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A symplectic manifold homeomorphic but not diffeomorphic to $\mathbb{CP}^2\#3\overline{\mathbb{CP}}^2$

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In this article we construct a minimal symplectic 4–manifold and prove it is homeomorphic but not diffeomorphic to $\mathbb{CP}^2\#3\overline{\mathbb{CP}}^2$.

57R17; 57M05, 54D05

1 Introduction

The main result of this article is the construction of a minimal symplectic 4–manifold that is homeomorphic but not diffeomorphic to $\mathbb{CP}^2\#3\overline{\mathbb{CP}}^2$.

The construction of manifolds homeomorphic but not diffeomorphic to $\mathbb{CP}^2\#k\overline{\mathbb{CP}}^2$s for $k \leq 9$ began with Donaldson's seminal example [8] that the Dolgachev surface $E(1)_{2,3}$ is not diffeomorphic to $\mathbb{CP}^2\#9\overline{\mathbb{CP}}^2$. In 1989, Dieter Kotschick [13] proved that the Barlow surface is homeomorphic but not diffeomorphic to $\mathbb{CP}^2\#8\overline{\mathbb{CP}}^2$. In 2004 Jongil Park [17] constructed the first exotic smooth structure on $\mathbb{CP}^2\#7\overline{\mathbb{CP}}^2$. Since then Park's results have been expanded upon by Ozsváth and Szabó [16], Stipsicz and Szabó [19], Fintushel and Stern [10] and J Park, Stipsicz and Szabó [18], producing infinite families of smooth 4–manifolds homeomorphic but not diffeomorphic to $\mathbb{CP}^2\#k\overline{\mathbb{CP}}^2$ for $k = 5, 6, 7, 8$. The $k = 5$ examples are not symplectic.

Akhmedov [1] describes a construction of a symplectic 4–manifold homeomorphic to but not diffeomorphic to $\mathbb{CP}^2\#5\overline{\mathbb{CP}}^2$. Our approach is indebted to his idea of using the symplectic sum construction along genus 2 surfaces to kill fundamental groups in an efficient way. Earlier approaches start with a simply connected manifold and kill generators of the second homology using the rational blowdown approach. Akhmedov and D Park announce a similar result to our main theorem in [2].

Using Luttinger surgery in addition to symplectic sums expands the palette of available symplectic constructions, and combined with Usher's theorem [22], verifying that a construction yields a minimal symplectic manifold is straightforward. This is the approach taken in investigating small symplectic manifolds in our previous article [5],

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which among other things contains examples of symplectic manifolds homeomorphic but not diffeomorphic to $\mathbb{CP}^2 \# 5\overline{\mathbb{CP}}^2$.

Many of our constructions have their origin in Fintushel and Stern [9], where symplectic sums of products of surfaces and surgery along nullhomologous tori are used to construct symplectic and nonsymplectic manifolds which are homeomorphic and in some cases not diffeomorphic.

Our experience, gleaned while working on [7; 5; 6], taught us that there are serious technical issues arising from working with fundamental groups and cut and paste constructions, which can easily lead to plausible but unverified or even incorrect calculations. As usual, base point issues are the culprit. Thus in writing the present article we take great care in performing fundamental group calculations. This is reflected in the length of the proof of Theorem 2, whose statement is perhaps not surprising in hindsight, but critical for what follows. At every stage of our constructions we must keep track not just of homotopy classes, but representative loops. We encourage the interested reader to start with the proof of our main result, Theorem 7, and to save the proof of Theorem 2 for last.

To summarize our construction, our example is the symplectic sum of two manifolds along genus 2 surfaces. The first manifold W is obtained from Luttinger surgery on a pair of Lagrangian tori in $T^4 \# 2\overline{\mathbb{CP}}^2$. The second manifold P is obtained by Luttinger surgery on four Lagrangian tori in $F_2 \times T^2$, where F_2 is a surface of genus 2. Recall from Gompf [12] that the symplectic sum is obtained by removing a neighborhood of a surface in each manifold and gluing the resulting manifolds along their boundary. Thus our approach is informed by the methods of knot theory: we essentially calculate the fundamental groups of the complement of a link of two tori and a genus 2 surface in $T^4 \# 2\overline{\mathbb{CP}}^2$ and the complement of a link of four tori and a genus 2 surface in $F_2 \times T^2$, as well as their meridians and longitudes with respect to paths from all the link components to the base point. It is this last point which makes the calculations challenging.

To make the exposition as concise as possible, we use the following strategy. To show a group is trivial, it suffices to show it is a quotient of the trivial group. More generally, one can view the Seifert–Van Kampen theorem as giving two pieces of information: first it provides generators and then identifies all relations. Since our goal is to show that the example is simply connected, it suffices to find all generators and sufficiently many relations for the building blocks to reach the desired conclusion. Thus we eschew the problem of finding a complete presentation of the fundamental groups of W and P, and content ourselves with establishing the relations we require for the proof.

We remark that the equation $\ell_2 = bab^{-1}$ which appears in the statement of Theorem 2 (rather than the perhaps expected $\ell_2 = a$) hints at the fact that calculations of fundamental groups of torus surgeries on Lagrangian tori in the product of surfaces are likely to be subtle. By stating Theorem 2 as we did (ie in the product of punctured tori) it will be very useful in other contexts when small symplectic manifolds are to be constructed, since, for example, one can build products of closed surfaces starting with the product of punctured tori.

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2 Fundamental group calculations

Let H be an oriented genus 1 surface with one boundary component. Let x, y be oriented embedded circles representing a symplectic basis of $H_1(H)$ so that x and y intersect transversally and positively in one point, which we denote by h. Denote the corresponding based homotopy classes in $\pi_1(H, h)$ also by x and y.

Now let K be another oriented genus 1 surface with one boundary component. Let a, b be oriented embedded circles representing a symplectic basis of $H_1(K)$ so that a and b intersect transversally and positively in one point, which we denote by k.

The image of the loops x, y, a, b under the inclusion $H \times \{k\} \cup \{h\} \times K \subset H \times K$ define homotopy classes which we as usual denote by $x, y, a, b \in \pi_1(H \times K, (h, k))$. The base point (h, k) for $H \times K$ is to be understood throughout this section.

Let X be a push off of x in H to the right with respect to the orientations on H and x. Let Y be a parallel push off of y to the left. Thus x and X are disjoint parallel curves on H.

Now let A_1 be a parallel push off of a in K to the right of a. Let A_2 be a further parallel push off of A_1, to the right of A_1. Thus a, A_1 and A_2 are parallel curves in K.

Figure 1 illustrates all the curves on the surfaces H and K.

We define two disjoint tori T_1, T_2 in $H \times K$ as follows.

$$T_1 = X \times A_1 \quad \text{and} \quad T_2 = Y \times A_2.$$

Fix a product symplectic form on $H \times K$. (Typically we think of H and K as codimension 0 submanifolds of closed tori $\widehat{H}$ and $\widehat{K}$ and restrict the standard product symplectic form on $\widehat{H} \times \widehat{K}$ to $H \times K$.)

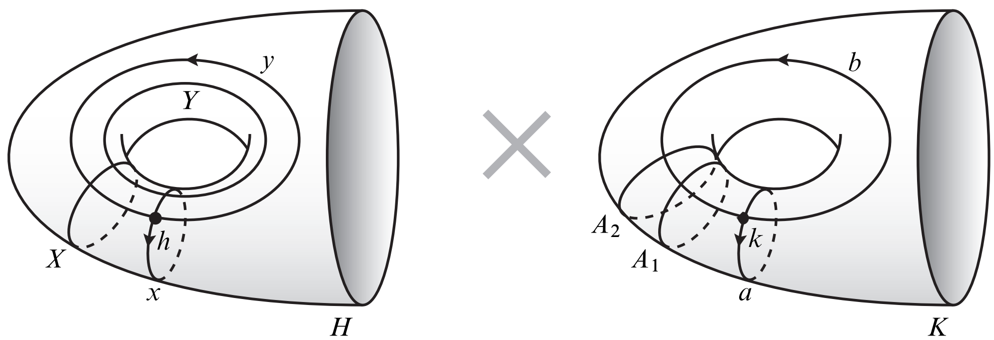

Figure 1: The surface $H \times K$

The proof of the following proposition is simple.

Proposition 1 *The tori T_1 and T_2 are Lagrangian and the surfaces $H \times \{k\}$ and $\{h\} \times K$ are symplectic. Moreover, T_1 and T_2 are disjoint and disjoint from $H \times \{k\}$ and $\{h\} \times K$.* □

Notice that every torus of the form $C \times D \subset H \times K$, (where $C \subset H$ and $D \subset K$ are embedded curves) is Lagrangian. Recall that a Lagrangian torus T in a symplectic 4–manifold M has a canonical framing called the *Lagrangian framing*. In fact, the Darboux–Weinstein theorem [15] implies that a tubular neighborhood of T can be identified with $T \times D^2$ in such a way that the parallel tori in M corresponding to $T \times \{d\}$ in this framing are also Lagrangian for every $d \in D^2$. In particular, given any such neighborhood and any $d \in \partial D^2$, we will call the torus $T \times \{d\}$ in the boundary of a tubular neighborhood of T a *Lagrangian push off* of T, and if $\gamma \subset T$ is a curve we call the curve corresponding $\gamma \times \{d\}$ the *Lagrangian push off of* γ.

The following theorem is the critical step in our constructions. Before we state it, we begin with an observation and a warning. First the observation: the torus T_2 intersects the torus $x \times b$ transversally in one point. Together with the remarks about the Lagrangian framing discussed above, one concludes without much trouble that in $\pi_1(H \times K - (T_1 \cup T_2))$, the meridian of T_2 takes the form $[\tilde{x}, \tilde{b}] = \tilde{x}\tilde{b}\tilde{x}^{-1}\tilde{b}^{-1}$, and the Lagrangian push off of the curves Y and A_2 take the form $\tilde{y}$ and $\tilde{a}$ respectively, where for $z \in \pi_1(H \times K - (\bigcup_i T_i))$ we let $\tilde{z}$ denote some conjugate of z.

Put another way, consider the three circles that lie on the boundary of a tubular neighborhood of T_2, namely the boundary of a meridian disk $\{t\} \times D^2$, and the Lagrangian push offs of the curves Y and A_2 with respect to a normal Lagrangian vector field. These curves are freely homotopic to (respectively) the triple $[x,b]$, y and a in $H\times K-(T_1\cup T_2)$.

But *they need not be equal to this triple* when the boundary of the tubular neighborhood is joined by a path to the base point (h,k) in $H\times K-(T_1\cup T_2)$. There is some freedom in the choice of path to simultaneously conjugate all three. But to expect that there exists a path to the base point so that $([\tilde{x},\tilde{b}],\tilde{y},\tilde{a})=([x,b],y,a)$ in $\pi_1(H\times K-(T_1\cup T_2))$ is in general too much to hope for, and has led to some confusion and mistakes which we need to avoid.

The configuration is nevertheless sufficiently explicit in our situation to prove the following theorem.

Theorem 2 *There exist paths in* $H\times K-(T_1\cup T_2)$ *from the base point to the boundary of the tubular neighborhoods* $T_1\times\partial D^2$ *and* $T_2\times\partial D^2$ *with the following property. Denote by* $\mu_i, m_i, \ell_i\in\pi_1(H\times K-(T_1\cup T_2))$ *the loops obtained by following the chosen path to the boundary of the tubular neighborhood of* T_i*, then following (respectively) the meridian of* T_i *and the two Lagrangian push offs of the generators on* T_i*, then returning to the base point by the chosen path. Then these loops are given by the following formulae:*

$$\mu_1=[b^{-1},y^{-1}],\qquad m_1=x,\qquad \ell_1=a,$$

and

$$\mu_2=[x^{-1},b],\qquad m_2=y,\qquad \ell_2=bab^{-1}.$$

where $x,y,a,b\in\pi_1(H\times K-(T_1\cup T_2))$ *are the loops which lie on the surfaces* $H\times\{k\}$ *and* $\{h\}\times K$ *described above.*

Moreover, $\pi_1(H\times K-(T_1\cup T_2))$ *is generated by* x,y,a,b *and the relations*

$$[x,a]=1,\qquad [y,a]=1,\qquad [y,bab^{-1}]=1$$

as well as

$$[[x,y],b]=1,\qquad [x,[a,b]]=1,\qquad [y,[a,b]]=1$$

hold in $\pi_1(H\times K-(T_1\cup T_2))$.

Remark Note that we are not assuming any particular orientation convention on the meridians, or even that the two meridians are oriented by the same convention. Looking ahead, when we perform Luttinger surgeries below we are free to do either 1 or -1 surgeries, and we will pick the sign that introduces the relation we require.

Proof Figure 2 and Figure 3 will guide the reader through the argument. View a torus as a square with opposite sides identified, thus T^4 can be thought of as a quotient of the product of two squares. Equivalently, we think of it as a quotient of the cube

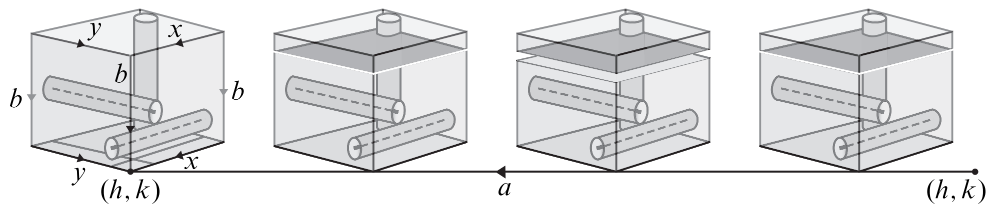

Figure 2: $H \times K$

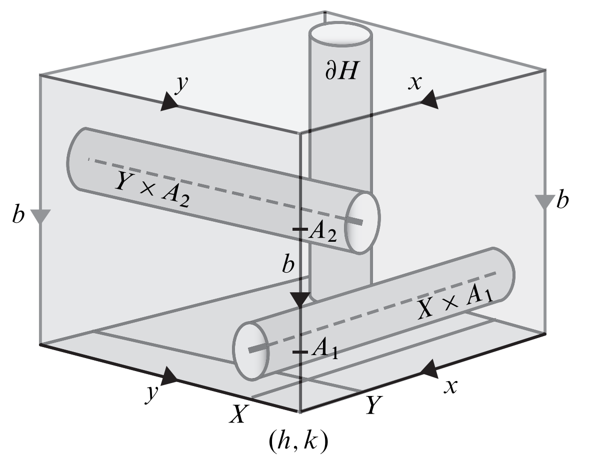

Figure 3: The slice $a = 1$

with coordinates x, y, b and an interval corresponding to the a coordinate. Since $H \times K \subset T^4$, we visualize $H \times K$ as a subset of the 4–cube.

We start with the easy torus T_1 first. Let $p \in H$ denote the intersection point of X and y. Let α be the following path from the base point to the boundary of the tubular neighborhood of T_1. Starting at (h, k), let α_1 denote the path traced out by traveling backwards along y in $H \times \{k\}$ until you hit X at the point (p, k). Then let α_2 denote the path obtained by traveling in $\{p\} \times K$ backwards along b until just before you hit A_1. This defines the path $\alpha = \alpha_1 * \alpha_2$ in $H \times K - (T_1 \cup T_2)$ from the base point to the point (p, q), where $\{p\} = X \cap y$ and q is a point on b just to the right of A_1.

The square $[0, 1] \times [0, 1]$ maps to $H \times K - T_2$ by

$$y^{-1} \times b^{-1} \colon I \times I \to H \times K - T_2.$$

The interior of this square intersects T_1 transversally once. Moreover, the path α lies on this square starting at the image of $(0,0)$. It follows that the meridian of T_1 is (based) homotopic to the boundary of this square, starting at $(0,0)$, ie $\mu_1=[b^{-1},y^{-1}]$.

Next consider the loop m_1 which follows α to (p,q), then follows the loop $X\times\{q\}$ around back to (p,q), and finally returns to the base point along α^{-1}. This is the Lagrangian push off of X since the second coordinate q is held fixed as one moves along X.

We show that the loop m_1 is based homotopic to x in $H\times K-(T_1\cup T_2)$. First, there is an annulus in H with boundary x and X which contains the arc from h to p following y backwards. This determines an annulus F_1 in $H\times\{k\}\subset H\times K$ which misses $T_1\cup T_2$ with (h,k) on one boundary circle, (p,k) on the other, and the arc α_1 spanning these two points. There is another annulus F_2 of the form $X\times\alpha_2$ which contains the arc α_2 and misses $T_1\cup T_2$. Gluing F_1 to F_2 along their common boundary $X\times\{k\}$ yields a homotopy from x to m_1 which is base point preserving since it contains the path α spanning the two boundary components.

Next, consider the loop ℓ_1 which first follows α to (p,q), then follows the loop $\{p\}\times A_1^+$ where A_1^+ is the parallel copy of A_1 in K that passes through q, and finally returns to the base point along α^{-1}. As explained above, $\{p\}\times A_1^+$ is the Lagrangian push off of $A_1\subset T_1$ since it is the push off of A_1 in K.

We show that the loop ℓ_1 is based homotopic to a. We argue similarly as above. This time there is an annulus F_3 which lies in $H\times a$ with boundary the curves $\{h\}\times a$ and $\{p\}\times a$ which contains the path α_1 spanning its boundary components. There is an annulus F_4 in $\{p\}\times K$ with boundary the curves $\{p\}\times a$ and $\{p\}\times A_1^+$ which contains the path α_2. This proves that a and ℓ_1 are based homotopic.

We now turn to the other torus T_2. The attentive reader will realize that the difficulty here is that the analogue of the path α_2 we would want to use intersects T_1. The solution presents itself from this consideration: we will need to travel *forwards* along b until we approach A_2.

Proceeding in earnest now, let $r\in H$ denote a point on x close to and to the right of Y (and left of y.) Let $s\in K$ denote the intersection point of A_2 with b. Let β_1 be the path in $\{h\}\times K$ which starts at (h,k) and moves forward along $\{h\}\times b$ to the point (h,s). Let β_2 be the path in $H\times\{s\}$ starting at (h,s) and moving along x backwards until the point (r,s) in the boundary of the tubular neighborhood of T_2 is reached. The path $\beta=\beta_1*\beta_2$ is our path from the base point to the boundary of the tubular neighborhood of T_2.

To compute μ_2, we notice that there is a map of a square

$$x^{-1} \times b\colon I \times I \to H \times K - T_1$$

which intersects T_2 transversely once and contains the path β, starting at $(0,0)$. Thus μ_2 can be read off the boundary of the square, and hence $\mu_2 = [x^{-1}, b]$.

Next, consider the loop m_2 which follows β to (r,s), then follows $Y^+ \times \{s\}$ and returns to the base point along β^{-1}, where Y^+ is the push off of Y in H which passes through r. This is the Lagrangian push off of Y, since $Y^+ \times \{s\}$ is a Lagrangian curve. There is an annulus F_5 with boundary $y \times \{k\}$ and $y \times \{s\}$ which contains the path β_1. There is an annulus $F_6 \times \{s\}$ with boundary $y \times \{s\}$ and $Y^+ \times \{s\}$ which contains the path β_2. These glue to give a base point preserving homotopy of m_2 to y.

We saved the most difficult calculation for last, and it is here that Figure 2 becomes most helpful. Consider the loop ℓ_2 which follows β to (r,s), then follows $\{r\} \times A_2$ and then returns along β^{-1}. There is a surface F_7 in $\{h\} \times K$ (a punctured annulus) with three boundary components: $\{h\} \times a$, $\{h\} \times A_2$, and $\{h\} \times \partial K$ which contains the path β_1. There is an annulus F_8 of the form $\beta_2 \times A_2$ with boundary $\{h\} \times A_2$ and $\{r\} \times A_2 = \ell_2$.

Cut a slit in F_7 along an arc of the form $\{h\} \times \gamma$, where γ is a path in K from k to the boundary. Then the commutator $bab^{-1}a^{-1}$ is homotopic to the composite of γ, the loop that follows the boundary, and then γ^{-1}. Cutting F_7 along β_1 and γ and reading the word on the boundary one finds $\beta_1 * A_2^{-1} * \beta_1^{-1} * bab^{-1}a^{-1} * a$ and gluing on F_8 one concludes that

$$\ell_2 = bab^{-1}a^{-1}a = bab^{-1}.$$

(For the benefit of the reader, we sketch an alternative way to see this, referring to Figure 3. Let β_3 be the path following b forwards starting at $\beta_1(1)$, so that $\beta_1 * \beta_3 = b$. The square of the form $\beta_2^{-1} \times a$ glues to the square $\beta_3 \times a$ to give a homotopy from ℓ_2 to bab^{-1}.)

We now turn to the assertions about $\pi_1(H \times K - (T_1 \cup T_2))$. The surface K decomposes into two surfaces: an annulus K_1 with boundary A_1 and A_2 and its complement, a 3–punctured sphere with boundary the disjoint union $\partial K \cup A_1 \cup A_2$.

We take the preimages of the K_i via the projection to K. Precisely, let $\Phi\colon H \times K \to K$ denote the projection and define

$$W_1 = \Phi^{-1}(K_1) \cap (H \times K - \mathrm{nbd}(T_1 \cup T_2))$$

and

$$W_2 = \Phi^{-1}(K_2) \cap (H \times K - \mathrm{nbd}(T_1 \cup T_2)).$$

Notice that W_1 is homeomorphic to $H\times K_1$ and W_2 is homeomorphic to $H\times K_2$. Thus $W_1\cup W_2 = H\times K - \mathrm{nbd}(T_1\cup T_2)$. The intersection $W_1\cap W_2$ has two components: one of them is

$$\Phi^{-1}(A_1)\cap(H\times K-\mathrm{nbd}(T_1\cup T_2)) = H\times A_1-\mathrm{nbd}(T_1) = (H-\mathrm{nbd}(X))\times A_1.$$

The other one is

$$\Phi^{-1}(A_2)\cap(H\times K-\mathrm{nbd}(T_1\cup T_2)) = H\times A_2-\mathrm{nbd}(T_2) = (H-\mathrm{nbd}(Y))\times A_2.$$

To apply the Seifert–Van Kampen theorem requires the intersection to be connected, so we take the usual approach (eg taken when computing fundamental groups of bundles over S^1) and change W_1 and W_2 slightly to make their intersection connected, as follows.

Let τ denote the arc in $\{h\}\times K$ which starts at the base point (h,k) and travels along b *backwards*, passes through A_1, and ends at the intersection point of b with A_2.

We let $W_1' = W_1\cup\tau$, this is just W_1 with a small hair attached connecting it to the base point. Define three loops in W_1' based at (h,k) as follows. Let k' denote a point on b between A_1 and A_2. Follow the arc τ from (h,k) to (h,k'), then take the loop $x\times\{k'\}$, then return to (h,k) along τ^{-1}. Call this loop x'. Similarly define the loop y'. Finally, define the loop a' to be the loop obtained by following τ from (h,k) to (h,k'), then following a loop parallel to and between $\{h\}\times A_1$ and $\{h\}\times A_2$ in $\{h\}\times K$, and finally returning to (h,k) along τ^{-1}.

Since W_1 is homeomorphic to $H\times K_1$ (always taking the base point (h,k)),

$$\pi_1(W_1') = \langle x',y'\rangle\oplus\mathbb{Z}a'.$$

We then let $W_2' = W_2\cup\tau$. This is W_2 with an arc attached spanning the boundary components corresponding to A_1 and A_2. Notice that the loops $a,b,x,$ and y all lie in W_2' (recall that these are the explicit loops on $H\times\{k\}\cup\{h\}\times K$ which we claim generate $\pi_1(H\times K-(T_1\cup T_2))$). Denote by c the loop in K_2 based at k which travels to the boundary ∂K, goes around once, and returns to k (thus, in $\pi_1(K)$, c represents the commutator $[a,b]$). We consider the loop $c' := \{h\}\times c$ in $\{h\}\times K$ based at (h,k), this is also a loop in W_2'.

Then because W_2' is obtained from $W_2\cong H\times K_2$ by adding the arc τ, it is clear that the five loops a,b,x,y,c' generate $\pi_1(W_2')$. We will not need this, but note that a and c' commute with x,y and that b generates a free factor.

We now apply the Seifert–Van Kampen theorem to conclude that the fundamental group $\pi_1(H\times K-\mathrm{nbd}(T_1\cup T_2))$ is generated by the loops

$$a',x',y',a,b,x,y,c'.$$

Thus to establish our claim that x, y, a, b generate $\pi_1(H\times K-\mathrm{nbd}(T_1\cup T_2))$, we must show that the based homotopy classes a', x', y', and c' in $\pi_1(H\times K-\mathrm{nbd}(T_1\cup T_2))$ can be expressed in terms of a, b, x, and y.

Since a' lies on $\{h\}\times K$, which misses $T_1\cup T_2$, it is obvious that a' and a represent the same class. Equally easy is the observation that $c'=[a,b]$ in $\pi_1(H\times K-\mathrm{nbd}(T_1\cup T_2))$.

This leaves the classes x' and y'. Consider first x'. We claim it is based homotopic to x. We can give an explicit formula for such a homotopy. Let β denote the path from k to k' in K that follows b backwards. For $s\in[0,1]$, let β_s denote the path $t\mapsto\beta((1-s)t)$ (so $\beta_0=\beta$ and β_1 is the constant path at k).

Then the homotopy

$$s\mapsto(\{h\}\times\beta_s)*(x\times\{\beta(1-s)\})*(\{h\}\times\beta_s)^{-1}$$

is a based homotopy from x' to x *that misses* $T_1\cup T_2$. This is because ,when passing through $\Phi^{-1}(A_1)\cap(H\times K-\mathrm{nbd}(T_1\cup T_2))=(H-\mathrm{nbd}(X))\times A_1$ (ie when $\beta_s(1)=\beta(s)$ lies on A_1), the curve x is parallel to X and hence misses it.

We can similarly show that y' is based homotopic in $H\times K-\mathrm{nbd}(T_1\cup T_2)$ to a loop represented by a word in a, b, x, y. This time we need to push across A_2 instead of A_1. Since we have already noticed that any based loop in W_2' can be expressed in terms of a, b, x, y, it is easiest just to slide y' along τ past A_2. This time the fact that Y is parallel to y and $\Phi^{-1}(A_2)\cap(H\times K-\mathrm{nbd}(T_1\cup T_2))=(H-Y)\times A_2$ allows us to conclude that y' can be expressed in terms of a, b, x, y, c', and hence a, b, x, and y.

Thus $\pi_1(H\times K-\mathrm{nbd}(T_1\cup T_2))$ is generated by the loops a, b, x, y, as claimed.

To finish the proof, we establish the stated commutator relations. The torus $x\times a$ contains the base point (h,k) and the curves x and a, and misses T_1 and T_2. Hence $[x,a]=1$. Similarly the torus $y\times a$ shows that $[y,a]=1$. If e denotes the loop in H that goes from the base point to the boundary of H, travels around the boundary, then returns to h (avoiding the curves X and Y) then the mapped in torus $e\times b$ misses $T_1\cup T_2$ and hence $[[x,y],b]=1$ in $\pi_1(H\times K-\mathrm{nbd}(T_1\cup T_2))$. Similarly $[x,[a,b]]=1$ and $[y,[a,b]]=1$.

The other commutator relation is a consequence of the fact that μ_i, m_i, and ℓ_i commute, since they live on the boundary of a tubular neighborhood of T_i, a 3–torus.

This completes the proof of Theorem 2. □

There are a few other relations in $\pi_1(H\times K-\mathrm{nbd}(T_1\cup T_2))$ which we did not mention in the statement of Theorem 2, eg $[[b^{-1},y^{-1}],x]$, $[[b^{-1},y^{-1}],a]$, $[[x^{-1},b],y]$, and

$[[x^{-1},b],bab^{-1}]$. These follow from the fact that they correspond to loops on the boundary of the tubular neighborhoods of T_1 and T_2. We will not need these relations in our argument.

In Theorem 2 we worked with the product of two punctured tori not for generality's sake, but because we will need to use the same construction in three different contexts later:

(1) H is the complement of a disk in a (closed) torus $\widehat{H}$. Thus we will be interested in the two Lagrangian tori T_1,T_2 in $\widehat{H}\times K$, the fundamental group of the complement $\pi_1(\widehat{H}\times K-(T_1\cup T_2))$, and the corresponding $\mu_i\, m_i,\ell_i$.

(2) K is the complement of a disk in a (closed) torus $\widehat{K}$. Thus we will be interested in the two Lagrangian tori T_1,T_2 in $H\times\widehat{K}$, the fundamental group of the complement $\pi_1(H\times\widehat{K}-(T_1\cup T_2))$, and the corresponding $\mu_i\, m_i,\ell_i$.

(3) K and H are both complements of disks in a (closed) tori. Thus we will be interested in the two Lagrangian tori T_1,T_2 in the four torus T^4, the fundamental group of the complement $\pi_1(T^4-(T_1\cup T_2))$, and the corresponding $\mu_i\, m_i,\ell_i$.

(Cases (1) and (2) are inequivalent due to the asymmetry of the pair X,Y and the pair A_1,A_2). The effect on fundamental groups in these three cases is clearly to impose the appropriate commutator relation.

Scholium 3 *In the three cases enumerated above, the statement of Theorem 2 remains true if we replace $H\times K$ by $\widehat{H}\times K$, $H\times\widehat{K}$, and T^4 respectively. Moreover, in the three cases, there is a further relation in the fundamental groups:*

(1) *The relation $[x,y]=1$ holds in $\pi_1(\widehat{H}\times K-(T_1\cup T_2))$.*

(2) *The relation $[a,b]=1$ holds in $\pi_1(H\times\widehat{K}-(T_1\cup T_2))$.*

(3) *The relations $[x,y]=1$ and $[a,b]=1$ hold in $\pi_1(T^4-(T_1\cup T_2))$.* □

Recall that given a Lagrangian torus T in a symplectic 4–manifold M, with meridian μ, and Lagrangian push offs m and ℓ in $\pi_1(M-T)$, *Luttinger surgery* is the process which removes a neighborhood $T\times D^2$ from M and glues it back in by a diffeomorphism which takes a disk $\{t\}\times D^2$ to a curve of the form $\mu m^{kp}\ell^{kq}$ where p,q are relatively prime integers and k is an integer. To specify the choices, we say the resulting manifold is obtained *by $1/k$ Luttinger surgery along the curve $pm+q\ell$*. Luttinger [14] (see also Auroux, Donaldson and Katzarkov [3]) proved that for any integer k and any choice of p,q, the result of Luttinger surgery admits a symplectic structure in which the core $T\times\{0\}$ is also Lagrangian, and so that the symplectic structure is unchanged in the complement of the tubular neighborhood of T.

We include the following well-known lemma for completeness.

Lemma 4 *The fundamental group of the manifold obtained by $1/k$ Luttinger surgery on M along $pm+q\ell$ is the quotient*

$$\pi_1(M-T)/N(\mu m^{kp}\ell^{kq})$$

where $N(\mu m^{kp}\ell^{kq})$ denotes the normal subgroup of $\pi_1(M-T)$ generated by $\mu m^{kp}\ell^{kq}$.

Proof The 2–torus has a handle structure with one 0–handle, two 1–handles, and one 2–handle. Thus the product $T^2\times D^2$ has a handle structure with one 0–handle, two 1–handles, and one 2–handle. Looking from the outside in, one sees that attaching $T^2\times D^2$ can be accomplished by attaching one 2–handle, two 3-handles, and one 4–handle. Attaching the 2–handle has the stated effect on fundamental groups, and attaching 3 and 4 handles does not further affect the fundamental group. □

Call the relations in Theorem 2 and Scholium 3 *universal relations* since they hold for any Luttinger surgery, and indeed, in the complement of $T_1\cup T_2$. The relations of Lemma 4 coming from Luttinger surgery will be called *Luttinger relations*.

We end this section with one lemma which will be used to establish minimality of the manifolds we construct.

Lemma 5 *Let M be obtained from the 4–torus $T^4=\hat{H}\times\hat{K}$ by $1/k_1$ Luttinger surgery on T_1 along x and $1/k_2$ surgery on T_2 along a. Then $\pi_2(M)=0$, and hence M is minimal.*

Proof First, $1/k_1$ surgery on T_1 along x transforms T^4 into $N\times S^1$, where N is the 3–manifold that fibers over S^1 with monodromy the k_1–th power of the Dehn twist on $\hat{H}$ along x. This follows from the well-known fact for fibered 3–manifolds that changing the monodromy by a Dehn twist corresponds to a Dehn surgery along a curve in a fiber. One can find a careful explanation in [3, p 189].

View $N\times S^1$ as a trivial circle bundle over N. Removing a neighborhood of T_2 and regluing has the effect of changing this trivial S^1 bundle to a nontrivial bundle. Explicitly one removes a neighborhood of y in N and its preimage in $N\times S^1$, then reglues in such a way that $k_2[y]$ becomes the divisor of the resulting S^1 bundle. Details can be found in the paper [4] by the first author. In any case one can check directly from the construction that M has a free circle action which coincides with the action on $N\times S^1$ away from T_2.

Thus M is an S^1 bundle over a fibered 3–manifold N with fiber a torus. It follows from the long exact sequence of homotopy groups that $\pi_2(M) = 0$, and hence M contains no essential 2–spheres. In particular, M is minimal. □

3 The building blocks

3.1 The manifold W

Consider the 4-torus $T^4 = S^1 \times S^1 \times S^1 \times S^1 = T^2 \times T^2$. Denote the coordinate circles respectively by s_1, t_1, s_2, t_2. So for example $s_2 = \{1\} \times \{1\} \times S^1 \times \{1\}$. These determine loops in T^4. Let $\Phi\colon T^4 \cong \widehat{H} \times \widehat{K}$ be a base point preserving diffeomorphism (in fact linear map) that takes the circles s_1, t_1, s_2, t_2 to x, y, a, b respectively. Pulling back the tori T_1, T_2 via the symplectomorphism Φ gives a pair of Lagrangian tori in T^4 which, by abuse of notation, we also denote by T_1 and T_2.

It is helpful to call T_1 the $s_1 \times s_2$ *torus* and T_2 the $t_1 \times s_2$ *torus* to remember what (conjugacy) classes in the fundamental group they carry. The nomenclature can be confusing, since T_2 is pushed farther away than T_1 from the loop a, due to the fact that A_1 and A_2 are different curves in K. In particular, the Lagrangian push offs are only specified up to conjugacy by this notation: for example, Theorem 2 states that the Lagrangian push off of the curve on T_2 represented by s_2 curve is $\ell_2 = t_2 s_2 t_2^{-1}$.

Theorem 2, Scholium 3, and Lemma 4 allow us to conclude that the fundamental group of the manifold V obtained by -1 Luttinger surgery on the $s_1 \times s_2$ torus along s_1 and -1 surgery on the $t_1 \times s_2$ torus along s_2 is generated by s_1, t_1, s_2, t_2 and the Luttinger relations

$$[t_2^{-1}, t_1^{-1}] = s_1, [s_1^{-1}, t_2] = t_2 s_2 t_2^{-1}$$

as well as the universal relations

$$[s_1, t_1] = 1, [s_2, t_2] = 1, [s_1, s_2] = 1, [t_1, s_2] = 1, [t_1, t_2 s_2 t_2^{-1}] = 1$$

hold. Note that by conjugating by t_2^{-1} we may simplify the second Luttinger relation to

$$[t_2^{-1}, s_1^{-1}] = s_2.$$

The last universal relation reduces to the (redundant) relation $[t_1, s_2] = 1$.

Thus $\pi_1(V)$ is a quotient of the group with presentation

$$\langle s_1, t_1, s_2, t_2 \mid [s_1, t_1],\ [s_2, t_2],\ [s_1, s_2],\ [t_1, s_2],\ [t_2^{-1}, t_1^{-1}] s_1^{-1},\ [t_2^{-1}, s_1^{-1}] s_2^{-1} \rangle.$$

Remark It is critical in these calculations that the loops s_1, t_1 are to be understood as explicit loops in the symplectic surface $\widehat{H} = T^2 \times \{(1,1)\} \subset T^4 - (T_1 \cup T_2)$ and the loops s_2, t_2 are to be understood as loops in the symplectic surface $\widehat{K} = \{(1,1)\} \times T^2 \subset T^4 - (T_1 \cup T_2)$, all based at $(1,1,1,1)$.

Lemma 5 shows that V can be described as an S^1–bundle over a 3–manifold that fibers over a circle with genus one fibers, and so V is a minimal symplectic 4–manifold.

The symplectic tori $\widehat{H} = T^2 \times \{(1,1)\}$ and $\widehat{K} = \{(1,1)\} \times T^2$ in T^4 miss neighborhoods of T_1 and T_2, and hence determine symplectic tori in V that we continue to call $\widehat{H}$ and $\widehat{K}$. Notice that $\widehat{H}$ and $\widehat{K}$ intersect once transversally and positively at the base point $p = (h,k) = (1,1,1,1)$.

Symplectically resolve this intersection point as explained in Gompf [12]. This is a local modification in a small neighborhood of p which replaces $\widehat{H} \cup \widehat{K}$ by a smooth symplectic surface G.

The topological description of this process is as follows. In a small 4–ball around p, a pair of intersecting 2–disks in $\widehat{H} \cup \widehat{K}$ are removed and replaced by an annulus so that the resulting closed genus 2 surface G is oriented consistently with the orientations of $\widehat{H}$ and $\widehat{K}$. Thus one can choose a base point p' inside this annulus, based loops s_1', t_1', s_2', t_2' on G satisfying $[s_1, t_1][s_2, t_2] = 1$ in $\pi_1(G, p')$, and a small arc in the 4–ball from p' to p so that the inclusion $\pi_1(G, p') \to \pi_1(V, p')$ followed by the identification $\pi_1(V, p') \cong \pi_1(V, p)$ given by the small arc takes s_i', t_i' to s_i, t_i. Therefore we can safely rename $p' = p, s_i' = s_i$, $t_i' = t_i$ and the fundamental group calculations are unchanged.

Now blow up V twice at two distinct points on G, obtaining a symplectic manifold

$$W = V \# 2\overline{\mathbb{CP}}^2.$$

The proper transform of G is a symplectic surface in W [12] which we continue to call G. It has the same fundamental group properties as it did in V, but, in addition, $G \subset W$ has a trivial normal bundle and intersects each exceptional sphere transversally once. Moreover, since $\pi_2(V) = 0$, it follows from the Hopf sequence that the spherical classes in $H_2(W)$ are generated by the exceptional spheres, and hence G intersects every -1 spherical class.

Fix a push off $G \to W - \mathrm{nbd}(G)$ and give $W - G$ the base point which is the image of p via this push off. Use a path in a meridian disk to identify based loops in $W - G$ and based loops in W. Since the surface G intersects a sphere (either of the two exceptional spheres) transversally in one point, the meridian of G in $W - G$ is nullhomotopic. Moreover, the inclusion $W - G \subset W$ induces an isomorphism on fundamental groups, since every loop in W can be pushed off G and every homotopy that intersects G can be replaced by a homotopy that misses G (using the exceptional sphere and the fact that G is connected). Therefore we conclude the following lemma. As before $N(S)$ denotes the normal subgroup generated by a set S.

Lemma 6 (1) *The closed symplectic 4–manifold W contains a closed symplectic genus 2 surface G with trivial normal bundle. There are based loops s_1, t_1, s_2, t_2 on G representing a standard symplectic generating set for $\pi_1(G, p)$ (thus satisfying $[s_1, t_1][s_2, t_2] = 1$) such that these loops generate $\pi_1(W, p)$ and, in $\pi_1(W, p)$, the relations*

$$1 = [s_1, t_1] = [s_2, t_2] = [s_1, s_2] = [t_1, s_2] = [t_2^{-1}, t_1^{-1}]s_1^{-1} = [t_2^{-1}, s_1^{-1}]s_2^{-1}$$

hold. The inclusion $W - G \subset W$ induces an isomorphism on fundamental groups.

(2) *Let R be any 4–manifold containing a genus 2 surface F with trivialized normal bundle. Let $\phi\colon G \to F$ be a diffeomorphism, and set $g_i = \phi_*(s_i), h_i = \phi_*(t_i)$ in $\pi_1(R)$. Given a map $\tau\colon G \to S^1$, let $\widetilde{\phi}\colon G\times S^1 \to F\times S^1$ be the diffeomorphism given by $\widetilde{\phi}(a, s) = (\phi(a), \tau(a)\cdot s)$. Form the sum:*

$$S = (R - \mathrm{nbd}(F)) \cup_{\widetilde{\phi}} (W - \mathrm{nbd}(G)).$$

Then the quotient group

$$\pi_1(R)/N([g_1, h_1], [g_2, h_2], [g_1, g_2], [h_1, g_2], [h_2^{-1}, h_1^{-1}]g_1^{-1}, [h_2^{-1}, g_1^{-1}]g_2^{-1})$$

surjects to $\pi_1(S)$.
Moreover, the Euler characteristic of S, $e(S)$, equals $e(R)+6$ and the signature $\sigma(S)$ equals $\sigma(R)-2$.

Proof The first assertion is explained in the paragraph that precedes the statement of Lemma 6.

For the second assertion, the statements about the fundamental group of S are a straightforward consequence of the Seifert–Van Kampen theorem applied to the decomposition $S = (R - \mathrm{nbd}(F)) \cup_{\widetilde{\phi}} (W - \mathrm{nbd}(G))$, using the fact that the meridian of G bounds a disk in W (the punctured exceptional sphere) and that $\pi_1(G) \to \pi_1(W - G)$ is surjective (because its composite with the isomorphism $\pi_1(W - G) \to \pi_1(W)$ is surjective).

The only remaining unverified assertions are the claims about Euler characteristic and signature. The Euler characteristic of S is computed using the formula $e(A\#_H B) = e(A) + e(B) - 2e(H)$, which is true for any sum of 4–manifolds along surfaces. Therefore $e(S) = e(W) + e(R) + 4 = 2 + e(R) + 4 = e(R) + 6$. Novikov additivity can be used to compute the signature, so $\sigma(S) = \sigma(W) + \sigma(R) = \sigma(R) - 2$. □

In Lemma 6, suppose further that R is symplectic and F is a symplectic genus 2 surface in R. Then S admits a symplectic structure [12]. Finally, if R is minimal, and not an S^2 bundle over F, then S is minimal by Usher's theorem [22]. This follows since every embedded -1 sphere in W intersects the surface G.

3.2 The manifold P

The second building block P will be the symplectic sum along a torus of two manifolds constructed in the same manner as V. Alternatively, P can be described as the result of Luttinger surgeries on four Lagrangian tori in the product of a genus two surface with a torus. There are three perspectives for the reader to keep in mind:

(1) To apply the calculations of Theorem 2, one should view P as the union along their boundary of two manifolds obtained by Luttinger surgeries on the product of a punctured torus with a torus, and then apply the Seifert–Van Kampen theorem.

(2) To conclude that P is symplectic one should view P as the symplectic sum of two manifolds obtained by Luttinger surgeries on $T^4 = T^2 \times T^2$.

(3) To conclude that P is minimal one should view P as the symplectic sum of two minimal symplectic manifolds.

Since the fundamental group calculation is the most delicate, we take the first perspective, and trust that the reader can follow the claims about symplectic structure and minimality.

We therefore build P as the union of two manifolds P_1 and P_2 along their boundary. Give each torus which appears in the following construction the standard symplectic form (ie as the quotient $\mathbb{R}^2/\mathbb{Z}^2$). A punctured torus should be given the restricted symplectic form, and the product of two (punctured or unpunctured) tori should be given the product symplectic form.

For P_1, start with a product $\hat{H}_1 \times K_1$ of a torus with base point h_1 and a punctured torus with base point k_1. Label the loops on $\hat{H}_1$ generating $\pi_1(\hat{H}_1)$ by x_1, y_1 and the loops in K_1 generating $\pi_1(\hat{K}_1)$ by s_1, t_1. Let $\hat{H}$ and K be as in Theorem 2 and Scholium 3.

Let $\psi_1\colon \hat{H}_1 \to \hat{H}$ be the diffeomorphism of the torus which rotates the square by angle $\pi/2$. Thus ψ_1 preserves base points, is orientation preserving, and induces the isomorphism $x_1 \mapsto y$ and $y_1 \mapsto x^{-1}$ on fundamental groups. Similarly, let $\psi_2\colon K_1 \to K$ be the diffeomorphism of the punctured torus which rotates the punctured square by angle $\pi/2$. Thus ψ_2 preserves base points, is orientation preserving, and induces the isomorphism $s_1 \mapsto b$ and $t_1 \mapsto a^{-1}$.

Since rotation by $\pi/2$ induces an area-preserving map on the torus, the diffeomorphism $\Psi = \psi_1 \times \psi_2\colon \widehat{H}_1 \times K_1 \to \widehat{H} \times K$ is a symplectomorphism which takes the loops x_1, y_1, s_1, t_1 to y, x^{-1}, b, a^{-1} respectively. We do $+1$ Luttinger surgery on $\Psi^{-1}(T_1)$ (the $y_1^{-1} \times t_1^{-1}$ torus) along y_1^{-1} and $+1$ Luttinger surgery on $\Psi^{-1}(T_2)$ (the $x_1 \times t_1^{-1}$ torus) along t_1^{-1}. Then Theorem 2 and Scholium 3 imply that the fundamental group of the resulting manifold P_1 is generated by

$$x_1, y_1, s_1, t_1$$

and the Luttinger relations

$$y_1 = [s_1^{-1}, x_1^{-1}], s_1 t_1 s_1^{-1} = [y_1, s_1]$$

as well as the universal relations

$$[y_1^{-1}, t_1^{-1}] = 1,\ [x_1, t_1^{-1}] = 1,\ [x_1, s_1 t_1^{-1} s_1^{-1}] = 1,\ [x_1, y_1] = 1$$

hold. We rewrite the second Luttinger relation as

$$t_1 = [s_1^{-1}, y_1].$$

For P_2, start with a product $\widehat{H}_2 \times K_2$ of a torus and a punctured torus. Label the loops on $\widehat{H}_2$ generating $\pi_1(\widehat{H}_2)$ by x_2, y_2 and the loops in K_2 generating $\pi_1(\widehat{K}_2)$ by s_2, t_2.

As above, choose a symplectomorphism $\Psi_2\colon \widehat{H}_2 \times K_2 \to \widehat{H} \times K$ which takes the generators x_2, y_2, s_2, t_2 to y, x^{-1}, b, a^{-1} respectively.

We do $+1$ Luttinger surgery on $\Psi_2^{-1}(T_1)$ (the $y_2^{-1} \times t_2^{-1}$ torus) along t_2^{-1} and -1 Luttinger surgery on $\Psi_2^{-1}(T_2)$ (the $x_2 \times t_2^{-1}$ torus) along x_2. Then Theorem 2 and Scholium 3 imply that the fundamental group of the resulting manifold P_2 is generated by

$$x_2, y_2, s_2, t_2$$

and the Luttinger relations

$$t_2 = [s_2^{-1}, x_2^{-1}], x_2 = [y_2, s_2]$$

as well as the universal relations

$$[y_2^{-1}, t_2^{-1}] = 1, [x_2, t_2^{-1}] = 1, [x_2, s_2 t_2^{-1} s_2^{-1}] = 1, [x_2, y_2] = 1$$

hold.

Denote by M_1 and M_2 the symplectic manifolds obtained by the same construction as P_1 and P_2 but starting with closed tori, ie $\widehat{H}_i \times \widehat{K}_i \cong T^4$. Denote by z_1 and z_2 the centers of the disks removed from $\widehat{K}_i$ to obtain K_i. As a smooth manifold, the symplectic sum, P, of M_1 and M_2 along the symplectic tori with trivial normal bundles $\widehat{H}_1\times\{z_1\}$ and $\widehat{H}_2\times\{z_2\}$ [12], is the union of P_1 and P_2 along their boundary 3–tori. We use the diffeomorphism of the tori along which the symplectic sum is performed so that x_1 is identified with x_2 and y_1 is identified with y_2.

More precisely, there exists an arc β in K_1 which starts at a point $k_1'\in\partial K_1$ and ends at k_1 and which misses $\psi_2^{-1}(A_i), i=1,2$, since cutting the surface K along $a\cup b\cup A_1\cup A_2$ does not disconnect k from ∂K. This arc β should be (and can be) chosen so that the loop traced out by the boundary is homotopic rel endpoint to $\beta*[s_1,t_1]*\beta^{-1}$ in K_1. The arc $\widetilde{\beta}=\{h_1\}\times\beta\subset\widehat{H}_1\times K_1$ misses $T_1\cup T_2$, since β misses $A_1\cup A_2$, and hence can be viewed as a path in P_1.

Conjugating by $\widetilde{\beta}$ induces an isomorphism $\pi_1(P_1,(h_1,k_1))\cong\pi_1(P_1,(h_1,k_1'))$ so that the loops x_1,y_1,s_1,t_1 are sent to loops we temporarily call x_1',y_1',s_1',t_1'. Obviously, all the relations listed above involving the x_1,y_1,s_1,t_1 also hold for the x_1',y_1',s_1',t_1'.

The loop $x_1'=\widetilde{\beta}*x_1*\widetilde{\beta}^{-1}$ is homotopic rel endpoint into the boundary of P_1. In fact, the one parameter family of loops $x_1'(s)=\widetilde{\beta}_s*(x_1\times\beta(s))*\widetilde{\beta}_s^{-1}$ (where $\widetilde{\beta}_s(t)=\widetilde{\beta}(st)$) gives a homotopy of x_1' to the loop $x_1\times\{k_1'\}$ in the boundary 3–torus $\widehat{H}_1\times\partial K_1$ of P_1. (Note that this uses the fact that β misses $\psi_2^{-1}(A_1\cup A_2)$.) A similar comment applies to y_1'. The loop $[s_1',t_1']$ lies entirely on $\{h_1\}\times K_1\subset\widehat{H}_1\times K_1-\text{nbd}(T_1\cup T_2)\subset P_1$. Hence the loop $\{h_1\}\times\partial K_1\subset\partial P_1$ maps to $[s_1',t_1']$ via the inclusion $\pi_1(\partial P_1,(h_1,k'))\to\pi_1(P_1,(h_1,k'))$.

Thus we abuse notation slightly and rename $x_1=x_1'(1)$, $y_1=y_1'(1), s_1=s_1'$, and $t_1=t_1'$. These loops are based at the base point (h_1,k_1') on the boundary of P_1, generate $\pi_1(P_1)$, and all the relations listed above hold. Moreover the three loops x_1,y_1, and $c=\{h_1\}\times\partial K_1$ all lie on the boundary ∂P_1, generate $\pi_1(\partial P_1,(h_1,k_1'))$, and c is sent to $[s_1,t_1]$ in $\pi_1(P_1)$.

A similar comment applies to P_2, so we end up with the same presentation, but with base point $(h_2,k_2')\in\partial P_2$, and the loops x_2,y_2, and $[s_1,t_1]$ in ∂P_2 generating $\pi_1(\partial P_2,(h_2,k_2'))$.

We glue P_1 to P_2 using a base point preserving diffeomorphism which takes x_1 to x_2, y_1 to y_2, and $[s_1,t_1]$ to $[s_2,t_2]^{-1}$. (This last gluing actually follows from the first two and the fact that we are forming the symplectic sum of M_1 and M_2 to build P.) Note that we can arrange this to be the relative symplectic sum [12] of $(M_1,\{h_1\}\times\widehat{K}_1)$ and $(M_2,\{h_2\}\times\widehat{K}_2)$ so that the surfaces $\{h_1\}\times K_1$ and $\{h_2\}\times K_2$ line up along their boundary, yielding a closed symplectic genus 2 surface F in P. The loops s_1,t_1,s_2,t_2 lie on F and these form the standard set of generators of the fundamental group of F.

This fact allows us to apply Lemma 6 in the proof of Theorem 7 below. The Seifert–Van Kampen theorem implies that $\pi_1(P)$ is generated by $x_1, y_1, s_1, t_1, x_2, y_2, s_2, t_2$.

The definition of P, the calculations for P_1 and P_2 given above, and the Seifert–Van Kampen theorem imply that the relations

$$y_1 = [s_1^{-1}, x_1^{-1}], \quad t_1 = [s_1^{-1}, y_1], \quad [y_1^{-1}, t_1^{-1}] = 1, \quad [x_1, t_1^{-1}] = 1, \quad [x_1, s_1 t_1^{-1} s_1^{-1}] = 1, \quad [x_1, y_1] = 1 \tag{1}$$

$$t_2 = [s_2^{-1}, x_2^{-1}], \quad x_2 = [y_2, s_2], \quad [y_2^{-1}, t_2^{-1}] = 1, \quad [x_2, t_2^{-1}] = 1, \quad [x_2, s_2 t_2^{-1} s_2^{-1}] = 1, \quad [x_2, y_2] = 1 \tag{2}$$

$$x_1 = x_2, \quad y_1 = y_2 \tag{3}$$

hold in $\pi_1(P)$. The additional relation $[s_1, t_1][s_2, t_2] = 1$ also follows from the Seifert–Van Kampen theorem, but we will not need it below.

The closed symplectic manifolds M_1 and M_2 have trivial second homotopy group, and hence are minimal, by Lemma 5. Thus by Usher's theorem [22] their symplectic sum P is also minimal.

The Euler characteristic is $e(P) = e(M_1) + e(M_2) + 0 = 0$ and the signature $\sigma(P) = \sigma(M_1) + \sigma(M_2) = 0$.

4 Assembly: an exotic symplectic $\mathbb{CP}^2\#3\overline{\mathbb{CP}}^2$

Let X be the symplectic sum of P and W along the genus 2 surfaces $F \subset P$ and $G \subset W$,

$$X = (P - \mathrm{nbd}(G)) \cup_{\tilde{\phi}} (W - \mathrm{nbd}(G))$$

using a diffeomorphism $\phi\colon F \to G$ that identifies generators in $\pi_1(F)$ with their namesakes in $\pi_1(G)$.

By Lemma 6 and the text that immediately follows it, X is a symplectic 4–manifold with $e(X) = 6$ and $\sigma(X) = -2$. Furthermore, X is minimal by Usher's theorem [22] since P is, and since $W - G$ contains no -1 spheres.

Once we show X is simply connected, then Freedman's theorem [11] implies that X is homeomorphic to $\mathbb{CP}^2\#3\overline{\mathbb{CP}}^2$. It cannot be diffeomorphic $\mathbb{CP}^2\#3\overline{\mathbb{CP}}^2$ however, since X is minimal, and by results of Taubes [20; 21], a minimal symplectic 4–manifold cannot contain a smoothly embedded -1 sphere, but $\mathbb{CP}^2\#3\overline{\mathbb{CP}}^2$ contains smoothly embedded -1 spheres, namely, the exceptional spheres.

Theorem 7 *The minimal symplectic manifold X is simply connected, hence homeomorphic but not diffeomorphic to $\mathbb{CP}^2\#3\overline{\mathbb{CP}}^2$.*

Proof Since the loops s_1, t_1, s_2, t_2 lie on F, Lemma 6 implies that the fundamental group of X is a quotient of $\pi_1(P)/N$ where N is the normal subgroup generated by

$$(4) \qquad [s_1,t_1],\ [s_2,t_2],\ [s_1,s_2],\ [t_1,s_2],\ [t_2^{-1},t_1^{-1}]s_1^{-1},\ [t_2^{-1},s_1^{-1}]s_2^{-1}.$$

Denote by *relations 1–20* the 14 relations listed for the fundamental group of P in Equations (1), (2), and (3) and the six additional relations of Equation (4). Recall that $[r,s]^{-1} = [s,r]$.

To start, observe that relations 1 and 19 imply

$$y_1 = [s_1^{-1}, x_1^{-1}] = [[t_1^{-1}, t_2^{-1}], x_1^{-1}].$$

Relation 4 implies that x_1 commutes with t_1 and relations 10 and 13 imply that x_1 commutes with t_2. This implies that $y_1 = 1$.

The rest of the generators are rapidly killed. Relation 14 implies $y_2 = 1$. Relation 2 implies $t_1 = 1$. Relation 19 implies that $s_1 = 1$. Relation 20 implies that $s_2 = 1$. Relation 7 now shows that $t_2 = 1$ and Relations 8 and 13 imply that $x_1 = x_2 = 1$.

Thus Lemma 6 says that $\pi_1(P)$ is a quotient of the trivial group, hence is trivial. As explained above this implies that X is homeomorphic to, but not diffeomorphic to $\mathbb{CP}^2\#3\overline{\mathbb{CP}}^2$. □

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