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On the Nature of Connectivity Types

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Abstract

A connectivity type is defined by the connections between pairs of boundary nodes in a graph G. The purpose of this study is to gather more information on critical circular planar graphs. Critical circular planar graphs are useful in the study of inverse problems, where a graph being *critical* implies that the inverse problem is solvable. It is proved in this paper that any critical graph with *1-connections* forms a *complete intersection.* Hopefully this will be of substantive aid when trying to come up with a proof to the conjecture of the complete intersection property for any critical circular planar graph.

1 Introduction

Why study connectivity types one may ask? A decent response is that it allows for the classification of certain networks that may be potentially useful application-wise or theory-wise. By itself, connectivity types is not a subject studied to an end, but rather as a means to an end. A *connectivity type* involves the *sets of connections* in a graph. Given a graph G, we can determine its connectivity type by counting the types and number of connections the graph possesses. The study of such connections and how it relates to a graph G is our main focus in this paper. Related to graphs are networks, since networks are graphs with certain conditions imposed on it. A fitting example would be a conductivity function that measures the conductance of each edge. As such, the study of connectivity types is closely tied to the *inverse problem* over a resistor network. To gain an overview of what the topic on inverse problem is about, [1] would be a good place to begin, but, roughly speaking, the inverse problem is to find the conductance of each edge in a graph G from given measurements of voltages and currents at the boundary nodes. It is characterized by a linear map which takes the boundary voltages to boundary currents. So given a situation where voltages and currents are placed on the boundary nodes of a graph G, the main question one asks is whether a graph is *recoverable*, that is, whether it is possible to recover the conductances of each edge in G. This definition of recoverability is analogous to the solvability of the inverse problem. For certain graphs no matter what the imposed currents and voltages are at the boundary nodes the conductance on the edges cannot be recovered. Recoverable graphs have certain physical and theoretical uses in the study of electrical networks.

The problem itself originated in 1980 through Calderon's foundational paper [2], presented at a seminar in Rio de Janeiro, and subsequently followed up by papers from Curtis and Morrow [3]. As presented by Calderon, the inverse problem may be broken into four questions: uniqueness of the map, a given characterization of the map, an algorithm to calculate conductances based on the given map, and continuity of the

map. Curtis and Morrow looked for ways to find resistors in a network, and then gave certain characterizations of networks that solved the inverse problem. To the four questions they added a fifth: whether the graph G could be deduced. Morrow has spent over fifteen years doing research related to these five questions, and the majority of the results are presented in [1]. However, this paper will only deal with connectivity types. All of the research conducted for this project relates to *connectivity types* in one way or another. As such, the inverse problem only serves as a motivation tool to lead into the main topic of this paper.

2 Networks and Connections

Some terminology will be defined to make the inverse problem explicit. We will start by defining a network $\Gamma = (G, \gamma)$ over which the study of connectivity types is based upon.

Definition 1. A *graph with boundary* is a triple $G = (V, V_B, E)$, where V is the set of nodes and E is the set of edges for a finite graph, and V_B is a nonempty subset of V called the set of *boundary nodes.* The set $I = V - V_B$ is the set of *interior nodes.* The graph G is allowed to have multiple edges or loops. [1]

As a remark, any mention of a graph G from this point on will be a graph with boundary.

Definition 2. If u and v are distinct boundary nodes, a *path α from u to v* through G, consists of a sequence of edges : $e_0 = ur_1, e_1 = r_1r_2, ..., e_{h-1} = r_{h-1}r_h, e_h = r_hv$ such that $r_1, r_2, ...r_h$ are *distinct interior nodes* of G. An edge e between two distinct boundary nodes u and v is allowed as a path from u to v through G.

We will use the terms node and vertex interchangeably depending on occasion.

Definition 3. A *conductivity* on a graph G is a function γ which assigns to each edge e in G a positive real number $\gamma(e)$, called the *conductance* of the edge e.

Definition 4. A *resistor network* $\Gamma = (G, \gamma)$ is a graph G together with a conductivity function γ.

The term resistor network is the standard term for a graph with resistors as edges. The conductance of a resistor is the reciprocal of the resistance, and for algebraic reasons, is more convenient than the resistance. If u is a function defined on all the nodes of a resistor network Γ, and e is an edge of G, with endpoints p and q, the current $c(e)$ through edge e is defined by *Ohm's Law*:

$$c(e) = \gamma(e)[(u(p) - u(q)].$$

If there is one or more edges joining p to q in G, $\gamma_{p,q}$ is defined to be the sum of the conductances of edges joining p to q. The current from p to q is:

$$c_{p,q} = \gamma_{p,q}[u(p) - u(q)].$$

For each node p in G, the set of nodes q for which there is an edge joining p to q is called the set of *neighbors* of p and is denoted $N(p)$. A function u defined on the nodes of G is said to be *γ-harmonic* at p if the algebraic sum of the currents from p to the neighboring nodes is 0. This means that the value of u at each interior node is the weighted average of the values of u at the neighboring nodes. Explicitly, we mean:

$$\sum_{q \in N(p)} \gamma_{p,q}[u(p) - u(q)] = 0.$$

If u is γ-harmonic at each of the interior nodes, u is said to be a γ-harmonic function. At a node where u is not γ-harmonic, Kirchhoff's Law says that they current $\phi(p)$ into the network at p must equal to the sum of the currents from p to its neighboring nodes. That is,

$$\phi(p) = \sum_{q \in N(p)} \gamma_{p,q}[u(p) - u(q)].$$

However, if we sum $\phi(p)$ for all nodes p in G, and observing that the current across each edge occurs twice with opposite signs, we get

$$\sum_{p \in G} \phi(p) = 0.$$

The algebraic sum of the currents into Γ at all nodes is 0. And if u is a γ-harmonic function on Γ with boundary, then $\phi(p) = 0$ at all interior nodes, and hence the sum of $\phi(p)$ over all nodes p on the boundary has to be 0. Suppose that the resistor network has n boundary nodes and d interior nodes. If there is a function f defined on the boundary nodes, there is a unique function u that agrees with f at the boundary nodes, and is γ-harmonic at the interior nodes. The function u on Γ is called the *potential*, and the resulting current due to the potential u at the boundary nodes is called the *network response*. The linear map $\Lambda = \Lambda_\gamma$ which takes the boundary voltage f to the boundary current ϕ is called the *response map* because it gives the current to

any potential imposed on the boundary. The response map will be known when the potential is found for each boundary function f and the resulting boundary current ϕ is calculated. Using standard basis, we can represent the response map by an $n \times n$ matrix, also denoted Λ, called the *response matrix.* Entries in the response matrix will be of the form $\lambda_{i,j}$, designating the ith by jth entry in the matrix. The matrix has the following three properties:[1]

1. Λ is symmetric : $\lambda_{i,j} = \lambda_{j,i}$

2. The sum of the entries in each row equals 0

3. $\forall\, i \neq j,\ \lambda_{i,j} \leq 0$

The *Kirchhoff matrix* on the other hand stores the values of the conductances of the edges of the network. For a graph G with n boundary nodes and d interior nodes, and hence $m = n + d$ total nodes, the Kirchhoff matrix is an $m \times m$ matrix, denoted K. Entries in the Kirchhoff matrix for a network are defined by the following two properties:

1. $\forall\, i \neq j,\ K_{i,j} = -\gamma_{i,j}$

2. $K_{i,i} = \sum_{i \neq j} \gamma_{i,j}$

The matrix can be written in block form:

$$K = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}. \qquad (1)$$

Submatrix $A_{n \times n}$ stores the conductances of edges between boundary nodes, $C_{d \times d}$ stores conductances of edges between interior nodes, $B_{n \times d}$ and its transpose $B^T_{d \times n}$ store conductances between edges joining boundary to interior nodes.

2.1 The Inverse Problem

Let $K(I;I)$ be the submatrix of the Kirchhoff matrix K with index set I, where I denotes the set of interior nodes in matrix K. In the matrix above, the submatrix $K(I;I)$ is identically the submatrix C. As such, the response matrix Λ can be obtained using the submatrix $K(I;I)$ to block reduce K in the following manner:

$$\Lambda = K/K(I;I) = A - BC^{-1}B^{\top}. \tag{2}$$

This is equivalent to Gaussian elimination, while using block structures, and is termed the *Schur Complement.* If I is the empty set, then $K/K(I;I)$ is defined to be K, and $\Lambda_\gamma = K$. Given a graph G along with the conductances along each edge, finding the response matrix is the *forward problem.* The analogy of this is applying the Schur Complement on matrix K to get matrix Λ. On the other hand, if given Λ and the underlying graph G, a network Γ is recoverable if $\Gamma = (G, \gamma)$ is recoverable for any positive γ.

The *inverse problem* is to:

1. determine when Γ is recoverable,
2. compute Γ in terms of Λ and G when Γ is recoverable, or
3. compute the set of all γ that yield Γ on G when Γ is *not* recoverable.

The motivation for this paper follows the first, that is, to determine when the network is recoverable. As we can see, it amounts to being able to go backwards from the Λ matrix to the K matrix.

Now our focus will be to define connections and finally show how it relates to the inverse problem.

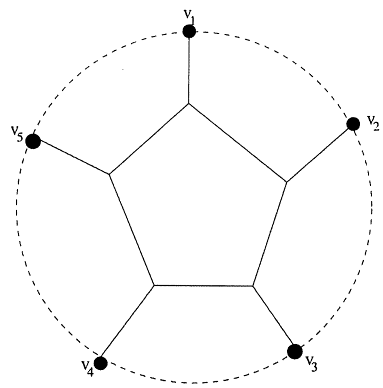

Figure 1: A circular planar graph

2.2 Connections and Connectivity Types

Definition 5. A *circular planar graph* is a graph G with boundary which is embedded in a disc D in the plane so that the boundary nodes lie on the circle C which bounds D, and the rest of G is in the interior of D. The nodes on the boundary will aptly be named boundary nodes, and they are labeled as $v_1, v_2, ..., v_n$ in a clockwise order around C, and the nodes in the interior likewise will be named interior nodes. A *circular pair* is a pair $(P; Q) = ((p_1, ..., p_k), (q_1, ..., q_k))$ of sequences of boundary nodes such that p_1, ... , p_k, q_1, ... , q_k are all distinct and appear on C in a clockwise order. (Figure 1 gives an example of a circular planar graph)

Remark 1. Any subsequent mention of a graph will be a circular planar graph G with boundary nodes.

Definition 6. Given a graph, $G = (V, E)$, suppose $P = \{p_1, ..., p_k\}$, and $Q = \{q_1, ..., q_k\}$ are two sets of boundary nodes on a disc D such that $(p_1, ... , p_k, q_1, ... , q_k)$ appear in clockwise order. P and Q are *connected through* G if for $i = 1, ..., k$, there exists a path α_i from p_i to q_{k+1-i} through G, and such that $\alpha_1, ..., \alpha_k$ are pairwise vertex-disjoint. To say the paths $\alpha_1, ..., \alpha_k$ are disjoint means that $m \neq n$ implies α_m and α_n have no vertices in common. The set $\alpha_1, ..., \alpha_k$ is a k-connection from P to Q. A path which joins one boundary node to another boundary node is a 1-connection.

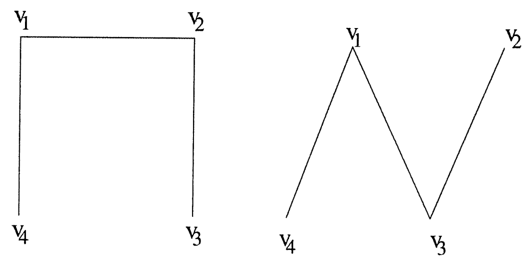

Figure 2: Two embeddings, two different *connectivity types*

Remark 2. It is clear from the definition that there may be more than one path α_i between any fixed sequences of pairs of node in P and Q, and hence more than one k-connection. However, for practical purposes, the existence of one k-connection suffices, for $k \leq n/2$.

As an example, there exists a 2-connection between pairs of nodes (v_1, v_2) and (v_3, v_5), shown in Figure 1. There exists separate, disjoint paths from v_1 to v_5 and from v_2 to v_3. Clearly there does not exist another 2-connection between those two pairs of nodes, or those two paths would not be vertex disjoint; that is, they would have to intersect at some vertex, thus breaking the rule of disjoint vertices.

Under this terminology, a *connectivity type* of a certain graph G is understood to be the set of all possible k-connections of G, given one fixed circular ordering of the graph. There is a slight subtlety to this definition. Two graphs can possess the same counts of connections, that is, both may have two 2-connections and three 1-connections, but they do not possess the same *connectivity type*, because they have different circular pairs. Figure 2 illustrates two graphs with the same counts of connections, but different connectivity types.

Now we would like a property to remove edges in a graph G so that we could possibly break k-connections. We need to define two operations for the removal of an edge e in a graph G, denoted $G - e$:

1. (Boundary node deletion): By deleting an edge e that is joining two boundary nodes or

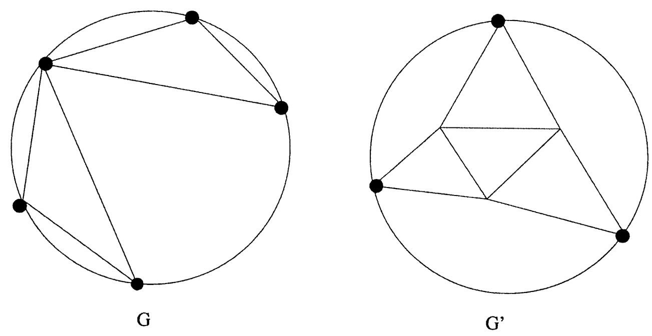

Figure 3: G is critical, G' is not critical

2. (Boundary edge contraction): By contracting an edge e to one of its endpoints

For the second operation, an edge joining two boundary nodes is not allowed to be contracted to a single node. Hence only boundary to interior node contractions are justified, and the remaining node becomes a boundary node.

Finally, we state the definition that motivates us to study connectivity types.

Definition 7. A graph G is *critical* if for all edges $e \in G$, $G - e$ breaks a certain k-connection between two boundary sets P and Q. In other words, if there exists some k-connection from P to Q through G, the resulting graph G' from by the removal of any edge e in the original graph G does not have that k-connection.

Figure 3 illustrates two graphs G and G'. G is critical and G' is not. It is possible to remove any edge incident to any two interior nodes of G' and not break a k-connection in the graph.

Definition 8. A circular planar graph G is *well-connected* if for every circular pair $(P;Q) = \{p_1, ..., p_k; q_1, ..., q_k\}$ of sets of boundary nodes, there is a k-connection from P to Q, and such that for any edge e in G, $G - e$ breaks some k-connection. A well-connected graph on n boundary nodes is denoted G_n.

Remark 3. For a fixed n, there are possibly many well-connected graphs with n boundary nodes. However, we will see in fact that all such G_n's are essentially equivalent graphs that have undergone a transformation that will be stated later.

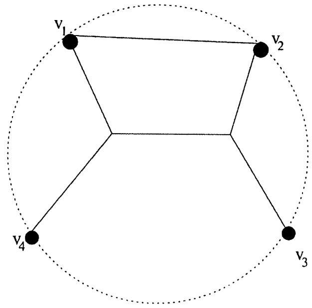

Figure 4: A well-connected 4-boundary node graph G

Figure 4 illustrates a well-connected 4 boundary node graph, which will be studied in more detail later. Note that the n boundary node well-connected graph is the smallest possible n boundary node graph that has all possible k-connections between pairs of sequences of boundary nodes. Hence it has $\binom{n}{2}$ edges, exactly the number of entries in the upper triangle of the response matrix.

Now we will define a term and give a lemma to help prove our main theorem.

Definition 9. Suppose a graph G contains four vertices p, q, r, and w, and three edges pw, qw, and rw. All vertices except vertex w are boundary nodes of G, and there are no other edges incident to w. A $Y-\Delta$ *transformation* in G consists in eliminating the vertex w, deleting the edges incident to w, and inserting edges pq, qr, and rp to form the Δ. [1]

Definition 10. A graph G' obtained by a $Y-\Delta$ transformation from G is said to be $Y-\Delta$ *equivalent* to G.

Lemma 1. Two critical graphs G and G' are $Y-\Delta$ equivalent if and only if they possess the same connectivity type.

The proof has been given on page 168 in [1].

It has been shown in [1] that the inverse problem is solvable under the condition that the underlying graph is critical, so the motivation for studying connectivity types is to attempt to gain a stronger foundation for these critical graphs. In essence by studying

connectivity types we would like to gain another property for critical graphs, and these would then solve the inverse problem. So in a nutshell, studying connectivity types on critical graphs allows us to gain more structure on the types of inverse problems that are solvable.

Now it is time to show the link between connections and determinants in a graph G. The theory here is strictly work done by Curtis and Morrow [1].

Lemma 2. Suppose $\Gamma = (G,\gamma)$ is a circular planar resistor network and $(P;Q)$=$(p_1, ..., p_k; q_1, ..., q_k)$ is a circular pair of boundary nodes.

1. If $(P;Q)$ are not connected through G, then $det\Lambda(P;Q) = 0$
2. If $(P;Q)$ are connected through G, then $(-1)^k \cdot det\Lambda(P;Q) > 0$.

The proof is given on page 52 in [1]. We can readily deduce that $\Lambda(P;Q)$ is non-singular if and only if P and Q are connected through G. In particular, any connected n-boundary node graph has the property that $\Lambda(P;Q)$ is non-singular. The theorem states that there is a connection between a pair of sequences of nodes $(u_i, ..., u_k)$ from P and $(v_i, ..., v_k)$ from Q if and only if the determinant of the submatrix formed by $(u_i, ..., u_k)$ and $(v_i, ..., v_k)$ is always nonzero. Letting the nodes of P be the row space and Q be the column space of the Λ matrix, the matrix formed by a subset of nodes $(u_i, ..., u_k)$ and $(v_i, ..., v_k)$ from P and Q is exactly the matrix formed by taking the u_jth and v_jth entries from the rows and columns, where $i \leq j \leq k$.

3 Complete Intersection

In order for us to deal with a certain n-boundary node critical circular planar graph, we would like a comprehensive list of all n-boundary node critical circular planar graphs up to a certain embedding class for $n \geq 3$. By definition, the well-connected n-boundary node graph G_n retains all possible k-connections for a given n. So it would seem reasonable if by a sequence of steps of edge removals from G_n and its $Y-\Delta$ equivalent

graphs, we would gain the whole class of critical graphs for a fixed n. We are allowed to do this since removing an edge gives us a new graph that is also critical. This motivates the next set of definitions.

Definition 11. An *algebraic variety* is a generalization to n dimensions of algebraic curves. It is a reduced scheme of finite type over a field K. An algebraic variety V in $\mathbf{R^n}$ is defined as the set of points satisfying a system of polynomial equations $f_i(x_1, ..., x_n) = 0$ for $i = 1, 2, ...$ formally written as $V \subset \mathbf{R^n}$. According to the Hilbert Basis Theorem, a finite number of equations suffices for the set of points that satisfy the system of polynomial equations. [4]

In a graphical sense, our system of homogeneous polynomial equations $f_i = 0$ for $i = 1, 2, ...$ are the equations of broken k-connections. Since we know if there is no k-connection from a set P to set Q, the determinant of that submatrix is zero. This determinant is the homogeneous polynomial equation we will be dealing with.

Definition 12. The *dimension* of the variety V, denoted dim V, is the number of elements in a basis of homogeneous equations of a given graph G, denoted $f_1 = f_2 = ...f_j = 0$, that spans all homogeneous equations of G, denoted $f_1 = f_2 = ...f_j = ...f_k = 0$, where $j \leq k$.

We can think of $\mathbf{R^n}$ as defined above to be the space of entries in the upper triangle of our Λ matrix, with n entries corresponding to n edges. We will define the variety V for a given graph G with n edges to be the set of all edges remaining after a finite number of boundary edge deletions or boundary node contractions from. The variety V itself will be the set of homogeneous equations for a given graph G with a set amount of edges. Let m denote the number of elements in the set of all edges remaining in V. Then dim $V = m$. Likewise, the number of edges that have been removed from these two operations will be codim $V = n - m$.

Since the set of homogeneous equations corresponds to the set of k-connections in the response matrix, knowing whether or not the connections exist can give us information on the homogeneous equation. More precisely, from Lemma 3, the set

of connections that do not exist between a circular pair $(P;Q)$ is exactly the set of homogeneous equations. Where a connection does not exist between pairs of boundary nodes, i.e. where an edge has been removed from the well connected graph G_n, the determinant of those pairs of boundary nodes is zero. This implies that every edge removal or edge contraction gains us at least one more homogeneous equation. Let us denote the number of edges in a critical circular planar graph G by $|E|$. Given the well-connected graph G_n has $\binom{n}{2}$ edges, the number of steps to reach a certain n boundary node graph G with edge size $|E|$ using a sequence of edge removals by deletion or contraction will be $\binom{n}{2}$ - $|E|$. For the dimension, dim $V = |E|$, the number of edges in the current graph G, and codim $V = \binom{n}{2}$ - $|E|$, or the number of edges removed from G_n. For the proofs, it will be necessary to use the latter criterion, but for theoretical purposes we will refer to the the dimension of V as m and codimension as $n - m$ to keep it short and simple.

Observation 1. The critical graph G with size $|E|$ satisfies at least $\binom{n}{2}$ - $|E|$ homogeneous equations. By our definition of a variety V, this set of equations is our V.

Now we can state the rudimentary ideas of this paper. Suppose we are given a critical circular planar graph G and we know the number of homogeneous equations it satisfies. The natural questions that arise are: does there exist a basis of homogeneous equations that spans all homogeneous equations, and if so, what is the minimum bound on the elements of this basis that spans the space of homogeneous equations, or V? Furthermore, does this basis apply to all critical circular planar graphs G with n boundary nodes?

Definition 13. If the minimal number of equations needed for a basis B to span an ideal defining the algebraic variety V is equal to the codimension of V, then the space of response matrices (Λ) forms a *complete intersection.*

Certainly, if we have the set of all critical circular planar graphs for a fixed n, it would be possible to check whether or not each one of these graphs G forms a complete

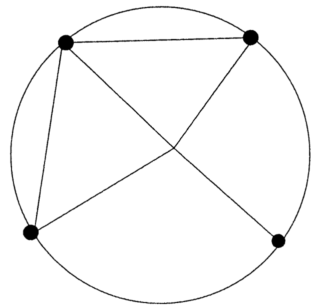

Figure 5: A $Y - \Delta$ equivalent graph of G_4

intersection by a few turns of algebra or in extreme cases by using the power of the computer. Using methods such as these, we have the following result.

Theorem 1. For any n boundary node critical circular planar graph G such that $n \leq 4$, the space of response matrices is a complete intersection.

Proof. The proof goes by way of exhaustion. In each case, a basis of homogeneous equations that spans V will be shown. The proof for the two boundary node case is trivial. It is the minimal connected graph, which implies a graph G such that $G - e$ is disconnected, for any edge $e \in G$. The proof for the three boundary node critical circular planar graphs is obvious, since breaking any edge of the Δ graph produces one homogeneous equation, and we are left with a minimal connected graph. For the four boundary node case, a comprehensive list of all such 4-boundary node graphs is given, and it remains to check whether each graph is a complete intersection. Let us give an example. Take the $Y - \Delta$ equivalent graph of G_4 illustrated in Figure 4. Figure 5 illustrates one of its $Y - \Delta$ equivalent graphs G_4' after applying a $Y - \Delta$ transformation on the original graph G_4.

Labeling the nodes in a clockwise manner starting at the top on the left hand side, we label v_1, v_2, v_3, v_4. By contracting v_3 to its interior node, and relabeling the interior node as v_3, the connection between v_2 and v_4 is broken, and hence $\lambda_{2,4} = 0$. This one connection trivially spans itself, and we have a complete intersection. Deleting another the edge between v_2 and v_3, we break the connection between v_2 and v_3, i.e. $\lambda_{2,3} = 0$. Furthermore another connection is broken, a 2-connection between

pairs (v_1, v_2), and pairs (v_3, v_4). This connection is represented by $det\lambda(1,2;3,4) = 0$. By our conjecture, we would need two homogeneous equations that span these three homogeneous equations. It is readily apparent that we can use a basis of the two 1-connections to span itself and the third 2-connection. That is, $\lambda_{2,3} = 0$ and $\lambda_{2,4} = 0$ $\Rightarrow det\lambda(1,2;3,4) = 0$. To see this in a response matrix, we have:

$$\Lambda = \begin{bmatrix} -\sum\limits_{j\neq 1} \gamma_{1,j} & \gamma_{1,2} & 0 & \gamma_{1,4} \\ \gamma_{2,1} & -\sum\limits_{j\neq 2} \gamma_{2,j} & 0 & 0 \\ 0 & 0 & -\sum\limits_{j\neq 3} \gamma_{3,j} & \gamma_{3,4} \\ \gamma_{4,1} & 0 & \gamma_{4,3} & -\sum\limits_{j\neq 4} \gamma_{4,j} \end{bmatrix} \tag{3}$$

The fact that we have a row or columns of two zeros automatically shows that the determinant of any 2×2 matrix is also zero. Using this method on all 4-boundary node critical circular planar graphs G, we can see that in all instances, a complete intersection is formed. Figure 6 illustrates a comprehensive list of all 4-boundary node critical circular planar graphs.

□

We now state the main theorem of this paper, and prove it later.

Theorem 2. For any n-boundary node critical graph G with k-connections such that $k \leq 1$, the space of response matrices is a complete intersection.

Definition 14. A graph G such that $V(G)$ can be partitioned into two subsets U and W, called partite sets, where every edge of G joins a vertex of U to a vertex of W, and where every vertex in U is adjacent to every vertex in W is called a *complete bipartite graph.* A complete bipartite graph is denoted $K_{s,t}$, where $|U| = s$ and $|W| = t$. When $s = 1$ or $t = 1$, $K_{s,t}$ is a *star.* [5]

Definition 15. The *root* of a star is the vertex that is adjacent to all other vertices.

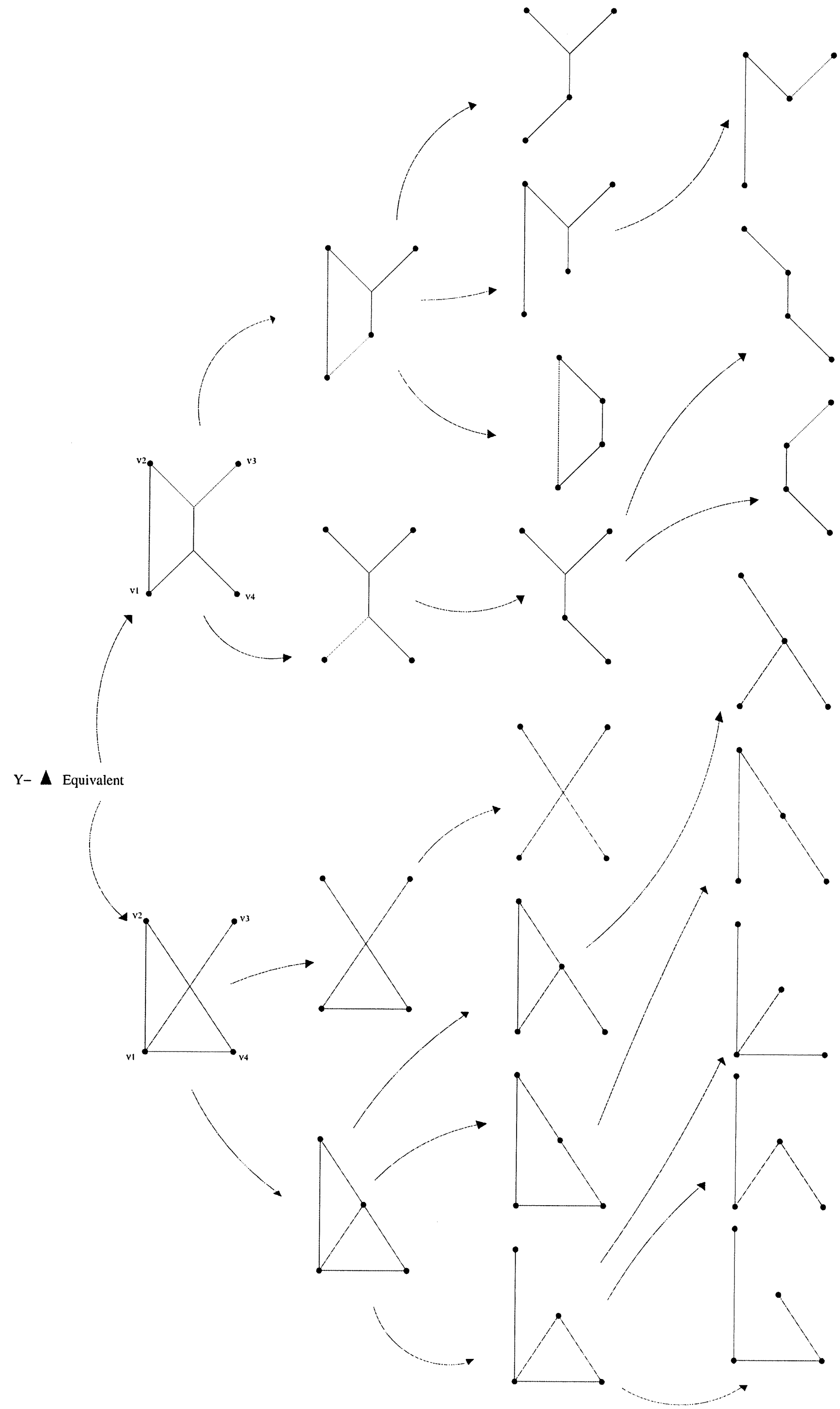

Figure 6: All 4-Boundary Node Critical Circular Planar Graphs

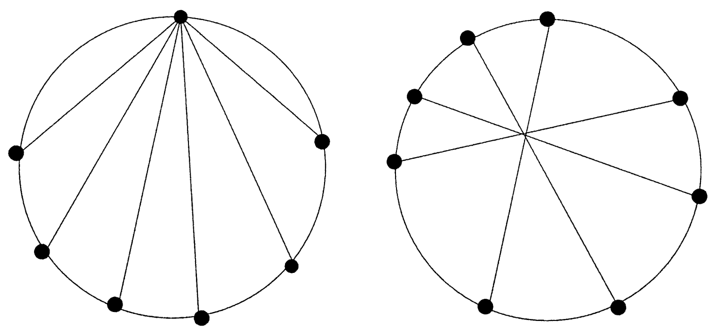

Figure 7: Left: Star with zero interior nodes Right: Star with one interior node

We want to characterize all n-boundary node critical circular planar graphs G with 1-connections to two independent cases, so that we only have to prove the general proposition for these two cases.

Proposition 1. For any n-boundary node critical graph G with k-connections such that $n \geq 4$ and $k \leq 1$, the graph G is

1. an n-boundary node star with zero interior nodes, or

2. an n-boundary node star with one interior node. (Figure 7)

Proof. [contradiction] The case where $k = 0$ implies that the graph G is disconnected, and hence is trivially true, so it remains to prove for $k = 1$. Fix an n boundary node graph G. For $k = 1$ and by connectedness of G, the graph G can be connected only through a series of interior nodes, or $n - 1$ boundary nodes must be connected to the nth boundary node such that the degree of the $n - 1$ nodes is one, and degree of the nth node is $n - 1$. In the latter case, we have satisfied our first requirement. Hence we need to show that there can be at most one interior node and that graph G is a star. Let's prove it must be a star first. Assume G is not a star. There exist two disjoint subsets of boundary nodes U and W on the circle C such that $|U| \geq 2$ and $|W| \geq 2$. Since every node of U is joined to every node W in G, then the pair $(P; Q)$, where P= (u_i, u_{i+1}) and Q = (v_i, v_{i+1}), would have a 2-connection, where the pair of paths u_i

to v_{i+1} and u_{i+1} and v_i are vertex-disjoint. Contradiction. Hence every graph G with 1-connection must be a star.

Now we want to show that there can be at most one interior node. Assume that there can be more than one interior node. Fix an n boundary node graph G. Without loss of generality, assume there are two nodes u_1 and u_2. Let u_1 be the root of the star, where u_1 is an interior node. All boundary nodes v_j, $1 \leq j \leq n$ have to be connected by a unique path to u_1. By criticality of G, u_2 cannot lie on any path from any boundary node v_j to u_1. By symmetry of G, we can place u_2 in between the v_kth and v_{k+1}th nodes, $\forall\ 1 \leq k \leq n$. Since our graph G is connected, there must exist a path to u_2 from boundary nodes v_k or v_{k+1}. Without loss of generality, assume v_k is that node. Then there can be two possibilities: there is a path from v_k to u_2 followed by a path from u_2 to u_1, or there exists a path from v_k to u_2, and u_2 to a node on the neighboring edge between v_{k+1} and u_1. The first cannot be, since our graph G is critical, and removing an edge along the path from v_k to u_2 would not break a k-connection. Likewise, the second scenario creates a 2-connection from v_k to v_{k+1} and a path between any other two boundary nodes through u_1. This is a contradiction, and hence we have proved our proposition.

$\square$

Let us now prove our main proposition, which is restated here:

For any n-boundary node critical circular planar graph G with k-connections such that $k \leq 1$, the space of response matrices is a complete intersection.

Proof. We have to prove that this is true for any n-boundary node star $K_{1,n-1}$ with either zero or one interior node. The property states that a graph G forms a complete intersection if codim V is equal to the size of the basis B. The codim $V = \binom{n}{2}$ - $|E|$, so the proof amounts to showing that our basis B has exactly $\binom{n}{2}$ - $|E|$ many elements. Since our graph is a star with one or no interior nodes, $|E| = n-1$ or $|E| = n$ respectively, and our basis should contain $\binom{n}{2}$ - $(n-1)$ or $\binom{n}{2}$ - n elements. The strategy will be to bound our basis B from below and above, ie.

$\binom{n}{2}$ - $|E| \leq |B| \leq \binom{n}{2}$ - $|E|$

So we have to show that a basis B of homogeneous equations cannot include more than $\binom{n}{2}$ - $|E|$ elements, or else it would not be a minimal spanning set. Likewise, if it had less than that many elements, there would not be a basis to span all homogeneous equations, and hence we indeed have a minimal basis B that spans the variety of homogeneous equations of a given graph G.

We first prove the upper bound. We have two cases: The n-boundary node star $K_{1,n-1}$ with either zero or one interior node. Let us prove the case for no interior node. Without loss of generality, let v_1 be the root of the star. Then all other nodes v_l, where $2 \leq l \leq n$, are connected to v_1 by a single edge. Our response matrix Λ will be of form:

$$\Lambda = \begin{bmatrix} -\sum\limits_{j\neq 1}\gamma_{1,j} & * & * & * & * \\ * & -\sum\limits_{j\neq 2}\gamma_{2,j} & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & 0 & 0 & -\sum\limits_{j\neq n-1}\gamma_{n-1,j} & 0 \\ * & 0 & 0 & 0 & -\sum\limits_{j\neq n}\gamma_{n,j} \end{bmatrix} \tag{4}$$

The symbol * above implies a non-zero entry, and will in subsequent matrices take the same definition. Given our graph $K_{1,n-1}$, our response matrix Λ has non-zero entries for its first row and first column. By inspection, the upper triangle in our response matrix clearly has $\binom{n}{2}$ - $(n-1)$ zeros. Thus taking our basis B to be the nonexisting 1-connections between pairs of nodes v_i and v_j, where $2 \leq i, j \leq n$, we will span all homogeneous equations, since the determinant of any submatrix of the Λ matrix is exactly zero. It can readily be seen that with more homogeneous equations (2-connections, 3-connections, etc.) we will still span all homogeneous equations, but we would have more elements in the spanning set, and hence it would not classify as a basis.

To show the case for one interior node, we note that our response matrix is different, since all 1-connections exist. Our response matrix will be of form:

$$\Lambda = \begin{bmatrix} -\sum\limits_{j\neq 1}\gamma_{1,j} & * & * & * & * \\ * & -\sum\limits_{j\neq 2}\gamma_{2,j} & * & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & * & -\sum\limits_{j\neq n-1}\gamma_{n-1,j} & * \\ * & * & * & * & -\sum\limits_{j\neq n}\gamma_{n,j} \end{bmatrix} \tag{5}$$

By inspection all entries are non-zero, and since our graph with one interior node has n edges, we have to find a basis that contains $\binom{n}{2}$ - n elements, our codim V. We are given that there cannot be k-connections for $k \geq 2$. This implies all 2×2 and larger subdeterminants are zero. Let us assign variables to our first row or column, since the column space is the row space by symmetry of the response matrix. Our matrix will be of form:

$$\Lambda = \begin{bmatrix} -\sum\limits_{j\neq 1}\gamma_{1,j} & \alpha & \beta & \gamma & \delta \\ \alpha & -\sum\limits_{j\neq 2}\gamma_{2,j} & * & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \gamma & * & * & -\sum\limits_{j\neq n-1}\gamma_{n-1,j} & * \\ \delta & * & * & * & -\sum\limits_{j\neq n}\gamma_{n,j} \end{bmatrix} \tag{6}$$

Similar to the previous proof for no boundary nodes, we will try to find a basis B with exactly $\binom{n}{2}$ - n elements that will span all homogeneous equations. Since we know all determinants of submatrices of the form $n \times n$ are zero, where $n \geq 2$, using our matrix above we can provide even more information:

$$\Lambda = \begin{bmatrix} -\sum_{j\neq 1}\gamma_{1,j} & \alpha & \beta & \gamma & \delta \\ \alpha & -\sum_{j\neq 2}\gamma_{2,j} & c\beta & c\gamma & c\delta \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \gamma & c\gamma & * & -\sum_{j\neq n-1}\gamma_{n-1,j} & * \\ \delta & c\delta & * & * & -\sum_{j\neq n}\gamma_{n,j} \end{bmatrix} \tag{7}$$

Certainly any 2×2 determinant is zero, and from this we have a collection of polynomial equations. Let $\lambda_{2,3} = x$ and $\lambda_{2,4} = y$. Then necessarily $y\beta = x\gamma$. Equivalently, $y = x\gamma/\beta$ or $x = y\beta/\gamma$. Letting $y/\gamma = c$ and $x/\beta = d$, but observing that $d = c$, we get entries $\lambda_{2,3}$ and $\lambda_{2,4}$ in the above matrix. Likewise, doing the same procedure for $\lambda_{2,5}$ gives us that entry and a new variable. In essence, entries in the second row are multiples of entries in the first row. By this method of introducing variables, we are able to span the collection of homogeneous equations. So we want to show that introducing $\binom{n}{2}$ - n variables allows us to span the homogeneous equations of the response matrix Λ.

Let us fix rows i, $i+1$ and column j such that $i+2 = j$ and $j \leq n-1$. This construction allows us to form 2×2 matrices by using entries $\lambda_{i,j}$ and $\lambda_{i+1,j}$ entries with subsequent column entries $\lambda_{i,j+s}$ and $\lambda_{i+1,j+s}$, where $1 \leq s \leq n-i-1$. We can find the missing variables for any 2×2 matrix. Assuming $n \geq 5$, for $i = 1$, $j = 3$, $j+s$ varies from 4 to n. Hence, we gain $n-3$ variables from the 2×2 matrices in the first two rows. Likewise, if $i = 2$, $j = 4$, $j+s$ varies from 5 to n, and we gain $n-4$ variables. Letting i vary from 1 to $n-2$, in total we gain $(n-3) + (n-4) + (n-5)$ + ... + 1 variables, or $(n-3)(n-2)$ / 2. However, we have to deal with entry $\lambda_{n-1,n}$. We observe that again we form 2×2 matrices formed using entries $\lambda_{n-2,n}$ and $\lambda_{n-1,n}$ along with subsequent columns j, where j varies from 1 to $n-3$. Clearly we can see that we gain $n-3$ variables, and using these along with our previous $(n-3)(n-2)$ /

2 variables, we have spanned the set of all 2-connections. If we tally these variables,

$$(n-3)(n-2)/2+(n-3)=n(n-3)/2.$$

which is exactly $\binom{n}{2}$ - n.

Finally, knowing any 2×2 determinants being zero implies for all $n \geq 3$, $n \times n$ determinants are zero by cofactor expansion, and we have all elements to span our variety V. Hence we have found a basis with the exact number of elements needed that completely determines the Λ matrix, and our proposition is proved for the upper bound. It remains to be shown that this basis is indeed a minimal spanning set of the variety V; that is, any smaller set would not span V.

$\square$

Although the complete intersection property has been proven for any n boundary node critical circular planar graph G with k-connections, $k \leq 1$, the problem is still open for critical graphs G with arbitrary k-connections. Hence we formulate the open problem.

Conjecture 1. For any n boundary node critical circular planar graph G, the space of response matrices is a complete intersection.

While no proof has been given of this phenomenon, no counterexample has been found so far.

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5 Vita

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