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WEATHER DERIVATIVES AND THE MARKET PRICE OF RISK

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Abstract. Weather derivatives are becoming prominent features in multi-asset class portfolios of alternative risk. The pricing of these securities is nonetheless challenging since it requires an incomplete market framework. We discuss pricing formulas for temperature-based weather derivative options, constructing mean reverting stochastic models for describing the dynamics of daily temperature with a constant speed of mean reversion for three cities. Truncated Fourier series are used to model the volatility, and assuming a constant market price of risk, we introduce a novel approach for estimating this constant, using Monte Carlo simulations.

1. Introduction

Natural monopolies and regulated markets are prominent features in the world's economies. Unlike deregulated markets, the government controls both entry into the market and the commodity prices in regulated markets. As a result, the balance of power in deregulated markets is tilted towards consumers who enjoy the competition between suppliers. Energy markets across the world are one such market wherein energy producers for a long time enjoyed the benefits of regulation. Indeed, though energy producers knew of the adverse effects weather had on the demand for energy and on their revenues, it never pushed them to confront this risk since they could adjust for poor returns by increasing prices, thereby shifting weather-related risk to the consumers. However, following the collapse of Wall Street in 1929, the US energy market was left with only eight companies bearing the burden of supplying energy to the entire country. This attracted a lot of attention from the government. Congress introduced the Public Utility Holding Company Act (PUHCA) in 1935, and started the deregulation of the US energy market. Since introducing this act, the energy crisis of the 1970s and other related events have kept the attention of Congress fixed on energy markets, resulting in several amendments to PUHCA and the introduction of other legislation such as the Energy Policy Act of 1992. This continued activity on energy legislation has led to complete or partial deregulation of the energy markets across the US, putting energy producers in the cross-hairs of weather-related risk.

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A *weather derivative* is a contract between two parties that stipulates how payments will be exchanged between the parties depending on certain meteorological conditions such as rainfall, temperature, humidity, or snowfall during the contract period. These derivatives are one in a handful of financial instruments that can be used to reduce risk associated with unexpected weather. Furthermore, weather derivatives are possible tools for portfolio diversification because the underlying weather indices have little or no correlation with other market indices. In addition, traditional insurance contracts predated weather derivative contracts and provided security against catastrophic weather events. The difference between such traditional weather insurance and weather derivatives is that the final settlement payment of a weather derivative contract is calculated from measured values of the weather itself rather than on an estimate of the loss sustained because of the weather. This means that weather derivatives are a more general tool than traditional insurance, because we can use them to cover the adverse effects of weather on intangibles such as profits and volume of sales [11].

Standardized futures contracts written on temperature indices have been traded on the Chicago Mercantile Exchange (CME) since October 2003, together with European call and put options written on these futures [1]. The valuation of temperature-based weather derivatives requires an incomplete market framework, since the underlying temperature indices on which the options are written are not tradable. To understand the risk involved in such trading, robust pricing models of both futures and options are called for [3]. Pricing formulas for temperature-based weather derivative contracts must take into account the dynamics of daily temperature. The challenge posed by this need to capture temperature dynamics is inherent in the high locality property of temperature which warrants us to construct a model for every city in which trading is of interest.

Previously used approaches for pricing weather derivatives include the Actuarial Method, Historical burn Analysis (HBA) and Index modelling. While the first two all turned out to be faulty, the latter was deemed to be computationally too expensive and lead to loss of data in both extreme and common events. Daily Average Temperature (DAT) modelling was introduced to counter the challenges of these methods, and outside of providing more accurate results, it proved to be computationally less demanding. DAT modelling is rooted in finding a stochastic process that describes the dynamics of daily average temperature. However, deriving an accurate model for the daily average temperature is not a straightforward process.

Several authors have opted to model the temperature dynamics with continuous processes which are mean reverting OU processes (see [5] for more about OU processes). The cyclical nature of the temperature time series justifies the use of a mean-reverting process in modelling its dynamics. [8] proposed the following model for the temperature :

$$dT_t = a(\theta_t - T_t)dt + \sigma dW_t \quad , \tag{1.1}$$

where $W = (W_t, t \geq 0)$, the driving noise process, was taken to be the Wiener process; a the speed of mean reversion; θ was the seasonal mean and σ was the volatility of the process.

To capture fully the mean-reveting dynamics of the temperature, it is important that we have:

$$\mathbb{E}(T_t) \approx \theta_t. \tag{1.2}$$

However, on solving (1.1) with the requirement that $T_s = \theta_s = C$, we find that:

$$T_t = \theta_t - \int_s^t e^{-a(t-k)} d\theta_k + \int_s^t \sigma e^{-a(t-k)} dW_k. \tag{1.3}$$

Consequently, leveraging the fact that the expectation of the Itô integral is zero, we arrive at:

$$\mathbb{E}(T_t) = \theta_t - \int_s^t e^{-a(t-k)} d\theta_k, \tag{1.4}$$

which does not check (1.2).
To address this issue, the authors of [8] suggested that the term $\frac{d\theta_t}{dt}$ be added to the model so that its resulting form be:

$$dT_t = \frac{d\theta_t}{dt} + a(\theta_t - T_t)dt + \sigma dW_t \ . \tag{1.5}$$

It is then easy to see (see [12] or [8] for proof) that the solution of (1.5) is given by:

$$T_t = \theta_t + \int_s^t \sigma e^{-a(t-k)} dW_k, \tag{1.6}$$

which checks (1.2).

Following the introduction of this model, much of the work in continuous process stochastic modelling of DAT has been around the nature of a, the functional form of θ_t that best captures the trend and seasonality in the temperature time series, and the functional form of σ. There has also been some discussion around the driving noise process in this model, with some authors arguing in favour of fractional Brownian motion and others a general Lévy process.

Indeed, while Dornier and Querel in [8] assumed a constant speed of mean reversion and a constant volatility, Alaton et al. in [7] model $\sigma = \sigma(t)$ as a piece-wise constant function representing a monthly variation in volatility. Their argument for using a Wiener process as the driving noise process comes from the observation that the temperature differences are close to normally distributed. However, they do not provide a statistical test for normality, and the authors admit that the empirical frequency of small temperature differences are higher than predicted by the fitted normal distribution. Furthermore, both papers do not show a study of the possible time dependencies in the residuals observed from the regression model [3].The model proposed by Brody et al. in [6] where the Wiener process in equation (1.1) is replaced by a fractional Brownian motion, was later criticized in [3] where it was shown that a fractional Brownian dynamics does not seem to be an appropriate model when considering the Norwegian temperature data. In [3], the authors propose an Ornstein-Uhlenbeck model with seasonal mean and volatility, where the residuals are generated by a Lévy process rather than a Brownian motion. They suggest the class of generalized hyperbolic Lévy processes, a flexible class of Lévy processes capturing the semi-heavy tails and skewness. Their model pushes the framework of Alaton et al. and the seasonal volatility is modeled by a continuous form that repeats annually. The model (1.5) resurfaces in [4] and then in [12] with the volatility modeled as a stochastic process. Gyamerah et al. in [9], proposed the so-called novel time-varying mean-reversion Levy regime-switching model for the dynamics of the deseasonalized temperature. The authors, however, did not use the proposed model for pricing weather derivative contracts.

[2] returned to the framework in [8], and captured the seasonality and volatility with truncated Fourier series. According to [1], using truncated Fourier series, a good fit for both the seasonality and the variance component can be obtained while keeping the number of parameters relatively low. The Fourier series simplifies the needed calculations for the estimation of the parameters and for the derivation of the pricing formulas. Zapranis and Alexandridis in [1] follow up on the work in [2] and model the seasonal cycle using an extension and a combination of discrete Fourier Transform approach and the regression method. Zapranis and Alexandridis use a technique called Wavelet Analysis (WA) to decompose the temperature time series.

We follow [1, 2, 7] to construct temperature models for Seattle, New York and Cincinnati. Thereafter, we employ Girsanov's theorem together with the risk neutral valuation formula, to find prices for weather derivative options. In view of this, the rest of this paper is organised as follows: in Section 1.1 we discuss the rudimentary notions of the weather derivatives market and temperature indices, and follow this with Section 2 where we construct temperature models for all three cities using 25 year temperature data for each city to estimate the parameters of the ensuing models. The temperature data sets we use are sourced from both Kaggle[1] and the National Oceanic and Atmospheric Administration (NOAA) website[2]. All data sets contained daily temperature reading from 1990 to 2016 and with the exception of Seattle's set, they contained missing values which we fill using the approach outlined in [1]. We use 25 years of the time series to train the model and the remaining 1 year to test the model. The pricing of weather derivatives options is carried out in Section 3.1 prior to outlining how Monte Carlo simulations were used to estimate the constant Market Price of Risk (MPOR) in Section 3.3. The paper concludes with a summary and discussion of the results, in Section 4.

1.1. The Weather Derivatives Market

Since the underlying indices of weather derivatives is temperature which cannot be traded, weather events must be clearly defined in order to make clear the terms of the contract. Temperature indices are the *degree-day* indices. These are representative values of the difference between the temperature of a given day and the reference temperature.

Definition 1.1 (Temperature)**.** For a given day x, let T^x_{max} and T^x_{min} be, respectively, the maximum and minimum temperatures for the day, measured in degree

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[1]The seattle data set is the set used for the *Did it rain in Seattle* challenge on Kaggle and can be found on *https://www.kaggle.com/chandraroy/seattle-weather-forecast-using-logistic-regression*

[2]The recordings for New York are taken from the New Brookswick weather station while those for Cincinnati are from the Brooksville weather station. Both sets can be found on *https://gis.ncdc.noaa.gov/maps/ncei/summaries/daily*

Celsius. The average temperature for the day is:

$$T_x = \frac{T^x_{max} + T^x_{min}}{2}$$

Definition 1.2 (Degree Days)**.** Denote by T_{ref} the reference temperature for a particular region and period. For a given day i, let T_i be the temperature for that day:

(1) A heating degree day (HDD$_i$) generated on day i is:

$$HDD_i = max\{T_{ref} - T_i, 0\}$$

(2) A cooling degree day (CDD$_i$) generated on day i is:

$$CDD_i = max\{T_i - T_{ref}, 0\}.$$

The HDD index is a measure of cold in a winter period while the CDD index is a measure of heat in a summer period.

Typically the HDD season runs from November to March while the CDD season is from May to September with April and October standing out as the "shoulder months" [7]. The payout from a weather derivative contact is usually contingent on the accumulation of some weather index over the specified contract period. For example, given a contract period of n days, the accumulated HDD index over the period is defined by

$$H_n = \sum_{i=1}^{n} HDD_i$$

Consequently, for an uncapped HDD call option with strike K and tick size η, the payout $\mathcal{X}$ is

$$\mathcal{X} = \eta\ max\{H_n - K, 0\}$$

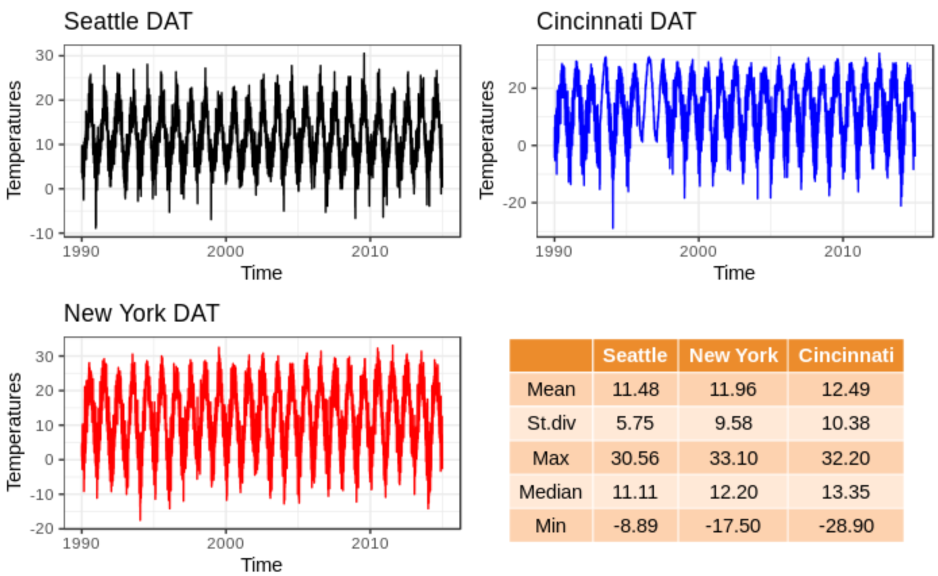
Usually, the buyer of a HDD call pays the seller a premium at the beginning of the contract. At the end of the contract period, if the number of HDDs for the contract period is greater than the predetermined strike level, the buyer will receive a payout $\mathcal{X}$ as defined above. In contrast, the seller of a put option pays the buyer a premium, and if the number of HDDs for the contract period is below the strike, the buyer pays the seller. The pay out function for a put option is:

$$\mathcal{X} = \eta\ max\{K - H_n, 0\}$$

2. Modelling Daily Average Temperatures

In this section, we find a continuous stochastic process for describing the evolution of temperature. Our approach is consistent with the underlying ideas in [2, 7]. Figure 1 shows plots of the tidy data and a table of summary statistics for all three cities. From it, we observe that temperature follows a predicted cycle and moves around a seasonal mean which is affected by urbanization trends. Consequently, following [8], we model the dynamics of DAT with the mean reverting Ornstein-Uhlenbeck process:

$$dT_t = \left[\frac{dS(t)}{dt} + a(S(t) - T_t)\right] dt + \sigma(t) dB_t. \tag{2.1}$$



	Seattle	New York	Cincinnati
Mean	11.48	11.96	12.49
St.div	5.75	9.58	10.38
Max	30.56	33.10	32.20
Median	11.11	12.20	13.35
Min	-8.89	-17.50	-28.90

FIGURE 1. Daily Average Temperature for all three cities for the years from 1990 to 2015 and a table of summary statistics of the respective temperature time series.

In what follows, we find a functional form of $S(t)$ and $\sigma(t)$ and use the linear approach from [1] to estimate the value of the constant a.

2.1. Deterministic Seasonal Mean

A closer look at the "denoised" temperature time series reveals that temperatures have uniform peaks and very weak rising trends. This inspires us to model the trend with a linear function and the seasonality with a single sine function. Thus, the deterministic seasonal temperature is modeled by:

$$S(t) = A + Bt + Csin(\omega t + \varphi), \tag{2.2}$$

where the period $\omega = \frac{2\pi}{365}$ and the phase shift φ captures the fact that the minimum and maximum temperatures do not occur on the first of January, or the first of July, respectively.

Notice that considering a general periodic function in (2.2) will be misleading. Indeed, it will suggest that temperature follows a fixed periodic cycle, hitting its annual minimum or maximum at the same date of the year. Moreover it will suggest that the annual maximum and minim have fixed values, both of which are not true.

Using historical data, we estimate these parameters using the method of least squares. That is, observing that Equation (2.2) can be rewritten as:

$$S(t) = a_1 + a_2 t + a_3 sin(\omega t) + a_4 cos(\omega t),$$

we find the vector $\xi = (a_1, a_2, a_3, a_4)$ that minimizes the estimation errors. Then

$$A = a_1, \qquad B = a_2, \qquad C = \sqrt{a_3^2 + a_4^2}, \qquad \varphi = tan^{-1}\left(\frac{a_4}{a_3}\right) - \pi.$$

Consequently, we have:

$$S(t) = \begin{cases} 11.7 - (0.00005)t + 7.12sin(\omega t - 1.99) & \text{in Seattle} \\ 11.4 + (0.0001)t + 12.14sin(\omega t - 1.99) & \text{in New York} \\ 12.9 - (0.00006)t + 12.14sin(\omega t - 1.93) & \text{in Cincinnati} \end{cases}$$

The equations further emphasize the weak linear trend in the temperature time series with the remarkably small values coefficients of t.

2.2. Estimating the Speed of Mean Reversion

We follow the linear approach to estimate the speed of mean reversion. First, observe that using the Euler discretization scheme, the SDE (2.1) can be transformed to

$$T_t - T_{t-1} = T_t^m - T_{t-1}^m + a(T_t^m - T_t)dt + \sigma(t)\xi(t),$$

where $\xi(t) \sim N(0,1)$. Using the transformation $\tilde{T}_t = T_t - T_t^m$ (this is equivalent to detrending and deseasonalizing the DAT) we get:

$$\tilde{T}_t = \alpha \tilde{T}_{t-1} + e(t), \tag{2.3}$$

where

$$a = 1 - \alpha, \qquad and \qquad e(t) = \sigma(t)\xi(t).$$

Equation (2.3) is an AR(1) model of the detrended and the deseasonalized temperature. Using the historical data, we find that:

$$a = \begin{cases} 0.2757, & \text{in Seattle} \\ 0.3499, & \text{in New York} \\ 0.2832, & \text{in Cincinnati} \end{cases}.$$

2.3. Temperature Volatility

Here, we break away from the framework in [7] and follow somewhat the likes of [1, 2], who use Fourier series to model the volatility of temperature. The volatility models they proposed were

$$\sigma(t) = p + \sum_{i=1}^{I} c_i sin(i\omega t) + \sum_{j=1}^{J} d_j cos(j\omega t).$$

We replace the constant p with a linear term to capture the trend in the annual volatility, i.e the fact that there is higher volatility in the winter months than in the summer months. Consequently, the volatility is modeled by:

$$\sigma(t) = V + Ut + \sum_{i=1}^{I} c_i sin(i\omega t) + \sum_{j=1}^{J} d_j cos(j\omega t).$$

Using historical data, one can then find plausible values for I, J and the accompanying constants. To achieve this, the average variance for each day of the year for the 24 years of the data is calculated and then the best model for this values is

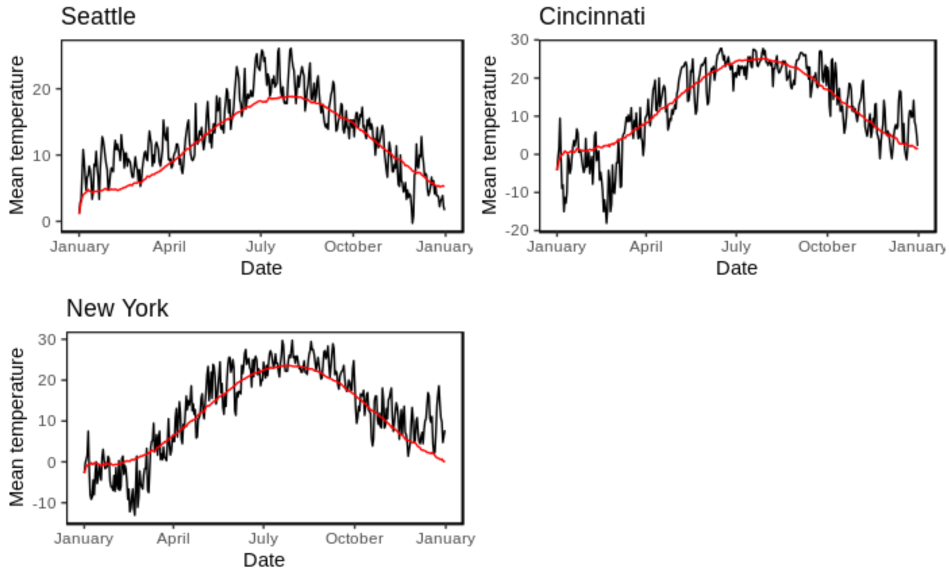


FIGURE 2. Results of a thousand Monte-Carlo simulations plotted over real data.

obtained using the method of least squares. The models for volatility for Seattle, New York and Cincinnati are respectively the following :

$$\sigma(t) = \begin{cases} 7.48 + 0.64sin(\omega t) + 1.56cos(\omega t) + 2cos(2\omega t) + 0.79cos(3\omega t), \\ 20.2 - 0.02t + 2.25sin(\omega t) + 8.25cos(\omega t), \\ 27.5 - 0.02t + 3.74sin(\omega t) + 11.5cos(\omega s), \end{cases}$$

The above results show that while the volatility frame works in [1, 2] are sufficient to describe the volatility in Seattle, while the introduction of the linear term improves the results for both Cincinnati and New York.

3. Pricing HDD Call/Put Options

Consider the SDE :

$$dT_t = \left(\frac{dS(t)}{dt} + a(S(t) - T_t)\right) dt + \sigma_t dB_t \quad , \tag{3.1}$$

and let

$$g(t, x) = \exp(at)(x - S(t)).$$

Then it holds first that:

$$\frac{dg}{dx} = \exp(at) \quad ; \quad \frac{d^2g}{dx^2} = 0 \quad ; \quad \frac{dg}{dt} = a\exp(at)(x - S(t)) - \exp(at)\frac{dS(t)}{dt},$$

and then a straight forward application of the Itô formula yields:

$$dg(t, x) = \left[\frac{dg}{dt} + \mu\frac{dg}{dx} + \frac{\sigma^2}{2}\frac{d^2g}{dx^2}\right] dt + \sigma\frac{dg}{dx}dB_t,$$

where μ and σ are the drift and diffusion coefficients respectively. On further simplification we get that:

$$dg(t,T_t) = \left(\mu e^{(at)} + e^{(at)}\left\{a(x - S(t)) - \frac{dS(t)}{dt}\right\}\right)dt + \sigma(t)e^{(at)}dB_t.$$

But

$$\mu = \frac{dS(t)}{dt} + a(S(t) - T_t).$$

Hence,

$$dg(t,T_t) = \left(\mu e^{(at)} - \mu e^{(at)}\right)dt + \sigma(t)e^{(at)}dB_t = \sigma(t)e^{(at)}dB_t.$$

Integrating both sides from s to t, and using the fact that $T_s = x$ we find that

$$e^{(at)}\left(T_t - S(t)\right) - e^{as}\left(x - S(s)\right) = \int_s^t \sigma(k)e^{(ak)}dB_k$$

which simplifies to

$$T_t = S(t) + (x - S(s))e^{-a(t-s)} + \int_s^t \sigma(k)e^{-a(t-k)}dB_k. \tag{3.2}$$

Equation (3.2) then provides that:

$$Var[T_t|\mathcal{F}_s] = \int_s^t \sigma^2(k)e^{-2(t-k)}dt \tag{3.3}$$

and

$$\mathbb{E}^{\mathbb{P}}[T_t|\mathcal{F}_s] = S(t) + (x - S(s))e^{-a(t-s)}.$$

Now, denote by λ the market price of risk and $\mathbb{Q}$ the risk-neutral measure characterized by λ. Further, let $W = (W_t, t \geq 0)$ be the $\mathbb{Q}$-standard Brownian motion. Then Equation (2.1) can be transformed to:

$$dT_t = \left(\frac{dS(t)}{dt} + a(S(t) - T_t) - \lambda\sigma(t)\right)dt + \sigma_t dW_t.$$

Consequently, it holds:

$$T_t = S(t) + (T_s - S(s))e^{-a(t-s)} - \lambda\int_s^t \sigma(t)e^{-a(t-k)}dk + \int_s^t \sigma(t)e^{-a(t-k)}dW_k \quad ,$$

and taking expectations on both sides we get:

$$\mathbb{E}^{\mathbb{Q}}[T_t|\mathcal{F}_s] = \mathbb{E}^{\mathbb{P}}[T_t|\mathcal{F}_s] - \lambda\int_s^t \sigma(t)e^{-a(t-k)}dk. \tag{3.4}$$

Further, we find that the covariance and variance are related by:

$$Cov[T_tT_u|\mathcal{F}_s] = e^{a-(u-t)}Var(T_t|\mathcal{F}_s).$$

3.1. Winter Valuation of HDD and CDD Contracts

The reference temperature for weather derivatives contracts traded at the CME is 18^oC and as a result, the payoff function for a degree day contract is defined:

$$\mathcal{X} = \alpha max\{H_n - K, 0\},$$

where α is the tick size, K the strike level, n the number of days in the contract period and H_n the number of heating degree days in the contract period defined by

$$H_n = \sum_{i=1}^{n} max\{18 - T_{t_i}, 0\}.$$

From the temperature time series, we observe that for the winter months,

$$\mathbb{P}(max\{18 - T_{t_i}, 0\} = 0) \approx 0.$$

This observation was first made in [7]. Outside the fact that it eliminates the maximum function from Equation (3.1), it allows us to leverage the fact that the OU process is Gaussian to conclude accurately that under the risk-neutral measure $\mathbb{Q}$,

$$T_t \sim N(\mu_t, \sigma_t^2)$$

with the quantities μ_t and σ_t^2 given by the Equations (3.4) and (3.3), respectively. In light of this, we can rewrite equation (3.1) as

$$H_n = 18n - \sum_{i=1}^{n} T_{t_i} \quad ,$$

whence we get:

$$u_t = \mathbb{E}^{\mathbb{Q}}[H_n|\mathcal{F}_t] = 18n - \sum_{i=1}^{n} \mathbb{E}^{\mathbb{Q}}[T_{t_i}|\mathcal{F}_t] \ ,$$

and

$$\begin{aligned} v_t = Var[H_n|\mathcal{F}_t] &= Var\left(\sum_{i=1}^{n}[T_{t_i}|\mathcal{F}_t]\right) \\ &= \sum_{i=1}^{n} Var[T_{t_i}|\mathcal{F}_t] + 2\sum_{i=1}^{n}\sum_{i<j} Cov[T_{t_i}, T_{t_j}|\mathcal{F}_t] \ . \end{aligned}$$

That is:

$$H_n \sim N(u_t, v_t) \ .$$

It follows from the fundamental theorem of asset pricing that the price $C(t)$ at any time t of a winter weather derivative call option with tick size η and strike level K is

$$\begin{aligned} C(t) &= e^{-r(t_n - t)}\mathbb{E}_{\mathbb{Q}}[\eta \max\{H_n - K, 0\}|\mathcal{F}_t] \\ &= \eta e^{-r(t_n - t)} \int_K^{\infty} (x - K) f_H(x) dx. \\ &= \sigma_n \eta e^{-r(t_n - t)} \left((K - \mu_n)\Phi(-\alpha_n) + \frac{\sigma_n}{\sqrt{2\pi}} e^{-\frac{\alpha_n^2}{2}} \right) \ , \end{aligned}$$

where ϕ is the density function of the standard normal distribution.

Moreover, using the same contract parameters for a put option and the above transformations, the price of a put option is given by:

$$\begin{aligned} P(t) &= e^{-r(t_n-t)}\mathbb{E}_{\mathbb{Q}}[\eta \max\{K-H_n,0\}|\mathcal{F}_t] \\ &= \eta e^{-r(t_n-t)}\int_0^K (K-x)f_H(x)dx \\ &= \sigma_n \eta e^{-r(t_n-t)}\left((K-\mu_n)\left\{\Phi(\alpha)-\Phi(-\frac{\mu_n}{\sigma_n})\right\}\right. \\ &+ \left.\frac{\sigma_n}{\sqrt{2\pi}}\left\{e^{-\frac{\alpha_n^2}{2}}-e^{-\frac{\mu_n^2}{2\sigma_n^2}}\right\}\right). \end{aligned}$$

3.2. Valuation Using Monte Carlo Simulations

The formulation of pricing formulas considered in the previous section and its accompanying assumptions do not support the pricing of weather derivative contracts in all seasons and in the event of high volatility (as is characteristic of cold seasons) we may get highly inaccurate prices. The alternative to this method that overcomes this risk of inaccurate pricing, and which works for contracts in all seasons, is the use of Monte-Carlo simulations. To use the Monte Carlo method, we generate a set of paths and calculate the payoff for each path. Then, the average of the payoffs from all the generated paths is used to represent the expected price of the derivative [13]. In this regard, the price at time t of an HDD call option in a time space of length n, with constant interest rate r is given by

$$C_{HDD}(t) = e^{-r(t_n-t)}\frac{1}{N}\sum_{i=1}^{N}\eta\mathbb{E}\left[max\{H_n^i-K,0\}\right], \tag{3.5}$$

where η is the tick size (or principal nominal) and N is the number of paths generated.

3.3. Estimating the Market Price of Risk

The discussion in the previous section suggests that we need to simulate temperature trajectories under the risk neutral measure. That is we need to simulate the temperature dynamics from Equation (3.5). To do this, we must first find estimates for the parameter λ which characterizes the risk preferences of the weather traders. As suggested by [7], the value of λ can be estimated by comparing the market prices for some weather derivatives contracts and settling for the best value of λ for which our model gives prices similar to these. The drawback is that the weather derivatives market is relatively underdeveloped and therefore there are few contracts that can support this approach. However, this suggestion implies that the best estimate for the value of λ is that for which the predicted degree days are close to the observed degree days.

Using our test data, we find the approximate values of this parameter through Monte Carlo simulations. Guided by the values estimated in [7], we perform two thousand Monte Carlo simulations of temperature trajectories for the month of

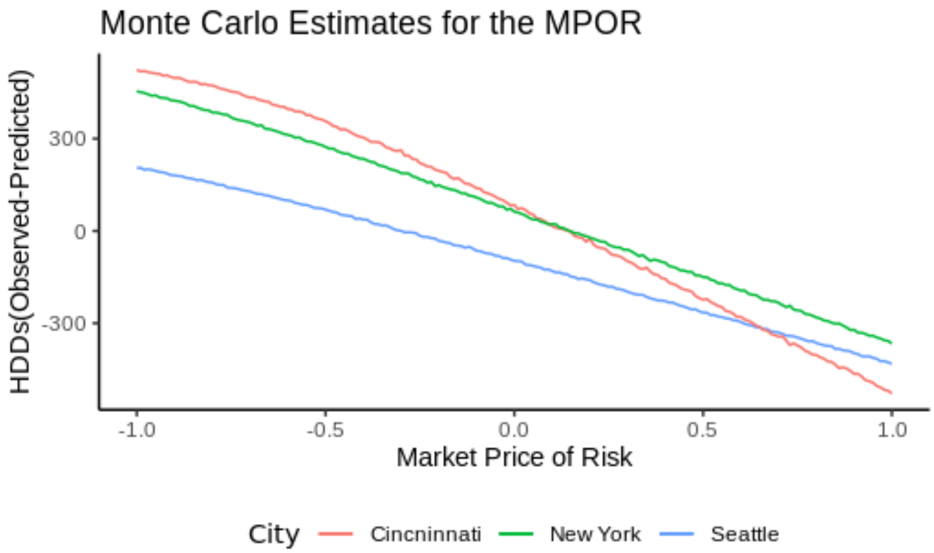


FIGURE 3. Plot of the difference in HDDs predicted via two thousand Monte Carlo simulations for the different values of the parameter λ showing a negative linear relationship between the variables.

January in all three cities for each:

$$\lambda \in \{-1.0, \dots, -0.02, -0.01, 0.0, 0.01, 0.02, \dots, 1.0\}.$$

The difference between the predicted HDDs and the observed HDDs is then computed and compared for each value of λ. Figure 3 shows the results of these simulations and depicts the inverse relationship between an increase in the value of λ and the accuracy of the predicted HDDs. In Seattle, we found that the only value of λ which gave accurate HDD predictions within a ± 1 error margin was -0.3. Moreover, we found that for values of the parameter in the ranges $[0.14, 0.16]$ and $[0.13, 0.16]$ for New York and Cincinnati, respectively, the error in the prediction was between -1 and 1. Therefore, the value of λ for the latter cities can be chosen within these ranges.

4. Conclusion

Weather derivatives have become prominent on the CME in the last decade, attracting attention from both hedgers and speculators. These derivatives are attractive to investors because they have little correlation with other market variables and they provide excellent coverage against uncertainty in daily weather. The nature of the underlying in these derivatives, however, has been a major drawback in their pricing and trading.

In this work, we set out to find pricing formulas for weather derivatives in three cities. The attained aim of our work was to seek continuous stochastic processes for describing the evolution of temperature, and then using the Girsanov theorem, find an equivalent martingale measure which we subsequently use together with the risk neutral pricing formula to find close form pricing formulas for call and put options written on HDD indices. We assumed a constant market price of risk, and we introduced the use of Monte Carlo simulations to estimate these constants. To the best of our knowledge, this is the first time this has been done. In future work, we will explore this in detail and follow up on the suggestions in [10] which allude to a possible time dependence of the market price of risk, whilst exploring the possibility of modelling volatility with piece-wise non-constant functions with each piece capturing volatility for each season, and also the use of splines to model the volatility would be considered.

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References

1. Alexandridis, Antonis and Zapranis, Achilleas D. *Weather derivatives: modeling and pricing weather-related risk*. Springer Science & Business Media, 2012.
2. Benth, Fred Espen and Benth, Jūratė šaltytė. The volatility of temperature and pricing of weather derivatives. *Quantitative Finance*, 7(5):553–561, 2007.
3. Benth, Fred Espen and Benth, Jūratė šaltytė. Stochastic modelling of temperature variations with a view towards weather derivatives. *Applied Mathematical Finance*, 12(1):53–85, 2005.
4. Bhowan A. Temperature derivatives. *University of the Wiwatersrand*, 2003.
5. Bibbona, E., Panfilo, G., and Tavella, P. The OrnsteinUhlenbeck process as a model of a low pass filtered white noise. *Metrologia*, 45(6), S117. 2008.
6. Brody, Dorje C, Syroka, Joanna, Mihail, Zervos., et al. Dynamical pricing of weather derivatives. *Quantitative Finance*, 2(3):189–198, 2002.
7. Djehiche, Boualem, Alaton, Peter, and Stillberger, David. On modelling and pricing weather derivatives. *Applied Mathematical Finance*, 9:1–20, 2002.
8. Dornier, Q. and Querel, M. Caution to the wind. energy and power risk managment. *Weather risk special report*, 2000.
9. Gyamerah, Samuel Asante, Ngare, Philip, and Ikpe, Dennis. Regime-switching temperature dynamics model for weather derivatives. *International Journal of Stochastic Analysis*, 2018, 2018.
10. Härdle, Wolfgang Karl and Cabrera, Brenda López. The implied market price of weather risk. *Applied Mathematical Finance*, 19(1):59–95, 2012.
11. Jewson, Stephen. Introduction to weather derivative pricing. *Available at SSRN 557831*, 2004.
12. Mraoua, Mohammed. Temperature stochastic modeling and weather derivatives pricing: empirical study with moroccan data. *Afrika Statistika*, 2(1), 2007.
13. Wang, Zhiliang, Li, Peng, Li, Lingyong, Huang, Chunyan, and Liu, Min. Modeling and forecasting average temperature for weather derivative pricing. *Advances in Meteorology*, 2015, 2015.

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