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MARKOV SEMIGROUPS AND ESTIMATING FUNCTIONS, WITH APPLICATIONS TO SOME FINANCIAL MODELS

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Abstract. We consider the probabilistic approach to the problems treated in [7]. We focus on the diffusion models generated by $L_{\vec{\theta},a}\,u(x) := \theta_2\,x^{2a}\,u''+ (\,\theta_2\,a\,x^{2a-1}+\theta_1\,x^a)\,u',\ \vec{\theta}=(\theta_1,\theta_2)^T\in\mathbb{R}\times(0,+\infty)$, when $a=\frac{1}{2}$ or $a=1$ and face the problem of finding optimal (in the asymptotic sense) estimators of the unknown parameter vector $\vec{\theta}$.

1. Introduction

In [7] we studied different realizations of the operators

$$L_{\theta,a}\;u(x)\;:=\;x^{2a}u''(x)\;+\;(ax^{2a-1}+\theta x^a)u'(x),$$

where $\theta\in\mathbb{R}$, $a\in\mathbb{R}$, acting on suitable spaces of real valued continuous functions. For $0\le a\le 1$ we obtained explicit representations of the semigroups generated by $L_{\theta,a}$ via perturbations of squares of suitable generators of groups.

In this paper we focus on the diffusion models generated by

$$L_{\vec{\theta},a}\,u(x) := \theta_2\,x^{2a}\,u''(x) + (\,\theta_2\,a\,x^{2a-1}+\theta_1\,x^a)\,u'(x),$$

where $\vec{\theta}=(\theta_1,\theta_2)^T\in\mathbb{R}\times(0,+\infty)$, and either $a=\frac{1}{2}$ or $a=1$. Such a choice of a is motivated by the applications to genetics and financial mathematics. The problem of finding optimal (in the asymptotic sense) estimators of the unknown parameter vector $\vec{\theta}$ is considered for the case $a=1$ (the case $a=\frac{1}{2}$ is studied in [10]).

2. A probabilistic analysis of diffusion models with applications

We will now consider the class of one-dimensional diffusion processes that are solutions of the following stochastic differential equation (SDE):

$$dX_t\;=\;\left(\,\theta_2\,a\,X_t^{2a-1}+\theta_1\,X_t^a\,\right)\,dt+\sqrt{2\,\theta_2}\,X_t^a\,dW_t, \tag{2.1}$$

where $0\le a\le 1$, $W=\{W_t,\,t\ge 0\}$ is a standard one-dimensional Wiener process, and $\vec{\theta}=(\theta_1,\theta_2)^T$ is an unknown parameter vector in $\Theta=\mathbb{R}\times(0,+\infty)$ to be estimated (T denotes transpose of a vector or matrix). The SDE (2.1) is associated to the operator:

$$L_{\vec{\theta},a}\;=\;\theta_2\,G_a^2\,u+\theta_1\,G_a\,u\;=\;\theta_2\,x^{2a}\,u''+(\,\theta_2\,a\,x^{2a-1}+\theta_1\,x^a)\,u'$$

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($G_a\, u := x^a\, u', 0 \le a \le 1$), with domain on the space $C[0,+\infty]$ defined on Theorem 3.1 of [7].

Case 1: $a = \dfrac{1}{2}$. The SDE (2.1) becomes:

$$dX_t \;=\; \left(\frac{\tau^2}{4} - k\,\sqrt{X_t}\right) dt + \tau\,\sqrt{X_t}\, dW_t, \tag{2.2}$$

with $\tau := \sqrt{2\,\theta_2} > 0$ and $k := -\theta_1 \in \mathbb{R}$. When $k > 0$, the SDE (2.2) is known in financial literature as Longstaff's model (see [9] and [11, Section 12.3]). This model postulates the dynamics for the "short-term interest rate" of zero-coupon bonds, in absence of arbitrage, to be governed by the SDE (2.3). It is referred to as the "Double Square-Root" (DSR) interest rate process and it is a modified version of the well known CIR or "Square-Root" interest rate process:

$$dX_t \;=\; (a - b\,X_t\,)\,dt + \sigma\,\sqrt{X_t}\,dW_t,$$

where a, b and σ are strictly positive constants (for more details see [11, Section 12.3] and [5]). The main difference with the CIR model is that zero-coupon bond's yield is a non-linear function of the short-term interest rate.

Let $\vec{\gamma} := (k, \tau)^T$ denote the unknown parameter vector in (2.2). We will assume that $\vec{\gamma} \in \Gamma = (0,+\infty)^2$, as Longstaff's model requires. Let $\mathbb{P}_{\vec{\gamma}}$ denote the law of the corresponding diffusion process $X = \{X_t,\ t \ge 0\}$, the unique solution to (2.2). The state space of X is the interval $I = [0,+\infty)$, where the endpoint 0 is attainable and an *instantaneously reflecting* barrier: each sample path of X returns immediately to positive values when the origin is hit. The transition probability density $p_{\vec{\gamma}}(t, y, x)$ of X at time t, say X_t (i.e. the conditional density under $\mathbb{P}_{\vec{\gamma}}$ of X_t given the initial condition $X_0 = x$), is obtained from the density of the square of a reflected Brownian motion (see [9, Section 2]) and approaches a unique invariant density as $t \to \infty$, which is the density function of the Weibull distribution. Then X is an *ergodic* process for any $\vec{\gamma} \in \Gamma$.

Case 2: $a = 1$. The SDE (2.1) becomes

$$dX_t = \alpha\, X_t\, dt + \beta\, X_t\, dW_t, \tag{2.3}$$

with $\alpha := (\theta_1 + \theta_2) \in \mathbb{R}$ and $\beta := \sqrt{2\,\theta_2} > 0$ unknown parameters. From now on, let $\vec{\vartheta} = (\alpha, \beta) \in \Theta = \mathbb{R} \times (0,+\infty)$ denote the unknown parameter vector, and $\mathbb{P}_{\vec{\vartheta}}$ the law of the corresponding diffusion process $X = \{X_t,\ t \ge 0\}$, the unique solution to (2.3).

The SDE (2.3) is used in population genetics as a model to describe the evolution of certain population growth processes with environmental effects which vary randomly in time (see [8, Ch. 15]), and is well known in financial mathematics as the *Black-Scholes equations with constant volatility* (see [4], [6], [11, Ch. 5]) to model the price of assets, say, shares of common stocks, that are traded in a perfect market.

The diffusion process X is the so-called *geometric Brownian motion* with sample paths given "explicitly" by

$$X_t = X_0\,\exp\left[\left(\alpha - \frac{\beta^2}{2}\right) t + \beta\, W_t\right], \qquad t \ge 0. \tag{2.4}$$

Note that if the initial condition $X_0 \geq 0$ almost surely (a.s.) in (2.4) holds, then $X_t \geq 0$ a.s. for all $t \geq 0$. In this case, according to the relevant applications of (2.3), we can assume the interval $I = (0, +\infty)$ as the state space of X. Moreover the transition probability density $p_{\vec{\vartheta}}(t, y, x)$ of X_t, given the initial condition $X_0 = x \in I$, is given by

$$p_{\vec{\vartheta}}(t, y, x) = \frac{1}{\beta\, y \sqrt{2\,\pi\, t}}\, exp \left\{ -\, \frac{(\log \frac{y}{x} \;-\; (\alpha - \frac{\beta^2}{2})\, t)^2}{2\, \beta^2\, t} \right\}, \quad y \in I. \tag{2.5}$$

The diffusion X is non-ergodic for any $\vec{\vartheta} \in \Theta$.

We are now interested in finding optimal (in the asymptotic sense) estimators of either the unknown parameter vector $\vec{\gamma}$ $(a = 1/2)$ and $\vec{\vartheta}$ $(a = 1)$ from observations of the corresponding diffusion process X. We assume X to be discretely observed at equidistant time points $0 = t_0 < t_1 < \ldots < t_n < t_{n+1} < \ldots$, with step size $\Delta = t_n - t_{n-1}$ to be fixed. We consider the integer

$$n_T \;=\; \max\{\, n \in \mathbb{N} : t_n \leq T\,\} \;=\; \left[\frac{T}{\Delta}\right],$$

where $[\cdot]$ denotes the integer part of a real number, and $[0, T]$, $T > \Delta$, is a time interval. Let $X_{t_0}, X_{t_1}, \ldots, X_{t_{n_T}}$ denote the observations of X at times $t_0, t_1, \ldots, t_{n_T}$, respectively.

Our main goal is to use a very recent estimation approach (see [2]) for discretely observed diffusion-type models that consists in constructing well-chosen estimating functions, either martingales or not, in order to get rather high efficient estimators for the unknown parameters of the model. This approach provides a useful alternative to the Maximum Likelihood (ML) method when the likelihood function is not available or difficult to calculate, as in the case $a = 1/2$. We also point out that in the setting of diffusion type-models the estimating functions approach provides highly efficient estimators when the diffusion is an ergodic process, since in this case specific conditions for the existence of consistent and asymptotically normal estimators are given (see [2, Section 2.3] and [12, Theorem 3.6]). In the case of non-ergodic diffusion-type models, such conditions, including the Central Limit Theorem, become more general and are more difficult to be satisfied.

A systematic study of the estimation problem for Longstaff's model (case $a = 1/2$) is in the recent paper [10].

For the case $a = 1$, let $\vec{\vartheta}_0 = (\alpha_0, \beta_0)^T$ be the element of Θ that we wish to estimate, and $\mathbb{P}_0$, the probability law of X with respect to $\vec{\vartheta}_0$. Note that the ML estimator of $\vec{\vartheta}_0$, say $\vec{\vartheta}^{ML}_{n_T} = (\alpha^{ML}_{n_T}, \beta^{ML}_{n_T})^T$, is computable and reads as

$$\begin{aligned} \alpha^{ML}_{n_T} &= \frac{1}{n_T \Delta} \sum_{i=1}^{n_T} \log \frac{X_{t_i}}{X_{t_{i-1}}} \;+\; \frac{{\beta^{ML}_{n_T}}^2}{2}, \\ \beta^{ML}_{n_T} &= \sqrt{\frac{1}{n_T \Delta} \left[\sum_{i=1}^{n_T} \left(\log \frac{X_{t_i}}{X_{t_{i-1}}} \right)^2 - \frac{1}{n_T} \left(\sum_{i=1}^{n_T} \log \frac{X_{t_i}}{X_{t_{i-1}}} \right)^2 \right]}. \end{aligned} \tag{2.6}$$

From [1, Section 2.4, Corollary 1] it follows that as $n_T \to \infty$, i.e. the number of observations is expanded by extending the time interval $[0, T]$, the sequence $\{\vec{\vartheta}^{ML}_{n_T}\}$ is consistent, say $\vec{\vartheta}^{ML}_{n_T} \xrightarrow{\mathbb{P}_0} \vec{\vartheta}_0$ and asymptotically normal, i.e.

$$\sqrt{n_T}\,(\vec{\vartheta}^{ML}_{n_T} - \vec{\vartheta}_0) \xrightarrow{\mathcal{D}} N\left(0,\, Q(\vec{\vartheta}_0)^{-1}\right),$$

where the asymptotic covariance matrix $Q(\vec{\vartheta}_0)$, which in our case reads as

$$Q(\vec{\vartheta}_0) = \begin{pmatrix} \dfrac{\Delta}{\beta_0^2} & -\dfrac{\Delta}{\beta_0} \\ -\dfrac{\Delta}{\beta_0} & \dfrac{2 + \Delta\,\beta_0^2}{\beta_0^2} \end{pmatrix}, \tag{2.7}$$

is the smallest possible among those of all consistent and asymptotically normal estimators of $\vec{\vartheta}_0$. This means that $\vec{\vartheta}^{ML}_{n_T}$ is an efficient estimator, that is it attains the maximal possible concentration about the true value $\vec{\vartheta}_0$. We are going now to show that the estimating functions approach can also be applied successfully to non-ergodic diffusion processes, by proving the existence of consistent and asymptotically normal estimators of $\vec{\vartheta}_0$. Furthermore, we will analyze the degree of efficiency of the constructed estimators by comparing them with the asymptotically efficient maximum likelihood estimator given in (2.6).

Then, we consider the so-called *quadratic estimating function* of $\vec{\vartheta}$ (see [2, Section 5.1]) defined as follows

$$\begin{aligned} G_{n_T}(\vec{\vartheta}) = \sum_{i=1}^{n_T} \{ & a(X_{t_{i-1}}; \vec{\vartheta})[X_{t_i} - F(X_{t_{i-1}}; \vec{\vartheta})] \\ & + b(X_{t_{i-1}}; \vec{\vartheta})[(X_{t_i} - F(X_{t_{i-1}}; \vec{\vartheta}))^2 - \phi(X_{t_{i-1}}; \vec{\vartheta})]\}, \end{aligned}$$

where the coefficients a and b depend on the first four moments of the transition distribution of X

$$\begin{aligned} F(x; \vec{\vartheta}) &= E_{\vec{\vartheta}}[X_\Delta | X_0 = x], \quad \phi(x; \vec{\vartheta}) = E_{\vec{\vartheta}}[(X_\Delta - F(x; \vec{\vartheta}))^2 | X_0 = x], \\ \eta(x; \vec{\vartheta}) &= E_{\vec{\vartheta}}[(X_\Delta - F(x; \vec{\vartheta}))^3 | X_0 = x], \\ \psi(x; \vec{\vartheta}) &= E_{\vec{\vartheta}}[(X_\Delta - F(x; \vec{\vartheta}))^4 | X_0 = x] - \phi(x; \vec{\vartheta})^2. \end{aligned}$$

An estimator of $\vec{\vartheta}_0$ can be found as a solution, if there exists one, to the estimating equation $G_{n_T}(\vec{\vartheta}) = 0$.

Note that the stochastic process $\{G_n(\vec{\vartheta}_0) : n \geq 1\}$ is a $\mathbb{P}_0$-martingale, i.e. $E_0[G_n(\vec{\vartheta}_0) | \mathcal{F}_{n-1}] = G_{n-1}(\vec{\vartheta}_0)$, with respect to the filtration generated by the observations of X, say $\mathcal{F}_n = \sigma\,(X_{t_0}, X_{t_1}, ..., X_{t_n})$, $n \geq 1$, under the model given by the true parameter value $\vec{\vartheta}_0$ ($G_0 = 0$ and $\mathcal{F}_0$ is the trivial σ-field). Moreover, $E_0\left(G_n(\vec{\vartheta}_0)\right) = 0$ for any $n \geq 1$ (E_0 denotes expectation under $\mathbb{P}_0$).

From (2.4) the four conditioned moments of X can be found explicitly

$$\begin{aligned} F(x;\vec{\vartheta}) &= xe^{\Delta\,\alpha}, \quad \phi(x;\vec{\vartheta}) \;=\; x^2 e^{2\Delta\,\alpha}(e^{\Delta\,\beta^2}-1), \\ \eta(x;\vec{\vartheta}) &= x^3 e^{3\Delta\,\alpha}(e^{\Delta\,\beta^2}-1)^2(e^{\Delta\,\beta^2}+2), \\ \psi(x;\vec{\vartheta}) &= x^4 e^{4\Delta\,\alpha}(e^{\Delta\,\beta^2}-1)^2(e^{4\Delta\,\beta^2}+2e^{3\Delta\,\beta^2}+3e^{2\Delta\,\beta^2}-4). \end{aligned}$$

Hence, after some simplifications, we obtain the following bivariate quadratic estimating function of the vector $\vec{\theta}$

$$\begin{aligned} G_{n_T}(\vec{\vartheta}) &= \sum_{i=1}^{n_T} H(X_{t_{i-1}}, X_{t_i}, \vec{\vartheta}) \\ &= \sum_{i=1}^{n_T} \begin{pmatrix} \dfrac{X_{t_i}^2}{X_{t_{i-1}}^2} - e^{\Delta\,(\alpha+\beta^2)}\,(e^{\Delta\,\beta^2}+1)^2\,\dfrac{X_{t_i}}{X_{t_{i-1}}} + e^{2\Delta(\alpha+\beta^2)}\,(e^{\Delta\,\beta^2}+2) \\ \\ -\dfrac{X_{t_i}^2}{X_{t_{i-1}}^2} + e^{\Delta\,(\alpha+\beta^2)}\,(e^{\Delta\,\beta^2}+1)\,\dfrac{X_{t_i}}{X_{t_{i-1}}} - e^{2\Delta\,(\alpha+\beta^2)} \end{pmatrix}. \end{aligned} \tag{2.8}$$

Then by equating to zero $G_{n_T}(\vec{\vartheta})$, we find the following explicit estimators of the parameters α_0, β_0:

$$\begin{aligned} \alpha_{n_T}^{MEF} &= \frac{1}{\Delta}\,\log\left(\frac{1}{n_T}\sum_{i=1}^{n_T}\frac{X_{t_i}}{X_{t_{i-1}}}\right), \\ \beta_{n_T}^{MEF} &= \sqrt{\frac{1}{\Delta}\,\log\left(\frac{n_T\displaystyle\sum_{i=1}^{n_T}\left(\frac{X_{t_i}}{X_{t_{i-1}}}\right)^2}{\left(\displaystyle\sum_{i=1}^{n_T}\frac{X_{t_i}}{X_{t_{i-1}}}\right)^2}\right)}. \end{aligned} \tag{2.9}$$

The asymptotic properties of the sequence of estimators $\{\vec{\vartheta}_{n_T}^{MEF}\}$, having denoted by $\vec{\vartheta}_{n_T}^{MEF} = (\alpha_{n_T}^{MEF}, \beta_{n_T}^{MEF})^T$, will be proved in the following Theorem.

Theorem 2.1. *The sequence of estimators $\{\vec{\vartheta}_{n_T}^{MEF}\}$ of $\vec{\vartheta}_0$ is consistent, i.e.*

$$\vec{\vartheta}_{n_T}^{MEF} \xrightarrow{\mathbb{P}_0} \vec{\vartheta}_0 \tag{2.10}$$

as $n_T \to \infty$, and asymptotically normal

$$\sqrt{n_T}\,(\vec{\vartheta}_{n_T}^{MEF} - \vec{\vartheta}_0) \xrightarrow{\mathcal{D}} N\left(0,\, W(\vec{\vartheta}_0)^{-1}\,\Sigma_{\vec{\vartheta}_0}\,(W(\vec{\vartheta}_0)^{-1})^T\right) \tag{2.11}$$

as $n_T \to \infty$, where $\Sigma_{\vec{\theta}_0} = E_0\left(H(X_{t_{i-1}}, X_{t_i}, \vec{\vartheta}_0)\,H(X_{t_{i-1}}, X_{t_i}, \vec{\vartheta}_0)^T\right)$ is the covariance matrix for the vector H in (2.8) and $W(\vec{\vartheta}_0) = E_0\left(\partial_{\vec{\vartheta}}\,G_{n_T}(\vec{\vartheta}_0)\right)$ is the expected Jacobian matrix of G_{n_T}.

Proof. The property (2.10) can be proved directly by using the explicit expressions of the estimators $\alpha_{n_T}^{MEF}$ and $\beta_{n_T}^{MEF}$ given in (2.9). Indeed, from (2.4) with $\alpha = \alpha_0$ and $\beta = \beta_0$, for any integer $k \geq 1$ we can write

$$\left(\frac{X_{t_i}}{X_{t_{i-1}}}\right)^k = e^{k\Delta\left(\theta_{0,1} - \frac{\theta_{0,2}^2}{2}\right)} \cdot e^{Y_i}, \tag{2.12}$$

and so

$$\frac{1}{n_T} \sum_{i=1}^{n_T} \left(\frac{X_{t_i}}{X_{t_{i-1}}}\right)^k = e^{k\Delta\left(\alpha_0 - \frac{\beta_0^2}{2}\right)} \frac{1}{n_T} \sum_{i=1}^{n_T} e^{Y_i},$$

where $Y_1, \ldots, Y_{n_T}$ are independent and normally distributed random variables with zero mean and variance equal to $\Delta\, k^2\, \beta_0^2$. As a consequence, $e^{Y_1}, \ldots, e^{Y_{n_T}}$ are independent and identically distributed (i.i.d.) random variables with expected value $E_0\left[e^{Y_i}\right] = e^{\frac{\Delta\, k^2\, \beta_0^2}{2}}$. The classical Strong Law of Large Numbers implies that

$$\frac{1}{n_T} \sum_{i=1}^{n_T} \left(\frac{X_{t_i}}{X_{t_{i-1}}}\right)^k \xrightarrow{a.s.} E_0\left[\frac{1}{n_T} \sum_{i=1}^{n_T} \left(\frac{X_{t_i}}{X_{t_{i-1}}}\right)^k\right] = e^{k\,\Delta(\alpha_0 + \frac{(k-1)\,\beta_0^2}{2})} \tag{2.13}$$

as $n_T \to \infty$. Hence, from (2.9) the property (2.10) holds.

In order to prove (2.11), first observe that the estimating function $G_{n_T}(\vec{\vartheta})$ given in (2.8) is continuously differentiable with respect to $\vec{\vartheta}$ for any $\vec{\vartheta} \in \Theta$, then we can define the Jacobian matrix of $G_{n_T}(\vec{\vartheta})$ with respect to $\vec{\vartheta}$, say

$$J_{n_T}(\vec{\vartheta}) = \left(\partial_{\vartheta_j}\, G_{n_T,i}(\vec{\vartheta})\right)_{1 \leq i,j \leq 2}$$

(by this expression we mean that the ith row of the matrix consists of the partial derivatives with respect to $\vec{\vartheta}$ of the ith coordinate of G_{n_T}). From (2.8) the elements of the Jacobian matrix read

$$\partial_\alpha\, G_{n_T,1}(\vec{\vartheta}) = \Delta\, e^{\Delta(\alpha+\beta^2)} \left[2\, n_T\, e^{\Delta(\alpha+\beta^2)}\, (e^{\Delta\beta^2} + 2) - (e^{\Delta\beta^2} + 1)^2 \sum_{i=1}^{n_T} \frac{X_{t_i}}{X_{t_{i-1}}}\right],$$

$$\partial_\beta\, G_{n_T,1}(\vec{\vartheta}) = 2\beta\, \Delta\, e^{\Delta(\alpha+\beta^2)} \Bigg[n_T\, e^{\Delta(\alpha+\beta^2)}\, (3e^{\Delta\beta^2} + 4) - (e^{\Delta\beta^2} + 1)\, (3e^{\Delta\beta^2} + 1) \sum_{i=1}^{n_T} \frac{X_{t_i}}{X_{t_{i-1}}}\Bigg],$$

$$\partial_\alpha\, G_{n_T,2}(\vec{\vartheta}) = \Delta\, e^{\Delta(\alpha+\beta^2)} \left[-2\, n_T\, e^{\Delta(\alpha+\beta^2)} + (e^{\Delta\beta^2} + 1) \sum_{i=1}^{n_T} \frac{X_{t_i}}{X_{t_{i-1}}}\right],$$

$$\partial_\beta\, G_{n_T,2}(\vec{\vartheta}) = 2\beta\, \Delta\, e^{\Delta(\alpha+\beta^2)} \left[-2\, n_T\, e^{\Delta(\alpha+\beta^2)} + (2e^{\Delta\beta^2} + 1) \sum_{i=1}^{n_T} \frac{X_{t_i}}{X_{t_{i-1}}}\right].$$

For technical reasons, we define for $\vec{\vartheta}^{\,(i)} \in \Theta\, (i = 1, 2)$, a second matrix by

$$J_{n_T}(\vec{\vartheta}^{(1)}, \vec{\vartheta}^{(2)}) := \begin{pmatrix} \partial_\alpha G_{n_T,1}(\vec{\vartheta}^{(1)}) & \partial_\beta G_{n_T,1}(\vec{\vartheta}^{(1)}) \\ \partial_\alpha G_{n_T,2}(\vec{\vartheta}^{(2)}) & \partial_\beta G_{n_T,2}(\vec{\vartheta}^{(2)}) \end{pmatrix} \tag{2.14}$$

Let us consider now the Taylor expansion

$$0 = G_{n_T}(\vec{\vartheta}_{n_T}^{MEF}) = G_{n_T}(\vec{\vartheta}_0) + J_{n_T}(\vec{a}_{n_T}^{(1)}, \vec{a}_{n_T}^{(2)})\,(\vec{\vartheta}_{n_T}^{MEF} - \vec{\vartheta}_0),$$

where each $\vec{a}_{n_T}^{(i)} \in \Theta$ is a convex combination of $\vec{\vartheta}_{n_T}^{MEF}$ and $\vec{\vartheta}_0$. By rearranging the terms, we get

$$\frac{1}{n_T} J_{n_T}(\vec{a}_{n_T}^{(1)}, \vec{a}_{n_T}^{(2)}) \sqrt{n_T}\,(\vec{\vartheta}_{n_T}^{MEF} - \vec{\theta}_0) = -\frac{1}{\sqrt{n_T}} G_{n_T}(\vec{\vartheta}_0). \tag{2.15}$$

We have to prove that $\dfrac{1}{\sqrt{n_T}} G_{n_T}(\vec{\vartheta}_0)$ converges in distribution as $n_T \to \infty$. Note that when the underlying diffusion is an ergodic process, this condition has been proved in [3]. In the case of a non-ergodic diffusion process, the multivariate Central Limit Theorem for martingales (see, e.g., [2, Theorem 2.3]) would be applied.

In this case it is enough to apply the Classical Central Limit Theorem to $G_{n_T}(\vec{\vartheta}_0) = \sum_{i=1}^{n_T} H(X_{t_{i-1}}, X_{t_i}, \vec{\vartheta}_0)$, since from (2.8) and (2.12) it follows that

$$\{H(X_{t_{i-1}}, X_{t_i}, \vec{\vartheta}_0)\}_{1 \le i \le n_T}$$

is a sequence of i.i.d. $\mathbb{R}^2$-valued random variables with zero-mean vector and covariance matrix

$$\begin{aligned} \Sigma_{\vec{\vartheta}_0} &= E_0\left(H(X_{t_{i-1}}, X_{t_i}, \vec{\vartheta}_0)\, H(X_{t_{i-1}}, X_{t_i}, \vec{\vartheta}_0)^T\right) \\ &= e^{\Delta(4\alpha_0 + 3\beta_0^2)}\,(e^{2\Delta\beta_0^2} - 1) \begin{pmatrix} e^{2\Delta\alpha_0^2} + 2e^{\Delta\beta_0^2} - 1 & -(e^{\Delta\beta_0^2} - 1) \\ -(e^{\Delta\beta_0^2} - 1) & e^{\Delta\beta_0^2} - 1 \end{pmatrix}, \end{aligned}$$

which depends only on the true parameter $\vec{\vartheta}_0$. Then the following result holds

$$\frac{1}{\sqrt{n_T}} G_{n_T}(\vec{\vartheta}_0) \xrightarrow{\mathcal{D}} N\left(0, \Sigma_{\vec{\vartheta}_0}\right) \tag{2.16}$$

as $n_T \to \infty$.

Coming back to the equation (2.15), the property (2.11) is completely proved if we show that

$$\frac{1}{n_T} J_{n_T}(\vec{a}_{n_T}^{(1)}, \vec{a}_{n_T}^{(2)}) \xrightarrow{\mathbb{P}_0} W(\vec{\vartheta}_0), \tag{2.17}$$

as $n_T \to \infty$, where

$$W(\vec{\vartheta}_0) = E_0\left(\partial_{\vec{\vartheta}} G_{n_T}(\vec{\vartheta}_0)\right) = \begin{pmatrix} e^{2\Delta\beta_0^2} + 2e^{\Delta\beta_0^2} - 1 & -2\beta_0 \\ 1 - e^{\Delta\beta_0^2} & 2\beta_0 \end{pmatrix},$$

which is an invertible and, in our case, a non-random matrix (when a diffusion process is non-ergodic, the limiting matrix W might in general be random).

Then, for all $\varepsilon > 0$ we consider the set

$$M_{n_T}^{(\varepsilon)}(\vec{\vartheta}_0) = \left\{ \vec{\vartheta} \in \Theta : \left\| \vec{\vartheta} - \vec{\vartheta}_0 \right\| \leq \frac{\varepsilon}{\sqrt{n_T}} \right\},$$

which shrinks to $\vec{\vartheta}_0$ as $n_T \to \infty$. The convergence in (2.13) implies that for all $\varepsilon > 0$ and $\vec{\vartheta}^{(i)} \in M_{n_T}^{(\varepsilon)}(\vec{\vartheta}_0)$ $(i = 1, 2)$,

$$\frac{1}{n_T} J_{n_T}(\vec{\vartheta}^{(1)}, \vec{\vartheta}^{(2)}) \xrightarrow{a.s.} W(\vec{\vartheta}_0),$$

as $n_T \to \infty$. From the consistency property (2.10) of the estimators $\vec{\vartheta}_{n_T}^{MEF}$ it follows that for all $\varepsilon > 0$, $\vec{\vartheta}_{n_T}^{MEF} \in M_{n_T}^{(\varepsilon)}(\vec{\vartheta}_0)$, as $n_T \to \infty$. This implies that $\vec{a}_{n_T}^{(i)} \in M_{n_T}^{(\varepsilon)}(\vec{\vartheta}_0)$, as $n_T \to \infty$, so the convergence (2.17) holds. □

The degree of efficiency of the MEF estimator, $\vec{\vartheta}_{n_T}^{MEF}$, can be evaluated comparing it with the corresponding ML estimator, $\vec{\vartheta}_{n_T}^{ML}$, in terms of their asymptotic bias and standard error. To this end, we performed a simulation study of the finite sample behavior of both the estimators. Samples paths of the geometric Brownian motion X solving (2.3) with the true parameter value $\vec{\vartheta}_0 = (0.11, 0.33)^T$, have been simulated in different time intervals $[0, T]$, which we assumed to be measured in years. Each simulated sample of observations of X, obtained for different values of the sampling interval Δ and of the sample size n_T, produces an MEF and an ML estimate of the true parameters α_0 and β_0. We point out that the choice of the values $\Delta = 1/252$, $\Delta = 1/52$ and $\Delta = 1/12$ has not been casual, since they correspond to considering daily ($\Delta = 1/252$, assuming on average 252 trading days in a year), weekly ($\Delta = 1/52$) and monthly ($\Delta = 1/12$) rate observations, respectively, on U.S. stocks. This procedure was repeated $M = 500$ times, so that the first two moments (mean and standard error) of the finite sample distribution of $(\vec{\vartheta}_{n_T}^{MEF} - \vec{\vartheta}_0)$ and $(\vec{\vartheta}_{n_T}^{ML} - \vec{\vartheta}_0)$, averaged over $M = 500$ realizations, were computed.

The results in Table 1 show that the asymptotic performance of both the estimators is very close. For each fixed value of Δ, their slight bias (sample mean) decreases according to an increasing number of observations and their sample standard error is in accordance with the asymptotic one.

Figure 1 compares the asymptotic standard error of both the estimators (the value of the sampling interval Δ on the graph ranges from 0 to 1/12) and shows a very slight increasing variance of the MEF estimator with respect to the variance of the corresponding ML estimator when $\Delta > 0.04$.

TABLE 1. Sample Mean, sample Standard Error (SE) and Asymptotic Standard Error (ASE) of $(\vec{\vartheta}_{n_T}^{MEF} - \vec{\vartheta}_0)$ and $(\vec{\vartheta}_{n_T}^{ML} - \vec{\vartheta}_0)$. The true parameter value is $\vec{\vartheta}_0 = (0.11, 0.33)^T$ and the initial point is $X_0 = 0.05$.

Δ	n		Mean	SE	ASE
1/252	252 (T=1)	$\alpha_{n_T}^{MEF} - \alpha_0$	0.278873e-1	0.33157373	0.33003565
		$\alpha_{n_T}^{ML} - \alpha_0$	0.278874e-1	0.33157150	0.33003565
		$\beta_{n_T}^{MEF} - \beta_0$	-0.103294e-2	0.154142e-1	0.147089e-1
		$\beta_{n_T}^{ML} - \beta_0$	-0.103032e-2	0.153935e-1	0.146994e-1
	1260 (T=5)	$\alpha_{n_T}^{MEF} - \alpha_0$	-0.357115e-2	0.14723438	0.14759643
		$\alpha_{n_T}^{ML} - \alpha_0$	-0.356991e-2	0.14723508	0.14759643
		$\beta_{n_T}^{MEF} - \beta_0$	-0.337780e-3	0.656871e-2	0.657802e-2
		$\beta_{n_T}^{ML} - \beta_0$	-0.325920e-3	0.655500e-2	0.657376e-2
	2520 (T=10)	$\alpha_{n_T}^{MEF} - \alpha_0$	-0.704671e-2	0.982509e-1	0.10436644
		$\alpha_{n_T}^{ML} - \alpha_0$	-0.704649e-2	0.982512e-1	0.10436644
		$\beta_{n_T}^{MEF} - \beta_0$	-0.418480e-3	0.501039e-2	0.465136e-2
		$\beta_{n_T}^{ML} - \beta_0$	-0.416340e-3	0.499760e-2	0.464835e-2
1/52	52 (T=1)	$\alpha_{n_T}^{MEF} - \alpha_0$	-0.153488e-1	0.34591419	0.33017285
		$\alpha_{n_T}^{ML} - \alpha_0$	-0.153466e-1	0.34593007	0.33017273
		$\beta_{n_T}^{MEF} - \beta_0$	-0.475364e-2	0.327390e-1	0.324609e-1
		$\beta_{n_T}^{ML} - \beta_0$	-0.471358e-2	0.326671e-1	0.323592e-1
	520 (T=10)	$\alpha_{n_T}^{MEF} - \alpha_0$	-0.799075e-2	0.10701036	0.10440982
		$\alpha_{n_T}^{ML} - \alpha_0$	-0.800203e-2	0.10700596	0.10440978
		$\beta_{n_T}^{MEF} - \beta_0$	-0.123242e-2	0.101972e-1	0.102650e-1
		$\beta_{n_T}^{ML} - \beta_0$	-0.133100e-2	0.101167e-1	0.102329e-1
	1300 (T=25)	$\alpha_{n_T}^{MEF} - \alpha_0$	-0.533449e-2	0.675528e-1	0.66034600
		$\alpha_{n_T}^{ML} - \alpha_0$	-0.533049e-2	0.675529e-1	0.66034500
		$\beta_{n_T}^{MEF} - \beta_0$	-0.161840e-3	0.646761e-2	0.649219e-2
		$\beta_{n_T}^{ML} - \beta_0$	-0.125340e-3	0.646761e-2	0.647183e-2
1/12	12 (T=1)	$\alpha_{n_T}^{MEF} - \alpha_0$	-0.892490e-2	0.32962385	0.33075010
		$\alpha_{n_T}^{ML} - \alpha_0$	-0.897525e-2	0.32962935	0.33074783
		$\beta_{n_T}^{MEF} - \beta_0$	-0.257143e-1	0.670721e-1	0.682830e-1
		$\beta_{n_T}^{ML} - \beta_0$	-0.257645e-1	0.660678e-1	0.673610e-1
	120 (T=10)	$\alpha_{n_T}^{MEF} - \alpha_0$	0.388242e-2	0.10452034	0.10459237
		$\alpha_{n_T}^{ML} - \alpha_0$	0.389285e-2	0.10453860	0.10459165
		$\beta_{n_T}^{MEF} - \beta_0$	-0.672800e-3	0.215848e-1	0.215930e-1
		$\beta_{n_T}^{ML} - \beta_0$	-0.516180e-3	0.210362e-1	0.213014e-1
	300 (T=25)	$\alpha_{n_T}^{MEF} - \alpha_0$	-0.118733e-2	0.669703e-1	0.661500e-1
		$\alpha_{n_T}^{ML} - \alpha_0$	-0.119962e-2	0.669653e-1	0.661496e-1
		$\beta_{n_T}^{MEF} - \beta_0$	0.326940e-3	0.144745e-1	0.136566e-1
		$\beta_{n_T}^{ML} - \beta_0$	0.217880e-3	0.144591e-1	0.134722e-1

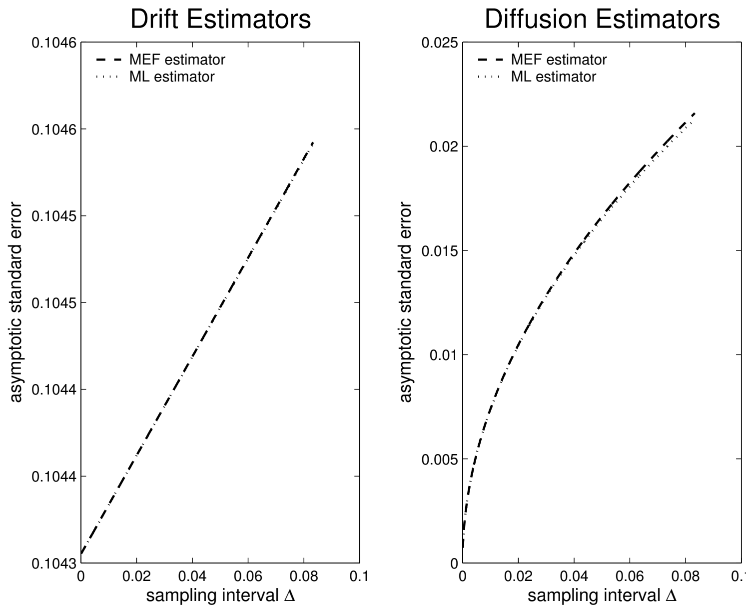

FIGURE 1. Comparison of the asymptotic standard errors of the estimators.

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