Seminar on Continuity in Semilattices

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SCS 21: ≤ (n)

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SEMINAR ON CONTINUITY IN SEMILATTICES (SCS)

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TOPIC $\leq(n)$

REFERENCE Lawson SCS Memo dated 7/12/76 Other Memos quoted herein
Mislove SCS memo dated 8/13/76

Now that everyone seems to agree that "way below" should have as its definition:

(a) $x \ll y$ iff $y \leq \sup A$ implies $x \leq \sup F$ for some finite $F \subseteq A$.

rather than the original:

(b) x is way below y iff $y = \sup A$ implies $x \leq \sup F$ for some finite $F \subseteq A$.

our seminar would suggest yet a new notation and term for this concept. Before everyone curses us out loud, please hear us out.

In his memo of 7/12/76, Lawson defines $\lll$ by

(c) $x \lll y$ iff $y \leq \sup A$ implies $x \ll \sup F$ for some finite $F \subseteq A$.

and proves that, for compact S, $x \lll y$ iff $y \in (\uparrow x)^\circ$. He also asserts (Corollary 5) that $x \in \Lambda(S) = \{z \in S \mid S$ has small semilattices at $z\}$ iff $x = \sup\{y \in S \mid y \ll x\}$. No proof is given and, in fact, the assertion remains an open question. In a personal correspondence received here on 10/5/76 Lawson did establish the assertion for compact metric S.

West Germany: TH Darmstadt (Gierz, Keimel)
U. Tübingen (Mislove, Visit.)

England: U. Oxford (Scott)

USA: U. California, Riverside (Stralka)
LSU Baton Rouge (Lawson)
Tulane U., New Orleans (Hofmann, Mislove)
U. Tennessee, Knoxville (Carruth, Crawley)

Next, Mislove (3/18/76) used this open assertion to construct an example of a compact S such that $L(S) = \Lambda(S)$ is not a CL - object. First of all, there is a typographical error in the definition of T in step 1. The set Q should have been defined as $F \cup P^{-1} \cup \Delta$, where

$$P = \{((n,1),(n+1,0)) \mid n \in \mathbb{N}\} .$$

(Note: $((n,1),(n+1,0))$, not $((n,0),(n+1,1))$.) This example can also be viewed as a generalized hormos with $X = \{1 - 1/n \mid n \in \mathbb{N}\} \cup \{1\}$ in an obvious fashion. Next, he correctly observes that $1 \in L(T)$, as $\{1\} = \bigcap_{n \in \mathbb{N}} (\uparrow[n,0])^{\circ}$ and $\uparrow[n,0]$ is a subsemilattice. He then incorrectly "shows" that each semilattice neighborhood of $[n,r]$ must contain $[0,0]$. Both of his assertions "each semilattice neighborhood of $[n,r]$ must contain $[n,0]$ " and "each semilattice neighborhood of $[n,0]$ must contain $[n-1,0]$ " are correct, however, it does not follow that "each semilattice neighborhood of $[n,r]$ must contain $[n-1,0]$". In fact, if this were true, each semilattice neighborhood of 1 would necessarily contain $[0,0]$, yielding $1 \notin L(T)$. The problem, of course, is that each semilattice neighborhood of $[n,r]$ must contain $[n,0]$ but it need not be a neighborhood of $[n,0]$. In fact, if $r \neq 0$, $\uparrow[n,0]$ is a semilattice neighborhood of $[n,r]$ (which is not a neighborhood of $[n,0]$) (unless $n=0$). It can, however, be easily argued that $L(T) = \{[0,0], 1\}$ anyway.

Now, in step 2, Mislove concludes that $(1,1) \in L(S)$ by using the fact that $(1,1) = \sup\{z \in S \mid z \ll_S (1,1)\}$. As we have observed, the sufficiency of this condition remains in question. However, one sees that $(1,1) \in \uparrow([n,0],[n,0])^{\circ}$ as before and so $(1,1) \in L(S)$ (or a glutton for punishment might use Lawson's observation that the condition is sufficient for compact metric S). Similar remarks apply to the "proof" that $([n,0],1) \in L(S)$.

We hasten to say that the example is quite nice. Along these lines, is it obvious that $L(S)$ is always a closed subspace of S ? Is it even true ?

Now to the topic of this memo. Throughout, S will be a complete lattice. We also adopt the convention that D will always be an arbitrary up-directed subset of S and d will always denote an element of D which, when written, is asserted to exist. Hence, we know that:

(d) $x \ll y$ iff $y \leq \sup D$ implies $x \leq d$; and

(e) $x \lll y$ iff $y \leq \sup D$ implies $x \ll d$.

Let us define:

(1) $x \leq(1)\ y$ iff $x \leq y$;

(2) $x \leq(2)\ y$ iff $y \leq \sup D$ implies $x \leq(1)\ d$;

(3) $x \leq(3)\ y$ iff $y \leq \sup D$ implies $x \leq(2)\ d$;

and recursively define:

(n) $x \leq(n)\ y$ iff $y \leq \sup D$ implies $x \leq(n-1)\ d$.

One might read $x \leq(2)\ y$ as "x is two-below y", and, in general, $x \leq(n)\ y$ as "x is n-below y". Recall that $x \prec y$ means $y \in (\uparrow x)^{\circ}$.

Observation 1.1. (a) For a compact S, $\leq(3) = \prec$ according to Lawson (7/12/76);

(b) For a compact Lawson S, $\leq(2) = \prec$ and so $\leq(2) = \leq(3)$ by (a).

Problem. Is $\leq(2) = \leq(3)$ for all compact S? Equivalently, is $\leq(2) = \prec$ for all compact S?

Surely almost everyone has tried to prove this in one form or another and it appears to be quite difficult.

We offer some observations about $\leq(n)$. Perhaps these relations have been studied for complete lattices in the literature, but we are not aware of any such studies.

Proposition 1.2. $P = \{n \in \mathbb{N} \mid \leq(n+1) \subseteq \leq(n)\} = \mathbb{N}$.

Proof. First we show $1 \in P$. Fix $(x,y) \in \leq(2)$. Let $D = \{y\}$. Then $y \leq(1) \sup D = y$ implies $x \leq(1)\ d = y$. Hence, $x \leq(1)\ y$, $\leq(2) \subseteq \leq(1)$, and $1 \in P$.

Assume that $k \in P$. That is, assume that $\leq(k+1) \subseteq \leq(k)$. Fix $(x,y) \in \leq(k+2)$ and fix D such that $y \leq(1) \sup D$. Then, since $x \leq(k+2)\ y$, we have $x \leq(k+1)\ d$. Now, by the induction hypothesis, $x \leq(k)\ d$. Hence, by definition, $x \leq(k+1)\ y$ so that $\leq(k+2) \subseteq \leq(k+1)$ and $k+1 \in P$. ∎

Corollary 1.3. $\leq(n)$ is antisymmetric for all $n \in \mathbb{N}$. ∎

Example 1.4. Let $D = \{1 - 1/n \mid n \in \mathbb{N}\}$ and let $S = D \cup \{1\}$ with the usual ordering. Then, $1 \leq(1) \sup D$, but $1 \not\leq(1)\ d$ for any $d \in D$. Hence, $1 \not\leq(2)\ 1$, and so $\leq(n)$ is not reflexive for any $n \geq 2$ (in view of 1.2).

The next proposition proves to be indispensible in the study of $\leq(n$

Proposition 1.5. $\leq(n) \circ \leq(m) \subseteq \leq(n+m-1)$.

Proof. The proof is by induction on n .

$n = 1$: For $m = 1$ the assertion is clear since $\leq(1)$ is transitive. Assume for $m = k$ and fix $x \leq(1)\ y \leq(k+1)\ z$. Now, fix D such that $z \leq(1) \sup D$. Then $y \leq(k)\ d$ and by the I. H. , $x \leq(k)\ d$. Hence, $x \leq(k+1)\ z$ ..

Assume for $n = k$: i.e. $\leq(k) \circ \leq(m) \subseteq \leq(k+m-1)$ for all $m \in \mathbb{I}$
We proceed to show that $\leq(k+1) \circ \leq(m) \subseteq \leq(k+m)$ for all $m \in \mathbb{N}$.

$m = 1$: Fix $x \leq(k+1)\ y \leq(1)\ z$ and D such that $z \leq(1) \sup D$. Then $y \leq(1) \sup D$ implies $x \leq(k)\ d$. Hence, $x \leq(k+1)\ z$ as was to be shown .

Assume for $m = j$: i.e. $\leq(k+1) \circ \leq(j) \subseteq \leq(k+j)$.
Fix $x \leq(k+1)\ y \leq(j+1)\ z$ and D such that $z \leq(1) \sup D$. Then, $y \leq(j)\ d$ and by the I. H. $x \leq(k+j)\ d$. Hence, $x \leq(k+j+1)\ z$. ∎

Corollary 1.6. (a) $\leq(n)$ is transitive for all $n \in \mathbb{N}$.

(b) $\Delta \cap \leq(n) = \Delta \cap \leq(m)$ for all $n,m \geq 2$. *Menge der kompakten Elemente*

(c) If $1 \leq n < m$ and $\leq(n) = \leq(m)$, then $\leq(n) = \leq(k)$ for all $k \geq n$.

(d) If $n \geq 2$ and $\leq(n) \subseteq \leq(n) \circ \leq(n)$ (i.e. $\leq(n)$ has the interpolation property), then $\leq(n) = \leq(n+1)$.

(e) If $n \geq 2$ and $\Delta \subseteq \leq(n)$, then $\leq(k) = \leq(1)$ for all $k \in \mathbb{N}$. *Dies heißt: Alle Elemente sind kompakt, d.h. alle Ketten mit größtem El^t sind endlich.*

(f) If $\leq(2)$ has the interpolation property, then $\leq(2) = \leq(3)$

Proof: (a) For all $n \in \mathbb{N}$, $n+n-1 \geq n$ and so we simply apply Proposition 1.5 and then Proposition 1.2.

(b) It suffices to show that, for $n \geq 2$, $\Delta \cap \leq(n) = \Delta \cap \leq(n+1)$. The containment from right to left follows from 1.2. Now, fix $x \in S$ such that $(x,x) \in \leq(n)$. Then, by 1.2, $(x,x) \in \leq(2)$ also. Hence, by 1.5, $(x,x) \in \leq(n+1)$.

(c) Certainly $\leq(n) = \leq(k)$ for all k such that $n \leq k \leq m$, in view of 1.2. Hence, we need only show that if $\leq(n) = \leq(n+1)$, then $\leq(n+1) = \leq(n+2)$. Fix $x \leq(n+1)\ y$ and $D \subseteq S$ such that $y \leq(1) \sup D$. Then, $x \leq(n)\ d$ and, since we are assuming $\leq(n) = \leq(n+1)$, $x \leq(n+1)\ d$. Therefore, $x \leq(n+2)\ d$ as was to be shown.

(d) Since $n \geq 2$, $2n-1 \geq n+1$. Hence,

$$\begin{aligned} \leq(n) \subseteq \leq(n) \circ \leq(n) &\subseteq \leq(2n-1) && \text{by } 1.5 \\ &\subseteq \leq(n+1) && \text{by } 1.2 \\ &\subseteq \leq(n) && \text{by } 1.2\,. \end{aligned}$$

(e) It suffices to show that $\leq(1) \subseteq \leq(2)$, in view of (c) above. First observe that $\Delta \subseteq \leq(2)$, in view of 1.2.

$$\begin{aligned} \text{Now, } \leq(1) = \leq(1) \circ \Delta \subseteq \leq(1) \circ \leq(2) &\subseteq \leq(2) && \text{by } 1.5 \\ &\subseteq \leq(1) && \text{by } 1.2\,. \end{aligned}$$

(f) This follows immediately from (d) and is only isolated as a special case for emphasis. ∎

Corollary 1.7. (a) If S is compact, then $\leq(3) = \leq(n)$ for all $n \geq 3$

(b) If S is compact Lawson, then $\leq(2) = \leq(n)$ for all $n \geq 2$.

In view of Corollary 1.7, perhaps only the pure algebraists will be interested in the general study of $\leq(n)$. However, at least the question of whether $\leq(2) = \leq(3)$ for compact S remains for the topological algebraists. Moreover, we proceed to discuss relationships between $\leq(n)$ and $\prec$ in the non-compact topological setting which should be of interest to the topological people. First we state the following proposition. For $n = 2$, this is essential in proving that $\leq(2)$ has the interpolation property in a continuous lattice (Scott - Continuous Lattices, Observation on page 110 without proof; Isbell - Meet-continuous Lattices, Istituto Nazionale di Alta Matematica Symposia Mathematica 1975, Proposition 2.3 on page 46; Scott - SCS Memo dated 3/30/76 Page 5).

Proposition 1.8. $x \leq(n)\, z$ and $y \leq(n)\, w$ imply $x \vee y \leq(n)\, z \vee w$.

Proof. The proof is by induction on n, is not difficult, and is omitted. ∎

Proposition 1.9. Let $(S, \mathcal{T}, \leq)$ have the property that $(S, \mathcal{T})$ is a topological space and $(S, \leq)$ is a complete lattice. We define, as always, $x \prec y$ iff $y \in (\uparrow x)^{\circ}$. Then:

(a) $(\uparrow x)^{\circ} \cap (\uparrow y)^{\circ} = [\uparrow(x \vee y)]^{\circ}$.

(b) For all $y \in S$, $D_y = \{s \in S \mid s \prec y\}$ is closed under finite sups and hence is up-directed;

(c) $\leq(n) \circ \prec \subseteq \prec$ for all $n \in \mathbb{N}$.

If m is separately continuous, then:

(d) $\prec \circ \leq(n) \subseteq \prec$ for all $n \in \mathbb{N}$.

(e) $(\uparrow x)^\circ$ is an upper end; i.e. $\uparrow[(\uparrow x)^\circ] \subseteq (\uparrow x)^\circ$.

If $(S, \mathcal{T})$ is compact T_2 and m is continuous, then:

(f) $\prec \subseteq \leq(n)$ for all $n \in \mathbb{N}$. (Previously observed.)

Proof. Once again, the proofs are straightforward and are omitted. ∎

Ron Wilson observes that (f) of 1.9 can also be proved for Scott's relation $\prec$. In fact, both results follow from a more general relation theoretic result in the spirit of Mike Smith, and the Gierz, Hofmann, Keimel, Mislove SCS Memo dated 8/1/76.

Proposition 1.10. Let $\sqsubset$ be a binary relation on S such that:

(a) $\sqsubset \subseteq \leq(1)$;

(b) $\sqsubset \circ \leq(1) \subseteq \sqsubset$;

(c) $a \sqsubset \sup D$ implies $a \sqsubset d$.

Then, $\sqsubset \subseteq \leq(n)$ for all $n \in \mathbb{N}$.

Proof. The case $n = 1$ is simply (a). Assume for $n = k$. Fix $x \sqsubset y$ and $D \subseteq S$ such that $y \leq(1) \sup D$. Then, by (b), $x \sqsubset \sup D$ and, by (c), $x \sqsubset d$. Now, by the I.H., $x \leq(k)\, d$ and so $x \leq(k+1)\, y$. ∎

Along these same lines, consider the category CSRIP. If we modify the assumptions somewhat, while retaining the interpolation property, we can obtain $\sqsubset \subseteq \leq(n)$ for all $n \in \mathbb{N}$.

Proposition 1.11. Suppose $(S, \sqsubset)$ satisfies:

(1) S is a complete lattice; and

(2) $\sqsubset$ is a binary relation on S such that:

(2.1) $\sqsubset \subseteq \sqsubset \circ \sqsubset$; and

(2.2) $\sqsubset \subseteq \leq(2)$.

Then, $\sqsubset \subseteq \leq(n)$ for all $n \in \mathbb{N}$.

Proof. For $n = 1$ use 2.2 and 1.2 . For $n = 2$ use 2.2 . Assume for k with $k \geq 2$. Fix $x \sqsubset y$ and $D \subseteq S$ such that $y \leq(1) \sup D$. By two applications of 2.1 we obtain elements a and b such that $x \sqsubset a \sqsubset b \sqsubset y \leq(1) \sup D$. Hence, $x \leq(k)\ a \leq(k)\ b \leq(1) \sup D$ by the I.H. and it follows that $x \leq(k)\ a \leq(k-1)\ d$. Therefore, $x \leq(2k-2)\ d$ by 1.5 and, since $k \geq 2$, $2k-2 \geq k$ which implies $x \leq(k)\ d$. Thus, $x \leq(k+1)$;

This seems to be a good place to make an observation due to Ron Wilson. The reference is the Gierz, Hofmann, Keimel, Mislove SCS Memo dated 8/1/76 . On page 20 , the question "Is $w_S = v_S$?" is asked. Unless we are missing something, the answer is yes and it follows from the following:

Observation 1.12. $\ll^* = \prec^*$ for each compact S .

Proof. It suffices to show that $\ll$ dyadic chains are $\prec$ dyadic chains, and conversely. If φ is a $\ll = \leq(2)$ dyadic chain, then

$r < s$ implies $\varphi(r) \leq(2) \varphi(\frac{r+s}{2}) \leq(2) \varphi(s)$ which implies $\varphi(r) \leq(3) \varphi(s)$ by 1.5 . Now, by Lawson's observation that $\leq(3) = \prec$, we have $\varphi(r) \prec \varphi(s)$. That each $\prec$ dyadic chain is a $\leq(2)$ dyadic chain follows from 1.9(f) . ∎

This last observation indicates, it would seem, that the relations $\leq(n)$ can be useful, even though they collapse to various degrees in the compact setting. As to the purely algebraic setting, the next proposition and its corollary, due to C. E. Clark, are of interest.

<u>Proposition 1.13</u>. Let S be a complete lattice and fix $x \in S$. Let $T = (\mathbb{N} \times S) \cup \{0,p,1\}$ and define a partial order on T by:

(i) $\uparrow(i,y) = [\{i\} \times (\uparrow y)] \cup \{(j,1) \mid j \geq i\} \cup \{1\}$;

(ii) $\uparrow 0 = T$; $\uparrow 1 = \{1\}$;

(iii) $\uparrow p = \bigcup_{i=1}^{\infty} [\{i\} \times (\uparrow x)] \cup \{p\} \cup \{1\}$.

Then, $(T, \leq)$ is a complete lattice and the following properties hold:

(1) $x \leq(n) y$ in S implies $p \leq(n) (i,y)$ in T $\forall i \in \mathbb{N}$;

(2) $x \not\leq(n) y$ in S implies $p \not\leq(n) (i,y)$ in T $\forall i \in \mathbb{N}$.

(3) $x \leq(n) 1$ in S and $x \not\leq(n+1) 1$ in S imply

$p \leq(n+1) 1$ in T and $p \not\leq(n+2) 1$ in T .

<u>Proof</u>. First we show that T is a complete lattice. Fix $A \neq \square$ in T . (a) If $1 \in A$, then $\sup A = 1$. In the remaining cases we assume $1 \notin A$.

(b) If $A \cap (\{i\} \times S) \neq \square$ for infinitely many i , then $\sup A = 1$.

(c) If $A \subseteq \{0,p\}$, then $\sup A \in \{0,p\}$.

(d) If $A \subseteq \{0,p\} \cup (\{i\} \times S)$ for some i and $A \cap (\{i\} \times S) \neq \square$, then $\sup A = (i, \sup \pi_2(A))$ if either $\sup \pi_2(A) \geq x$ or $p \notin A$, and $\sup A = (i, (\sup \pi_2(A)) \vee x)$ otherwise.

(e) If $A \cap (\{i\} \times S) \neq \square \neq A \cap (\{j\} \times S)$ with $i \neq j$, then $\sup A = (n,1)$, where $n = \max \{k \in \mathbb{N} \,/\, A \cap (\{k\} \times S) \neq \square\}$.

These cases are exhaustive and so $(T, \leq)$ is a complete lattice.

Proof of (1): For $n = 1$ the assertion follows immediately from the definition of $\uparrow p$. Assume for $n = k$. Suppose $x \leq(k+1)\ y$. Fix $i \in \mathbb{N}$ and $D \subseteq T$ such that $(i,y) \leq(1) \sup D$. If $1 \in D$, then $x \leq(k+1)\ y \Rightarrow x \leq(k)\ y \Rightarrow p \leq(k)\ (i,y)$ by the I.H. $\Rightarrow p \leq(k)\ 1$ by 1.5 $\Rightarrow p \leq(k)\ d$. If $1 \notin D$, then $D \cap (\{j\} \times S) \neq \square$ for some $j \geq i$ (since $\sup(\{1,\ldots,i-1\} \times S) = (i-1,1)$). If D meets at least one additional slices, then $(j,1) \in D$ for some $j \geq i$. Hence, again, $x \leq(k+1)\ y \Rightarrow x \leq(k)\ y \Rightarrow p \leq(k)\ (j,y)$ by the I.H. $\Rightarrow p \leq(k)\ (j,1)$ by 1.5 $\Rightarrow p \leq(k)\ d$. Finally, if D meets only the j- slice, then $y \leq(1) \sup(\pi_2[(\{j\} \times S) \cap D])$. (If $j = i$, this is easy, whereas if $j > i$, $\sup(\pi_2[(\{j\} \times S) \cap D]) = 1$ $(1) \geq y$.) Now, since $x \leq(k+1)\ y$, we have $x \leq(k)\ \pi_2(d)$ for some $d \in \{j\} \times S$. Hence, $p \leq(k)\ (j, \pi_2(d)) = d$. Therefore, in any case, $p \leq(k)\ d$ for some $d \in D$, so that $p \leq(k+1)\ (i,y)$ and we are finished.

Proof of (2): For $n = 1$ the assertion follows immediately from the definition of $\uparrow p$. Assume for $n = k$. Suppose $x \not\leq(k+1)\ y$. Fix $i \in \mathbb{N}$. Then there exists D such that $y \leq(1) \sup D$ but

$x \not\leq (k)\ d$ for any $d \in D$. Now, $\{i\} \times D$ is an up-directed set in T, $(i,y) \leq (1)\ (i,\sup D) = \sup[\{i\} \times D]$. But, $x \not\leq (k)\ d$ for all $d \in D$ implies, by the I.H., $p \not\leq (k)\ (i,d)$ for all $d \in D$. Hence, $p \not\leq (k+1)\ (i,y)$.

Proof of (3): First we show that $p \leq (n+1)\ 1$ in T. Let D be an up-directed set in T such that $\sup D = 1$. If $1 \in D$, then $x \leq (n)\ 1$ in S implies (by (1)) $p \leq (n)\ (i,1)$ for all $i \in \mathbb{N}$ and hence $p \leq (n)\ 1$ (by 1.5). If $1 \notin D$, then there exist i,j with $i \neq j$ such that $D \cap (\{i\} \times S) \neq \square \neq D \cap (\{j\} \times S)$. It follows that D contains an element of the form $d = (\ell,1)$ for some $\ell \in \mathbb{N}$. But, $x \leq (n)\ 1$ implies $p \leq (n)\ (\ell,1)$ by (1). Hence, $p \leq (n+1)\ 1$. Next we show that $p \not\leq (n+2)\ 1$ in T. Let $D = \{(i,1) \mid i \in \mathbb{N}\}$. Then $\sup D = 1$. But $x \not\leq (n+1)\ 1$ in S implies $p \not\leq (n+1)\ (i,1)$ for all $i \in \mathbb{N}$ by (2). Therefore, $p \not\leq (n+2)\ 1$ in T. ∎

Corollary 1.14. There exists a complete lattice S such that $\leq (n+1) \neq \leq (n)$ for all $n \in \mathbb{N}$.

Proof. According to 1.13 and 1.4, there is, for each $n \in \mathbb{N}$, a complete lattice S_n such that $\underset{S_n}{\leq}(n+1) \neq \underset{S_n}{\leq}(n)$. Stack such a collection almost any way you like (adding 0 and/or 1 if necessary) and a suitable example is obtained. For example, if we stack them up (with no identifications) and add a 1 to obtain T, then one can show that $\underset{T}{\leq}(n) \cap (S_i \times S_i) = \underset{S_i}{\leq}(n)$ for all $i,n \in \mathbb{N}$ from which it follows that $\underset{T}{\leq}(n+1) \neq \underset{T}{\leq}(n)$ for all $n \in \mathbb{N}$. ∎