie Mellon, Pittsburgh, Pa. 15213).

A categorist on the loose independently and without any reference known to any of you rediscovers CL à la Alan Day.