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ON AN ASSET MODEL OF HOBSON-ROGERS TYPE

NARN-RUEIH SHIEH*

Abstract. In this article, we consider a risky asset X for which evolution follows a model motivated by D.G. Hobson and L.C.G. Rogers [9]. We assume that the volatility of X depends on the ratio of the present value and the exponentially weighted average of the past value. Using the Markovian modeling of the enlarged two-dimensional process, we shall discuss the option pricing with X as the underlying asset. We will also discuss it as a model for the regime switching in financial markets.

1. Introduction

In this article, we consider a diffusion-type non-Markovian process $X(t)$ in $\mathbb{R}_+$, which obeys a certain geometric Brownian motion and is motivated by Hobson and Rogers [9]. Firstly we review briefly that, the model in [9] is a certain asset model with the stochastic volatility and with the infamous volatility smile; yet it has the advantage that the model preserves the market completeness of which the usual SV model is lack. This model has been under some numerical and market-data tests in [14], and a P.D.E. approach for the model appears in [4]. Meanwhile, the option pricing properties in the continuous non-Markovian setting has appeared in Section VI. B of [3].

This article is organized as follows. In Section 2, we formulate our asset model, and will call it as *an asset model of Hobson-Rogers Type*, since our model is mainly motivated by [9]. In Section 3, we discuss the option pricing for our model, with a section conclusion on the novel situation. Section 4 is the discussion on a relevant perspective, namely the regime switching theory. The proofs of our results in §3 are given in the final Section 5.

2. The Asset's Model

§2.1. We assume that the market has a unique constant risk-free interest rate $r > 0$ in the time horizon $[0, T]$ or $[0, \infty)$, and assume that the underlying risky asset $X(t)$ is with this interest rate r, and is with a non-constant volatility function $\sigma(\cdot)$ depending solely on the ratio of the present value $X(t)$ and the exponentially

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weighted past value $Y(t)$;

$$Y(t) := \lambda \int_0^\infty e^{-\lambda s} X(t-s)\, \mathrm{d}s. \tag{2.1}$$

The parameter $\lambda > 0$ for the exponential averaging, and the continuous deterministic past-memory

$$X(s) = \xi(s), \quad s \in (-\infty, 0], \tag{2.2}$$

are pre-given. Thus, $X(t)$ obeys the following geometric Brownian motion equation:

$$\mathrm{d}X(t) = rX(t)\,\mathrm{d}t + \sigma(Z(t))X(t)\,\mathrm{d}B(t), \ t \geq 0, \tag{2.3}$$

in which Z is the ratio process

$$Z(t) := \frac{X(t)}{Y(t)}.$$

We notice that the Y has the differential

$$\mathrm{d}Y(t) = \lambda(X(t) - Y(t))\,\mathrm{d}t. \tag{2.4}$$

Under suitable Lipschitz and growth conditions for the $\sigma(\cdot)$, the $X(t)$ exists uniquely as the strong solution for (2.3). Notice that $X(t)$ itself is non-Markovian, since the defining equation brings the memory of the past-values of $X(\cdot)$ into the present-value $X(t)$. We refer to [12] for the detailed discussion on the general theory for stochastic differential equations with memory. By the two-variate Itô formula, see for example §6.6 of [16] , we have the following differential for Z,

$$\mathrm{d}Z(t) = (r + \lambda - \lambda Z(t))Z(t)\,\mathrm{d}t + \sigma(Z(t))Z(t)\,\mathrm{d}B(t). \tag{2.5}$$

In view of (2.3) and (2.5), we see that the two-dimensional process $(X(t), Z(t))$ constitutes a Markovian diffusion (strong Markov process with continuous-paths) in $\mathbb{R}^2$. We should remark that, the $X(t)$ itself is not Markovian and the two-dimensional enlargement $(X(t), Z(t))$ is Markovian; this is due to our choice of the past-value $Y(t)$. The prevailing memory-effect choice of $Y(t) = X(t - t_0)$ for some instant t_0 does not have such a Markovian enlargement; though $X(t)$ itself can be regarded as an *infinite-dimensional* Markov process, in the sense to regard $X(t)$ as a Markov process taking vlaues in the path space $C[0, T]$, as discussed in [12].

§2.2. Our standing assumption on the volatility function $\sigma(z)$, besides the standard Lipschitz continuity, is the following non-constant boundedness:

Assumption 1.

$$0 < \sigma_2 = \inf \sigma(z) < \sup \sigma(z) = \sigma_1 < \infty.$$

This is a reasonable assumption for the non-constant volatility function; see [9, 2] for more detailed discussions.

§2.3. The log-process $\ln Z$ has a certain mean-reverting property; indeed we have, by applying Itô formula to Z and $\ln z$, we can solve that

$$Z(t) = Z(0) \exp\big\{ \int_0^t [-\lambda(Z(s) - (1 + \frac{r}{\lambda} - \frac{\sigma^2(Z(s))}{2\lambda}))]ds + \int_0^t \sigma(Z(s))\,\mathrm{d}B(s)\big\}. \tag{2.6}$$

By the Assumption 1, it could be seen that $\ln Z(t)$ is mean-reverting to a zone $[1+\frac{r}{\lambda}-\frac{\sigma_1^2}{2\lambda}, 1+\frac{r}{\lambda}-\frac{\sigma_2^2}{2\lambda}]$. We remark that, a diffusion which exhibits such mean-reverting to a zone, rather than to a constant or to a time-curve, seems to be a new class, if we compare with the usual mean-reverting Ornstein-Uhlenbeck processes; a precise discussion on such "zone-reverting" diffusions is under working [15].

§2.4. To compare with the model [9], the authors considered the logarithm of the discounted asset price, and the deviation (called the offset function in that article) of the present value and the past value, based on this log-price. We would propose that the process $Z(t)$ defined directly as the ratio of the present asset price and the past asset price (without referring to the logarithm) should be more natural to serve as an index for the asset price change; see §4.3 of Section 4 for a potential goodness of this proposal.

3. The Option Pricing

§3.1. From now on, we consider the non-Markovian asset process $X(t)$, and the two-dimensional enlargement $(X(t), Z(t))$, which is a Markovian diffusion in $\mathbb{R}_+^2$. We notice that the market is complete (namely only one underlying BM in the defining S.D.E.'s), and thus the martingale fair-pricing theory is valid in our model. In [9], the authors use the Feynman-Kac formula to derive the two-variate Black-Scholes P.D.E. for the European option in their model. The general multidimensional Feynman-Kac formula in Section 6.6 of [16] allows us to write down a similar P.D.E. for our (X, Z). We will not address it here, since the methodology is entirely the same.

§3.2. Now, we discuss the fair price of the American put option of the risky asset X in our model. The defining equation (2.3) of X tells that X is an Itô process (though it is *not* an Itô diffusion), and, since the market is complete (there is only one underlying Brownian motion $B(t)$), the time-value $V(t)$ of our American put option in the horizon $t \in [0, T]$, with the striking price K, is given by

$$V(t) = \operatorname{esssup}_{\tau\in\mathbb{T}_{t,T}} \mathbb{E}\big[e^{-r(\tau-t)}(K - X(\tau))^+ | \mathcal{F}_t\big], \tag{3.1}$$

where the $\{\mathcal{F}(t), t \geq 0\}$ denotes the Brownian filtration, and the $\mathbb{T}_{t,T}$ is the class of all stopping times, w.r.t. the Brownian filtration, valued in the $[t, T]$. This general formula is more analytically tractable, via the two-dimensional Markovian $(X(t), Z(t))$, and it is

$$V(t, x, z) = \sup_{\tau\in\mathbb{T}_{t,T}} \mathbb{E}\big[e^{-r(\tau-t)}(K - X(\tau))^+ | X(t) = x, Z(t) = z\big], \tag{3.2}$$

we refer to Chapter 8 of [16].

A fundamental result for American put options is that, for the constant volatility model, there is a monotone increasing curve $t \mapsto s(t), t \in [0, T]$, with $X(0) < s(0) < K, s(T) = K$, which separates the $[0, T] \times (0, \infty)$ into the *continuation region* and the *stopped region*; see the two books cited above. The contribution of this article is to address this issue for our asset X, which is not Markovian by itself.

We state two lemmas on the $V(t, x, z)$.

Lemma 3.1. *Under Assumption 1, for each fixed $t \in [0,T]$, it has*

$$V_2(t,x) \le V(t,x,z) \le V_1(t,x), \quad \forall\,(x,z) \in \mathbb{R}_+^2,$$

where $V_i(t,x)$ is the time-t price of the American put option in the time-horizon [0,T], with the same striking price K, associated with the standard GBM,

$$\mathrm{d}X_i(t) = rX_i(t)\,\mathrm{d}t + \sigma_i X_i(t)\,\mathrm{d}B(t), \quad X_i(t) = x, \quad i = 1,2.$$

Lemma 3.2. *We have, for each fixed $t \in [0,T]$,*

$$(K-x)^+ \le V(t,x,z) \le K, \quad \forall (x,z);$$

moreover, for each $z > 0$, the mapping

$$x \mapsto V(t,x,z)$$

is convex, continuous, and decreasing in $x \in [0,\infty)$.

The method used in the proof of Lemma 3.2 can be applied to obtain the following result, which meets a financial knowing that the higher option's value for higher volatility. There have been papers to discuss such a monotonicity and "mis-pricing" issue, and we refer to [10] (Theorem 2.1 and Remark 2.3 there).

Lemma 3.3. *For each $t \in [0,T]$ and each given x, the mapping $z \mapsto V(t,x,z)$ is monotone nondecreasing in z, if we assume that $\sigma(z)$ is monotone increasing in z.*

Now we present our results as follows. The first one is a proposition to assert, for each fixed time-instant, the existence of the parametric curve to separate the whole region $(x,z) \in \mathbb{R}^+ \times \mathbb{R}^+$ into the "high region" and the "low region". The second one asserts that the skewness of a time-parameter striking curve $t \mapsto s(t)$ under a monotone assumption of the volatility function $\sigma(\cdot)$ mentioned in Lemma 3.3. The third one is the perpetual case, in which the mapping $t \mapsto s(t)$ fluctuates in a parallel zone; this may provide some insight on the "zone-reverting" of the diffusion $Z(t) = X(t)/Y(t)$.

Proposition 3.4. *Under Assumption 1, for each fixed $t \in [0,T)$, in the (x,z)-plane there exists a continuous curve $x = b_t(z)$ such that the "high region"*

$$C_t = \{(x,z) \in \mathbb{R}_+^2 \colon V(t,x,z) > (K-x)^+\},$$

which is open in the (x,z)-plane, has the parametric boundary

$$\partial C_t = \{(x,z) \in \mathbb{R}_+^2 \colon x = b_t(z)\}.$$

For each $(x,z) \in \partial C_t$, it has $V(t,x,z) = (K-x)^+$, and $z \to b_t(z)$ is also the boundary of the "low region"

$$D_t = \{(x,z) \in \mathbb{R}_+^2 \colon V(t,x,z) = (K-x)^+\},$$

which is closed in the (x,z)-plane.

Remark: Since it always has $V(t,x,z) \ge (K-x)^+$, those two terms "the high" and "the low" in Proposition 3.4 are not only to indicate the relative positions in the (x,z)-plane, but also to indicate the relative values of the option at the time instant t. Furthermore, in Proposition 3.4, it would be interesting (and must be such expected) to prove that the optimal stopping time, that is, the one which maximizes the $V(t,x,z)$ defined by (3.2), is

$$\begin{aligned}\tau^*(\omega) &= \inf\{s\in[t,T]:\quad X(s,\omega)=b_t(Z(s,\omega))\}\\ &= \inf\{s\in[t,T]:\quad (X(s,\omega),Z(s,\omega))\in D_t\},\end{aligned}$$

which is the time-instant that the asset's price down from the "high region" to touch the boundary. See Remark 2 below Theorem 3.5 for this aspect.

Proposition 3.4 is for each fixed t; now we let t be varying over the time-horizon $[0,T]$. Then we present the result as follows

Proposition 3.5. *In the (t,x)-plane (time vs. the asset's value), we define a time-parameter mapping*

$$\begin{aligned}s:t\in[0,T]\mapsto s(t) &= \sup\{x<K:\quad (x,z)\quad \textit{below the curve}\ \ b_t(\cdot)\quad \textit{for some}\ \ z\}\\ &= \sup\{x<K: V(t,x,z)=(K-x)^+\quad \textit{for some}\ \ z\},\end{aligned}$$

where, for each t, $z\mapsto b_t(z)$ is the separation curve of the (x,z)-plane in Proposition 3.4.

Besides the standing Assumption 1, we assume that the non-constant volatility function $z\mapsto\sigma(z)$ is monotone increasing in z. It has, $s_2(t)\le s(t)\le s_1(t)$, where $s_i(t),\ \ t\in[0,T]$, denotes the increasing convex striking curve determined by the two constant-volatility GBM asset X_i. However, the mapping $t\mapsto s(t)$ is not monotone.

Remarks: 1. The $t\mapsto s(t)$ is the striking curve for the option in our model; it is the curve to separate the (t,x)-plane to the continuation region in which the holder is holding the option (at time-t, the option value $V(t,x,z)>(K-x)^+$, for any z) and the stopped region in which the option has been exercised (at time-t, the option value is then $V(t,x,z)=(K-x)^+$, for some z). We should remark that the curve is crucially dependent on the process $Z(t)$. Indeed, in our model, the option value $V(t,x,z)$ defined by the display (3.2) is conditioning on both $X(t)=x$ *and* $Z(t)=z$; this does not appear in the usual constant volatility case (no role of the variable z).
2. However, the rigorous proof that $t\mapsto s(t)$ is the optimal striking curve, which would be equivalent to prove that the τ^* mentioned in the remark below Proposition 3.4 is the optimal stopping time, is not achieved in this article. The optimality of the striking curve in the usual constant volatility case, for which the option value is $V(t,x)$ (no role of the variable z), is achieved by the PDE free-boundary theory in (t,x); which may not be easy for the present $V(t,x,z)$. We refer to [5, 6] for the relevant discussions.
3. In §4.2 of [9], the volatility function is supposed to be

$$\sigma(z)=\eta\sqrt{1+\epsilon z^2}\wedge N,$$

for the simulation of the volatility smile under their model. Such a volatility function satisfies the conditions of Theorem 3.5.

Now, suppose that the option is perpetual, that is the time-horizon is $t\in[0,\infty)$. We say that a real-valued curve $t\mapsto f(t), t\in[0,\infty)$, is *everywhere non-constant* if, for each given sub-interval $[t_1,t_2]$, its values are not constant on $[t_1,t_2]$; alternatively, the graph of $f(\cdot)$ has no horizontal segments.

Proposition 3.6. *For the perpetual case, the mapping $t \mapsto s(t)$ in Proposition 3.5 is then a everywhere non-constant curve lying between a parallel zone with band determined by the two GBM's X_1, X_2.*

§3.3. We give a concluding remark of this section as follows. Nowadays, it is well-known that the option pricing theory is at the beginning to be viewed from the martingale aspect. Then it is more important to move to view options in the Markov process aspect; since only then the BS P.D.E. for European options and the striking curve for American options can be discussed. For the latter, it is the parametric curve to separate the region in which the owner of the option holds and waits, and the region in which the owner exercises and gets the (positive) reward. In the basic (that is, the constant volatility is assumed) BSM theory, the striking curve is a monotone increasing and convex curve across the time horizon $[0, T]$; see Chapter 8 of [16]. In this article, our contribution is that, if we assume the volatility is the ratio of the asset's present value $X(t)$ and historical value $Y(t)$, with the choice of the historical values being exponentially averaged, then we still have a striking curve, which appears to be a non-monotone curve, as shown in Propositions 3.5 and 3.6. This would assert that the striking curve is *skewed*, due to the historical value of the asset. To our knowledge, this result is not noticed and discussed in the literature.

4. Regime-switching in Financial Markets

§4.1. This section briefs a proposal for the regime-switching (RS) mechanism. The RS theory of financial economics has been a huge literature since the pioneering works of J.D. Hamilton [7, 8]; see a good review [1] (the discrete-time setting there can be well-adapted to the continuous-time setting). The RS dynamics is usually specified by an exogenous finite-states Markov chain, and one important issue is the triggering of RS; see the beginning subsection of Section 4 of [1], Now we propose a RS aspect for our asset model as follows. The underlying assumption for the volatility function, instead of the previous Lipschitz continuity and Assumption 1 in §2, $\sigma(\cdot)$ is now assumed to be

Assumption 2. *The $\sigma(\cdot)$ is an rcll (right-continuous with left-limits everywhere) function on $\mathbb{R}^+$, and there are N values $0 < \sigma_1 < \cdots < \sigma_N < \infty$ and $N+1$ points such that $\sigma(z) = \sigma_j$, $z \in [z_j, z_{j+1})$, $z_1 < \cdots < z_{N+1}$ in $\mathbb{R}^+$.*

§4.2. Firstly, we mention that, under Assumption 2, the solution of S.D.E. (2.3) for such $\sigma(\cdot)$ must be considered as a *weak* solution. Now, we consider the ratio process $Z(t)$ as an indicator, as follows. It would be well to say that the asset $X(t)$ is in the regime σ_j at time t, if the realized ratio $z(t) := x(t)/y(t)$ is in $[z_j, z_{j+1})$, and that a RS happens at t, whenever $z(t)$ moves from one value to another one. The random pre-image $t \in [0, T]$ for which the realized $\sigma(z(t)) = \sigma_j$ is a partition of the time $[0, T]$, as the σ_j varies. This partition *cannot be* time-sequential; means that the regime could return in the time-dynamics. This is one feature of RS mechanism. Notice the possible similarity of such-formulated RS mechanism with the well-known Itô-McKean excursion theory.

§4.3. As it is mentioned in §4.1 that the RS time-dynamics is usually specified by an exogenous RS indicator as a finite-states Markov chain independent on the asset. While, in view of our proposal as described in §4.1 and §4.2, the RS indicator Z is an endogenous driving mechanism. Such study of endogenous RS theory, to our knowledge, is not noticed and discussed in the literature. Of course this proposal is mathematical, and it entirely demands the empirical study to verify the possible real significance in financial economics.

5. Proofs

Proof of Lemma 3.1. We prove the case $t = 0$, and we skip the $t = 0$ from the notation $V(t, x, z)$; the any given t case can be obtained in a parallel way.

We use the time-change technique; see, for example, §5.1 of [13]. Define,

$$T(t,\omega) = \Big(\frac{1}{\sigma_2^2}\int_0^t \sigma^2(Z(u,\omega))\,\mathrm{d}u\Big) \wedge T, \quad t \in [0,T].$$

which is strictly increasing in $t \in [0,T]$, and $T(t) \uparrow T$, $a.s.$ as $t \uparrow T$, by our uniform lower bound assumption on σ, namely Assumption **??**. The inverse, writing in them of θ,

$$\hat{T}(\theta,\omega) = \inf\{0 \le t \le T : \ T(t,\omega) = \theta\}, \quad \theta : \theta \in [0, (\frac{\sigma_1^2}{\sigma_2^2})T],$$

is well-defined, and

$$\int_0^{\hat{T}(\theta,\omega)} \sigma^2(Z(u,\omega))\,\mathrm{d}u = \theta, \quad \theta \ge 0;$$

moreover, $\hat{T}(\theta,\omega)$ is also strictly increasing in θ, and $\hat{T}(\theta) \uparrow T$, $a.s.$ as $\theta \uparrow (\frac{\sigma_1^2}{\sigma_2^2})T$. Define the time-changed motion

$$\hat{B}(\theta,\omega) = \int_0^{\hat{T}(\theta,\omega)} \sigma(Z(u,\omega))\,\mathrm{d}B(u,\omega), \quad \theta \ge 0.$$

Then, as §5.2 [13] shows, the process $\theta \mapsto \hat{B}(\theta,\omega)$ is a standard Brownian motion w.r.t. the filtration $\hat{\mathcal{F}}(\theta) = \mathcal{F}(\hat{T}(\theta))$, $\theta \ge 0$.

Writing in them of θ, we have

$$X(\theta) = X(0)e^{\hat{T}(\theta)\cdot r - \frac{1}{2}\theta + \hat{B}(\theta)}. \tag{5.1}$$

While for $X_i(\theta)$, $i = 1, 2$, we have

$$X_i(\theta) = X(0)e^{\frac{\theta}{\sigma_i^2}\cdot r - \frac{1}{2}\theta + B_i(\theta)}, \tag{5.2}$$

in which $B_i(\theta))$ is the standard Brownian motion obtained from the scaling $\theta \to \sigma_i B(\frac{\theta}{\sigma_i^2})$. We notice that

$$\frac{\theta}{\sigma_1^2} \le \hat{T}(\theta,\omega) \le \frac{\theta}{\sigma_2^2}. \tag{5.3}$$

Since $t \rightsquigarrow \theta$ is a one-to-one transformation, we have

$$V(x,z) = \sup_{\tau'} \mathbb{E}_{(x,z)}\big[e^{-r\tau'}(K - X(\tau'))^+\big],$$

where τ' is ranging over the class of all stopping times over the scaled time-horizon $[0, (\frac{\sigma_1^2}{\sigma_2^2})T]$,
w.r.t. the time-changed Brownian filtration $\hat{\mathcal{F}}(\theta)$; so are for the $V_i(x), \quad i = 1, 2$.

In term of θ, $V(x, z)$ and $V_i(x)$ are all driven by the standard Brownian motion. We may compare the first term of the three exponentials in (5.1) and (5.2), together with (5.3), and conclude that, for all $\theta' > 0$,

$$\mathbb{E}_x\big[e^{-r\theta'}(K - X_2(\theta'))^+\big] \leq \mathbb{E}_{(x,z)}\big[e^{-r\theta'}(K - X(\theta'))^+\big] \leq \mathbb{E}_x\big[e^{-r\theta'}(K - X_1(\theta'))^+\big].$$

Substituting θ' by $\tau'(\theta)$, we have the desired bound. □

Proof of Lemma 3.2. We again prove the case $t = 0$, and skip the $t = 0$ from the notation $V(t, x, z)$. Taking $\tau = 0$ in the defining equality of $V(x, z)$, we have $V(x, z) \geq (K - x)^+$; that $V(x, z) \leq K$ is obvious. Using the (2.3), we can write $V(x, z)$ explicitly as

$$V(x, z) = \sup_\tau \mathbb{E}\big[e^{-r\tau'}\big(K - x\exp\{r\tau - \frac{1}{2}\int_0^\tau \sigma^2(u)\,\mathrm{d}u + \int_0^\tau \sigma(u)\,\mathrm{d}B(u)\}\big)^+\big],$$

in which $\sigma(u) = \sigma(Z(u)), X(0) = x, Z(0) = z$. From this display, it is seen that, for each z, $x \mapsto V(x, z)$ is convex, continuous, and decreasing in x. □

Proof of Lemma 3.3. Let $0 < z < z'$. We regard $V(t, x, z)$ and $V(t, x, z')$ as the time-t values of the same option for two stocks; the two stocks are with the same observed price x and with the observed volatilities $\sigma(z)$ and $\sigma(z')$ respectively. Then, we can apply the time-change argument presented in the proof of Lemma 3.1 to obtain the assertion. Alternatively, we can use Theorem 2.1 and Remark 2.3 of [10] to obtain the assertion, with the following remark. If one exercises an American put option at the time t, then the payoff is $K - X(t)$, and in this sense it meets the the non-path-dependence phrase used in the statement in the Theorem 2.1 of [10]. □

Proof of Proposition 3.4. We define, for each $t \in [0, T)$, a mapping

$$b_t(z) = \sup\{x \leq K : \quad V(t, x, z) = (K - x)^+\}, \quad z \in (0, \infty).$$

Since, for each t, $V(t, x, z)$ is lower semi-continuous in (x, z) (due to the Feller property of $(X(t), Z(t))$), the regions C_t and D_t are respectively open and closed in the (x, z)-plane. Moreover, $z \mapsto b_(z)$ must be continuous, and for each (x, z) in the curve $b_t(\cdot)$, it has $V(t, x, z) = (K - x)^+$. Now, we claim that

$$V(t, x, z) = (K - x)^+, \quad \forall x : \quad x \leq b_t(z); \tag{5.4}$$

so that the curve $z \mapsto b(z)$ indeed defines the separation boundary of the high region C_t and the low region D_t.

Suppose, on the contrary, that, for some $(x_1, z) : 0 < x_1 < b_t(z)$,

$$V(t, x_1, z) > (K - x_1)^+.$$

Since $V(t, b_t(z), z) = K - b_t(z)$, we must have, for some $\beta > 1$,

$$\frac{V(t, b_t(z), z) - V(t, x_1, z)}{b(z) - x_1} = -\beta < -1.$$

We recall that, by Lemma 3.2, $x \mapsto V(t,x,z)$ is decreasing. By Lemma 3.2 again, $x \mapsto V(t,x,z)$ is convex, and thus we have,

$$\frac{V(t,b(z),z)-V(t,x,z)}{b_t(z)-x} \le -\beta, \quad \forall x \le x_1.$$

This will imply that

$$\begin{aligned} V(t,x,z) &\ge V(t,b_t(z),z)+\beta(b_t(z)-x) \\ &= (K-b_t(z))+\beta(b_t(z)-x) \\ &= K+(\beta-1)b_t(z)-\beta x, \end{aligned}$$

which implies that $V(t,x,z) > K$, whenever $x < (\beta-1)b_t(z)/\beta$. This contradicts to the fact that $V(t,x,z) \le K$ (Lemma 3.2). Therefore, the supposition must be false. Thus, for each $(x,z) : 0 < x < b_t(z)$, it must have $V(t,x,z) = (K-x)^+$, that is, $(x,z) \in D_t$. □

Proof of Propositions 3.5 and 3.6. Firstly, the mapping $t \mapsto s(t)$ is well-defined. Indeed, for each t, $(x,z) \mapsto V(t,x,z)$ is lower semi-continuous (see the proof of Proposition 3.4), and, for each x, $z \mapsto V(t,x,z)$ is monotone increasing by Lemma 3.3; thus the supremum in the definition of $s(t)$ is attained at some (x,z), and it must have $x = b_t(z)$. Next, by Lemma 3.1, the two increasing convex curves $s_i(t)$ which squeeze $s(t)$ are those striking curves for the two American options of each the underlying asset follows the standard GBM with constant volatilities σ_1 and σ_2 respectively. See, for example, Chapter 8 of [16]. We mention that,

$$s_2(0) < s_1(0) < K, \quad s_2(T) = s_1(T) = K.$$

To prove the non-monotone assertion of $t \mapsto s(t)$, we claim that, for any two time-instants $0 < t < t' < T$, $s(t)$ can be smaller or be greater than $s(t')$, subject to those are observed for Z at the t and the t' (we remark that, this does not happen for the constant $\sigma(\cdot)$ case, since the role of Z does not appear then). To this end, we claim that, for any two z, z', the following "anti-comonotone" property,

$$(b_t(z)-b_{t'}(z'))(V(t,x,z)-V(t',x,z')) < 0, \quad \text{for any } x \text{ between } b_t(z), b_{t'}(z').$$

Indeed, suppose that $x : b_t(z) < x < b_{t'}(z')$, then, from the definition of the separation curve given in Proposition 3.4, it has that $V(t,x,z) > (K-x)^+$ and $V(t',x,z') = (K-x)^+$; thus $V(t',x,z') < V(t,x,z)$. On the other hand, suppose that $x : b_t(z) > x > b_{t'}(z')$, then the same argument gives $V(t',x,z') > V(t,x,z)$.

Next, we consider two curves $z \to b_t(z)$ and $z' \mapsto b_{t'}(z')$, which appear in Proposition 3.4 for t and t'. In the below, we *identify* the two (x,z)-plane and (x',z')-plane, and we compare the relative positions of the two curves. We assume that , for the moment, in this plane the curve $z \to b_t(z)$ is always strictly below the $z' \mapsto b_{t'}(z')$, and thus $b_t(z) < b'_t(z')$ for all z, z'. This will imply that the curve $b_t(\cdot)$ is now a subset of the interior of the time-t' low region $D_{t'}$ (see Proposition 3.4), and thus there exist (x,z) and (x,z'), with z, z' being allowed to be arbitrarily large or small, so that $V(t',x,z') = (K-x)^+ < V(t,x,z)$. This would mean that the option, no matter the change of the volatility, is always kept at t and is always exercised at t'; this violates the market's viability; see, for example, Chapter 5 of [11] for discussions. While, on the contrary, we assume that the curve $z \to b_t(z)$ is always strictly above the $z' \mapsto b_{t'}(z')$. Then, by the above "anti-comonotone" property, we have $V(t, x, z) < V(t', x, z')$ for all x, z, z' such that $x : b_t(z) > x > b_{t'}(z')$, which cannot be valid by Lemma 3.3. Indeed, firstly, since $t < t' \leq T$, $V(t', x, z) \leq V(t, x, z)$ (from the definition of $V(t)$), and, secondly, by Lemma 3.3, $V(t', x, z') < V(t', x, z)$ when $z' < z$.

Thus, neither the curve $b_t(\cdot)$ nor the curve $b_{t'}(\cdot)$ can top on the other. Therefore there must exist $z_i, z'_i, i = 1, 2$, such that $b_{t'}(z'_1) > b_t(z_1), b_{t'}(z'_2) < b_t(z_2)$. This means that, $s(t)$ can be smaller or be greater than $s(t')$, subject to the observations for Z at the t and the t'; namely $s(t) < s(t')$ if z_1, z'_1 are observed, while $s(t) > s(t')$ if z_2, z'_2 are observed.

Now, we assume, for a moment, that $t \mapsto s(t)$ *was* monotone, then the only possibility is that it was monotone increasing, since it always locates between two monotone increasing curves with the same end (T, K) (namely, the two striking curves for the two stocks with constant volatilities $\sigma_i, i = 1, 2$). Thus, it always has $s(t) < s(t')$, no matter how z, z' are observed; this contradicts to the arguments given in the above. Therefore, we must conclude that $t \mapsto s(t)$ cannot be monotone.

In the perpetual case, the striking curve for the American put option under the GBM is a constant curve with the explicit expression; see for example Chapter 8 of [16]. The parallel zone in the statement follows from this fact. Assume that $t \mapsto s(t)$ is constant in some interval $[t_1, t_2]$. Then, for two $t_1 \leq t' < t^{''} \leq t_2$, $s(t') = s(t^{''})$. However, the arguments in the proof of Theorem 3.5 (in which the finite time-horizon is not essential) assert that the ordering of $s(t'), s(t^{''})$ is subject to the observed $Z(t)$ at $t', t^{''}$ (we again remark that this does not happen in the constant-volatility case). Thus, the assumption must be void. □

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